The Properties of Equally Weighted Risk Contribution Portfolios

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Optimal portfolio construction, the process of efficiently allocating wealth among asset classes and securities, has a long-standing history in the academic literature. Over 50 years ago, Markowitz [1952, 1956] formalized the problem in a mean-variance framework where one assumes that the rational investor seeks to maximize the expected return for a given volatility level. While powerful and elegant, this solution is known to suffer from serious drawbacks in its practical implementation. First, optimal portfolios tend to be excessively concentrated in a limited subset of the full set of assets or securities. Second, the mean-variance solution is overly sensitive to the input parameters. Small changes in those parameters, most notably in expected returns (Merton [1980]), can lead to significant variations in the composition of the portfolio.

Alternative methods to deal with these issues have been suggested in the literature, such as portfolio resampling (Michaud [1989]) or robust asset allocation (Tütüncü and Koenig [2004]), but have their own disadvantages. On top of those shortcomings is the additional computational burden forced upon investors because of the need to compute solutions across a large set of scenarios. Moreover, it can be shown that these approaches can be restated as shrinkage estimator problems (Jorion [1986]) and that their out-of-sample performance is not superior to traditional approaches (Scherer [2007a, 2007b]). Looking at the marketplace, it also appears that a large fraction of investors prefers more heuristic solutions, which are computationally simple to implement and are presumed robust as they do not depend on expected returns.

Two well-known examples of such techniques are the minimum-variance portfolio and the equally weighted portfolio. The former is a specific portfolio on the mean-variance efficient frontier. Equity funds applying this principle have been launched in recent years. This portfolio is easy to compute because the solution is unique. As the only mean-variance efficient portfolio not incorporating information on the expected returns as a criterion, it is also recognized as robust; however, minimum-variance portfolios generally suffer from the drawback of portfolio concentration. A simple and natural way to resolve this issue is to attribute the same weight to all the assets considered for inclusion in the portfolio. Equally weighted or "$1/n$" portfolios are widely used in practice (Bernartzi and Thaler [2001] and Windcliff and Boyle [2004]) and they have been shown to be efficient out of sample (DeMiguel, Garlappi, and Uppal [2009]). In addition, if all assets have the same correlation coefficient as well as identical means and variances, the equally weighted portfolio is the unique portfolio on the efficient frontier. The drawback is that it can lead to a very limited diversification of risks if individual risks are significantly different.

In this article, we analyze another heuristic approach, which constitutes a middle-ground between minimum-variance and equally weighted portfolios. The idea is to equalize the risk contributions from the different components of the portfolio.[1] The risk contribution of a component i is the share of total portfolio risk attributable to that component and is computed as the product of the allocation to component i and its marginal risk contribution. The marginal risk contribution equals the change in the total risk of the portfolio induced by an infinitesimal increase in holdings of component i. Dealing with risk contributions has become standard practice for institutional investors, under the label of *risk budgeting*. Risk budgeting is the analysis of the portfolio in terms of risk contributions rather than in terms of portfolio weights. Qian [2006] showed that risk contributions are not solely a mere (ex ante) mathematical decomposition of risk, but they have financial significance as they can be good predictors of the contribution of each position to (ex post) losses, especially for losses of large magnitude. Equalizing risk contributions is also a standard practice for multi-strategy hedge funds, such as CTAs, although these funds generally ignore the effect of correlation among strategies (more precisely, they are implicitly making assumptions about homogeneity of the correlation structure).

Investigating the out-of-sample risk–reward properties of equally weighted risk contribution (ERC) portfolios is interesting because they mimic the diversification effect of equally weighted portfolios while taking into account single and joint risk contributions of the assets. In other words, no asset contributes more than its peers to the total risk of the portfolio. The minimum-variance portfolio also equalizes risk contributions, but only on a marginal basis. That is, for the minimum-variance portfolio, a small increase in any asset will lead to the same increase in the total risk of the portfolio (at least on an ex ante basis). Except in special cases, the total risk contributions of the various components will be far from equal, so that in practice the investor often concentrates risk in a limited number of positions, foregoing the benefit of diversification. It has been shown repeatedly that the diversification of risks can improve returns (Fernholtz, Garvy, and Hannon [1998] and Booth and Fama [1992]). Another rationale for ERC portfolios is based on optimality arguments. Lindberg [2009] showed that the solution to Markowitz's continuous-time portfolio problem is given, when positive drift rates are considered in Brownian motions governing stocks prices, by the equalization of quantities related to risk contributions.

The ERC approach is not new and has already been exposed in two recent articles by Neurich [2008] and Qian [2005], however, neither studied the global theoretical issues linked to the approach pursued here. Note that the Most-Diversified Portfolio (MDP) of Choueifaty and Coignard [2008] shares with the ERC portfolio a similar philosophy based on diversification. But the two portfolios are generally distinct, except when the correlation matrix components are the same. We also discuss the optimality of the ERC portfolio within the scope of the maximum Sharpe ratio (MSR) reinvestigated by Martellini [2008].

The structure of the article is as follows. We first define ERC portfolios and analyze their theoretical properties. We then compare the ERC with competing approaches and provide empirical illustrations. We end with some conclusions.

DEFINITION OF ERC PORTFOLIOS

In this section we define marginal contribution and total risk contribution and provide a specification of the ERC strategy.

Definition of Marginal Contribution and Total Risk Contribution

We consider a portfolio $x = (x_1, x_2, \ldots, x_n)$ of n risky assets. Let σ_i^2 be the variance of asset i, σ_{ij} be the covariance between assets i and j, and Σ be the covariance matrix. Let $\sigma(x) = \sqrt{x^\top \Sigma x} = \sqrt{\sum_i x_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} x_i x_j \sigma_{ij}}$ be the risk of the portfolio. The marginal risk contribution, $\partial_{x_i} \sigma(x)$, is defined as follows:

$$\partial_{x_i} \sigma(x) = \frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sum_{j \neq i} x_j \sigma_{ij}}{\sigma(x)}$$

The adjective *marginal* qualifies the fact that those quantities give the change in volatility of the portfolio induced by a small increase in the weight of one component. If $\sigma_i(x) = x_i \times \partial_{x_i} \sigma(x)$, the total risk contribution of the ith asset, then one obtains the following decomposition:[2]

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i(x)$$

thus, the risk of the portfolio can be viewed as the sum of the total risk contributions.[3]

Specification of the ERC Strategy

Starting from the definition of the risk contribution, $\sigma_i(x)$, the idea of the ERC strategy is to find a risk-balanced portfolio such that the risk contribution is the same for all assets in the portfolio. We voluntarily restrict ourselves to cases without short selling, that is, $\mathbf{0} \leq x \leq \mathbf{1}$, because most investors cannot take short positions. Moreover, since our goal is to compare the ERC portfolio with other heuristic approaches, it is important to maintain similar constraints for all solutions in order to be fair. Indeed, by construction, the $1/n$ portfolio satisfies the positive weight constraint and it is well known that constrained portfolios are less optimal than unconstrained ones (Clarke, de Silva, and Thorley [2002]). Mathematically, the problem can be written as follows:

$$x^* = \left\{ x \in [0,1]^n : \Sigma x_i = 1, \; x_i \times \partial_{x_i} \sigma(x) = x_j \times \partial_{x_j} \sigma(x) \text{ for all } i,j \right\} \tag{1}$$

Using endnote 3 and noting that $\partial_{x_i} \sigma(x) \propto (\Sigma x)_i$, the problem then becomes

$$x^* = \left\{ x \in [0,1]^n : \Sigma x_i = 1, \; x_i \times (\Sigma x)_i = x_j \times (\Sigma x)_j \text{ for all } i,j \right\} \tag{2}$$

where $(\Sigma x)_i$ denotes the ith row of the vector issued from the product of Σ with x.

Note that the budget constraint $\Sigma x_i = 1$ is only acting as a normalization constraint. In particular, if the portfolio y is such that $y_i \times \partial_{y_i} \sigma(y) = y_j \times \partial_{y_j} \sigma(y)$ with $y_i \geq 0$ but $\Sigma y_i \neq 1$, then the portfolio x defined by $x_i = y_i / \Sigma_{i=1}^n y_i$ is the ERC portfolio.

THEORETICAL PROPERTIES OF ERC PORTFOLIOS

In this section we analyze the ERC portfolio in the two-asset case and the general case. We offer two numerical solutions to the optimization problem and also compare the ERC portfolio with $1/n$ and minimum variance portfolios.

The Two-Asset Case

We begin by analyzing the ERC portfolio in the bivariate case. Let ρ be the correlation and $x = (w, 1 - w)$ be the vector of weights. The vector of total risk contributions is

$$\frac{1}{\sigma(x)} \begin{pmatrix} w^2\sigma_1^2 + w(1-w)\rho\sigma_1\sigma_2 \\ (1-w)^2\sigma_2^2 + w(1-w)\rho\sigma_1\sigma_2 \end{pmatrix}$$

In this case, finding the ERC portfolio means finding w such that both rows are equal, that is, w verifying $w^2\sigma_1^2 = (1-w)^2\sigma_2^2$. The unique solution satisfying $0 \leq w \leq 1$ is

$$x^* = \left(\frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}}, \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right)$$

Note that the solution does not depend on the correlation ρ.

The General Case

In the more general case, where $n > 2$, the number of parameters increases quickly, with n individual volatilities and $n(n - 1)/2$ bivariate correlations.

Let us begin with a particular case where a simple analytic solution can be provided. Assume that we have equal correlations for every couple of variables, that is, $\rho_{i,j} = \rho$ for all i, j. The total risk contribution of component i thus becomes $\sigma_i(x) = (x_i^2\sigma_i^2 + \rho\Sigma_{j\neq i} x_i x_j \sigma_i \sigma_j)/\sigma(x)$, which can be written as $\sigma_i(x) = x_i\sigma_i((1-\rho)x_i\sigma_i + \rho\Sigma_j x_j\sigma_j)/\sigma(x)$. With the ERC portfolio defined by $\sigma_i(x) = \sigma_j(x)$ for all i, j, simple algebra shows that this is equivalent to $x_i\sigma_i = x_j\sigma_j$.[4] Coupled with the (normalizing) budget constraint, $\Sigma_i x_i = 1$, we deduce that

$$x_i = \frac{\sigma_i^{-1}}{\Sigma_{j=1}^n \sigma_j^{-1}} \tag{3}$$

The weight allocated to each component i is given by the ratio of the inverse of its volatility with the harmonic average of the volatilities. The higher (lower) the volatility of a component, the lower (higher) its weight in the ERC portfolio.

In other cases, it is not possible to find explicit solutions of the ERC portfolio. Let us, for example, analyze the case in which all volatilities are equal, $\sigma_i = \sigma$ for all i, but correlations differ. By the same line of reasoning as in the case of constant correlation, we deduce that

$$x_i = \frac{\left(\sum_{k=1}^{n} x_k \rho_{ik}\right)^{-1}}{\sum_{j=1}^{n}\left(\sum_{k=1}^{n} x_k \rho_{jk}\right)^{-1}} \tag{4}$$

The weight attributed to component i is equal to the ratio between the inverse of the weighted average of correlations of component i with other components and the same average across all the components. Notice that contrary to the bivariate case and to the case of constant correlation, for higher-order problems, the solution is endogenous because x_i is a function of itself, both directly and through the constraint that $\Sigma_i x_i = 1$. The same issue of endogeneity naturally arises in the general case where both the volatilities and the correlations differ. Starting from the definition of the covariance of the returns of component i with the returns of the aggregated portfolio, $\sigma_{ix} = \mathrm{cov}(r_i, \Sigma_j x_j r_j) = \Sigma_j x_j \sigma_{ij}$, we have $\sigma_i(x) = x_i \sigma_{ix}/\sigma(x)$. Now, let us introduce the beta, β_i, of component i with the portfolio. By definition, we have $\beta_i = \sigma_{ix}/\sigma^2(x)$ and $\sigma_i(x) = x_i \beta_i \sigma(x)$. As the ERC portfolio is defined by $\sigma_i(x) = \sigma_j(x) = \sigma(x)/n$ for all i, j, it follows that

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{n} \beta_j^{-1}} = \frac{\beta_i^{-1}}{n} \tag{5}$$

The weight attributed to component i is inversely proportional to its beta. The higher (lower) the beta, the lower (higher) the weight, which means that components with high volatility or high correlation with other assets will be penalized. Recall that this solution is endogenous because x_i is a function of the component beta, β_i, which, by definition, depends on the portfolio x.

Numerical Solutions

Although the previous Equations (4) and (5) allow for an interpretation of the ERC solution in terms of the relative risk of an asset compared to the rest of the portfolio, because of the endogeneity of the program, they do not offer a closed-form solution. Finding a solution thus requires the use of a numerical algorithm.

One approach is to solve the following optimization problem using a sequential quadratic programming (SQP) algorithm,

$$\begin{aligned} x^* &= \arg\min f(x) \\ \text{u.c. } &\mathbf{1}^\top x = 1 \text{ and } \mathbf{0} \leq x \leq \mathbf{1} \end{aligned} \tag{6}$$

where

$$f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i(\Sigma x)_i - x_j(\Sigma x)_j\right)^2$$

The existence of the ERC portfolio is ensured only when the condition $f(x^*) = 0$ is verified (i.e., $x_i(\Sigma x)_i = x_j(\Sigma x)_j$ for all i, j). Basically, the program minimizes the variance of the (rescaled) risk contributions.

An alternative to the previous algorithm is to consider the following optimization problem:

$$\begin{aligned} y^* &= \arg\min \sqrt{y^\top \Sigma y} \\ \text{u.c. } &\begin{cases} \sum_{i=1}^{n} \ln y_i \geq c \\ y \geq 0 \end{cases} \end{aligned} \tag{7}$$

with c an arbitrary constant. In this case, the program is similar to a variance minimization problem subject to a constraint of sufficient diversification of weights (as implied by the first constraint), an issue to which we will return later. This problem may be solved using SQP. The ERC portfolio is expressed as $x_i^* = y_i^*/\Sigma_{i=1}^{n} y_i^*$ (see Appendix A).

We recommend the first optimization problem, which is easier to solve numerically, because it does not incorporate a nonlinear inequality constraint, however we were able to find examples for which numerical optimization is tricky. If a numerical solution for the optimization problem (6) is not found, we recommend slightly modifying this problem as follows: $y^* = \arg\min f(y)$ with $y \geq 0$ and $\mathbf{1}^\top y \geq c$ with c an arbitrary positive scalar. In this case, the ERC portfolio is $x_i^* = y_i^*/\Sigma_{i=1}^{n} y_i^*$ for $f(y^*) = 0$. This new optimization problem is easier to solve numerically than problem (6) because the inequality constraint $\mathbf{1}^\top y \geq c$ is less restrictive than the equality constraint $\mathbf{1}^\top x = 1$. The formulation in system (7) has the advantage of showing that the ERC solution is unique when the covariance matrix Σ is positive-definite. Indeed, system (7) is defining the minimization program of a quadratic function (a convex function) with a lower bound (also a convex function). Finally, when the long-only constraint is relaxed, various solutions satisfying the ERC condition can be obtained.

Comparison with 1/*n* and Minimum-Variance Portfolios

As stated in the introduction, $1/n$ and minimum-variance (MV) portfolios are widely used in practice.

ERC portfolios are naturally located between both and thus appear as good potential substitutes for these traditional approaches.

In the two-asset case, the three portfolios ($1/n$, MV, and ERC) are identical when the two assets have the same volatility. In the general n-assets context, and with a unique correlation, the ERC portfolio coincides with the $1/n$ portfolio when all volatilities are the same. Moreover, we can show that the ERC portfolio corresponds to the MV portfolio when cross-diversification is the highest, that is, when the correlation matrix reaches its lowest possible value.[5] This result suggests that the ERC strategy produces portfolios with robust risk-balanced properties.

Let us skip now to the general case. If we sum up the situation from the point of view of the mathematical definitions of these portfolios, we have the following.

$$
\begin{aligned}
& x_i = x_j && (1/n) \\
& \partial x_i\, \sigma(x) = \partial x_j\, \sigma(x) && (\text{mv}) \\
& x_i \partial x_i\, \sigma(x) = x_j \partial x_j\, \sigma(x) && (\text{erc})
\end{aligned}
$$

We use the fact that MV portfolios are equalizing marginal contributions to risk (Scherer [2007b]). Thus, ERC portfolios may be viewed as a portfolio located between the $1/n$ and MV portfolios. To elaborate further on this point of view, let us consider a modified version of the optimization problem (7):

$$
x^*(c) = \arg\min \sqrt{x^\top \Sigma x}
$$

$$
\text{u.c.} \begin{cases} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}^\top x = \mathbf{1} \\ x \geq \mathbf{0} \end{cases} \tag{8}
$$

In order to get the ERC portfolio, the volatility of the portfolio is minimized subject to an additional constraint $\sum_{i=1}^{n} \ln x_i \geq c$ where c is a constant being determined by the ERC portfolio. The constant c can be interpreted as the minimum level of diversification among components that is necessary in order to get the ERC portfolio.[6] Two polar cases can be defined with $c = -\infty$ to generate the MV portfolio and $c = -n \ln n$ to generate the $1/n$ portfolio. In particular, the quantity $\Sigma \ln x_i$, subject to $\Sigma x_i = 1$, is maximized for $x_i = 1/n$ for all i. This reinforces the interpretation of the ERC portfolio as an intermediary portfolio between the MV and $1/n$ portfolios, that is, a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of component weights. Finally, starting from this new optimization program, we show in Appendix B that volatilities are ordered in the following way:

$$
\sigma_{\text{mv}} \leq \sigma_{\text{erc}} \leq \sigma_{1/n}
$$

This means that we have a natural order of the volatilities of the portfolios with the MV being, unsurprisingly, the less volatile, the $1/n$ being the more volatile, and the ERC located between them.

Optimality

Now we investigate when the ERC portfolio corresponds to the maximum Sharpe ratio (MSR) portfolio, also known as the tangency portfolio in portfolio theory, whose composition is equal to $\frac{\Sigma^{-1}(\mu - r)}{\mathbf{1}^\top \Sigma^{-1}(\mu - r)}$ where μ is the vector of expected returns and r is the risk-free rate (Martellini [2008]). Scherer [2007b] showed that the MSR portfolio is defined as the one in which the ratio of the marginal excess return to the marginal risk is the same for all assets constituting the portfolio and equals the Sharpe ratio of the portfolio,

$$
\frac{\mu(x) - r}{\sigma(x)} = \frac{\partial_x \mu(x) - r}{\partial_x \sigma(x)}
$$

We deduce that the portfolio x is the MSR portfolio if it verifies the following relationship:[7]

$$
\mu - r = \left(\frac{\mu(x) - r}{\sigma(x)} \right) \frac{\Sigma x}{\sigma(x)}
$$

We can show that the ERC portfolio is optimal if we assume a constant correlation matrix and that all assets have the same Sharpe ratio. Indeed, with the constant correlation coefficient assumption, the total risk contribution of component i is equal to $(\Sigma x)_i / \sigma(x)$. By definition, this risk contribution will be the same for all assets. In order to verify the previous condition, it is sufficient for each asset to have the same individual Sharpe ratio, $s_i = \frac{\mu_i - r}{\sigma_i}$. But when correlations differ or when assets' Sharpe ratios differ, the ERC portfolio will be different from the MSR portfolio.

ILLUSTRATIONS

A Numerical Example

We consider a universe of four risky assets. Volatilities are respectively 10%, 20%, 30%, and 40%. We first consider a constant correlation matrix. In the case of the $1/n$ strategy, the weights are 25% for all the assets. The solution for the ERC portfolio is 48%, 24%, 16%, and 12%, respectively. The solution for the MV portfolio depends on the correlation coefficient. With a correlation of 50%, the solution is $x_1^{\text{mv}} = 100\%$. With a correlation of 30%, the solution becomes $x_1^{\text{mv}} = 89.5\%$ and $x_2^{\text{mv}} = 10.5\%$. When the correlation is 0%, we get $x_1^{\text{mv}} = 70.2\%$, $x_2^{\text{mv}} = 17.6\%$, $x_3^{\text{mv}} = 7.8\%$ and $x_4^{\text{mv}} = 4.4\%$. Needless to say, the ERC portfolio is a portfolio more balanced in terms of weights than the MV portfolio.

Next, we consider the following correlation matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.00 & 0.00 & 1.00 & \\ 0.00 & 0.00 & -0.50 & 1.00 \end{pmatrix}$$

We have the following results:

- The solution for the $1/n$ rule is

$\sigma(x) = 11.5\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	25%	0.056	0.014	12.3%
2	25%	0.122	0.030	26.4%
3	25%	0.065	0.016	14.1%
4	25%	0.217	0.054	47.2%

$c_i(x) = \sigma_i(x)/\sigma(x)$ is the risk contribution ratio. We check that the volatility is the sum of the four risk contributions $\sigma_i(x)$,

$$\sigma(x) = 0.014 + 0.030 + 0.016 + 0.054 = 11.5\%$$

Even if the third asset has a high volatility of 30%, it has only a small marginal contribution to risk because of the diversification effect; it has a zero correlation with the first two assets and is negatively correlated with the fourth asset. The two main risk contributors are the second and fourth assets.

- The solution for the minimum-variance portfolio is

$\sigma(x) = 8.6\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	74.5%	0.086	0.064	74.5%
2	0%	0.138	0.000	0%
3	15.2%	0.086	0.013	15.2%
4	10.3%	0.086	0.009	10.3%

The marginal contributions of risk are all equal except for the zero weights. This explains that we have the property $c_i(x) = x_i$, meaning that the risk contribution ratio is fixed by the weight. This strategy has a smaller volatility than the $1/n$ strategy, but the portfolio is concentrated in the first asset, in terms of both weights and risk contributions (74.5%).

- The solution for the ERC portfolio is

$\sigma(x) = 10.3\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	38.4%	0.067	0.026	25%
2	19.2%	0.134	0.026	25%
3	24.3%	0.106	0.026	25%
4	18.2%	0.141	0.026	25%

Contrary to the minimum-variance portfolio, the ERC portfolio is invested in all assets, and its volatility is higher than the volatility of the MV, but lower than the volatility of the $1/n$ strategy. The weights are ranked in the same order for the ERC and MV portfolios, but the ERC portfolio is obviously more balanced in terms of risk contribution.

Empirical Simulations

We consider an empirical study with light agricultural commodities, which are listed in Exhibit 1. Descriptive statistics, computed over the period January 1979 to March 2008, are displayed in Exhibit 1. Typically, there is large heterogeneity in volatilities and similarity in correlation coefficients around low levels (0%–10%). We compare the three strategies: $1/n$, MV, and ERC. We build the historical simulations using a rolling sample approach by rebalancing the portfolios on the last trading day of the month. For the MV and ERC portfolios, we estimate the covariance matrix using daily returns and a rolling window period of one year.

EXHIBIT 1
Descriptive Statistics of the Agricultural Commodities Portfolio

	Annualized Return	Annualized Volatility	Correlation Matrix (%)							
CC	4.5%	21.4%	100	2.7	4.2	61.8	51.6	13.9	4.6	9.3
CLC	17.2%	14.8%		100	31.0	4.5	3.5	2.5	0.8	3.7
CLH	14.4%	22.6%			100	7.0	5.9	5.0	-0.7	3.1
CS	10.5%	21.8%				100	42.8	16.2	6.3	10.4
CW	5.1%	23.7%					100	10.9	5.6	7.9
NCT	3.6%	23.2%						100	3.4	7.3
NKC	4.2%	36.5%							100	6.6
NSB	-5.0%	43.8%								100

Note: The list of commodities comprises (acronyms into brackets) Corn (CC), Live Cattle (CLC), Lean Hogs (CLH), Soybeans (CS), Wheat (CW), Cotton (NCT), Coffee (NKC), and Sugar (NSB).

Historical simulation results are summarized in Exhibit 2. For each strategy, we compute the compound annual return, volatility, and corresponding Sharpe ratio, using the fed funds rate as the risk-free rate. We indicate the 1% Value at Risk and the drawdown for the three holding periods: one day, one week, and one month. The maximum drawdown is also reported. We also compute statistics measuring concentration, namely, the Herfindahl and Gini indices, and turnover. In Exhibit 2 we present the average values of these concentration statistics for both the weights (denoted as $\bar{H}_w$ and $\bar{G}_w$, respectively) and the risk contributions (denoted as $\bar{H}_\kappa$ and $\bar{G}_\kappa$, respectively). We indicate the average values of turnover across time. In general, we prefer low values of concentration and turnover.

When compared with the $1/n$ portfolio, the ERC portfolio dominates in terms of both returns and risk. The hierarchy of results is confirmed by the Value at Risk and drawdowns. While the ERC portfolio presents some concentration in terms of weights (see $\bar{H}_w$ and $\bar{G}_w$ statistics), the $1/n$ competitor is more concentrated in terms of risk contributions ($\bar{H}_\kappa$ and $\bar{G}_\kappa$). In terms of turnover, the ERC portfolio is higher, but reasonable, as less than 2% of the portfolio is modified each month. Compared to the MV portfolio, the ERC portfolio is dominated in terms of average return and volatility. We observe, however, that short-term drawdowns can be higher for the MV portfolio. But the major advantage of the ERC portfolio when compared with the MV portfolio is the stabilizing effect of the implied diversification constraint of the ERC optimization problem (8). The MV portfolio, by contrast, posts huge concentration (both in weights and risk contributions) in the less volatile components. So the MV portfolio composition can be drastically modified when the relative risk of components changes significantly resulting in higher turnover. Furthermore, the outperformance of the MV portfolio is largely due to one asset (live cattle) having both the lowest risk and the highest return.

The box plot graphs in Exhibit 3 represent the historical distribution of the weights (Panel A) and risk contributions (Panel B) for the three strategies. Though the $1/n$ portfolio is by definition balanced in weights, it is not balanced in terms of risk contributions. For instance, a large part of the $1/n$ portfolio risk is explained by the sugar (NSB) component, the most volatile commodity over the period. In contrast, the MV portfolio concentrates weights and risk in the less volatile commodities. On average, roughly 40% of the MV portfolio is invested in live cattle (CLC), while sugar (NSB) accounts for barely 5% of the portfolio risk. The ERC portfolio appears to be a middle-ground alternative balanced in risk and in weights.

Our experience with other empirical simulations leads to the same general conclusions:[8] the ordering of the realized volatility remains the same as the ex ante volatility, meaning that ERC volatility is located between the volatility of the minimum-variance and $1/n$ portfolios; the Sharpe ratio of the ERC portfolio is competitive; and the turnover of the ERC portfolio is much lower than the turnover of the MV portfolio.

EXHIBIT 2

Historical Simulation Statistics of the Three Strategies for the Agricultural Commodities Portfolio

	1/*n*	MV	ERC
Annualized Return	10.2%	14.3%	12.1%
Annualized Volatility	12.4%	10.0%	10.7%
Sharpe Ratio	0.27	0.74	0.49
VaR 1D 1%	-1.97%	-1.58%	-1.64%
VaR 1W 1%	-4.05%	-3.53%	-3.72%
VaR 1M 1%	-7.93%	-6.73%	-7.41%
Drawdown 1D	-5.02%	-4.40%	-3.93%
Drawdown 1W	-8.52%	-8.71%	-7.38%
Drawdown 1M	-11.8%	-15.1%	-12.3%
Maximum Drawdown	-44.1%	-30.8%	-36.9%
$\bar{H}_w$	0.00%	14.7%	2.17%
$\bar{G}_w$	0.00%	48.1%	19.4%
Average Turnover	0.00%	4.90%	1.86%
$\bar{H}_{rc}$	6.32%	14.7%	0.00%
$\bar{G}_{rc}$	31.3%	48.1%	0.00%

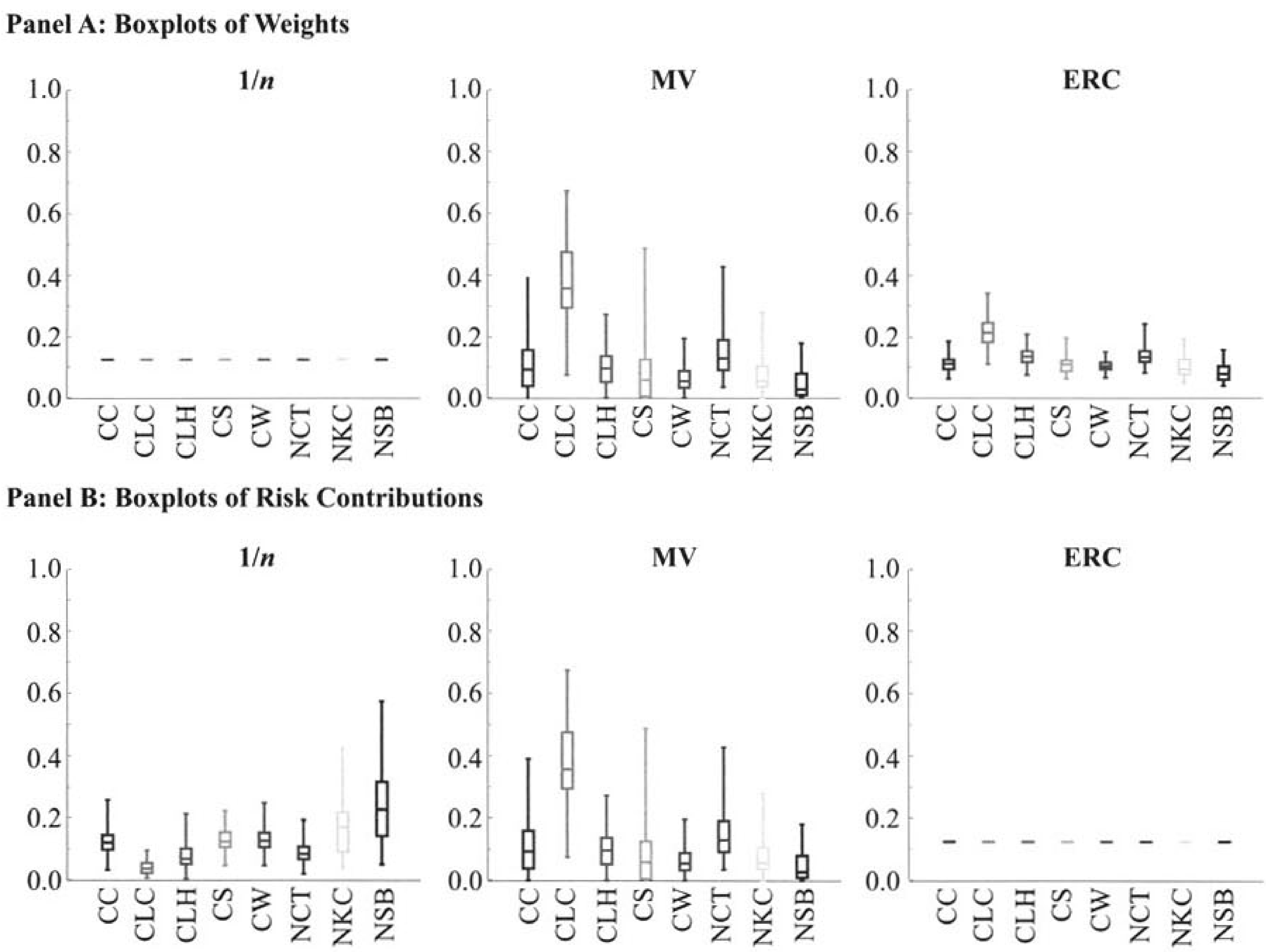
CONCLUSION

A perceived lack of robustness or discomfort with empirical results has led investors to be increasingly skeptical of traditional asset allocation methodologies that incorporate expected returns. From this perspective, emphasis has been put on minimum-variance (i.e., the unique mean-variance efficient portfolio independent of return expectations) and equally weighted (1/*n*) portfolios. Despite their robustness, both approaches have their own limitations—a lack of risk monitoring for 1/*n* portfolios and a dramatic asset concentration for minimum-variance portfolios.

We analyze an alternative approach based on equalizing the risk contribution from the various components of the portfolio. In this way, we try to maximize the dispersion of risk, applying a kind of 1/*n* filter in terms of risk. Our approach constitutes a special form of risk budgeting where the asset allocator is distributing the same risk budget to each component, so that none dominates (at least on an ex ante basis). This middle-ground positioning

EXHIBIT 3

Weights and Risk Contributions of the Three Strategies for the Agricultural Commodities Portfolio



is particularly clear from the hierarchy of volatilities. We derived closed-form solutions for special cases, such as when a unique correlation coefficient is shared by all assets; however, numerical optimization is necessary in most cases due to the endogeneity of the solutions. Determining the ERC solution for a large portfolio might be a computationally intensive task, something to keep in mind when compared with the minimum-variance portfolio and, even more, with the $1/n$ competitors. Empirical applications show that equally weighted portfolios are inferior in terms of performance and for any measure of risk. Minimum-variance portfolios might achieve higher Sharpe ratios due to lower volatility, but they can be exposed to higher drawdowns in the short run. They are also always much more concentrated and appear largely less efficient in terms of portfolio turnover.

Empirical applications could be pursued in various ways. One of the most promising would be comparing the behavior of equally weighted risk contribution portfolios with other weighting methods used by major stock indices. For instance, in the case of the S&P 500 Index, competing methodologies are already commercialized, such as capitalization-weighted, equally weighted, fundamentally weighted (Arnott, Hsu, and Moore [2005]), and minimum-variance-weighted (Clarke, de Silva, and Thorley [2002]) portfolios. The way an ERC portfolio would compare with these approaches for this type of equity index remains an interesting, open question.

APPENDIX A

The Relationship between the Optimization Problem (7) and the ERC Portfolio

We consider the optimization problem (7) considered in the text,

$$y^* = \arg\min \sqrt{y^\top \Sigma y}$$
$$\text{u.c.} \begin{cases} \sum_{i=1}^{n} \ln y_i \geq c \\ y \geq 0 \end{cases}$$

The Lagrangian function of this optimization problem is

$$f(y;\lambda,\lambda_c) = \sqrt{y^\top \Sigma y} - \lambda^\top y - \lambda_c \left(\sum_{i=1}^{n} \ln y_i - c \right)$$

The solution y^* verifies the following first-order condition:

$$\partial_{y_i}(y;\lambda,\lambda_c) = \partial_{y_i}\sigma(y) - \lambda_i - \lambda_c y_i^{-1} = 0$$

and the following Kuhn–Tucker conditions:

$$\begin{cases} \min(\lambda_i, y_i) = 0 \\ \min\left(\lambda_c, \sum_{i=1}^{n} \ln y_i - c\right) = 0 \end{cases}$$

Because $\ln y_i$ is not defined for $y_i = 0$, then $y_i > 0$ and $\lambda_i = 0$. We notice that the constraint $\Sigma_{i=1}^{n} \ln y_i = c$ is necessarily reached (because the solution cannot be $y^* = 0$), then $\lambda_c > 0$ and we have

$$y_i \frac{\partial\sigma(y)}{\partial y_i} = \lambda_c$$

We verify that risk contributions are the same for all assets. Moreover, we face a well-known optimization problem (minimizing a quadratic function subject to lower convex bounds), which has a solution. We then deduce the ERC portfolio by normalizing the solution y^* such that the sum of weights equals one. Notice that the solution x^* may be found directly from the optimization problem (8) by using a constant $c^* = c - n\ln(\Sigma_{i=1}^{n} y_i^*)$, where c is the constant used to find y^*.

APPENDIX B

The Relationship among σ_{erc}, $\sigma_{1/n}$, and σ_{mv}

Let us start with the optimization problem (8) considered in the text,

$$x^*(c) = \arg\min \sqrt{x^\top \Sigma x}$$
$$\text{u.c.} \begin{cases} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}^\top x = 1 \\ \mathbf{0} \leq x \leq \mathbf{1} \end{cases}$$

If $c_1 \leq c_2$, we have $\sigma(x^*(c_1)) \leq \sigma(x^*(c_2))$ because the constraint $\Sigma_{i=1}^{n} \ln x_i - c \geq 0$ is less restrictive with c_1 than with c_2. We notice that if $c = -\infty$, the optimization problem is exactly the MV problem, and $x^*(-\infty)$ is the MV portfolio. Because of the Jensen inequality and the constraint $\Sigma_{i=1}^{n} x_i = 1$, we have $\Sigma_{i=1}^{n} \ln x_i \leq -n\ln n$. The only solution for $c = -n \ln n$ is $x_i^* = 1/n$, which is the $1/n$ portfolio. Thus, the solution for the general problem with $c \in [-\infty, -n \ln n]$ satisfies

$$\sigma(x^*(-\infty)) \leq \sigma(x^*(c)) \leq \sigma(x^*(-n\ln n))$$

or

$$\sigma_{mv} \leq \sigma(x^*(c)) \leq \sigma_{1/n}$$

Using the result of Appendix A, there exists a constant c^* such that $x^*(c^*)$ is the ERC portfolio. It proves that the following inequality holds:

$$\sigma_{\mathrm{mv}} \leq \sigma_{\mathrm{erc}} \leq \sigma_{1/n}$$

ENDNOTES

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[1]We have restricted ourselves to the volatility of the portfolio as the risk measure, but the ERC principle can be applied to other risk measures as well. Theoretically, it is only necessary that the risk measure is linear-homogeneous in the weights in order for the total risk of the portfolio to be fully decomposed into its components. Under some hypotheses, this is the case for Value at Risk (Hallerback [2003]).

[2]The volatility σ is a homogeneous function of degree 1. It thus satisfies Euler's theorem and can be reduced to the sum of its arguments multiplied by their first partial derivatives.

[3]In vector form, noting Σ the covariance matrix of asset returns, the n marginal risk contributions are computed as $\frac{\Sigma x}{\sqrt{x^\top \Sigma x}}$. We verify that $x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sqrt{x^\top \Sigma x} = \sigma(x)$.

[4]We use the fact that the constant correlation verifies $\rho \geq -\frac{1}{n-1}$.

[5]Proof of this result may be found in the working paper version of this article, which is available from the authors upon request.

[6]In statistics, the quantity $-\Sigma x_i \ln x_i$ is known as the entropy. For the analysis of portfolio constructions using the maximum entropy principle, see Bera and Park [2008]. Notice however that the issue studied here is more specific.

[7]Because we have $\mu(x) = x^\top \mu, \sigma(x) = \sqrt{x^\top \Sigma x},\ \partial_x \mu(x) = \mu$, and $\partial_x \sigma(x) = \Sigma x / \sigma(x)$.

[8]See, for instance, the working paper version of this article where two additional historical simulations are presented, one on equity sector indices and the other on a multi-asset portfolio.

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