OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth., 2003; **24**:153–172 (DOI: 10.1002/oca.727)

Simulation of pedestrian flows by optimal control and differential games

Serge Hoogendoorn*,† and Piet H.L. Bovy

Delft University of Technology, Faculty of Civil Engineering and Geosciences, Transportation Planning and Traffic Engineering Section, Delft, The Netherlands

SUMMARY

Gaining insights into pedestrian flow operations and assessment tools for pedestrian walking speeds and comfort is important in, for instance, planning and geometric design of infrastructural facilities, as well as for management of pedestrian flows under regular and safety-critical circumstances. Pedestrian flow operations are complex, and vehicular flow simulation modelling approaches are generally not applicable to pedestrian flow modelling. This article focusses on pedestrian walking behaviour theory and modelling. It is assumed that pedestrians are autonomous predictive controllers that minimize the subjective predicted cost of walking. Pedestrians predict the behaviour of other pedestrians based on their observations of the current state as well as predictions of the future state, given the assumed walking strategy of other pedestrians in their direct neighbourhood. As such, walking can be represented by a (non-co-operative or co-operative) *differential game*, where pedestrians may or may not be aware of the walking strategy of the other pedestrians. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: walker model; differential games; feedback control; micro-simulation

1. BACKGROUND

Research of pedestrian behaviour started in the 1960s, when pedestrian flows in urban areas were studied. The main purpose of these early investigations was to provide guidelines for optimal design of walkway infrastructure. Weidmann [1] presents a concise overview of empirical facts about pedestrian walking behaviour, concerning among other things the relation between walking speed and energy consumption, the factors influencing walking speeds, and the use of space by pedestrians. In his review, Weidmann shows that pedestrian walking speeds are dependent on *personal characteristics of pedestrians* (age, gender, size, health, etc.), *characteristics of the trip* (walking purpose, route familiarity, luggage, trip length), *properties of the infrastructure* (type, grade, attractiveness of environment, shelter), and finally

*Correspondence to: S. Hoogendoorn, Delft University of Technology, Faculty of Civil Engineering and Geosciences, Transportation Planning and Traffic Engineering Section, Delft, The Netherlands.
†E-mail: s.p.hoogendoorn@ct.tudelft.nl

Contract/grant sponsor: Social Science Research Council (MaGW) of the Netherlands Organization for Scientific Research (NWO)

environmental characteristics (ambient and weather conditions). Besides the exogenous factors, the walking speed also depends on the pedestrian density. An important characteristic of pedestrian flows is that flows in opposing directions tend to separate, which is referred to as *dynamic lane formation* or *streaming* [2]. The formation of lanes is the main reason for the relative small loss of capacity in case of bidirectional pedestrian flows (in the range of 4–14.5%); see Reference [1]. It turns out that for walkways of moderate width, the lanes are formed on the right-hand side, irrespective of the customs or traffic regulations. Similar results have been established for crossing flows [3], albeit in the form of strips or moving clusters composed of pedestrians walking in the same direction. Pedestrian co-operation plays a very important role in pedestrian flows (see References [4–6]).

Different modelling approaches have been considered to model pedestrian walking behaviour and pedestrian flow dynamics. *Hydrodynamic models*, based on the analogy with gas dynamics, describe the dynamics of the spatial distribution of concentration and velocity using partial differential equations. *Microscopic models* describe the time–space behaviour of individual pedestrians. Examples are the social-forces model [7], cellular automata (CA) models [8], and the non-local stimulus-response model of [9]. Although these models appear to yield generally plausible results for most situations, their oversimplified or incomplete behavioural rules prohibit application to complex situations where the microscopic behaviour of pedestrians is important. Moreover, a generic theory of pedestrian flow behaviour pertaining to all relevant behavioural levels, and providing a consistent theoretical foundation for model development was lacking.

Motivated by the need for accurate pedestrian flow models, Hoogendoorn and Bovy (see References [10–12]) present a comprehensive theory and models for pedestrian activity scheduling, path determination in the two-dimensional space, and walking behaviour, under the assumption of the *pedestrian economicus*. The theory is operationalized in terms of behavioural models that are determined by application of mathematical optimal control theory. This article focuses on the walking behaviour, while assuming that the activity scheduling and path planning have already taken place.

In comparison to other models, this walker model has a *clear theoretical foundation* based on the *micro-economic notion of subjective utility maximization*. Describing human behaviour using the analogy with optimal controllers has been applied with success to model different types of task operation such as car driving. The model describes individual walking behaviour in multidirectional pedestrian flows, which can reproduce observed collective pedestrian flow phenomena. Moreover, important empirical and experimental findings on microscopic pedestrian behaviour, such as anisotropy, and pedestrians co-operation, are included in the game-theoretic modelling approach.

The resulting pedestrian simulation model can support infrastructure designers as well as public transport planners in their tasks thereby optimizing their design. Also the management of pedestrian flows demands understanding of both the collective pedestrian flows as well as the individual pedestrian movements in the flow.

2. GENERAL MODELLING APPROACH

The presented model distinguishes two components: (1) the physical model and (2) the control model. The *physical model* describes the forces acting upon the pedestrians, for instance, when pedestrians collide. Pedestrians are described as compressible (circular) particles upon which both normal forces and tangential forces (friction) act. Let p denote the considered pedestrian, and let $\mathbf{r}_p(t)$ and $\mathbf{v}_p(t)$, respectively, denote the location and velocity of p at time instant t. The physical model can then be described as follows:

$$\dot{\mathbf{r}}_p(t) = \mathbf{v}_p(t) \quad \text{and} \quad \dot{\mathbf{v}}_p(t) = \mathbf{a}_p(t) \tag{1}$$

where $\mathbf{a}_p(t)$ denotes the acceleration of p at instant t. The acceleration $\mathbf{a}_p(t)$ consist of a controllable part $\mathbf{u}_p(t)$ and a non-controllable part $\mathbf{w}_p(t)$. The non-controllable part $\mathbf{w}_p(t)$ reflects the aforementioned physical forces, expressed by the following equations:

$$\mathbf{w}_p = \sum_{q \neq p} k_p^0 (l_{pq} - D_{pq})^+ \mathbf{h}_{pq} + \kappa_p^0 ((\mathbf{v}_q - \mathbf{v}_p)' \mathbf{t}_{pq})(l_{pq} - D_{pq})^+ \mathbf{t}_{pq} \tag{2}$$

where $D_{pq} = \|\mathbf{r}_q - \mathbf{r}_p\|$ denotes the *gross distance* between p and q, and where

$$\mathbf{h}_{pq} = \frac{\mathbf{r}_q - \mathbf{r}_p}{D_{pq}} \tag{3}$$

denotes the unit vector pointing from the centre of pedestrian p to pedestrian q; $\mathbf{t}_{pq}$ denotes the unit vector perpendicular to $\mathbf{n}_{pq}$; κ_p^0 and k_p^0 are constant model parameters. Note that $(\mathbf{v}_q - \mathbf{v}_p)' \mathbf{t}_{pq}$ denotes the projection of the vector $(\mathbf{v}_q - \mathbf{v}_p)$ along $\mathbf{t}_{pq}$. Moreover, l_{pq} denotes the sum of the radius l_p and l_q of pedestrians p and q, and $(a)^+ = \max\{a, 0\}$. Equation (2) shows that when the gross distance D_{pq} between p and q is less than l_{pq} (physical contact between p and q), a certain normal force will repel p away from q in direction $\mathbf{n}_{pq}$. At the same time, friction occurs along $\mathbf{t}_{pq}$ (perpendicular to $\mathbf{n}_{pq}$). The size of the friction depends on the velocity difference $(\mathbf{v}_q - \mathbf{v}_p)$ in this direction $\mathbf{t}_{pq}$. For details, we refer to Reference [7].

The *control model* describes the control decisions made by the pedestrians, i.e. it prescribes the *controllable part* $\mathbf{u}_p(t)$ of the acceleration $\mathbf{a}_p(t)$, and thus complements the physical model. By application of the theory of differential games, the remainder of this article focuses on how a pedestrian p determines the controllable part of the acceleration.

3. THEORY OF PEDESTRIAN BEHAVIOUR

Hoogendoorn and Bovy [10, 11] present an integral theory of pedestrian behaviour where pedestrian behaviour is classified into three mutually dependent levels, namely: (1) activity choice behaviour and activity area choice, (2) wayfinding to reach activity areas and (3) walking behaviour. Given the activity set a pedestrian aims to perform, the theory asserts that pedestrians make a simultaneous path-choice/activity schedule decision optimizing expected subjective utility. Together with the expected traffic conditions, the result of this choice serves as input for the pedestrian walking behaviour process, described in detail in the remainder of this article. We thus assume that pedestrian p has determined the optimal (or desired) velocity $\mathbf{v}_p^*(t, \mathbf{r}_p)$ at each time instant t and location $\mathbf{r}_p$. For details on pedestrian route choice theory and modelling, we refer to References [10, 11].

3.1. Walking behaviour theory

The walker theory presented in this section describes the most essential microscopic processes in walking, while the models must be able to predict and explain the phenomena in pedestrian flow that are important from both a theoretical and practical point of view, given the envisaged model applications. Examples are the relation between density and speed (e.g. see Reference [1]), and the formation of lanes and clusters (see References [2, 3, 13]), given specific infrastructure design. The presented model is based on six behavioural hypotheses H1–H6, which are listed below:

1. Pedestrians continuously reconsider their walking choices by using current observations and the resulting predictions into the subjective utility optimization (rolling horizon). Pedestrians are thus feedback-oriented controllers.
2. Pedestrians are
   (a) anisotropic particles that react mainly to stimuli in front of them, which
   (b) are (to a certain extent) compressible;
3. walkers anticipate on the behaviour of other pedestrians by predicting their walking behaviour according to non-co-operative or co-operative strategies;
4. pedestrians have limited predicting possibilities, reflected by discounting utility of their actions over time and space, implying that they mainly consider pedestrians in their direct environment;
5. walkers will be more evasive when encountering a group of pedestrians than when encountering a single pedestrian (described by assuming additivity of the proximity costs);
6. pedestrians minimize *predicted* discounted costs resulting from
   (a) straying from the planned trajectory,
   (b) from the vicinity of other pedestrians and obstacles, and
   (c) applying control (differentiating between longitudinal acceleration and lateral acceleration (side stepping)).

Let us briefly reflect upon these hypotheses by considering some research findings from literature. Goffman [4] describes how the environment of the pedestrian (infrastructure and pedestrians) is continuously observed through a mostly subconscious process called scanning in order to side-step small obstructions on the flooring and handling encounters with other pedestrians (H1). The scanning area is an ellipse, narrow to either side of the individual and longest in front of him/her (H2). Moreover, the area of the ellipse changes constantly according to the prevailing traffic density. Goffman [4] also describes how pedestrians react to one another by bilateral, subconscious communication. This is how pedestrians are generally aware of the future behaviour of other pedestrians in their direct environment (H3). Other researchers also stressed the importance of co-operation between pedestrians. In illustration, Sobel and Lillith [14] report that pedestrians are reluctant to unilaterally withdraw from an encounter until the last moment, possibly even resulting in physical contact between the pedestrians. This 'brushing' sends a signal to the 'offender' to co-operate. Naturally, the predictions are limited with respect to the time–horizon as well in a spatial sense (H4). The latter is also due to the limited observation range. Not all encounters will result in a co-operative decision, as the amount of space granted by a pedestrian depends on the cultural, social and demographic characteristics of the interacting pedestrians [6, 14]. Willis *et al.* [15] found similar results for one-to-many interactions: individuals tend to move for groups (H5).

Hill [16] argues that a pedestrian can be seen as a *pilot of a very special vehicle*. In the remainder, we argue that the task of guiding this vehicle is similar to the task of guiding a car, and can hence be described in an analogous manner, e.g. see Reference [17]. Experience and knowledge have skilled the pedestrian in optimally performing the guidance subtask (H6a). It is obvious that pedestrians aim to walk along the route that best meets their walking objectives. However, pedestrians generally dislike walking too close to other pedestrians, yielding tension and irritation [18]. This holds equally for frequent or severe accelerations, since these will yield additional cost in terms of energy consumption. It is plausible that the resulting walking behaviour is a trade-off between these factors (H6b).

3.2. Behavioural parameters

Important *behavioural parameters* that will largely determine pedestrian behaviour are:

- temporal discount factor η_p (describing the rate at which costs are discounted over time), and the spatial discount factor R_p^0 (describing the proximity discomfort reduction rate as a function of the distance between two pedestrians);
- anisotropy factors c_p^+ and c_p^- reflecting (the difference in) behaviour when reacting to stimuli in front or behind, relative to stimuli from the side;
- relative weighting factors c_p^k for the respective walking cost components; relative cost of longitudinal acceleration θ_p and side stepping $1 - \theta_p$;
- physical dimensions of the pedestrians (radius l_p);
- control restrictions (i.e. the maximum acceleration $a_p^0(t, \mathbf{r}_p)$, given specific infrastructure and prevailing traffic conditions).

These parameters are to be estimated from either microscopic data (comparing microscopic pedestrian characteristics and behaviour) or macroscopic data (reproducing emerging flow characteristics, i.e. speed density curves, spatiotemporal patterns, etc.). The parameters are discussed in detail in the ensuing of this article.

4. CONCEPTUAL WALKING TASK MODEL

This section discusses a conceptual model to describe walking behaviour in terms of a closed feedback control system where the pedestrian predicts the behaviour of other pedestrians (referred to as *opponents* in game-theoretic terms), including the presumed opponents' reactions to the control decisions of the pedestrian. A pedestrian is thus modelled as an operator performing walking tasks. These tasks comprise both control and guidance of the 'pedestrian vehicle'. The *pedestrian vehicle control subtask* includes all activities pertaining to the split-second exchange of information between the brain ('the pilot'), the senses (eyes, ears), and the actuators (legs, arms). These actions are nearly always *skill based* and performed automatically with no conscious effort. The *pedestrian vehicle guidance subtask* describes the collection of decisions required to guide the pedestrian safely and comfortably over the available walking infrastructure and it's elements, as well as the proper behaviour when encountering other pedestrians.

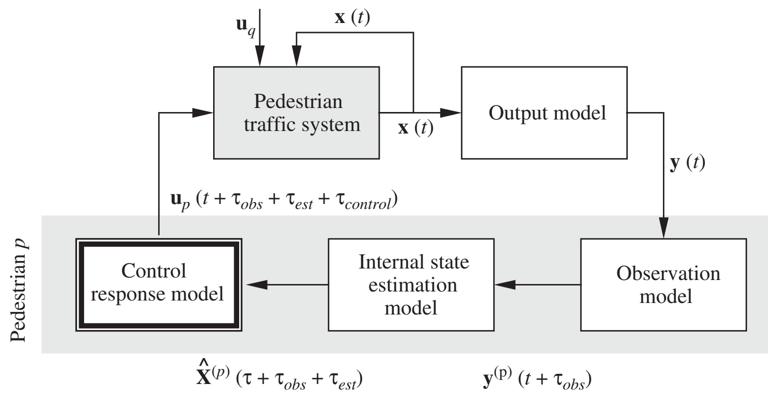

Figure 1. Continuous control loop of walking task execution for pedestrian p.

4.1. Walking as optimal feedback-oriented control

A pedestrian is assumed to use an internal model for determining an appropriate guidance control decision, thereby aiming to optimize his/her task performance. Experience and knowledge have skilled the pedestrian in performing this walking task, and hence the assumption of optimal behaviour is justified. The optimization includes the limitations of the pedestrians in terms of observation, information processing and internal state estimation, as well as processing times and reaction times. Moreover, the optimization includes the pedestrian walking strategy. Note that similar models have been proposed to describe execution of similar tasks, such as car driving [19, 20].

A pedestrian is constantly monitoring his/her position and velocity relative to the other pedestrians in the flow, as well as relative to obstacles and walls (H1). This monitoring allows the pedestrian to perform corrective actions, implying that walking is *feedback oriented*. A continuous feedback control system consists of a comparison between the input (positions, velocities) and the controlled output (acceleration, deceleration and direction changes).

Figure 1 depicts the continuous control loop for pedestrian $p \in \mathcal{Q}$, where $\mathcal{Q} = \{1, \ldots, n\}$ denotes the set of all pedestrians in the walking facility. The pedestrian traffic system describes the *actual pedestrian traffic operations*. The state $\mathbf{x}(t)$ of the system encompasses all information required to summarize the system's history, typically including the positions $\mathbf{r}_q(t)$ and the velocities $\mathbf{v}_q(t)$ of pedestrians q

$$\mathbf{x}(t) := (\mathbf{r}, \mathbf{v}) \quad \text{where } \mathbf{r} := (\mathbf{r}_1, \ldots, \mathbf{r}_n) \text{ and } \mathbf{v} := (\mathbf{v}_1, \ldots, \mathbf{v}_n) \tag{4}$$

The dynamics of the pedestrian traffic system are described by a system of coupled ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \tag{5}$$

where $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ denotes the control vector, reflecting how the pedestrians influence the state $\mathbf{x}$ (e.g. by accelerating and changing directions). The state dynamics equation (5) in fact reflects the physical model (1). The output model maps the state $\mathbf{x}(t)$ onto the observable system output $\mathbf{y}(t)$. Since both the location and the velocity can be observed externally, in the remainder of the contribution, we will assume that state can be observed directly, i.e. $\mathbf{y}(t) = \mathbf{x}(t)$.

In Figure 1, the shaded block containing the three boxes describes the walker model for pedestrian p, consisting of (1) the observation model, (2) the state estimation model and (3) the control response model. The *observation model* of pedestrian p describes which of the elements of the pedestrian traffic system's output $\mathbf{y}(t)$ can be observed by p. The output $\mathbf{y}^{(p)}(t)$ is generally a function of the system's output $\mathbf{y}(t)$, state $\mathbf{x}(t)$, the observation time delay τ_{obs}, and observation errors $\boldsymbol{\varepsilon}_{\text{obs}}$. The output $\mathbf{y}^{(p)}(t)$ typically describes the locations and velocities of pedestrians that p can observe. Pedestrian p will use his/her observation $\mathbf{y}^{(p)}$ as well as his/her experiences to estimate the state of the pedestrian traffic system, which is described by the *internal state estimation model* of Figure 1. Without going into details, we assume that the internal state estimate can be described by a function of the previous internal state estimate $\mathbf{x}^{(p)}(t)$, the observation $\mathbf{y}^{(p)}(t)$, and the estimation delay τ_{est} and error $\boldsymbol{\varepsilon}_{\text{est}}$. Here, $\hat{\mathbf{x}}^{(p)}(t)$ describes estimates of the locations and velocities of pedestrians q. Note that these estimates generally contain more information than the observations $\mathbf{y}^{(p)}(t)$. E.g. p may be aware of the presence of a pedestrian q, who is blocked from p's view (either by an obstacle or another pedestrian) by previous observations and prediction q's walking behaviour.

The *control response model* in Figure 1 reflects the process of determining the control actions of pedestrian p. Together with the non-controllable acceleration caused by physical encounters between pedestrians, the control actions describe the acceleration vector $\mathbf{a}_p(t)$ applied by p. We hypothesize that the control decision depends on the internal state estimate, the walking strategy, reflecting the walking objectives of the pedestrians, and the control delay τ_{control}. The control delay describes both the time needed to determine the control decision, and the time needed to start the control movement. In determining the walking actions, p will predict and optimize the walking cost $J^{(p)}$ incurred during some time interval $[t, t+T)$. The prediction of the internal state $\mathbf{x}^{(p)}$ will be described by the internal state dynamics (see Equation (9)). This state dynamics include the reactions of the pedestrians q as well. For any control path applied during $[t, t+T)$, p predicts the state dynamics using Equation (9), as well as the walking cost $J^{(p)}$ and chooses the optimal control path minimizing this cost (Equation (11)). The cost optimization process, its parameters, and its resulting control decisions differ between the pedestrians, due to different objectives, preferences, and physical abilities.

The control framework discussed here will be used to derive mathematical models. To this end, the following section specifies the pedestrian kinematics as well as the walking cost $J^{(p)}$ as functions of the internal state $\mathbf{x}^{(p)}$ and the control $\mathbf{u}^{(p)}$. In doing so, both observation errors and delays will be neglected. Furthermore, the observation and estimation models are specified such that pedestrian p is assumed to be able to estimate the locations and velocities of all pedestrians $q \in \mathcal{Q}$, implying that p has perfect information regarding the current state $\mathbf{x}(t)$ of the pedestrian traffic system. The model is specified such that p will only react to pedestrians in his/her direct environment and pedestrians that are directly in front of him/her.

5. PEDESTRIAN KINEMATICS AND WALKING COSTS

Walking is modelled by formulating the walking task of p in terms of an optimal control problem, where a performance function $J^{(p)}$ (walking cost) is minimized, subject to the kinematics of the pedestrian p as well as the other pedestrians $q \neq p$ (reflected by the internal prediction model used by p). These kinematics are reflected by ordinary differential equations describing the dynamics of the state $\mathbf{x}$ as a function of the controls of pedestrians p and q, and the state itself. In turn, the state comprises the current conditions of pedestrians p and q.

The performance function $J^{(p)}$ conveys different factors determining the walking cost. These factors are expressed in mathematical terms such that different theoretically and practically important issues of pedestrian behaviour, such as anisotropy, are properly described. We assume that walking costs are discounted, implying that costs incurred in the near future will have a stronger impact on walking than costs that might be experienced in the long run. This temporal cost discounting reflects the limited prediction horizon of pedestrians. Spatial cost discount reflects that nearby pedestrians and obstacles will have a stronger effect on walking behaviour than pedestrians and obstacles that are far away. While pedestrian p predicts the incurred walking cost, he or she will have to make some assumption on the walking strategy that the other pedestrians will adhere to.

5.1. Pedestrian kinematics and state dynamics

This section describes the internal model (or 'mental model') used by pedestrian p in order to predict the dynamics of the state $\mathbf{x}$. The model will be in line with the physical model described by Equation (1). Let $\mathbf{r}_q(t)$ and $\mathbf{v}_q(t)$ denote the location and the velocity of pedestrians q at time instant t, where q can reflect pedestrian p as well as his or her opponents. The prediction model used by pedestrian p is very straightforward and defined by the following ordinary differential equations, similar to Equation (1)

$$\dot{\mathbf{r}}_q = \mathbf{v}_q \quad \text{and} \quad \dot{\mathbf{v}}_q = \mathbf{a}_q \tag{6}$$

for all $q \in \mathcal{Q}$. In this formulation, $\mathbf{a}_p$ denotes the acceleration vector of pedestrian p, while $\mathbf{a}_q$ for $q \neq p$ denotes the *predicted acceleration vector of pedestrian* $q \neq p$. Similar to the physical model, we distinguish between non-controlled acceleration $\mathbf{w}_q$ and controlled acceleration $\mathbf{u}_q$ (i.e. $\mathbf{a}_q = \mathbf{u}_q + \mathbf{w}_q$); $\mathbf{u}_p$ then reflects the control of pedestrian p, while $\mathbf{u}_q$ for $q \neq p$, reflects the acceleration strategies of the opponents q *presumed* by pedestrian p.

In our optimal predictive control framework, it will turn out that an appropriate definition of the internal state $\mathbf{x}^{(p)}$ of pedestrian p enables straightforward application of the theory of differential games. We will use the following definition of the state of pedestrian p (equivalent with Equation (4))

$$\mathbf{x}^{(p)} = \mathbf{x} = (\mathbf{r}_1, \ldots, \mathbf{r}_n, \mathbf{v}_1, \ldots, \mathbf{v}_n)' \tag{7}$$

(For notational convenience, the superscript (p) is generally omitted in the remainder of the article.) Note that this definition describes the entire state of the system, and as such is also a valid state definition for the opponents q of p.

For all pedestrians $q \in \mathcal{Q}$, the state can be influenced either directly or indirectly by the controllable part $\mathbf{u}_q$ of the acceleration $\mathbf{a}_q$. The control vector thus becomes

$$\mathbf{u}^{(p)} = \mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)' \tag{8}$$

Note that generally pedestrian p can only directly influence $\mathbf{u}_p$; the extent in which u_q for $q \neq p$ can be determined by p depends on the walker strategies of the opponents q and the extent to which p is aware of these strategies.

We can easily determine the *state dynamics* using Equation (6) and the definitions of the state x and the control vector $\mathbf{u}$ (similar to Equation (5))

$$\dot{\mathbf{x}}(s) = \mathbf{f}(s, \mathbf{x}, \mathbf{u}) \quad \text{for } s > t \quad \text{with } \mathbf{x}(t) = \hat{\mathbf{x}}^{(p)}(t) \tag{9}$$

where $\hat{\mathbf{x}}^{(p)}(t)$ denotes pedestrian p's estimate of the state at instant t.

5.2. Pedestrian walking discomfort (resistance)

Pedestrians incur different types of discomfort (disutility or cost) while walking. These costs can be expressed as a function $J^{(p)}$ of the control vector $\mathbf{u}^{(p)}$ applied by the pedestrians during the planning period $[t, t+T)$, where t denotes the current time, and T denotes the terminal time. In optimal control theory, the cost function is generally determined by integrating the so-called running cost L_p over the planning period $[t, t+T)$. The limited prediction capabilities of the pedestrians are modelled using the concept of (temporal or time-) *discounted costs* (H4). The following specification of the cost functional $J^{(p)}$ will be used in the ensuing

$$J^{(p)}(\mathbf{u}_1, \ldots, \mathbf{u}_n | t, \hat{\mathbf{x}}^{(p)}(t)) = \int_t^{\infty} \mathrm{e}^{-\eta_p s} L_p(s, \mathbf{x}(s), \mathbf{u}_1(s), \ldots, \mathbf{u}_n(s))\, \mathrm{d}s \tag{10}$$

subject to the (predicted) state dynamics (9), where $\eta_p \geqslant 0$ is the (temporal) discount factor. Based on the notion of utility maximization (H6), pedestrian p will apply the acceleration function $\mathbf{u}_p^*$ that minimizes Equation (10)

$$\mathbf{u}_p^* = \mathbf{u}_p^*(t, \hat{\mathbf{x}}^{(p)}(t)) = \arg \min_{\mathbf{u}_p \in \mathscr{U}_p} \{J^{(p)}(\mathbf{u}_1, \ldots, \mathbf{u}_n | t, \hat{\mathbf{x}}^{(p)}(t))\} \tag{11}$$

where $\mathscr{U}_p$ denotes the *set of admissible controls* for pedestrian p, e.g.

$$\mathscr{U}_p = \{\mathbf{u}_p \text{ such that } \|\mathbf{u}_p\| \leqslant a_0(t, \mathbf{r}_p)\} \tag{12}$$

Here, $a_0(t, \mathbf{r}_p)$ is the time- and location-dependent maximum acceleration/deceleration.

5.3. Running cost components

The running cost L_p reflects a variety of factors k. Without loss of generality, we assume that the running cost is *linear in parameters*:

$$L_p(t, \mathbf{x}, \mathbf{u}) = \sum_k c_{p,k} L_{p,k}(t, \mathbf{x}, \mathbf{u}) \tag{13}$$

where $c_{p,k}$ denotes the relative weight of cost component $L_{p,k}$. Hypotheses H6a–c state that three (running) cost factors are considered: (a) cost $L_{p,1}$ of drifting from the planned trajectory; (b) cost $L_{p,2}$ of walking (too) near other pedestrians (and obstacles) and (c) cost $L_{p,3}$ of applying acceleration.

In References [10, 11], it was hypothesized that pedestrians at the *tactical planning level* determine an *optimal velocity functional* $\mathbf{v}_p^* = \mathbf{v}_p^*(t, \mathbf{r}_p)$ by minimizing the expected cost of walking and performing activities. At the operational level, H6a can be expressed by using a quadratic running cost function

$$L_{p,1} = \tfrac{1}{2}(\mathbf{v}_p^* - \mathbf{v}_p)'(\mathbf{v}_p^* - \mathbf{v}_p) \tag{14}$$

The cost caused by the presence of other pedestrians in the walking area will be referred to as the *proximity discomfort or proximity cost* (H6b). The discomfort incurred by p due to walking too close to pedestrian q is given usually by a function of the (relative) location and (relative) velocity of p with respect to q. $L_{p,2}$ will also depend on parameters specific for pedestrians p and q, such as their size (required to determine the net distance between pedestrian p and q). Moreover, the fact that pedestrians are *anisotropic* (H2a) is included in the proximity cost as well. H5 implies that the *total* proximity cost $L_{p,2}$ can be determined by the *sum of proximity costs* caused by pedestrians q, i.e.

$$L_{p,2} = \sum_{q \neq p} L_{pq,2}(\mathbf{x}) \tag{15}$$

We assume that pedestrians are to a certain extent compressible (albeit at a very high cost). This implies that under specific circumstances, the gross distance D_{pq} can be smaller than the sum of the radii l_{pq}. In these cases, the non-controllable part of the acceleration $\mathbf{w}_p$ describes how pedestrians are repelled from one another, as well as the frictional forces between them.

Let us consider the *effective locations* $\tilde{\mathbf{r}}_{pq}$, which are defined by

$$\tilde{\mathbf{r}}_{pq} = \mathbf{r}_p - l_p \mathbf{h}_{pq} \tag{16}$$

where $\mathbf{h}_{pq}$ is defined by Equation (3) and use this definition to define the *perceived distance* $\tilde{D}_{pq}$ *between p and q*

$$\tilde{D}_{pq} = \begin{cases} \sqrt{\psi^2[(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_p]^2 + [(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{n}_p]^2}, & D_{pq} > l_{pq} \\ 0, & D_{pq} \leqslant l_{pq} \end{cases} \tag{17}$$

where e_p is the unit direction vector of pedestrian p, and n_p is the unit vector perpendicular to e_p (i.e. according to Equation (20)).

The inner product $(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_p$ denotes the projection of the vector $(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})$ along $\mathbf{e}_p$. The function ψ will be defined such that in case q is in front of p, the cost $L_{pq,2}$ will be relatively high given the net distance $\|\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp}\|$ between p and q; when q is behind p, the cost will be relatively low. Mathematically, '*q is in front of p*' can be described by the requirement $(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_p \geqslant 0$; '*q* is behind *p*' implies that $(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_p < 0$. Thus, ψ can be expressed in terms of $(\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_{pq}$. The most straightforward form of ψ is the following:

$$\psi = \begin{cases} c_p^+ & \text{for } (\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_p \geqslant 0 \\ c_p^- & \text{for } (\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})'\mathbf{e}_p < 0 \end{cases} \tag{18}$$

where $0 < c_p^+ \leqslant 1 \leqslant c_p^-$, and $D_{pq} > l_p + l_q$. For $D_{pq} \leqslant l_p + l_q$, we have $\psi = 1$. From a behavioural point of view, c_p^- can be interpreted as describing the extent to which pedestrians react to other signals, received by *other senses than seeing* (e.g. hearing, feeling, smelling).

The proximity cost $L_{pq,2}$ is then defined by

$$L_{pq,2} = \exp(-\tilde{D}_{pq}/R_p^0) \tag{19}$$

where R_p^0 denotes a *spatial discount factor* for pedestrian p.

According to H6c, we need to distinguish (1) cost of *longitudinal acceleration* (increasing or decreasing speed) and (2) cost of *lateral acceleration* (side stepping perpendicular to the current direction $\mathbf{e}_p$). Let us define the vectors e_p and n_p

$$\mathbf{e}_p = (\mathbf{e}_p^1, \mathbf{e}_p^2)' = \mathbf{v}_p/\|\mathbf{v}_p\| \quad \text{and} \quad \mathbf{n}_p = (-\mathbf{e}_p^2, \mathbf{e}_p^1)' \tag{20}$$

respectively, denoting the unit vectors in the direction and perpendicular to the current direction of p. The total cost of acceleration is found by the weighted sum of the acceleration in the longitudinal direction $\mathbf{u}_p'\mathbf{e}_p$ and the acceleration in the lateral direction $\mathbf{u}_p'\mathbf{n}_p$

$$L_{p,3} = \theta_p(\mathbf{u}_p'\mathbf{e}_p)^2 + (1-\theta_p)(\mathbf{u}_p\mathbf{n}_p)^2 \tag{21}$$

where θ_p is a weighting factor, with $0 \leqslant \theta_p \leqslant 1$.

6. WALKER MODEL DERIVATION

The previous section discussed the state dynamics as well as the specification of the running cost factors $L_{p,k}$ that yield the optimization objective $J^{(p)}$. This section describes the walker model derived by application of optimal control and differential game theory [21]. For the sake of simplicity and mathematical tractability, we assume that pedestrians have perfect information regarding the current state $\mathbf{x}(t)$, and are memoryless.

6.1. *Walking as a non-co-operative differential game*

Let us first consider the case that pedestrian p predicts the behaviour of the opponents $q \neq p$ by assuming that the latter behave according to some feedback mechanism. That is, p assumes that opponent $q \neq p$ applies control $\mathbf{u}_p$ which can be expressed as a function of the state of the pedestrian system $\mathbf{x}(t)$, i.e. $\mathbf{u}_q(t) = \mathbf{u}_q(\mathbf{x}(t))$ for all $q \neq p$. In this case, the state dynamics reflect how pedestrian p predicts the behaviour of the state, and not the dynamics of the state *per se*. I.e. unless $\mathbf{u}_q(\mathbf{x}^{(p)})$ denotes the true behaviour of the opponents q, the true state $\mathbf{x}(s)$ and the state $\mathbf{x}^{(p)}(s)$ predicted by p will generally not be the same for $s > t$. Using the specifications of the state dynamics and the cost function, application of the *maximum principle* (MP) allows determining the following optimal acceleration law for pedestrian p.

To apply the MP, we first define the Hamilton function

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = L_p + \boldsymbol{\lambda}'\mathbf{f}(t, \mathbf{x}, \mathbf{u}) \tag{22}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) = \boldsymbol{\lambda}(t, \mathbf{x})$ denotes the so-called adjoint state variables, depicting the marginal costs of the state variables $\mathbf{x}$.

Let $\tilde{\mathbf{u}}_p = \{\mathbf{u}_q\}_{q \neq p} = \tilde{\mathbf{u}}_p(\mathbf{x})$ denote the controls of all opponents of p. Using the Hamilton function H, we can write down necessary conditions for optimality

$$H(t, \mathbf{x}, \mathbf{u}_p^*, \tilde{\mathbf{u}}_p(\mathbf{x})) \leqslant H(t, \mathbf{x}, \mathbf{u}_p, \tilde{\mathbf{u}}_p(\mathbf{x})) \quad \text{for all } \mathbf{u}_p \in \mathscr{U}_p \tag{23}$$

Inequality (23) holds for all admissible controls $\mathbf{u}_p \in \mathscr{U}_p$ that can be applied by pedestrian p. When the control is not restricted, the optimality condition can be expressed in terms of the partial derivative $\partial H/\partial \mathbf{u}_p = 0$, yielding the following expression for the optimal acceleration $\mathbf{u}_p^*$:

$$\mathbf{u}_p^* = -\mathbf{M}_p \frac{1}{c_{p,3}} e^{\eta_p t} \boldsymbol{\lambda}_{\mathbf{v}_p}(t, \mathbf{x}) \tag{24}$$

It thus turns out that the optimal acceleration $\mathbf{u}_p^*$ is a linear function of the marginal costs $\boldsymbol{\lambda}_{\mathbf{v}_p}$ of the velocity $\mathbf{v}_p$ of pedestrian p.

The MP states that the dynamics of the adjoint variables satisfy $-\dot{\boldsymbol{\lambda}} = \partial H/\partial \mathbf{x}$. Since the state dynamics and the running cost components are time independent, we can show easily that $\boldsymbol{\lambda}(t, \mathbf{x}) = e^{-\eta_p t}\boldsymbol{\Lambda}(\mathbf{x})$, which implies that $\eta_p \boldsymbol{\lambda}(t, \mathbf{x}) = \partial H/\partial \mathbf{x}$. The latter expression enables straightforward derivation of the marginal costs $\boldsymbol{\lambda}(t, \mathbf{x})$ for all state variables, and thus also for the marginal costs $\boldsymbol{\lambda}_{\mathbf{v}_p}$. It is left to the reader to verify that we find the following expression for the optimal control $\mathbf{u}_p^*$ applied by pedestrian p:

$$\mathbf{u}_p^* = \overbrace{\mathbf{M}_p \left[\mathbf{I} - \frac{1}{\eta_p}\left(\frac{\partial \mathbf{v}_p^*}{\partial \mathbf{r}_p}\right)'\right]\left(\frac{\mathbf{v}_p^* - \mathbf{v}_p}{\tau_p}\right)}^{(\mathbf{u}_p^*)_{\text{accel}}} - \overbrace{A_p^0 \mathbf{M}_p \left[\frac{\partial L_{p,2}}{\partial \mathbf{r}_p} + \eta_p \frac{\partial L_{p,2}}{\partial \mathbf{v}_p}\right]}^{(\mathbf{u}_p^*)_{\text{int},1}}$$

$$- \overbrace{A_p^0 \mathbf{M}_p \sum_{q \neq p} \left(\frac{\partial \mathbf{u}_q}{\partial \mathbf{r}_p} + \eta_p \frac{\partial \mathbf{u}_q}{\partial \mathbf{v}_p}\right)' \mathbf{S}_p^{-1} \left[\frac{\partial L_p^2}{\partial \mathbf{r}_q} + \eta_p \frac{\partial L_p^2}{\partial \mathbf{v}_q}\right]}^{(\mathbf{u}_p^*)_{\text{int},2}} \tag{25}$$

where τ_p denotes the *acceleration time*, defined by

$$1/\tau_p = \frac{c_{p,1}}{\eta_p c_{p,3}} \tag{26}$$

The acceleration time τ_p reflects the time needed by a pedestrian to accelerate or decelerate to the desired velocity $v_p^*(t, r_p)$. The definition shows how τ_p depends on the temporal discount factor η_p, and the weighing factors $c_{p,3}$ and $c_{p,1}$ of applying control and drifting from the planned (optimal) trajectory. Note that when η_p or c_p^3 increase, τ_p increases likewise, implying that when either short-term effects become more important or the cost of accelerating are high, pedestrians are less inclined to quickly adapt to the planned trajectory. At the same time, when the relative cost of drifting from the planned trajectory increases, the acceleration time τ_p decreases and the response is quicker (at the cost of high accelerations); A_p^0 denotes the *interaction constant*, defined by

$$A_p^0 = \frac{c_{p,2}}{(\eta_p)^2 c_{p,3}} \tag{27}$$

The interaction constant reflects the distance keeping behaviour between pedestrians: as A_p^0 increases, pedestrians are inclined to keep more distances between themselves and the other pedestrians. From the definition, it is clear how A_p^0 depends on the relative weight $c_{p,2}$ of walking near another pedestrian, the temporal discount factor η_p, and the relative cost of acceleration (applying control): when either $c_{p,2}$ increases, or η_p and $c_{p,3}$ decreases, A_p^0 increases. This implies that when the relative cost of walking near other pedestrians increases, or the discount factor decreases (implying that long-term effects become more important), and the cost of acceleration decreases, pedestrians will be more inclined to maintain larger spacings.

The matrix $\mathbf{M}_p$ equals

$$\mathbf{M}_p = \tfrac{1}{2}[\theta_p \mathbf{e}_p \mathbf{e}_p' + (1-\theta_p)\mathbf{n}_p \mathbf{n}_p']^{-1} \tag{28}$$

Let us briefly discuss the model: the optimal acceleration (25) of pedestrian p is governed by three terms. The first term $(\mathbf{u}_p^*)_{\text{accel}}$ can be considered an *acceleration term*, and describes how p aims to keep walking along the planned path described by the optimal velocity $\mathbf{v}_p^*$. This is in part determined by the weights $c_{p,1}$ and $c_{p,3}$ determining the acceleration time τ_p, reflecting, respectively, the cost of drifting from the planned velocity $\mathbf{v}_p^*$ and of applying control $\mathbf{u}_p$. The ratio $c_{p,1}/c_{p,3}$ reflects that when the cost of acceleration is relatively low (i.e. $c_{p,3}$ relatively small) or when the cost of drifting from the optimal velocity is relatively high (i.e. $c_{p,1}$ large), the acceleration time τ_p will be small while the acceleration term will be large. In this case, the pedestrian will be more inclined to keep walking at the optimal speed and in the optimal direction, even though this may imply large accelerations from time to time. The implications of the matrix $\mathbf{M}_p$ have already been discussed; it describes the preference for longitudinal acceleration and braking over side stepping. Neglecting the role of $\mathbf{M}_p$, the term

$$\left[\mathbf{I} - \frac{1}{\eta_p}\left(\frac{\partial \mathbf{v}_p^*}{\partial \mathbf{r}_p}\right)'\right]\left(\frac{\mathbf{v}_p^* - \mathbf{v}_p}{\tau_p}\right) \tag{29}$$

shows that the pedestrian will generally aim to walk in the direction $(\mathbf{v}_p^* - \mathbf{v}_p)$, i.e. into the direction where the drifting cost $L_{p,1}$ will decrease most rapidly. The matrix $\mathbf{I} - (1/\eta_p)(\partial \mathbf{v}_p^*/\partial \mathbf{r}_p)'$ corrects this direction in line with changes in the optimal velocity that are caused by changes in the location $\mathbf{r}_p$ of the pedestrian. In other words, it corrects for the fact that when the pedestrian's location r_p is changed, in general so will his/her optimal velocity $\mathbf{v}_p^* = \mathbf{v}_p^*(\mathbf{r}_p)$.

The second term $(\mathbf{u}_p^*)_{\text{int},1}$ of Equation (25) reflects what we will call *first-order interactions*. showing how pedestrian p reacts to the other pedestrians $q \neq p$. $(\mathbf{u}_p^*)_{\text{int},1}$ shows that the effect of multiple interactions is additive.

Considering the interaction of pedestrians p and q, $(\mathbf{u}_p^*)_{\text{int},1}$ shows that p will make the following trade-off in determining walking direction and speed. On the one hand, the term $\partial L_{pq,2}/\partial \mathbf{r}_p$ reflects the direction in which the proximity cost $L_{pq,2}$ changes most rapidly. On the other hand, the term $\eta_p \partial L_{pq,2}/\partial \mathbf{v}_p$ reflects changes in the proximity cost $L_{pq,2}$ due to changes in the velocity v_p of p. For instance, this can express the fact that the proximity cost will be relatively high when two pedestrians approach each other very quickly, even though the distance between two pedestrians is still considerable.

Finally, the third term $(\mathbf{u}_p^*)_{\text{int},2}$ of Equation (25) describes *second-order interactions*, reflecting the effects of changes in the locations $\mathbf{r}_q$ and velocities v_q of pedestrians $q \neq p$ that are expected by p, given the anticipated acceleration strategy $\mathbf{u}_q = \mathbf{u}_q(\mathbf{x})$ due to changes in the velocity $\mathbf{v}_p$ of p caused by acceleration.

6.2. Zero-acceleration opponents

Consider the case where p assumes that the opponents will *not respond to the control actions of* p, i.e. $\mathbf{u}_q = 0$ for all $q \neq p$. In practise, such situations can occur when pedestrian p catches up with pedestrian q from behind, and therefore assumes that q will not respond to p. Alternatively, q may be 'dominant' over p, at least from the viewpoint of p, for instance, due to societal reasons (see e.g. References [6, 14, 15]). Finally, it can occur that for some reason, p just has no idea on the response behaviour of q and thus has to rely completely on the information that p can observe (via the state $\mathbf{x}$). In this case, the optimal control law (25) simplifies to $\mathbf{u}_p^* = (\mathbf{u}_p^*)_{\text{accel}} + (\mathbf{u}_p^*)_{\text{int},1}$ (since $(\mathbf{u}_p^*)_{\text{int},2} = 0$).

Let us now consider the proximity cost function specification (19). For this particular case, we have

$$\mathbf{u}_p^* = \mathbf{M}_p \left[I - \frac{1}{\eta_p} \left(\frac{\partial \mathbf{v}_p^*}{\partial \mathbf{r}_p} \right)' \right] \left(\frac{\mathbf{v}_p^* - \mathbf{v}_p}{\tau_p} \right)$$

$$- A_p^0 \mathbf{M}_p \sum_{q \neq p} \frac{\psi^2((\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})' \mathbf{e}_p)\mathbf{e}_p + ((\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})' \mathbf{n}_p)\mathbf{n}_p}{\tilde{D}_{pq}} \mathrm{e}^{-\tilde{D}_{pq}/R_p^0} \chi_{D_{pq} > (l_p + l_q)} \tag{30}$$

where the step function χ is defined by $\chi_{x>a} = 0$ for $x \leqslant 1$ and $\chi_{x>a} = 1$ elsewhere. Equation (30) shows how the optimal acceleration is determined by the acceleration term, and by the first-order interaction term. With respect to the latter, the term

$$\frac{\psi^2((\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})' \mathbf{e}_p)\mathbf{e}_p + ((\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})' \mathbf{n}_p)\mathbf{n}_p}{\tilde{D}_{pq}}$$

describes the direction in which pedestrian p will walk away from opponent q, if the perceived distance $\tilde{D}_{pq}$ between p and q becomes too small. Note that for $\psi = 1$, this term will be equal to $\mathbf{h}_{pq}$ (i.e. the unit vector pointing from q to p; see Equation (16)). Note that the controllable part $\mathbf{u}_p$ of the acceleration $\mathbf{a}_p$ does not describe what happens when pedestrians physically interact; this is described by $\mathbf{w}_p$, see Equation (1).

It is interesting to see that upon choosing $\psi \equiv 1$ (no anisotropy) and $\theta_p = \frac{1}{2}$ (no distinction between longitudinal and lateral acceleration) our model yields the social-forces model of Reference [7]. The theory presented here thus provides a behavioural interpretation of the latter model.

6.3. Co-operative games

From empirical studies, it has been concluded that co-operation plays a very important role in case of interactions between pedestrians. Modelling this co-operation is straightforward, based on the idea that p does not only consider his/her own running cost, but also the running costs of opponents q. In contradiction to the non-co-operative case, the co-operative case does not really define a game situation, since the pedestrians do not have conflicting goals. Rather, the optimal acceleration can be determined easily by application of optimal control theory. It can be shown that the resulting acceleration strategy equals

$$\mathbf{u}_p^* = \mathbf{M}_p \left[I - \frac{1}{\eta_p} \left(\frac{\partial \mathbf{v}_p^*}{\partial \mathbf{r}_p} \right)' \right] \left(\frac{\mathbf{v}_p^* - \mathbf{v}_p}{\tau_p} \right) - A_p^0 \mathbf{M}_p \sum_{q' \in \mathcal{Q}} \alpha_{q'} \left(\frac{\partial L_{q',2}}{\partial \mathbf{r}_p} + \eta_p \frac{\partial L_{q',2}}{\partial \mathbf{v}_p} \right) \tag{31}$$

where $\alpha_{q'} > 0$ is some weighing parameter. This equation clearly shows that pedestrian p includes the proximity costs $L_{q'}^2$ of him/herself as well as of the 'opponents'. Let us note that when the costs incurred by opponent q' equal to the cost that p incurs (e.g. in case of a head-on interaction), the co-operative model is equal to the non-co-operative model described by Equation (30). In the remainder of this article, we will only consider the acceleration strategies (30).

7. ANALYSIS OF PEDESTRIAN FLOW PHENOMENA

This section discusses emerging model characteristics, as well as the relation between the characteristics of these phenomena and the model parameters (only in qualitative terms). To this end, we have applied a rather straightforward time-integration scheme based on *the improved Euler* (or Heun's method).

7.1. Fundamental diagram for unidirectional flows

The fundamental diagrams are established under the assumption that the *pedestrian flow is stationary and homogeneous*. Let us consider the case $\partial \mathbf{v}_p^* / \partial \mathbf{r}_p = 0$. As a consequence, the acceleration $a = 0$, implying that the equilibrium velocity $\mathbf{v}^e$ of pedestrian p equals (neglecting the effects of physical contact between pedestrians)

$$\mathbf{v}_p^e = \mathbf{v}_p^* - \tau_p A_p^0 \sum_{q \in \mathcal{Q}} \frac{\psi^2((\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})' \mathbf{e}_p) \mathbf{e}_p + ((\tilde{\mathbf{r}}_{pq} - \tilde{\mathbf{r}}_{qp})' \mathbf{n}_p) \mathbf{n}_p}{\tilde{D}_{pq}} \, \mathrm{e}^{-\tilde{D}_{pq}/R_0} \tag{32}$$

Let us consider a flow of pedestrians in a very narrow, ring-shaped hallway. The width of the hallway is such that pedestrians can only walk right behind each other (single lane). We assume that all pedestrians walk in the same direction, and have the same optimal speed. Under these assumptions, we can easily determine that the equilibrium velocity $\mathbf{v}_p^e = (v_1^e, v_2^e)$ equals the following analytical expression:

$$v_1^e = v_p^* - \tau_p A_p^0 \left[c_p^+ \frac{\mathrm{e}^{-c_p^+ D_1 / R_p^0}}{1 - \mathrm{e}^{-c_p^+ D_1 / R_p^0}} - c_p^- \frac{\mathrm{e}^{-c_p^- D_1 / R_p^0}}{1 - \mathrm{e}^{-c_p^- D_1 / R_p^0}} \right] \quad \text{and} \quad v_2^e = 0 \tag{33}$$

where $d_1 > 0$ denotes the mean gross distance between two following pedestrians, and v^* is the optimal (or free) speed, determined at the planning level; the pedestrian density is then equal to the width W of the hallway divided by D_1.

Equation (33) reveals the importance of including anisotropy in the model. In case anisotropy is not considered, we have $c_p^+ = c_p^-$, and thus $v_1^e = v_p^*$, which implies that the equilibrium speed equals the optimal speed, irrespective of the pedestrian density. Clearly, v_1^e will only be a monotonically decreasing function of the concentration when $c_p^+ < c_p^-$. Figure 2 shows some

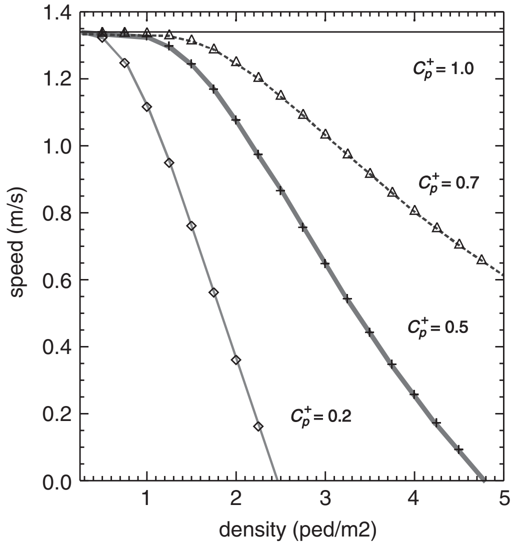

Figure 2. Equilibrium speed as a function of the density k in a narrow hallway for different c_p^+, with $A_p^0 = 25\ \mathrm{m/s^2}$, $R_p^0 = 0.08$ m, $v_p^* = 1.34$ m/s, $\tau_p = 0.5$ s and $c_p^- = 1$.

example speed—concentration curves derived for different values of c_p^+. Note that the model results are very similar to the empirical findings in Reference [1].

If we consider single directional flows in wide hallways (i.e. pedestrians are no longer required to walk behind each other), an analytical expression for the equilibrium speed cannot be determined. However, we can construct numerical approximations, which reveal similar results in case of wider hallways. Besides equilibrium speeds, homogeneous patterns result in which the relative locations of pedestrians are more or less uniform. The properties of the emerging patterns depend on, among other things, the density and the optimal speeds. However, we have observed that when optimal speeds are heterogeneous, as in real life, equilibrium (stationary and homogeneous) conditions may not result (especially for low pedestrian densities) due to the many overtaking opportunities.

7.2. *Lane formation in bidirectional opposing flow*

The derived model predicts that specific structures are self-formed. Among these structures are empty areas with no pedestrians (so-called bubbles), strips in crossing flows, and dynamic lanes. The latter two are important properties of real-life pedestrian flow operations, since it explains the observed equity [13] of efficiency of uni- and bidirectional pedestrian flows, since crossing or bidirectional flows are effectively disaggregated in unidirectional flow regions.

To illustrate the model behaviour, let us first consider a bidirectional flow. Besides their direction, all pedestrians are equal (in terms of their maximum speed). The parameter values are $R_p^0 = 0.08$ m, $\tau_p = 0.4$ s, $A_p^0 = 20$, and $c_p^{\pm} = 1$. The area is 40 by 8 m. Figure 3 shows the formation of lanes of pedestrians walking into the same direction ($\theta_p = \frac{1}{2}$). The lanes are uniform and narrow. We hypothesize that this is caused since pedestrians already walking behind a pedestrian with approximately the same free speed have little incentive to bypass. When the two

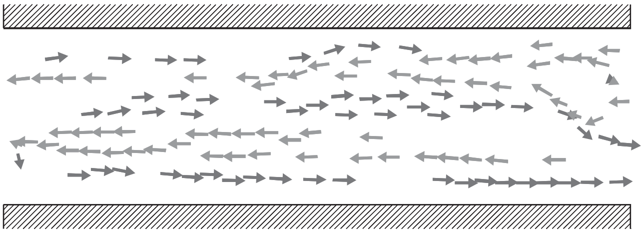

Figure 3. Lane-formation for homogeneous groups of pedestrians (equal maximum speeds) walking in opposite directions.

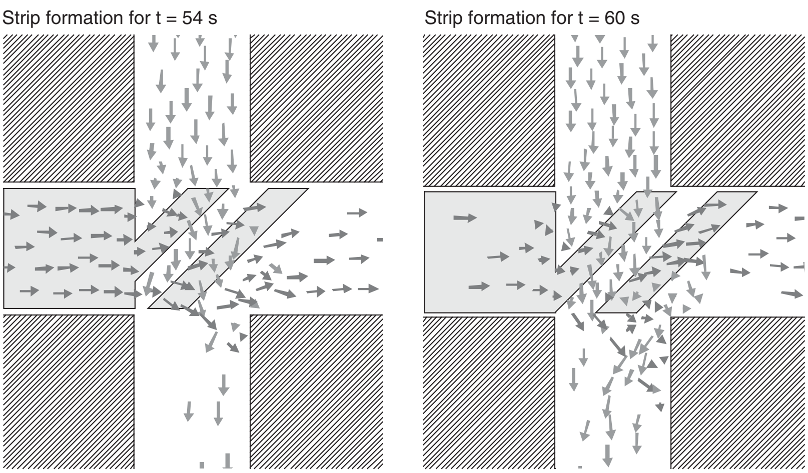

Figure 4. Formation of homogeneous strips in crossing pedestrian flows.

groups are not homogeneous with respect to their maximum speeds, the patterns that are formed are wider and have a more dynamic nature. When the maximum speed diverges even more, the model predicts that no lanes are formed, although it does show groups (clusters) of pedestrians walking into the same direction. Besides the distribution of the optimal speeds, the magnitude of the optimal speeds, and the relative cost of side stepping with respect to longitudinal acceleration and deceleration play a role as well.

7.3. Self-formation of strips in crossing pedestrian flows

The self-formation of spatiotemporal patterns is not confined to bidirectional pedestrian flows. In a number of Japanese studies, the formation of strips in case of crossing pedestrian flows have been described [13]. To study the behaviour of the model in this situation, the case of two homogeneous pedestrian flows which cross each other at an intersection has been considered. Figure 4 shows how the model described in this article predicts similar phenomena: at the crossing, homogeneous strips of approximately the same width are self-formed. The widths of the strips appear to vary depending on the number of pedestrians they consist of. The strips propagate at constant speed which is equal to the speeds of their constituent pedestrians.

Similar to the lane-formation process discussed in the previous section, the self-formation of strips depends critically on the model parameters. It turns out that when pedestrians are more inclined to temporarily decelerate rather than frequently changing their direction to fill up any gap emerging in the flow, strips are formed more frequently.

8. CONCLUSIONS AND FUTURE RESEARCH

In this article, we have put forward a new theory and models for pedestrian walking behaviour. The theory is based on the assumption that pedestrians are utility maximizers (or cost minimizers). An important role is played by the planned paths of the pedestrians, which are continuous functions in time and space. Being the focus of this article, the walking behaviour of the pedestrians is also based on the utility maximization concept. Important factors are the predicted cost due to applying control (accelerating and braking, side stepping), walking too close to other pedestrians and obstacles, and drifting from the planned path. Other important issues are the anisotropic nature of the pedestrians. In fact, pedestrians are described by optimal predictive controllers that minimize the predicted walking cost, given the presumed walking strategies of the other pedestrians. As such, different presumed strategies have been considered, varying from 'no reaction of the other pedestrians' to 'co-operative optimization of cost due to interacting'. To operationalize the theory (i.e. determine optimal decision variables), different techniques from mathematical optimal control theory have been applied, yielding pedestrian behaviour models at the tactical and operational levels. Only the latter level has been discussed in detail. Several model properties, such as the formation of dynamic lanes, homogeneous strips in crossing pedestrian flows, and emergent equilibrium behaviour were studied: the model is able to predict these empirically observed macroscopic characteristics of pedestrian flows.

We have also contemplated upon the relation between these macroscopic flow properties and the different model parameters. For one, we have observed how the self-formation of dynamic lanes in pedestrian flow depends critically on the homogeneity of the pedestrian population, in terms of their optimal walking speeds. Moreover, the inclination to make an evasive manoeuvre (side stepping) rather than to decelerate upon interaction has a negative influence of the lane formation, as well as on the formation of strips in crossing pedestrian flows.

NOMENCLATURE

General

p, q	indices for pedestrian $p \in \mathcal{Q}$ and (opponent) $q \in \mathcal{Q}$
$\mathcal{Q}$	set of all pedestrians $\{1, \ldots, n\}$
t, t_0	(initial) time instant (s)
n	number of pedestrians in system

Pedestrian variables

$\mathbf{r}_p(t)$	location vector of pedestrian p (m)
$\mathbf{v}_p(t)$	velocity vector of pedestrian p (m/s)

$v_p(t)$	$(=\|\mathbf{v}_p(t)\|)$ speed of pedestrian p (m/s)
$\mathbf{e}_p(t)$	$(=\mathbf{v}_p(t)/\|\mathbf{v}_p(t)\|)$ unit direction vector of pedestrian p
$\mathbf{n}_p(t)$	unit vector perpendicular to $\mathbf{e}_p(t)$ (m)
$\mathbf{a}_p(t)$	acceleration vector of pedestrian p (m/s^2)
$a_p(t)$	$(=\|\mathbf{a}_p(t)\|)$ gross acceleration of pedestrian p (m/s^2)
$\mathbf{w}_p(t)$	non-controllable part of acceleration of pedestrian p (m/s^2)
$\mathbf{u}_p(t)$	controllable part of acceleration of pedestrian p (m/s^2)
$\mathbf{v}_p^*(t,\mathbf{r}_p)$	optimal velocity of pedestrian p (m/s)
$a_p^0(t,\mathbf{r}_p)$	maximum acceleration applied by p (m/s^2)

Pedestrian parameters

$1/R_p^0$	spatial discount factor (m^{-1})
η_p	temporal discount factor (s^{-1})
τ_p	relaxation or acceleration time (s)
A_p^0	interaction factor
$c_p^{\pm}$	anisotropy factors
θ_p	weight of longitudinal acceleration cost relative to lateral acceleration
$\mathbf{M}_p$	matrix expressing effect of longitudinal/lateral acceleration distinction
l_p	radius of pedestrian p (m)
T	time horizon of planning period (s)

Interaction parameters

l_{pq}	$(=l_p+l_q)$ sum of radii l_p and l_q of pedestrians p and q (m)
k_p^0	repellent force factor for pedestrian p (1/s^2)
κ_p^0	friction force factor for pedestrian p (1/ms)
$\mathbf{S}_p$	second-order interaction matrix

Interaction variables

D_{pq}	$(=\|r_p-r_q\|)$ gross distance between p and q (m)
$\tilde{D}_{pq}$	perceived net distance between p and q (m)
$\mathbf{h}_{pq}$	unit vector pointing from p to q
$\mathbf{t}_{pq}$	vector perpendicular to $\mathbf{h}_{pq}$
$\tilde{\mathbf{r}}_{pq}$	$(=\mathbf{r}_p-l_p\mathbf{h}_{pq})$ effective location of p with respect to q

Optimal control model

$\mathbf{x}(t)$	pedestrian system state at instant t (physical system/internal model)
$\mathbf{y}(t)$	observable part of pedestrian system state $\mathbf{x}(t)$
$\mathbf{u}_p(t)$	control applied by pedestrian at time instant t (controllable acceleration) (m/s^2)
$\mathbf{f}(t,\mathbf{x},\mathbf{u})$	right-hand side of state equation for entire pedestrian system
$\mathbf{y}^p(t)$	state observations made by pedestrian p at instant t
$\boldsymbol{\lambda}(t)$	adjoint variables of state $\mathbf{x}$

$\tau_{\mathrm{obs}}, \tau_{\mathrm{est}}, \ldots$	delays (observation, estimation, ...) (s)
$\boldsymbol{\varepsilon}_{\mathrm{obs}}, \boldsymbol{\varepsilon}_{\mathrm{est}}, \ldots$	errors (observation, estimation, ...)
$\mathscr{U}_p$	set of admissible controls
$J^{(p)}$	expected walking cost (expressed in terms of travel time) (s)
$c_{p,k}$	relative weight of cost factor k for pedestrian p
L_p	running cost $(= \sum_k c_p^k L_p^k)$
$L_{p,1}$	cost of drifting from planned velocity $v_p^*(t, \mathbf{r}_p)$
$L_{pq,2}$	cost incurred by p of walking too close to q
$L_{p,2}$	$(= \sum_{q \neq p} L_{pq,2})$ total cost of walking too close to other pedestrians
$L_{p,3}$	cost of accelerating/decelerating

ACKNOWLEDGEMENTS

This research is funded by the Social Science Research Council (MaGW) of the Netherlands Organization for Scientific Research (NWO). The authors wish to acknowledge the critical but constructive comments of the anonymous reviewers.

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