Transportation Model

Taha textbook, Chapter 5

After this session, you should …

… be able to describe the transportation model.

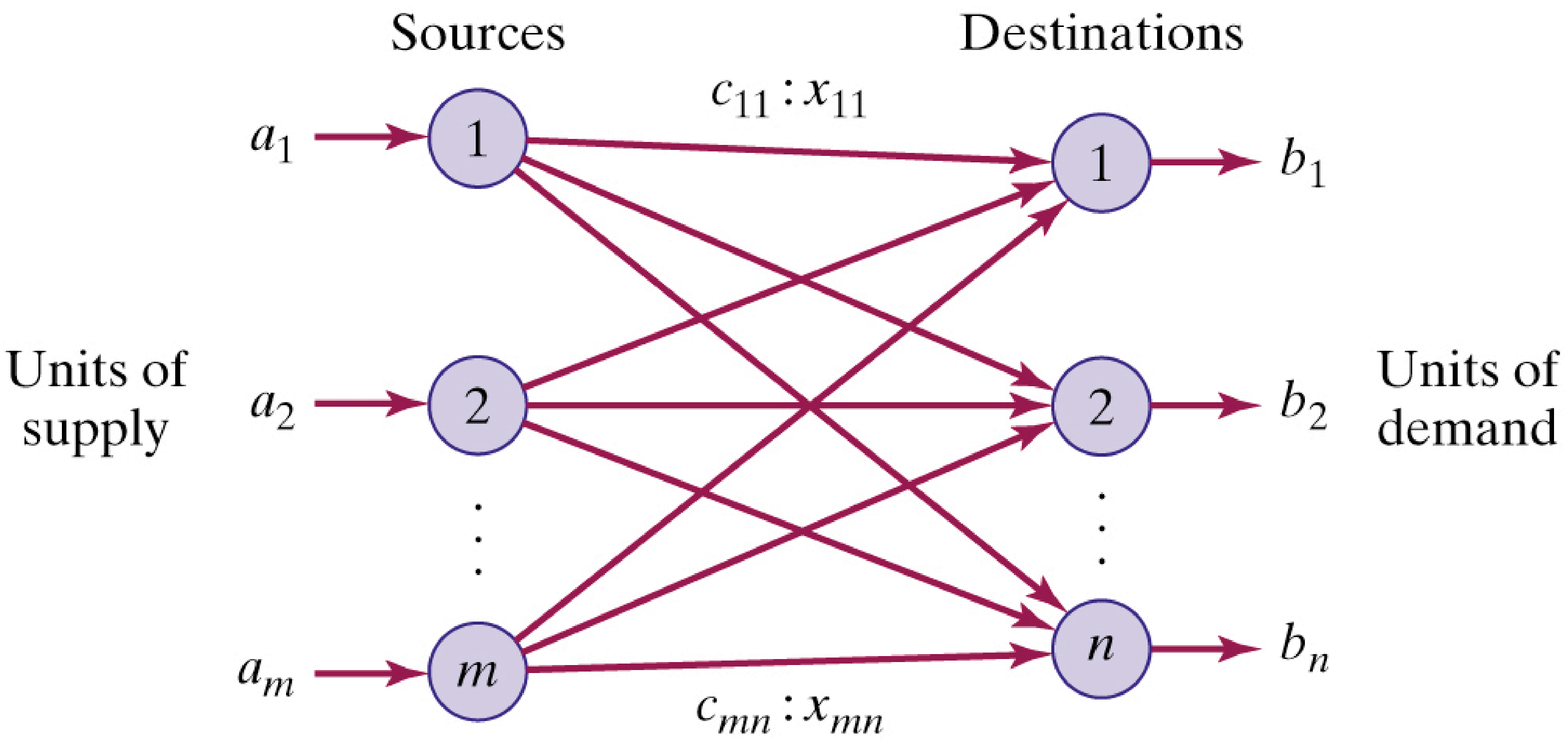
… know how to solve transportation models by hand.

… know how to use Excel solver to find optimal solutions.

Agenda

1. Defining the transportation model
2. The transportation algorithm
3. Special cases
4. Questions

The transportation model refers to a stylized situation



Objective: minimize transportation cost

MG Auto: an example

MG Auto has three plants in Los Angeles, Detroit, and New Orleans and two major distribution centers in Denver and Miami. The quarterly capacities of the three plants are 1000, 1500, and 1200 cars, and the demands at the two distribution centers for the same period are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given in the table.

The trucking company in charge of transporting the cars charges 8 cents per mile per car. Thus, the transportation costs per car on the different routes, rounded to the closest dollar, are computed in the second table.

Mileage:

	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

Cost:

	Denver (1)	Miami (2)
Los Angeles (1)	\$80	\$215
Detroit (2)	\$100	\$108
New Orleans (3)	\$102	\$68

The optimization problem

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

s.t.

$$\begin{aligned}
x_{11} + x_{12} &= 1000 \\
x_{21} + x_{22} &= 1500 \\
x_{31} + x_{32} &= 1200 \\
x_{11} + x_{21} + x_{31} &= 2300 \\
x_{12} + x_{22} + x_{32} &= 1400 \\
x_{ij} \geq 0, i = 1{,}2{,}3, j = 1{,}2 &
\end{aligned}$$

All the constraints are equations because the total supply equals the total demand.

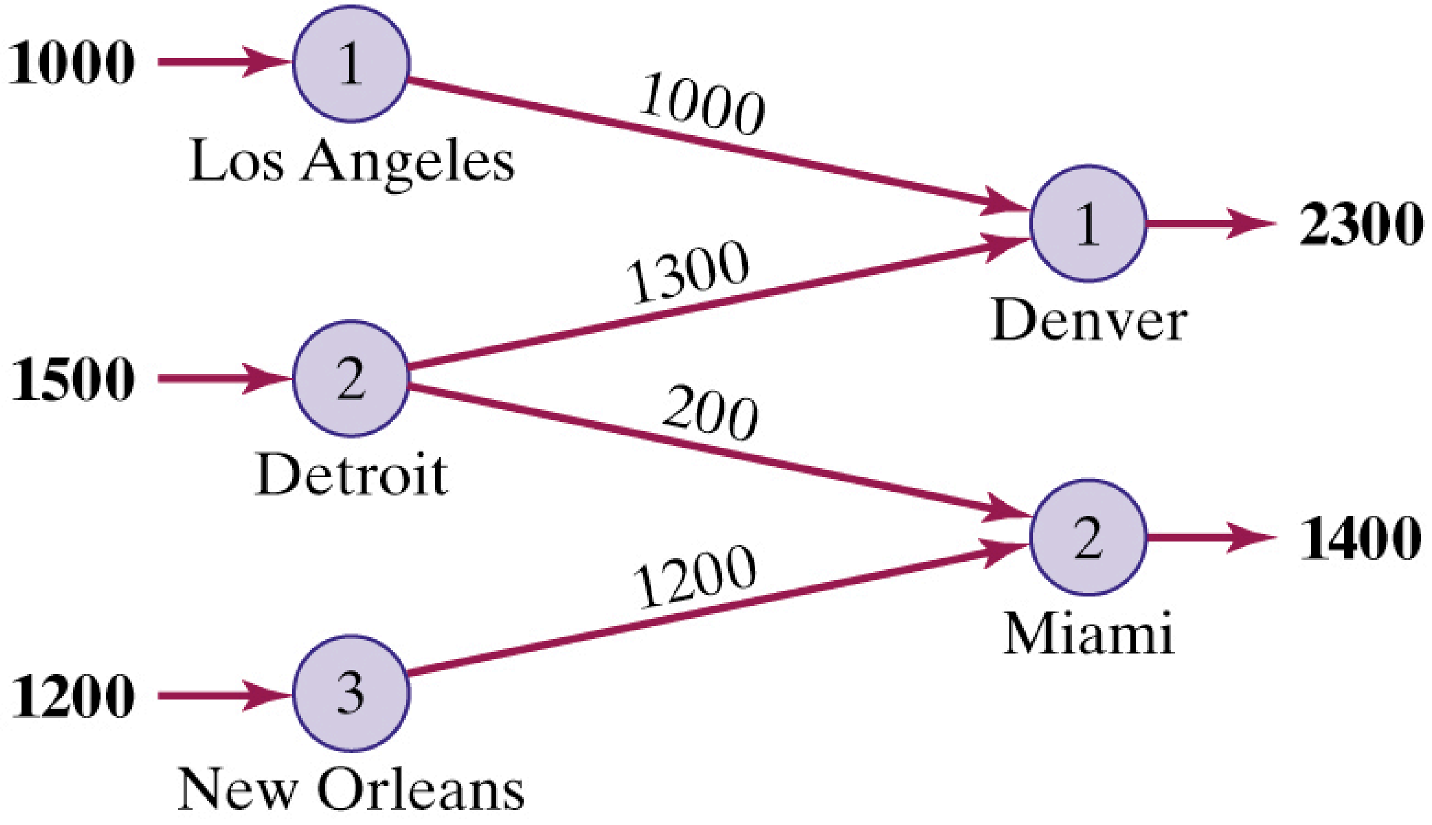
The transportation tableau for MG Auto

The special structure of the model allows for a convenient representation in a table.

The structure is not only common to transportation problems but may be used for many other applications.

	Denver	Miami	Supply
Los Angeles	80 x_{11}	215 x_{12}	**1000**
Detroit	100 x_{21}	108 x_{22}	**1500**
New Orleans	102 x_{31}	68 x_{32}	**1200**
Demand	**2300**	**1400**	

Optimal solution for MG Auto



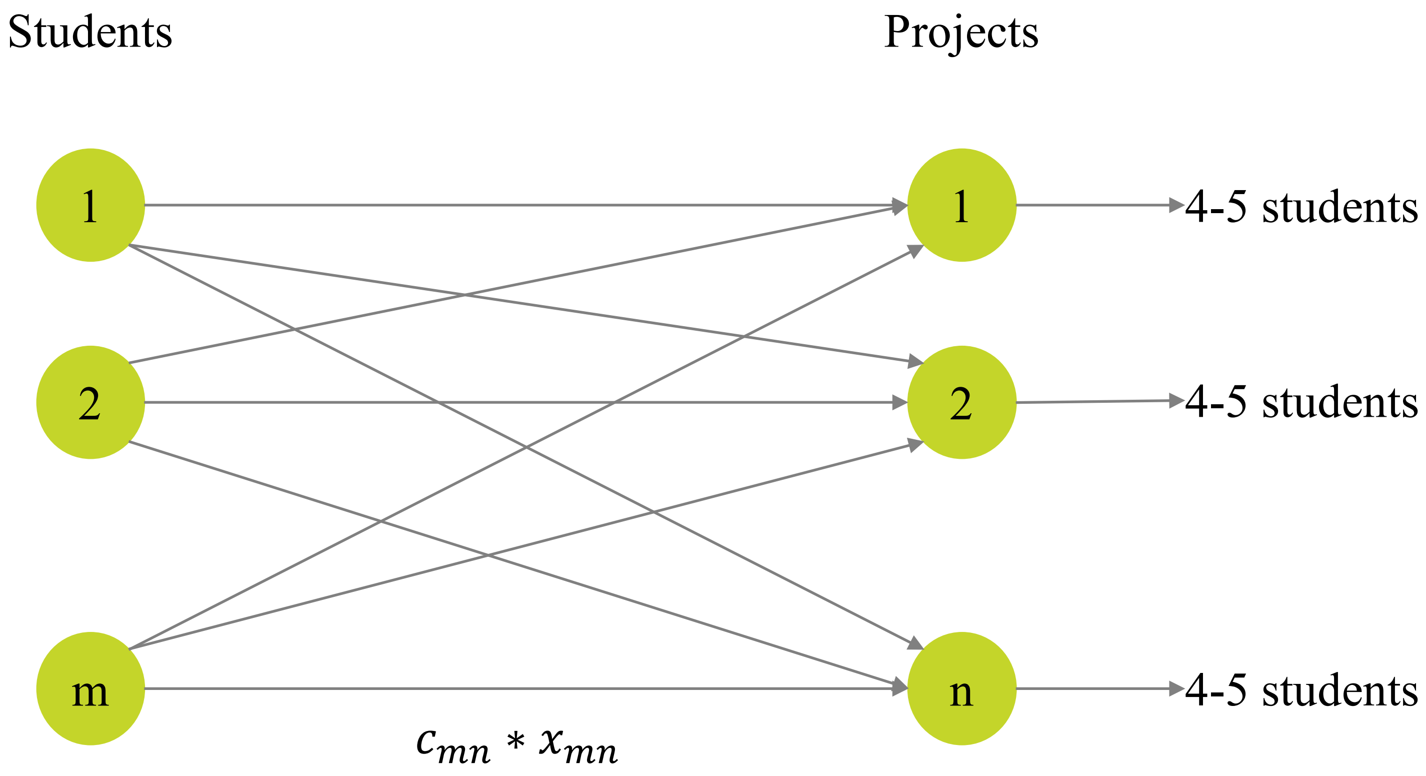
Dummy variables balance the model if needed

	Denver	Miami	Supply
Los Angeles	80 **1000**	215	**1000**
Detroit	100 **1300**	108	**1300**
New Orleans	102	68 **1200**	**1200**
Dummy Plant	0	0 **200**	**200**
Demand	**2300**	**1400**	

	Denver	Miami	Dummy	
Los Angeles	80 **1000**	215	0	**1000**
Detroit	100 **900**	108 **200**	0 **400**	**1500**
New Orleans	102	68 **1200**	0	**1200**
Demand	**1900**	**1400**	**400**	

To avoid the model to use the dummy additions, one assigns an extremely high cost to them.

Allocations follow a very similar structure to a TPP



Where c_{mn} is a variant of the priority of allocation, and x_{mn} is a binary variable “assign or not”

An “example” student allocation model

$$\text{Minimize } z = \sum c_{ij} * x_{ij}$$

s.t.

$$\sum_i x_{ij} \qquad = 1 \quad \forall j$$

$$\sum_j x_{ij} \qquad \geq 4 \quad \forall i$$

$$\sum_i x_{ij} \qquad \leq 5 \quad \forall i$$

$$x_{ij} \geq 0, \forall i, j$$

The transportation algorithm – enjoy the simplified simplex!

The entire operations are done on the transportation tableau.

1. Determine a starting basic feasible solution.
2. Use the optimality condition to determine the entering variable from among all the non-basic variables. If the optimality condition is satisfied, stop. Otherwise, continue.
3. Use the feasibility condition of the simplex method to determine the leaving variable from among all the current basic variables, and find the new basic solution. Return to step 2.

Example for transportation algorithm: SunRay Transport

SunRay Transport Company ships truckloads of grain from three silos to four mills. The model seeks the minimum cost shipping schedule between the silos and the mills.

Mill

Silo	1	2	3	4	Supply
1	10 x_{11}	2 x_{12}	20 x_{13}	11 x_{14}	**15**
2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	**25**
3	4 x_{31}	14 x_{32}	16 x_{33}	18 x_{34}	**10**
Demand	**5**	**15**	**15**	**15**	

Starting solutions

m sources, n destinations => m+n constraint equations.

The transportation model is always balanced => one constraint is redundant.

$\Rightarrow$ m+n-1 independent equations and thus m+n-1 basic variable

Two options for starting solutions:

- Northwest-corner method
- Least-cost method

Northwest corner method

1. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.
2. Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both are zero, cross out only one.
3. If exactly one row is left uncrossed out, stop. Otherwise, move to the cell right or below.

	1	2	3	4	Supply
1	10 **5**	2 **10**	20	11	**15**
2	12	7 **5**	9 **15**	20 **5**	**25**
3	4	14	16	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

For you: Northwest-corner method

	1	2	3	4	Supply
1	10	2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Demand	5	15	15	15	

Least-cost method

1. Assign as much as possible to the cell with the smallest unit cost.
2. The satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.
3. Select the uncrossed-out cell with the smallest unit cost and repeat until exactly one row of column is left uncrossed.

	1	2	3	4	Supply
1	10	(start) 2 **15**	20	11 **0**	**15**
2	12	7	9 **15**	(end) 20 **10**	**25**
3	4 **5**	14	16	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

For you: Least-cost method

	1	2	3	4	Supply
1	10	2	20	11	15
2	12	7	9	20	25
3	4	14	16	18	10
Demand	5	15	15	15	

Iterative computation of the transportation algorithm

Starting from a starting solution, we need to find an **entering variable**. To do so, we construct a set of equations:

$$u_i + v_j = c_{ij} \text{ for each basic } x_{ij}$$

Where u_i refers to row i, and v_j refers to column j.

The resulting set of equations

We arbitrarily set $u_i = 0$, and solve for all remaining variables.

Basic variable	(u, v)-Equation	Solution
x_{11}	$u_1 + v_1 = 10$	Set $u_1 = 0 \Rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 2$	$u_1 = 0 \Rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 2 \Rightarrow u_2 = 5$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 5 \Rightarrow v_3 = 4$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 5 \Rightarrow v_4 = 15$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 15 \Rightarrow u_3 = 3$

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With that, we find values for non-basic variables:

Nonbasic variable	Reduced cost $u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 15 - 11 = 4$
x_{21}	$u_2 + v_1 - c_{21} = 5 + 10 - 12 = 3$
x_{31}	$u_3 + v_1 - c_{31} = 3 + 10 - 4 = \mathbf{9}$
x_{32}	$u_3 + v_2 - c_{32} = 3 + 2 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 3 + 4 - 16 = -9$

Because the transportation model minimizes cost, the entering variable is the one having the most positive coefficient in the z-row.

Basic	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	$\mathbf{x_{31}}$	x_{32}	x_{33}	x_{34}
z	0	0	–16	4	3	0	0	0	**9**	–9	–9	0

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 **5**	2 **10**	20 –16	11 4	**15**
$u_2 = 5$	12 3	7 **5**	9 **15**	20 **5**	**25**
$u_3 = 3$	4 **?** 9	14 –9	16 –9	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

We can do the calculations straight in the tableau:

10 **5**	2 **10**	20	11
12	7 **5**	9 **15**	20 **5**
4	14	16	18 **10**

Remember

- for basic variables: $u_i + v_j = c_{ij}$
- for non-basic variables: $\overline{c_{ij}} = u_i + v_j - c_{ij}$

We find the leaving variable with a “loop”:

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 **5 − θ** −	2 **10 + θ** +	20 $\boxed{-16}$	11 $\boxed{4}$	**15**
$u_2 = 5$	12 $\boxed{3}$	7 **5 − θ** −	9 **15**	20 **5 + θ** +	**25**
$u_3 = 3$	4 **θ** + $\boxed{9}$	14 $\boxed{-9}$	16 $\boxed{-9}$	18 **10 − θ** −	**10**
Demand	**5**	**15**	**15**	**15**	

And now iterate, until we have no more positive values in the bottom right corners

	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10; −9	2; **15 − θ**; −	20; −16	11; **θ**; +; 4	**15**
$u_2 = 5$	12; −6	7; **0 + θ**; +	9; **15**	20; **10 − θ**; −	**25**
$u_3 = 3$	4; **5**	14; −9	16; −9	18; **5**	**10**
Demand	**5**	**15**	**15**	**15**	

Final solution:

	$v_1 = -3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 11$	Supply
$u_1 \equiv 0$	10 $\boxed{-13}$	2 **5**	20 $\boxed{-16}$	11 **10**	**15**
$u_2 = 5$	12 $\boxed{-10}$	7 **10**	9 **15**	20 $\boxed{-4}$	**25**
$u_3 = 7$	4 **5**	14 $\boxed{-5}$	16 $\boxed{-5}$	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

A summary of the optimal solution

From silo	To mill	Number of truckloads
1	2	5
1	4	10
2	2	10
2	3	15
3	1	5
3	4	5
Optimal cost = \$435		

So …

A transportation problem is an LP, that can be solved by software (making use of simplex)

↓

Why go through the effort of doing the calculations?

A problem with the structure of the transportation model can be solved very efficiently

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Making use of the special structure of a transportation problem can significantly decrease computation time for large-scale applications

The transshipment model

The assignment model

- Matching workers (with varying skills) to jobs.
- Idea: skill variation affects the cost of completing jobs.
- Objective: assign n workers to n jobs
- Specificity: One worker has one job and each job is served by exactly one worker

$$
\begin{array}{cc|cccc|c}
 & & & \text{Jobs} & & & \\
 & & 1 & 2 & \ldots & N & \\
\hline
 & 1 & c_{11} & c_{12} & \ldots & c_{1n} & \mathbf{1} \\
 & 2 & c_{21} & c_{22} & \ldots & c_{2n} & \mathbf{1} \\
\text{Worker} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & N & c_{n1} & c_{n2} & \ldots & c_{nn} & \mathbf{1} \\
\hline
 & & \mathbf{1} & \mathbf{1} & \ldots & \mathbf{1} &
\end{array}
$$

Hungarian Method is even more efficient to solve such a model.

Questions: Refineries

Three refineries with daily capacities of 6,5, and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is 10 cents per 1000 gallons per pipeline mile. The table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.

1. Construct the associated transportation model.
2. Determine the optimum shipping schedule in the network.

	Distribution area		
	1	2	3
Refinery 1	180	180	-
Refinery 2	300	800	900
Refinery 3	320	200	120

Questions: Transportation models repetition

Consider below transportation models.

1. Use the northwest-corer method to find starting solutions.
2. Develop the iterations that lead to the optimal solution.
3. Solve the problems in Excel.

(i)

$0	$2	$1	**6**
$2	$1	$5	**9**
$2	$4	$3	**5**
5	**5**	**10**	

(ii)

$10	$4	$2	**8**
$2	$3	$4	**5**
$1	$2	$0	**6**
7	**6**	**6**	

(iii)

—	$3	$5	**4**
$7	$4	$9	**7**
$1	$8	$6	**19**
5	**6**	**19**	

Questions: Penalties

In the table below, the total demand exceeds the total supply. Suppose that the penalty cost per unit of unsatisfied demand are $2, $5, and $3 for destinations 1, 2, and 3, respectively. Use the least-cost starting solution and compute the iterations leading to the optimum solution. Then, use Excel to verify your solution.

$5	$1	$7	**10**
$6	$4	$6	**80**
$3	$2	$5	**15**
75	**20**	**50**	

Take Away

- What is special about the transportation model?
- How to compute a solution for the transportation model by hand?
- How to find starting solutions?
- How to improve the starting solution to optimality?
- How to handle situation when demand exceeds supply (or vice versa).
- Why is the transportation problem “more” than a standard LP?