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Dynamic user-optimal assignment in continuous time and space ☆

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Abstract

To support planning, design, and management of pedestrian infrastructure, dynamic assignment models are useful tools. However, current models are network-based and presume that travelers can choose between a finite number of discrete route alternatives. For walking facilities, where pedestrians can choose their paths freely in two-dimensional space, applicability of these traditional network models is limited.

This article puts forward an approach for user-optimal dynamic assignment in continuous time and space. Contrary to network-based approaches, the theory allows the traffic units to choose from an infinite non-countable set of paths through the considered space. The approach consists of three interrelated steps, that is: determining the continuous paths using a path choice model, assigning the origin–destination flows, and calculating the resulting traffic conditions. The approach to determine a user-optimal assignment is heuristic and consists of a sequence of all-or-nothing assignments in continuous time-space. The article presents the mathematical problem formulation, solution approaches, and application examples.

Keywords: Dynamic assignment; Path-choice models; Pedestrian traffic; Continuous space models

1. Introduction

Traffic assignment models have been used for many years to support the planning, design and management of traffic networks. However, these models can only be used for discrete networks

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where the number of route alternatives is countable and finite. There are however instances where traffic units can move freely in continuous two-dimensional space. Typical examples are walking facilities, container terminals, and waterways. In dealing with the design of walking infrastructures such as public transport transfer stations, shopping malls, stadiums, etc., available analysis models are not generally applicable since pedestrians can move freely through the infrastructure, at least to a certain extent. Because of this, researchers such as Gipps (1986), and Hamacher and Tjandra (2001) describe pedestrian route choice through walking facilities by determining a finite number of routes through the walking infrastructure and applying basic *discrete choice modeling*.

Notwithstanding the practicality of assuming only a limited number of route alternatives, in real life pedestrians can choose between *an infinite* (*and in fact, non-countable*) *number of paths in the given space*. Using potential functions, Hughes (2002) accounts for this aspect, by describing the optimal walking direction to the destination (in terms of travel time) as a function of the current location x of the pedestrian. However, the approach prohibits including general path attributes, such as traveled distance, number of sharp turns, or stimulation of the environment, as well as uncertainty in the traffic conditions expected by the travelers.

Beckmann and Puu (1985) and Puu and Beckmann (1999) considered continuous space modeling (only static case); Yang et al. (1994) and Yang and Wong (2000) extended and applied the continuous space equilibrium approach for traffic assignment and determination of market areas of competitive facilities. Hughes (2002) uses a similar approach to model pedestrians flows. The concepts presented by the authors are similar to the dynamic concepts developed in Hughes' article.

The research described in this article has succeeded in remedying the issues mentioned in the introduction, while at the same time establishing a theoretical basis for continuous-space user-equilibrium path-choice. It puts forward a new theory for the dynamic assignment problem in continuous time and space, where traffic units (e.g. pedestrians) cannot improve their *experienced utility* (e.g. experienced or actual travel times instead of instantaneous travel times) by unilaterally changing their path choice. It goes without saying that the continuous approach is very different from the discrete approach. For one, traffic space is not represented as a directed graph consisting of nodes and links, but rather as a *continuous plane*. Origins (or *sources*) and destinations (or *sinks*) are *described by arbitrary areas* instead of points. Furthermore, the number of decision points in the continuous space is *infinite*, implying that individual traffic units can change their path at each location in the area. In fact, the number of path alternatives between an origin and a destination is infinite (and non-countable). It is emphasized that in this article only the case of inelastic demands is considered while only (dynamic) path choice is considered, that is departure time choice and activity scheduling are neglected for simplicity sake. In addition, uncertainty is not considered in this article. However, the proposed theory allows extending this deterministic inelastic case to more general conditions (Hoogendoorn and Bovy, 2004).

The new assignment model enables predicting future traffic conditions in the continuous infrastructure facility, facilitates comparisons of scenarios reflecting changes in the facility design, or to predict the effects of traffic management measures (e.g. signposting, control strategies for bi-directional escalators, pedestrian traffic lights, etc.). The theory and models proposed here provide an alternative to traditional discrete network assignment models considering user-equilibrium path choice through the infrastructure facilities: the outcomes of the model are time-dependent variables, including travelers' equilibrium path choices, equilibrium flows, and travel costs.

Although primarily developed for analyzing pedestrian flows, it is emphasized that the approach is not restricted to pedestrian traffic: travelers can also represent other types of traffic units moving in continuous space such as ships, cyclists, automated guided vehicles, cars in a parking lot, etc.

The article is outlined as follows. Section 2 introduces the notation and definitions used in the remainder of the paper. Section 3 presents the mathematical problem formulation of deterministic dynamic path equilibrium in continuous time and space. In Section 4, we introduce the mathematical model (path choice and infrastructure (network) loading). Sections 5 and 6 respectively consider the numerical solution approach and an application example. Section 7 provides the overall conclusions from the research presented here.

2. Notation and definitions

Before discussing the problem definition, and the mathematical and numerical solution approach, let us first present the notation and definitions used in the ensuing of the article.

2.1. Description of infrastructure

Infrastructure is described by an area $\Omega \subset \mathbb{R}^2$ in which the travelers move. The travelers enter the infrastructure at the origin areas $O_i \subset \Omega$, and leave via the destination areas $D_j \subset \Omega$. Both the origin and destination areas are described by closed sets. Note that travelers may use any point in the destination area to exit the facility. We assume that the time the traveler enters the facility is fixed. Within the infrastructure, obstacles $B_m \subset \Omega$ may be present. These obstacles reflect physical obstructions for the travelers, which is the reason why travelers will have to move around them while traveling to their destination.

2.2. Trajectories, velocities along trajectories, and paths

A *feasible trajectory* is any possible movement through continuous time and space, mathematically defined by a parameterized curve

$$x_{[t,T)} = \{x(s) \in \Omega | t \leqslant s \leqslant T,\ x(s) \notin B_m\} \tag{1}$$

where t denotes the departure time and T denotes the terminal time. Considering a traveler going from origin $O_i \subset \Omega$ to destination $D_j \subset \Omega$, we would have $x(t) \in O_i$ if t is the departure time. The final position $x(T)$ may either be in the destination area D_j or not. In the former case, T is the arrival time of the traveler at the destination area. In the latter case, T is the end of the planning period, or the time the destination area is not available anymore (e.g. departure time of a train).

At this point, we note that the approach described in this article is *destination-oriented*, which implies that behavior of travelers can be described by the location $x(t)$ and their destination, and is thus independent of the origin. Furthermore, sub-paths of optimal paths will turn out to be optimal as well. In solving the problem, we generally consider the optimal path of a traveler that has (somehow) arrived at some location $x(t')$ for $t' \geqslant t$ to a destination D_j.

A requirement is that a trajectory is a differentiable function of t. In other words, the derivative of x to t exists and is finite. Note that a trajectory is not necessarily loop-less: under specific circumstances, a traveler may decide to come back to a location visited at an earlier time. Consider for instance a traveler waiting for a while at a certain location that provides benefits in terms of comfort, shelter, etc. If the traveler can wait there, while still reaching his/her destination in time, loops in the trajectory are possible. This can occur since the model does not consider shortest paths in terms of travel time per se, but uses generalized cost instead.

Rather than the trajectories, the velocities $v_{[t,T)}$ along the trajectories $x_{[t,T)}$ will be used as the main decision variable of the travelers, for mathematical convenience only. These *velocity trajectories* are defined by

$$v_{[t,T)} = \{v(s) \in \Gamma | t \leqslant s < T\} \tag{2}$$

where Γ denotes the set of admissible velocities. Note that the *velocity* $v = eV \in \mathbb{R}^2$ of a traffic unit in the continuous case describes both its *speed* $V \in \mathbb{R}$ as well as its *(unit) direction* $e \in \mathbb{R}^2$, $|e| = 1$. The set of admissible velocities describes the constraints both caused by the infrastructure (travelers cannot walk into an obstacle, or in the direction opposite in the moving direction of an escalator), and by the flow conditions (traveler speed is less or equal to a density-dependent speed limit). In the remainder we assume that the traveler chooses the subjectively optimal velocity path $v^*_{[t,T)}$. Clearly, the trajectory $x_{[t,T)}$ is determined uniquely by the velocity trajectory $v_{[t,T)}$ (and vice versa), via the relations

$$x(t') = x(t) + \int_t^{t'} v(s)\,\mathrm{d}s \quad \text{and} \quad v(t) = \frac{\mathrm{d}}{\mathrm{d}s}x(t) \tag{3}$$

A *path* is a physically identifiable entity that describes the spatial characteristics of a trajectory. That is, it is the projection of the trajectory (1) on the space domain Ω. Fig. 1 aims to clarify further the concepts presented in this section. Note that a trajectory uniquely defines a path, but not vice versa: a path can be realized via an infinite number of trajectories.

2.3. Route cost, running cost and terminal cost

For each trajectory $x_{[t,T)}$ or velocity trajectory $v_{[t,T)}$, we assume that a traveler would experience a *subjective generalized travel cost*. We hypothesize that these costs can be described as follows

$$J_j(t, x(t) | v_{[t,T)}) := \int_t^T L(\tau, x(\tau), v(\tau))\,\mathrm{d}\tau + \phi_j(T, x(T)) \tag{4}$$

s.t. $\mathrm{d}x = v\,\mathrm{d}t$, where L and ϕ_j denote the so-called *running cost* and the *terminal cost* respectively for pedestrians having destination area D_j. We can specify the cost functional J_j either as a function of the trajectory $x_{[t,T)}$ or velocity trajectory $v_{[t,T)}$ without loss of generality. For the sake of clarity, in the remainder let $\overline{J}_j(t, x(t)|x_{[t,T)})$ and $J_j(t, x(t)|v_{[t,T)})$ respectively denote the generalized travel cost as a function of the trajectory or the velocity trajectory.

The running cost $L(\tau, x(\tau), v(\tau))$ reflects the costs incurred during a very small time period $[\tau, \tau + d\tau)$, given that the traveler is at $x(\tau)$ and is applying velocity $v(\tau)$ to change his position. The running costs express the impacts of various attributes of the path as well as the velocity needed to realize this path. As an example, *travel time* can be included in the path cost (4) by choosing $L = 1$. In substituting $L = 1$ into the integral (4), it is easily seen how the resulting contribution of L to the path cost equals the travel time $T - t$. Hoogendoorn and Bovy (2004) discuss in detail the

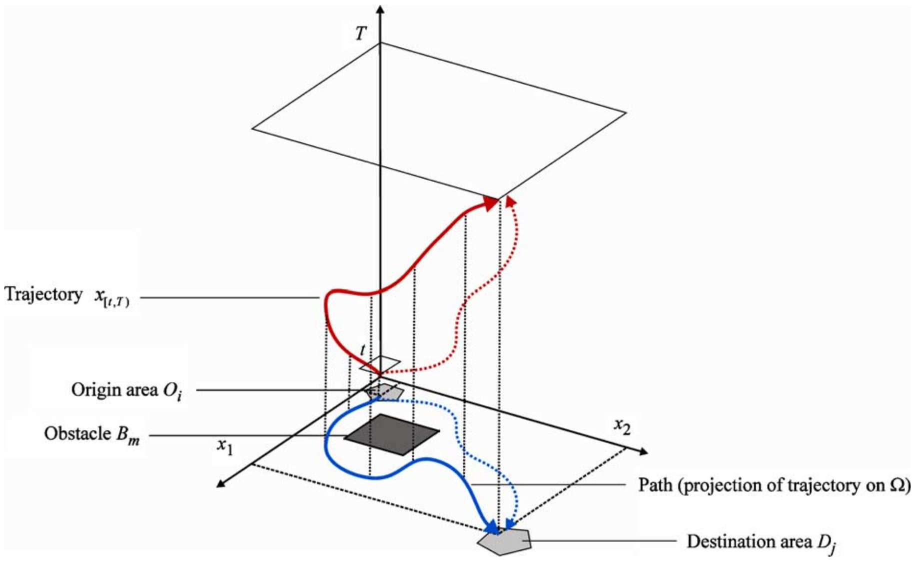

Fig. 1. Example illustrating concepts in continuous time and space path choice for a trip starting at instant t and ending at instant T. Example shows two alternative trajectories $x_{[t,T)}$.

different running cost factors relevant for pedestrian path choice, such as proximity of obstacles, energy consumption due to walking at a specific speed, etc.

The terminal cost $\phi_j(T, x(T))$ reflects the cost incurred by the traveler ending up at position $x(T)$ at the end time T. These costs typically reflect:

1. The penalty that may be incurred when the traveler does not arrive at the destination area D_j in time (e.g. between arrival and departure of a train).
2. The utility that is gained when arriving at a *specific time* or at a *specific location* at the destination area D_j (e.g. a pedestrian arriving at a ticket office).

In mathematical terms, point 1 is described by

$$\phi_j(t_1, x) = \phi_0 \tag{5}$$

where t_1 denotes the fixed end time of the planning period [1] (e.g. the departure time of the train), and where ϕ_0 denotes the penalty (e.g. of having missed the train). Point 2 is expressed mathematically by

$$\phi_j(T, x) = -U_j(T) \quad \text{for } x \in D_j \quad \text{for } T < t_1 \tag{6}$$

[1] Note that the planning period reflects the period within which the traveler plans his or her trip. The end of the planning period is generally not equal to the pre-specified or actual arrival time.

where $U_j(T)$ denotes the utility of arriving at destination D_j at time T. Note that this mathematical conduct allows also *penalizing early arrival*, e.g. arrival before the arrival time t_0 of the train, e.g.

$$U_j(T) = U_j^0 - \max(0, t_0 - T) \tag{7}$$

The generalized travel cost (4) depends among other things on personal preferences and abilities of the traveler, prevailing traffic conditions, etc. For one, this is specified by the specifications of the running cost L and the terminal cost ϕ. Secondly, the admissible velocity trajectories $v_{[t,T)}$ (and thus the admissible trajectories $x_{[t,T)}$) are dependent on the abilities of the traveler, as well as the prevailing traffic conditions. This dependence is explained in detail in the remainder of this section.

2.4. Choice sets and admissible velocities

In the remainder, we hypothesize that a traveler chooses the velocity trajectory $v^*_{[t,T)}$ minimizing total trip disutility (Hoogendoorn and Bovy, 2004). Subjective utility optimization implies that the traveler makes the following velocity trajectory choice (and the resulting path choice):

$$v^*_{[t,T)} = \{v^*(s) | t \leqslant s < T\} = \arg\min J_j(t, x(t) | v_{[t,T)}) \tag{8}$$

subject to

$$v^*(s, x^*(s)) \in \Gamma(s, x^*(s)) \tag{9}$$

The set $\Gamma(t,x)$ reflects the velocities that are possible (or admissible) at time t and location x, and is analogous to the objective choice-set used in discrete choice models. This set will depend among other things on the walking infrastructure and the prevailing traffic, ambient, and weather conditions. For instance, in case of pedestrian traffic, walking speeds will reduce when the density increases. Weidmann (1993) shows how the pedestrian speeds are dependent on the grade of the infrastructure. Standing on an escalator, pedestrian velocities are restricted to the speed and direction of the escalator, especially when there is no room to move. Finally, due to the presence of obstacles B_m, the travelers are unable to walk in arbitrary directions.

In order to mathematically conveniently describe the set of admissible velocities, we first consider the *set of admissible velocities under ideal* (*free flow*) *traffic conditions*, reflected by the set $\Gamma_0(t,x)$. This set includes the influence of infrastructure (obstacles, grades, stairs, escalators, etc.) at zero traffic density. Secondly, a traffic-condition and infrastructure type dependent *speed reduction factor* $0 \leqslant \alpha \leqslant 1$ is introduced, reflecting the reduction in possible walking speeds due to deteriorating traffic conditions. To keep matters simple, we assume that the speed reduction factor will depend only on the total density $\rho(t,x)$ prevailing at instant t and location x. Given the speed reduction factor α, the set of admissible velocities $\Gamma(t,x)$ is defined by

$$\Gamma(t,x) = \{\alpha(t,x)v_0(t,x) | \alpha(t,x) = \alpha^e(\rho(t,x)) \ \forall v_0(t,x) \in \Gamma_0(t,x)\} \tag{10}$$

where $\alpha^e(\rho)$ is *equilibrium speed reduction function* with $d\alpha^e/d\rho \leqslant 0$, $\alpha^e(0) = 1$ and $\alpha^e(\rho_{\text{jam}}) = 0$ for some jam-density value $\rho_{\text{jam}} > 0$. In illustration, Fig. 2 shows how the set $\Gamma(t,x)$ is composed from the set $\Gamma_0(t,x)$ and the equilibrium speed reduction factor.

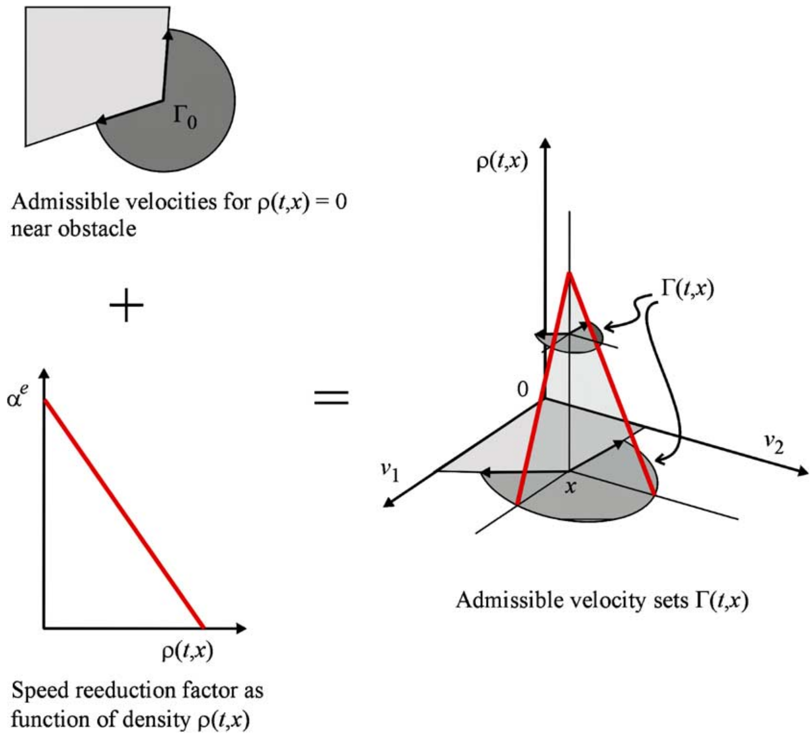

Fig. 2. Determination of admissible velocity set $\Gamma(t,x)$ near an obstacle.

2.5. Flow conservation

Predicting traffic flows and densities can be done using various approaches, for instance by simulation or analytically (see Bliemer (2001)). In this article, a traffic loading approach is proposed using the (deterministic) *conservation of flow equation*, and the directions and speeds determined by the route choice model. The network loading approach and the resulting equilibrium conditions are similar to the approach of Puu and Beckmann (1999) for the static case.

In the remainder of the paper, let $\rho_j(t,x)$ denote the density, $v_j(t,x) \in \mathbb{R}^2$ denote the velocity, and let $q_j(t,x) \in \mathbb{R}^2$ denote the flow of travelers having destination D_j, which are located at $x \in \Omega$ at instant t. The density $\rho_j(t,x)$ denotes the mean number of travelers per m^2 at instant t and location x, having destination D_j. The velocity $v_j(t,x)$ denotes the velocity of travelers along the routes used by travelers with destination j. These velocities are determined at the route-choice level, and are among other things dependent on the densities.

The fundamental relation between flow, density and velocity holds for each destination D_j, i.e.

$$q_j(t,x) = \rho_j(t,x)v_j(t,x) \tag{11}$$

The conservation of flow equation (for destination D_j) then equals

$$\frac{\partial}{\partial t}\rho_j(t,x) + \nabla q_j(t,x) = \frac{\partial}{\partial t}\rho_j(t,x) + \nabla(\rho_j(t,x) \cdot v_j(t,x)) = s_j(t,x) - r_j(t,x) \tag{12}$$

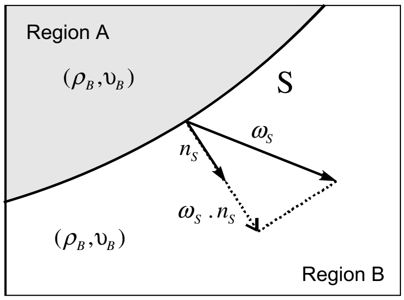

Fig. 3. Discontinuity or shock surface S separating regions A and B.

where $s_j(t,x)$ and $r_j(t,x)$ respectively denote the inflow and outflow of travelers with destination j at (t,x). The nabla-operator ∇ is defined by

$$\nabla := \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right]' \tag{13}$$

Besides the local flows $q_j(t,x)$, we will also consider the path flows $q_j(t,x|x_{[t,T)})$. These reflect the pedestrian flow along the path $x_{[t,T)}$ of pedestrians with destination D_j.

Under the presumption that the velocity $v_j(t,x)$ is determined by the prescribed direction (following from the route choice model) and an equilibrium speed determined by the local density $\rho(t,x)$, Eq. (12) describes a two-dimensional generalization of the kinematic wave model (Lighthill and Whitham, 1955), and hence the flow patterns that will emerge are similar. The model will predict the formation of acceleration fans when density decreases in the walking direction, describing how pedestrians will accelerate from a high into a low density region. The model will also predict the formation of shocks, when the density increases in the walking direction. The properties of these shocks are very similar to their one-dimensional counterparts: for instance, their speed can be determined by Rankine–Hugoniot relations that relate the dynamics of the shock S delineating regions A and B (see Fig. 3).

$$\omega_S \cdot n_S = \frac{\rho_B(v_B \cdot n_S) - \rho_A(v_A \cdot n_S)}{(\rho_B - \rho_A)} \tag{14}$$

Here n_S is the unit vector perpendicular to the shock-surface S, (ρ_A, v_A) and (ρ_B, v_B) respectively denote the densities and velocities at regions A and B, and ω_S denotes the velocity of the shock.

The consequences of shock formation on pedestrian route choice are not explored further in the manuscript. For practical applications, the numerical solution approach used to assign the pedestrian traffic on the continuous network introduces numerical viscosity, yielding smooth shocks rather than discontinuities.

3. Problem formulation and modeling assumptions

The preceding section has introduced the notation required to present the continuous time and space dynamic user-optimal assignment problem formulation. This is achieved in this section, as are the underlying behavioral assumptions.

3.1. Behavioral assumptions

The choice model presented in this section is based on the micro-economic assumption of human decision making by subjective expected cost minimization. This section briefly describes the consequent path (or rather, trajectory) choice model, based on the following behavioral assumptions:

1. Travelers make rational decisions based on the subjective expected path costs, from the set of admissible paths.
2. Travelers have full and perfect information on future travel conditions, and choose the best trajectory based on this information at the beginning of their trip.
3. Travelers have perfect knowledge about the infrastructure and the resulting path alternatives.
4. The path costs includes terminal costs upon arrival, or a penalty for not arriving at the destination within the planning period.
5. The traveler's departure time is fixed; the arrival time is free (inelastic demands).
6. Multiple optimal trajectories can exist.

3.2. Deterministic dynamic path equilibrium in continuous time and space

The definition of the user-equilibrium problem presented here is based on the definition of Friesz et al. (1993) of the *path integral equilibrium*, stating that "at equilibrium, for each origin–destination pair, the actual flow costs from time of departure to time of arrival on used paths, including any early or late arrival penalties, are identical and equal to the minimum path costs which can be realized from among all route options and departure time options" (Bliemer, 2001). Since departure time choice is not considered in this article, the following definition of the dynamic path equilibrium in continuous time and space is proposed:

> In a *dynamic path equilibrium in continuous space and time*, for each origin–destination pair and for each departure time, the actual travel costs on utilized paths are identical and equal to the minimum travel costs that can be realized from among the infinite set of path alternatives.

The *actual travel costs* describe the cost that a traffic unit departing from its origin at a certain instant t will actually experience. It is thus assumed that individuals maximize their utility, and choose the path with the minimum actual travel costs. The costs are dependent among other things on flows and densities, which in turn are following from by the chosen paths.

Below, the mathematical formulation of the dynamic user-optimal equilibrium in continuous time and space is given. To clearly indicate the differences with dynamic assignment in discrete networks, Table 1 depicts an overview of the main differences between the two approaches.

3.3. Mathematical formulation of dynamic user-optimal equilibrium

The *minimum actual path costs* are defined by considering the minimum path cost for all admissible trajectories $x_{[t,T)}$ (or equivalently, for all velocity trajectories $v_{[t,T)}$), i.e.

Table 1
Overview of main differences between dynamic assignment in discrete networks and continuous networks

Element	Network type	
	Discrete	Continuous
Infrastructure	Directed graph $G = (N, A)$	Area $\Omega \subset \mathbb{R}^2$; specification of walking surface and obstacles $B_m \subset \Omega$
Origins/destinations	Origin and destination nodes	Origin and destination areas O_i, $D_i \subset \Omega$
Routes or paths[a]	Ordered set of links $\{a_m\}$ from origin node to destination node	Continuous parameterized curve in Ω from origin area to destination area
Number of alternatives	Finite and countable	Infinite and non-countable
Choice variable	Route	Velocity
Trajectory	Defined by speeds on links (and decisions at nodes)	Defined by speeds *and* direction

[a] In this article, we have reserved the term 'routes' for the discrete networks, and the term 'paths' for the continuous networks.

$$W_j(t, x(t)) := \min_{x_{[t,T)}} \overline{J}_j(t, x(t)|x_{[t,T)}) = \min_{v_{[t,T)}} J_j(t, x(t)|v_{[t,T)}) \tag{15}$$

The mathematical conditions for a dynamic user-optimal equilibrium state are then given by

$$q_j(t, x(t)|x_{[t,T)}) > 0 \Rightarrow \overline{J}_j(t, x(t)|x_{[t,T)}) = W_j(t, x(t)) \tag{16}$$

Condition (16) implies that *any used trajectory has minimum actual travel cost*. We emphasize that in the remainder, trajectories are not considered explicitly, but only indirectly via the (optimal) velocities along the used paths.

4. Model formulation

In this section, the analytical model of the continuous time and space assignment problem is presented. To this end, we will first recall the route choice model from Hoogendoorn and Bovy (2004). Then, the network-loading model is briefly discussed as well as the consequent dynamic assignment model.

4.1. Path choice modeling by dynamic programming

For the path choice modeling in continuous time and space, let us recall the approach from Hoogendoorn and Bovy (2004). The key to the approach is that *instead* of determining the optimal paths $x^*_{[t,T)}$ of travelers with destination D_j explicitly, only the local choice behavior of the travelers is considered. That is, consider a traveler who is located at x at instant t and is traveling towards destination D_j. The dynamic programming approach from Hoogendoorn and Bovy (2004) will describe the optimal velocity $v^*_j(t, x)$ that the traveler needs to apply to reach the destination D_j as cheaply as possible.

This optimal velocity $v^*_j(t, x)$ of a traveler moving in a two-dimensional area Ω is a function of the minimum actual travel cost field $W_j(t, x)$—defined by Eq. (15)—of traveling from location x at instant t and reaching the destination D_j before instant T. In turn, $W_j(t, x)$ can be determined by solving the so-called *Hamilton–Jacobi–Bellman* (HJB) equation (Hoogendoorn and Bovy, 2004):

$$-\frac{\partial}{\partial t}W_j(t,x) = H(t,x,\nabla W_j) \tag{17}$$

with *terminal conditions* reflecting the penalty of not arriving at D_j before the end-time t_1

$$W_j(t_1,x) = \phi_0 \tag{18}$$

and *boundary conditions* describing the utility of arriving at the destination area D_j at time T

$$W_j(T,x) = -U_j(T) \quad \text{for } x \in D_j \quad \text{for } T < t_1 \tag{19}$$

The so-called *Hamilton function* H is defined by

$$H(t,x,\nabla W_j) = \min_{v\in\Gamma(t,x)}\{L(t,x,v) + v\cdot\nabla W_j\} \tag{20}$$

where $v \in \mathbb{R}^2$ denotes the velocity, and where L denotes the so-called running cost; $\Gamma(t,x)$ denotes the set of admissible velocities (i.e. admissible direction and speed, given prevailing flow conditions, e.g. densities $\rho_j(t,x)$, obstacles, etc.) at instant t and location x, defined by Eq. (10). The optimal velocity $v_j^*(t,x)$ chosen at instant t and location x can be expressed by the following relation

$$v_j^*(t,x) = \arg\min\{L(t,x,v) + v\cdot\nabla W_j|\text{subject to } v \in \Gamma(t,x)\} \tag{21}$$

The optimal path $x^*_{[t,T)}$ can then be determined by integration of the optimal speeds, i.e.

$$x_j^*(t') = \int_t^{t'} v_j^*(s, x_j^*(s))\,\mathrm{d}s \tag{22}$$

4.1.1. Existence of the solution

The HJB equation typically arises in *control until exit* problems in deterministic optimal control. The equation is derived under the implicit assumption that the actual minimal travel cost is continuously differentiable. In general, however, this will not be the case (e.g. in case of multiple optimal paths, see Fig. 4). In that case a notion of weak solutions to the HJB equation has been considered. This is discussed briefly in the subsequent paragraph.

Fleming and Soner (1993) show that any continuously differentiable function $W_j(t,x)$ that satisfies the HJB equation equals the minimum actual travel cost. Furthermore, they show that any control satisfying Eq. (21) is optimal.

4.1.2. Uniqueness of the solution

From the theory of dynamic programming, it is well known that classical solutions to the HJB equation may not exist. However, generalized or weak solutions (solutions that are piecewise continuously differentiable) exists but are generally not unique. To resolve this issue, the concept of *viscosity solutions* has been introduced by Crandall and Lions (1984). Notwithstanding the formal definition of Crandall and Lions (1984), its foundation is the following second-order partial differential equation

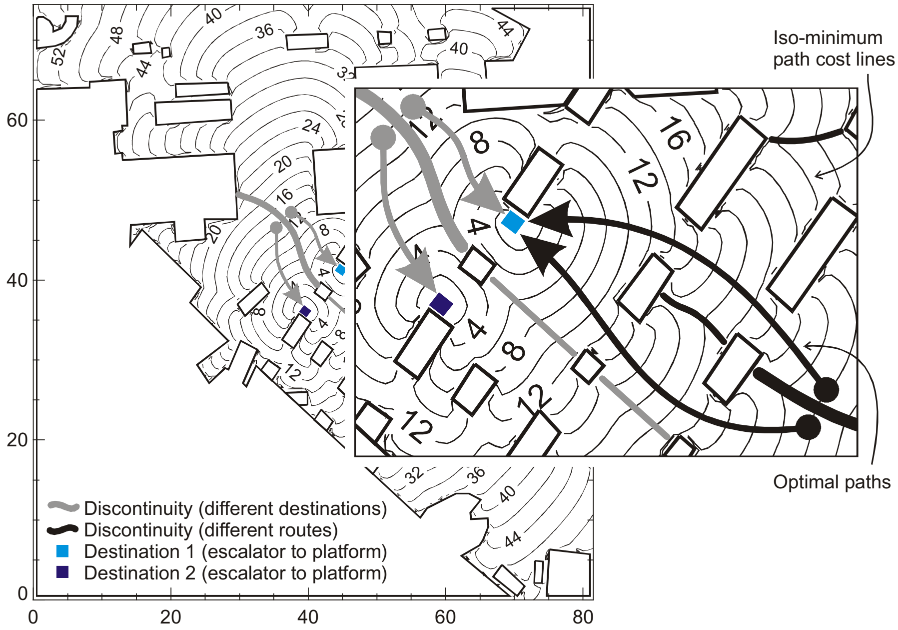

Fig. 4. Example of non-differentiable minimum path cost function. The thick black line indicates locations where W is not continuously differentiable and multiple paths to the *same destination* will exist. Two optimal paths (black arrows) are shown. The thick gray lines indicate locations where W is not continuously differentiable and multiple paths to *different destinations* will exist. The figure also shows optimal paths to two different destinations (gray arrows).

$$-\frac{\partial}{\partial t}W_j^\varepsilon = H(t,x,\nabla W_j^\varepsilon) + \varepsilon\Delta W_j^\varepsilon \tag{23}$$

for which we can prove existence and uniqueness. The so-called vanishing viscosity solution is defined as follows

$$W_j = \lim_{\varepsilon\downarrow 0} W_j^\varepsilon \tag{24}$$

In (Fleming and Soner, 1993), it is shown that in case H (Eq. (20)) satisfies

$$\left|\frac{\partial}{\partial t}H(t,x,p)\right| + |\nabla H(t,x,p)| \leqslant K(1+|p|) \quad \text{for all } p\in\mathbb{R}^2 \tag{25}$$

and

$$\left|\frac{\partial}{\partial p}H(t,x,p)\right| \leqslant K \quad \text{for all } p\in\mathbb{R}^2 \tag{26}$$

for some suitable constant $K>0$, the viscosity solution of the HJB equation is unique.

It can be shown that at points the (viscosity) solution $W_j(t,x)$ of Eq. (28) is continuously differentiable, the optimal velocity can be uniquely determined from $W_j(t,x)$; at points where derivative of $W_j(t,x)$ does not exist, multiple optimal velocities are possible, implying that multiple optimal trajectories exist.

4.2. Flow conservation

The velocities of the travelers in the conservation equation (12) are determined by both their direction and their speed, where it is assumed that the walking velocity equals the optimal velocity determined during route choice, i.e.

$$v_j(t,x) = v_j^*(t,x) \tag{27}$$

where the optimal velocity is defined by Eq. (21), in which $\Gamma(t,x)$ is defined by (10).

In the examples presented in the remainder of the article, it is assumed that the admissible velocities are determined by the total density $\rho(t,x) = \Sigma\rho_j(t,x)$ via the equilibrium speed reduction factor $\alpha^e(\rho)$. It is thus assumed that the flow interactions amongst different commodities (OD-pairs) are not taken into account. However, this assumption is made for practical reasons only, and can be relaxed without major alterations of the approach.

Note that contrary to the Lighthill–Whitham–Richards models used in vehicular traffic flow modeling, the velocities (speeds and directions) in the conservation of traffic equation do not depend directly on the density, but only *indirectly via the route choice model*. Describing the effect of deteriorating traffic conditions on travel speeds thus requires solving the route choice model as well as the conservation equation.

Traffic demand $Q_{ij}(t)$ describing the demand from origin O_i to destination D_j enters the infrastructure at origin area O_i, and is propagated through the walking infrastructure according to the conservation equation (12). The latter equation is a straightforward generalization of the one-dimensional bi-directional conservation of vehicle equation for vehicular traffic towards two-dimensional multi-directional flows. The inflows s_j and outflows r_j respectively reflect the *source* at the origin (pedestrians entering the facility) and the *sink* at the destination area D_j (pedestrians leaving the facility). The travel demand over time between origins O_i and destination areas D_j, is fixed and given (inelastic demands and no departure time choice).

4.3. Mathematical definition of user-equilibrium assignment

The path-choice mechanism described by Eq. (12) depends on the traffic conditions described by conservation Eq. (12) via the dependence of the set of admissible speeds $\Gamma(t,x)$ on the densities $\rho_j(t,x)$ and possibly the composition with respect to the travel directions (such as in the case of pedestrian traffic). Vice versa, the traffic conditions depend on the path-choices via the optimal velocity $v_j^*(t,x)$.

One of the characteristics of an equilibrium solution is that the velocity trajectory-choice pattern is consistent with the traffic conditions, and traffic conditions are consistent with the collective route choice behavior. Based on the results of the above sections, the equilibrium solution is characterized by a solution $\{\rho_j(t,x), \widetilde{W}_j(t,x)\}$ to the following dynamic program

$$\begin{aligned}&-\frac{\partial \widetilde{W}_j}{\partial t} = H(t,x,\nabla \widetilde{W}_j) \text{ s.t. } \widetilde{W}_j(t_1,x) = \phi_0 \quad \text{and} \quad \widetilde{W}_j(T,x) = -U_j(T) \quad \text{for } x \in D_j,\\ &T < t_1\end{aligned} \tag{28}$$

$$\frac{\partial}{\partial t}\tilde{\rho}_j(t,x) + \nabla(\tilde{\rho}_j(t,x)v_j^*(t,x)) = s_j(t,x) - r_j(t,x) \tag{29}$$

where

$$H(t,x,\nabla\widetilde{W}_j) := L(t,x,v^*) + v^*(t,x)\cdot\nabla\widetilde{W}_j \tag{30}$$

$$v^*(t,x) = \arg\min\{L(t,x,v) + v\cdot\nabla\widetilde{W}_j|\text{subject to } v\in\Gamma(t,x)\} \tag{31}$$

Section 5 presents a heuristic approach to determine the dynamic user-equilibrium flow pattern, as well as a numerical approximation approach to solve the HJB-Eq. (17).

5. Numerical solution procedure

This section presents an approach to determine the continuous-space user-equilibrium assignment. It is a heuristic iterative approach adopting several numerical solution approaches such as to solve the HJB-equation and the flow propagation. This is achieved by determining a numerical solution that satisfies the mathematical program defined by Eqs. (28)–(31).

5.1. General solution framework

Fig. 5 depicts the framework of the proposed continuous assignment model. It shows how the set of admissible velocities $\Gamma_0(t,x)$ under ideal conditions depends on the infrastructure and personal preferences and abilities of the travelers. In turn, the set of admissible velocities $\Gamma(t,x)$ is in part determined by $\Gamma_0(t,x)$. Given the specifications of the costs incurred by the travelers (running costs L and arrival cost ϕ), travelers choose their best paths. These paths are determined by solving the dynamic programming equation, yielding the minimum actual path cost $W_j(t,x)$.

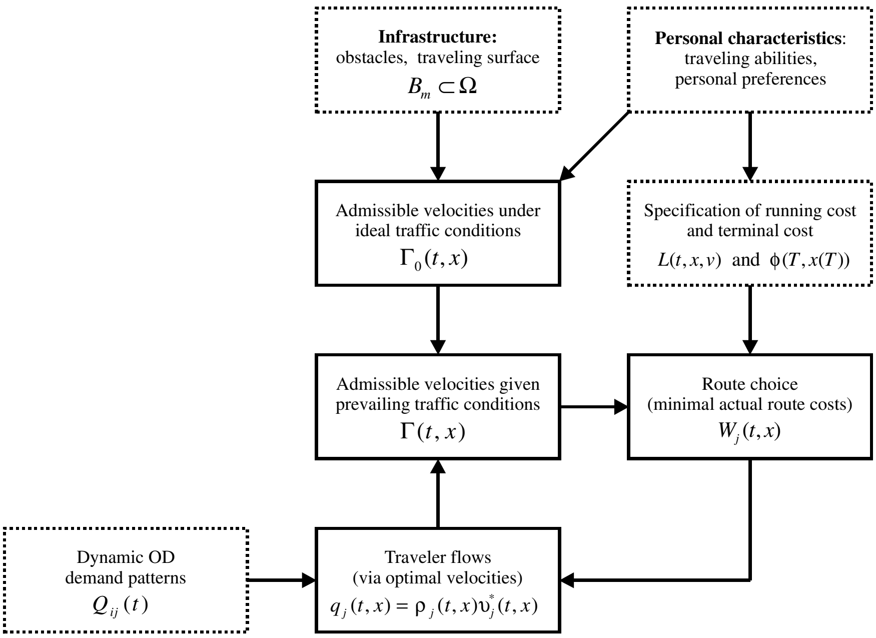

Fig. 5. Framework of the dynamic assignment model in continuous space.

The travelers move along the paths determined from $W_j(t,x)$ yielding the path flows and traffic densities that are determined by the flow propagation stemming from the conservation equation (12). These flows—or rather, the densities—lead to the set of admissible velocities $\Gamma(t,x)$. Fig. 5 shows that the densities and the set of admissible velocities $\Gamma(t,x)$ are interrelated. Finally, the admissible velocities determine the minimum actual route costs.

User-equilibrium is attained when convergence is reached in the loop, that is, when the flows through the infrastructure are consistent with the route choice behavior, and vice versa.

5.2. Heuristic approach to continuous dynamic assignment

Let us now briefly outline the heuristic approach used to determine a user-optimal equilibrium dynamic assignment. To solve the fixed-point problem an iterative procedure is used in which successively OD-flows are assigned to shortest paths the resulting densities of which are weighted between iteration steps until convergence. The approach is straightforward, and assures consistency between the actual traffic flows (and hence the actual travel cost) and the route choice in continuous time and space.

1. *Initialization*. At initialization, free flow conditions are assumed (zero densities and maximum speeds). That is, the numerical solution approach described in Section 5.4 is used to approximate the minimum actual walking cost $W_j^{(0)}(t,x)$ satisfying Eq. (17) s.t. Eqs. (18) and (19), for all destination areas D_j, while considering admissible set $\Gamma^{(0)}(t,x) = \Gamma_0(t,x)$. Set $k = 1$.
2. *Assignment*: based on the optimal route following from the value function $W_j^{(k-1)}(t,x)$ determined in step $k-1$, compute auxiliary densities $\hat{\rho}_j^{(k-1)}(t,x)$ for all x and t, subject to Eq. (12) using the numerical approximation scheme described in Section 5.4. The densities $\hat{\rho}_j^{(k-1)}(t,x)$ can be interpreted as resulting from the all-or-nothing assignment of the O–D flows based on the minimum actual travel cost function $W_j^{(k-1)}(t,x)$. For some value $0 < \lambda < 1$, update the densities by applying the following rule [2]

$$\rho_j^{(k)}(t,x) := (1-\lambda)\rho_j^{(k-1)}(t,x) + \lambda\hat{\rho}_j^{(k-1)}(t,x) \tag{32}$$

3. *Compute speed reduction factors*: based on the traffic conditions $\rho_j^{(k)}(t,x)$ for all destinations j, compute the maximum speed reduction factors $\alpha(t,x) = \alpha^e(\rho^{(k)}(t,x))$ from which the updated set of admissible velocities $\Gamma^{(k)}(t,x)$ can be determined using Eq. (10).
4. *Determine route-choice*: for each destination j, determine the value function $W_j^{(k)}(t,x)$ by numerically approximating Eq. (17) s.t. Eqs. (18) and (19) using the admissible velocity set $\Gamma^{(k)}(t,x)$ using the numerical approximation approach described in Section 5.4.
5. *Iterate*: as long as $|\rho_j^{(k)}(t,x) - \rho_j^{(k-1)}(t,x)| > \delta$ for some very small value $0 < \delta \leqslant \lambda$, set $k = k+1$ and go to step 2. When the difference between successive iteration steps is small enough, the procedure is stopped.

[2] Note that the scheme can be improved by making the factor λ variable during the iterations.

Application of the approach described in steps 1–5 above will yield a solution $\rho_j^{(k)}(t,x)$ that satisfies the stopping criterion, which, according to Eq. (32), implies that

$$\begin{aligned}\left|\rho_j^{(k)}(t,x) - \rho_j^{(k-1)}(t,x)\right| &= \left|(1-\lambda)\rho_j^{(k-1)}(t,x) + \lambda\hat{\rho}_j^{(k-1)}(t,x) - \rho_j^{(k-1)}(t,x)\right| \\ &= \lambda|\hat{\rho}_j^{(k-1)}(t,x) - \rho_j^{(k-1)}(t,x)| \leqslant \delta\end{aligned} \tag{33}$$

Eq. (33) shows that the assigned auxiliary densities $\hat{\rho}_j^{(k-1)}(t,x)$ (i.e. the flows based on the optimal routes following from the minimum actual travel cost function $W_j^{(k-1)}(t,x)$) is near to the final densities $\rho_j^{(k-1)}(t,x)$ on which the route choice behavior described by $W_j^{(k-1)}(t,x)$ is based. As a result, the stopping criterion implies consistency between route choice and traffic operations. This consistency in turn implies that the user-equilibrium solution has been determined (Bliemer, 2001).

Below, an outline follows of the numerical approaches adopted to solve the HJB-Eq. (7) describing the cost field W. For details see (Hoogendoorn and Bovy, 2004).

5.3. Numerical solution approach to solving the path choice model

To numerically approximate solutions of the cost function HJB Eq. (17) can be achieved by replacing the partial derivatives with the appropriate finite differences. The forward finite difference $\nabla_i^+ W_j$ and backward finite difference $\nabla_i^- W_j$ are defined by

$$\nabla_i^\pm W_j := \delta^{-1}[W_j(t,x\pm\delta e_i) - W_j(t,x)] \quad \text{for } i = 1,2 \tag{34}$$

In numerically approximating Eq. (17) the following solution approach is proposed (Fleming and Soner, 1993):

$$W_j(t-h,x) = W_j(t,x) - hH(x,\nabla_i^\pm W_j) \tag{35}$$

where the numerical Hamiltonian is defined by

$$H(x,\nabla_i^\pm W_j) = \min_{v\in\Gamma(t,x)}\left\{L(t,x,v) + \sum_i (v_i^+\nabla_i^+ W_j - v_i^-\nabla_i^- W_j)\right\} \tag{36}$$

To ensure numerical stability of approximation scheme (35), the following condition relating cell size, time step, and speed must hold (Fleming and Soner, 1993)

$$h|v| \leqslant \delta \quad \text{for all } v \in \Gamma(t,x) \tag{37}$$

For more details on the numerical approximation approach, see (Hoogendoorn and Bovy, 2004).

5.4. Numerical solution approach for flow propagation equation

Having numerically solved the HJB (17) and thus having determined the optimal routes to all destinations D_j from location x at instant t, approximation of the flow conservation equation is straightforward. Using the approximations (34) of the partial derivatives of W_j we can approximate the optimal velocities stemming from (21). Given the densities $\rho_j(t,x)$, the flows out of cell x

can be approximated by multiplying with the finite differences (34). The flow from cell x to cell $x \pm \delta e_i$ equals

$$q_{i,j}^{+} = \rho_j (v_j^*)^+ \quad \text{and} \quad q_{i,j}^{-} = \rho_j (v_j^*)^- \tag{38}$$

for $i = 1, 2$, where

$$v_j^*(t,x) = \arg\min \left\{ L(t,x,v) + \sum_i (v_i^+ \nabla_i^+ W_j - v_i^- \nabla_i^- W_j) \right\} \tag{39}$$

In most cases, explicit relations between the optimal velocity and the minimum actual route cost can be used instead of (39).

Having determined the outflows $Q_i^{\pm}$ for each cell x, the inflows $Q_i^{\pm}$ can be derived easily. The densities $\rho_j(t+h,x)$ in the following time step are then

$$\rho_j(t+h,x) = \rho_j(t,x) - \frac{h}{\delta} \left(\sum_{i=1,2} q_{i,j}^{\pm} + \sum_{i=1,2} Q_{i,j}^{\pm} \right) \tag{40}$$

Requirement (37) also ensures the stability of scheme (40).

6. Application example

This section illustrates the proposed model by an application example. The example pertains to pedestrians walking in Schiphol Plaza, which is a multi-purpose facility (shopping and multi-model transfer point) in Schiphol airport. In real-life, Schiphol Plaza is used by airline travelers departing or arriving at Schiphol Plaza, public transit travelers arriving or leaving Schiphol Plaza either by train or using private transport modes, etc. In our case here, only pedestrians are considered who leave Schiphol Plaza either via one of the escalators leading to the lower level train station, and pedestrians who leave Schiphol Plaza via either of the exits.

Fig. 6 shows a map of part of Schiphol Plaza, indicating the exits from Schiphol Plaza (exits 1–5 to the outdoor taxi stands and parking facilities and exits 6 and 7 to the underground train platform).

For the sake of simplicity, is it assumed that only the pedestrian density determines the maximum walking speeds, irrespective of the directions of the pedestrians the flow consists of. In practice, the maximum flow can be reduced to up to 14.5% (in case of bi-directional flows; Weidmann (1993)) due to bi-directionality or crossing flows. The following linear relation is used to describe the speed reduction factor used in the remainder of this example

$$\alpha(t,x) = \alpha^e(\rho(t,x)) = \max\{0, 1 - 0.185 \cdot \rho(t,x)\} \tag{41}$$

showing that the maximum pedestrian density equals 5.4 P/m^2, while the free speed of pedestrians is approximately 1.34 m/s. Furthermore, we assume that the admissible velocities under ideal

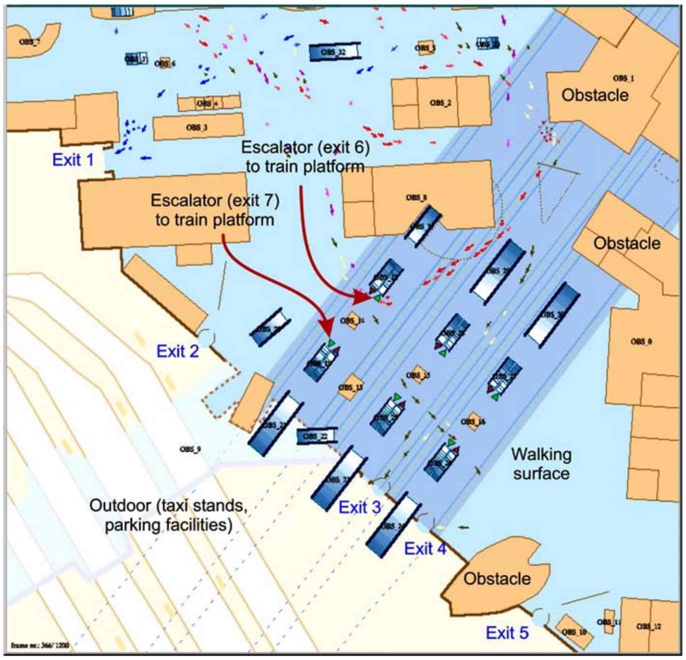

Fig. 6. Overview of Schiphol Plaza (output of NOMAD simulation model). Figure shows the exits 1–5 (to taxi stands and parking facilities), exits 6 an 7 (exits to underground train platforms), walking surface and example obstacles.

conditions are only dependent on the obstacles (i.e. not on the walking surface, grade, etc.). When not obstructed by an obstacle, the admissible velocities are determined as follows

$$\Gamma_0(t,x) := \{v | \|v\| \leqslant V_{\max}\} \tag{42}$$

where $V_{\max}$ is the maximum speed.

Fig. 7a–f shows the results of dynamic pedestrian assignment for flows of pedestrians heading towards either of the escalators. Fig. 7a–c shows the results of the first iteration of the assignment algorithm, i.e. when assuming that the network is empty. In this case, pedestrians choose the shortest route from their origin to their destination. In this case, the speeds are equal to the free speed of the pedestrians, irrespective of the pedestrian density. Fig. 7d–f shows the results after the dynamic assignment has converged. In this case, the impact of high pedestrian densities on the speed and the experienced travel times, and consequently on the route choice is taken into consideration. In this case, the expected traffic operations at escalator E6 cause some pedestrians to choose for escalator E7. As a result, pedestrian flows are distributed nearly evenly over both escalators.

Fig. 8a–f shows a similar result for pedestrians heading for the exits of Schiphol Plaza. Clearly, in the first iteration of the assignment algorithm (empty network) many pedestrians choose to use exit E1, causing the exit to become congested. As a result, the experienced travel times will increase to the extent that it will become beneficial to choose exit E2. This will not occur directly, but only after some time has elapsed. In illustration, compare Fig. 8e and f. At $t = 25$ s, a substantial number of pedestrians has seized using exit E1 and is using exit E2 instead.

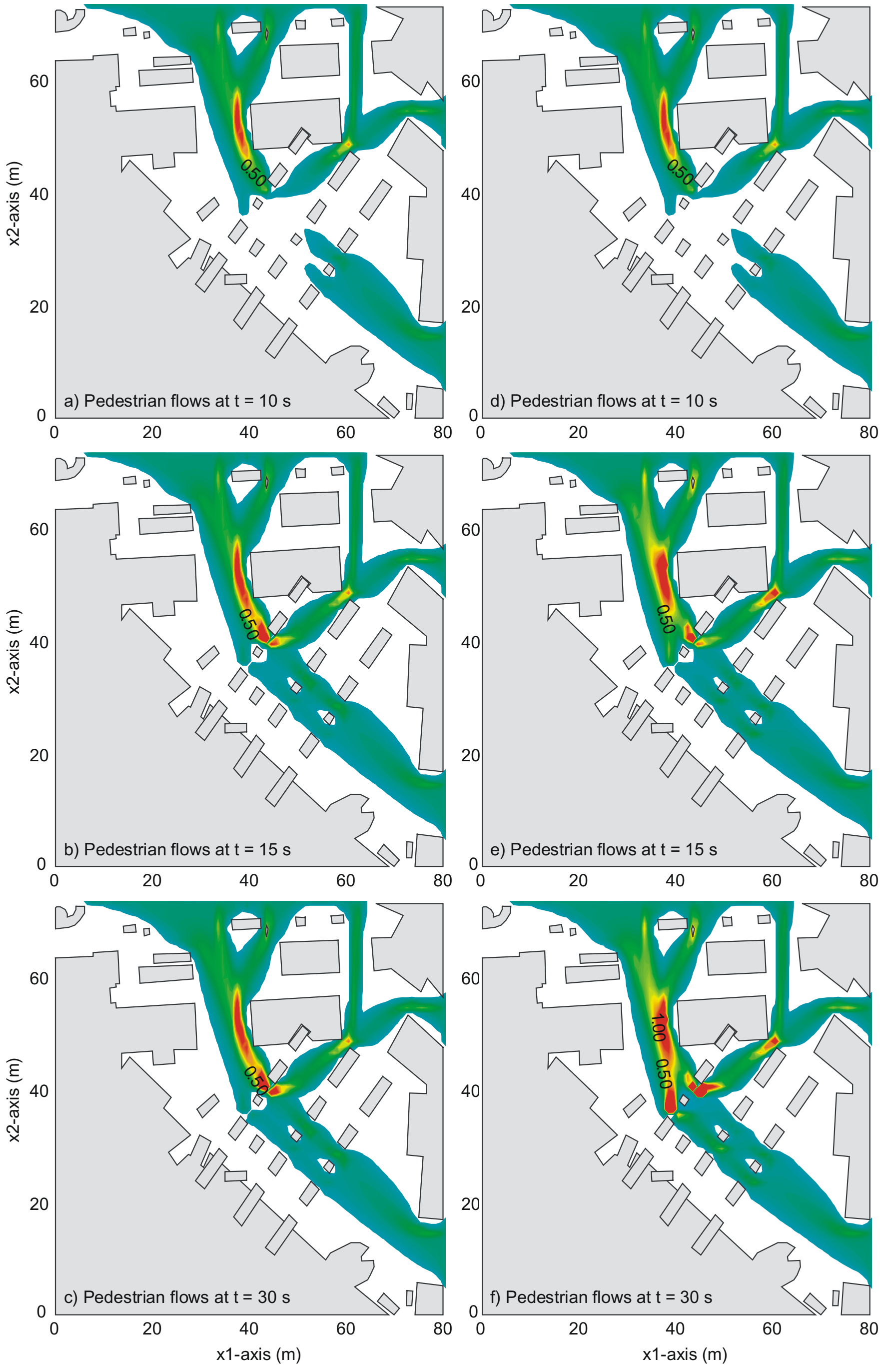

Fig. 7. Flows of pedestrians heading towards the escalators. Figures a–c show the results of the first iteration (i.e. assuming an empty network); figures d–f show the end results of the dynamic assignment. The figures show how assignment causes travelers to distribute more evenly over the escalators E6 and E7.

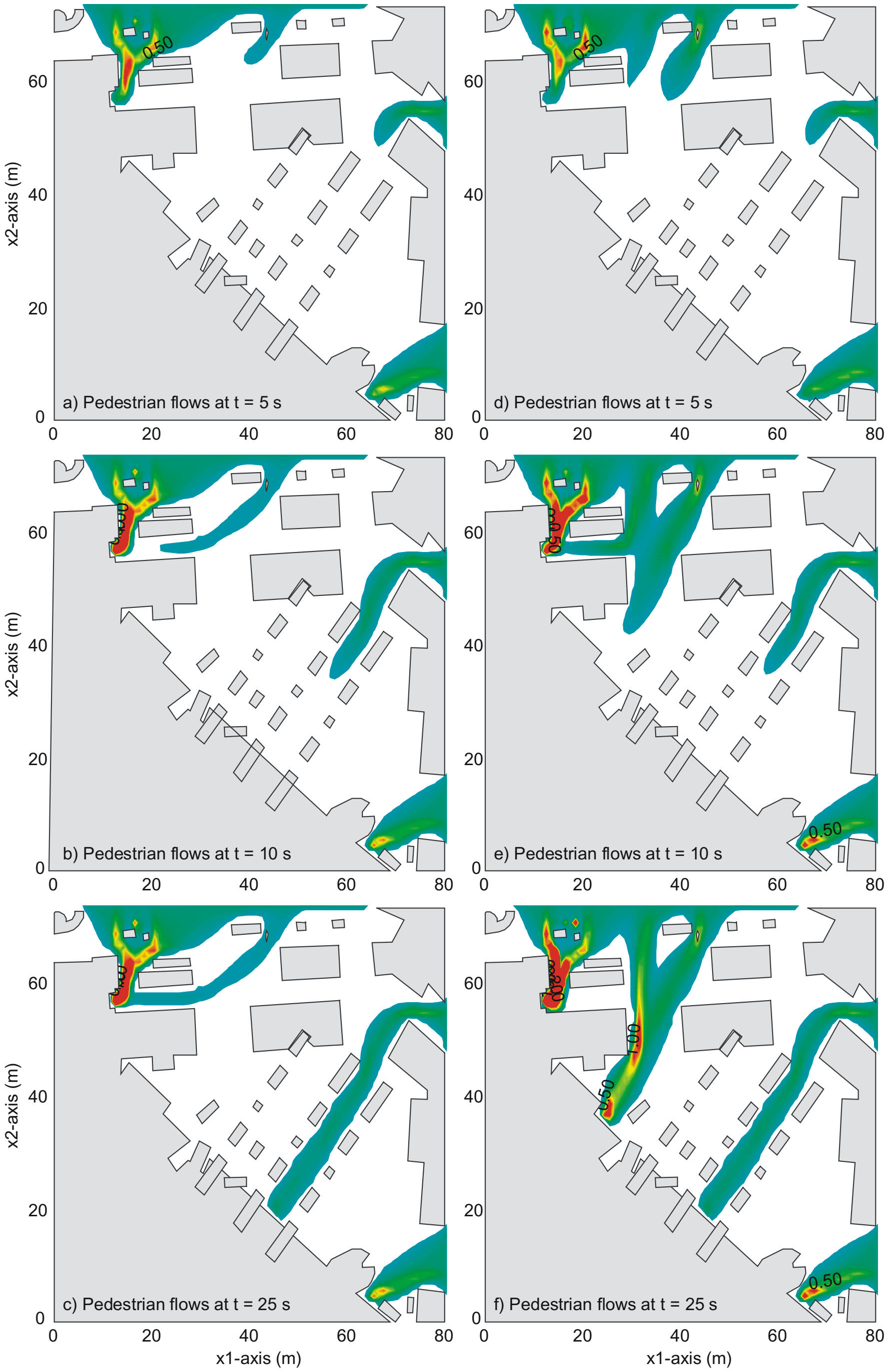

Fig. 8. Flows of pedestrians heading towards the exits of Schiphol Plaza. Figures a–c show the results of the first iteration (i.e. assuming an empty network); figures d–f show the end results of the dynamic assignment. Figures e and f clearly reveal pedestrians anticipating on large delay times at exit E1 choose exit E2, which in turn causes small delay times at E2.

7. Conclusion

This article proposed an approach to deterministic user-optimal dynamic assignment in continuous time and space. The approach is applicable for the assessment and planning of traffic networks where the available routes cannot be adequately described by a finite number of line-based paths along a fixed number of nodes, i.e. to networks where travelers can move freely around the present obstacles. Furthermore, the approach is applicable to assess the impacts of management measures on for instance pedestrian flows in uni-modal or multi-modal public transit transfer facilities, sports stadiums. While the model has been designed to study pedestrian traffic, the approach is generally applicable to other modes of transport or types of traffic as well, for instance cyclists, guided vehicles in container terminals, etc.

For simplicity sake, the article focuses on deterministic route choice, neglecting departure time and destination choice. More importantly, the introduction of intermediate destinations along the routes of the travelers has been neglected as well. Considering such intermediate destinations may be very important when considering public transport transfer stations, where travelers may have to buy a ticket (intermediate destination) before moving towards the train platform (final destination). Within the activity-based framework of (Hoogendoorn and Bovy, 2004), generalization of the approach is relatively straightforward.

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