Simplex Method and Sensitivity Analysis

Taha textbook, Chapter 3

After this session, you should …

… know how to calculate an LP solution by the simplex method by hand.

… be able to explain what the different steps in the simplex method mean, in particular by redrawing to the graphical solution.

… identify and know how to handle special cases of the simplex algorithm.

… know how to explore solutions for sensitivity analysis.

Agenda

1. Simplex method
2. Special cases
3. Sensitivity analysis

Requirements for simplex I: All constraints are equations with non-negative rhs

Converting ($\leq$) inequalities into equations with non-negative right-hand side (rhs) is achieved by adding a non-negative **slack variable *s*** to the lhs of the newly formed equation.

From Reddy Mikks:

$$6x_1 + 4x_2 \leq 24$$
$$6x_1 + 4x_2 + s_1 = 24, \qquad s_1 \geq 0$$

Requirements for simplex I: All constraints are equations with non-negative rhs

Converting ($\geq$) inequalities into equations with non-negative right-hand side (rhs) is achieved by subtracting a non-negative **surplus variable** $\boldsymbol{S}$ to the lhs of the newly formed equation.

From diet example:

$$x_1 + x_2 \geq 800$$
$$x_1 + x_2 - S_1 = 800, \qquad S_1 \geq 0$$

Enforcing a **non-negative rhs** can be achieved by multiplying the equation by **-1**.

Requirements for simplex II: All variables are non-negative

Most software packages will enforce this requirement internally. Still, we need it for the pen-and-paper calculation.

Sometimes, variables can assume negative values or positive values (for example hiring or firing extra staff). For such cases, we can **substitute** the single variable with **two variables** to ensure this constraint:

Let S_i denote the variation in headcount in staff:

$$S_i = S_i^- - S_i^+, \qquad S_i^- \geq 0, S_i^+ \geq 0$$

With this substitution, S_i^- is the number of workers hired and S_i^+ the number of workers fired. Only one of them can assume positive values in a solution.

From graphical to algebraic solution

Graphical Method	Algebraic Method
Graph all constraints, including nonnegativity restrictions	Represent the solution space by m equations in n variables and restrict all variables to nonnegative values, $m < n$
Solution space consists of infinity of **feasible** points	The system has infinity of **feasible** solutions
↓	↓
Identify **feasible corner points** of the solution space	Determine the **feasible basic solutions** of the equations
Candidates for the optimum solution are given by a *finite* number of **corner points**	Candidates for the optimum solution are given by a *finite* number of **basic feasible solutions**
↓	↓
Use the objective function to determine the **optimum corner point** from among all the candidates	Use the objective function to determine the **optimum basic feasible solution** from among all the candidates

Basic solution corresponds to the corner points in graphical solution.

We set $n - m$ variables equal to 0 and solve the m equations for the remaining m variables, provides the resulting solution is **unique**.

Thus, maximum number of corner points is

$$C_m^n = \frac{n!}{m!\,(n-m)!}$$

An example

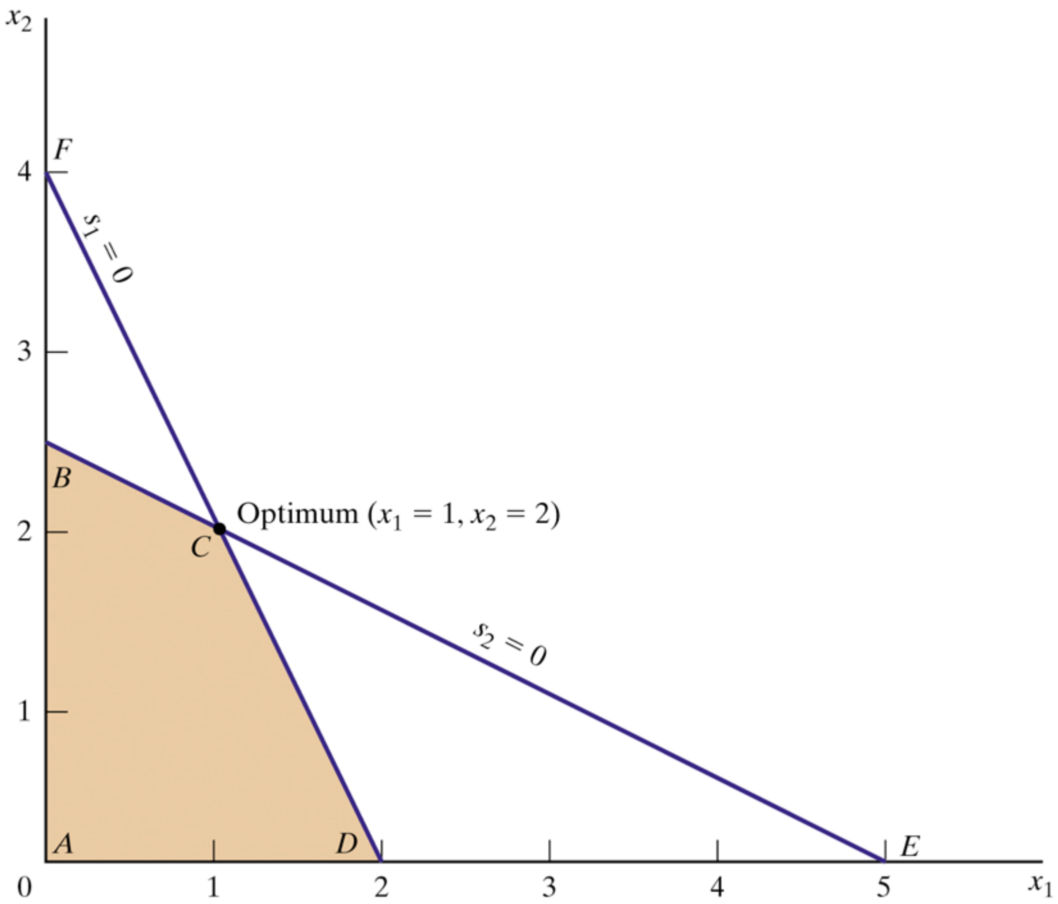
$$\text{Maximize } z = 2x_1 + 3x_2$$

s.t.

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Sketch the graphical solution and write the problem as a system of equations with non-negative rhs.

… and its graphical solution



… and the algebraic representation of the solution space

$$
\begin{aligned}
2x_1 + x_2 + s_1 \quad &= 4 \\
x_1 + 2x_2 + s_2 \quad &= 5 \\
x_1, x_2, s_1, s_2 \quad &\geq 0
\end{aligned}
$$

Thus, it has $m = 2$ equations and $n = 4$ variables.

Basic solutions: setting $n - m = 4 - 2 = 2$ variables equal to 0. Then, solve for the remaining $m = 2$ variables.

E.g., $x_1 = 0; x_2 = 0 => s_1 = 4; s_2 = 5$, objective value $z = 2x_1 + 3x_2 = 0$

What results when you set $s_1 = 0; s_2 = 0$?

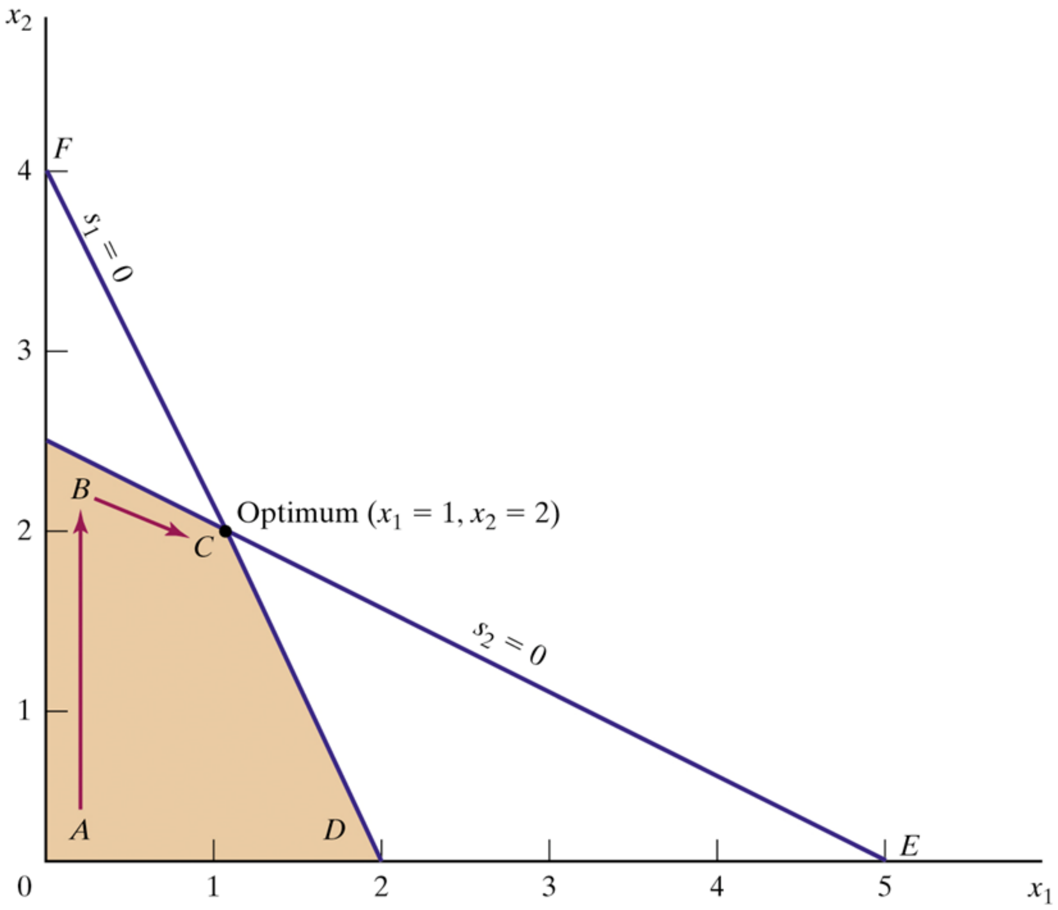
How to get there algebraically?

Nonbasic (zero) variables	Resulting equations	Solution	Corner point	Basic?	Feasible?	Objective value, z
$x_1 = x_2 = 0$	$s_1 = 4$ $s_2 = 5$	$s_1 = 4,\ s_2 = 5$	A	Yes	Yes	0
$x_1 = s_1 = 0$	$x_2 = 4$ $2x_2 + s_2 = 5$	$x_2 = 4,\ s_2 = -3$	F	Yes	No	–
$x_1 = s_2 = 0$	$x_2 = 4$ $2x_2 + s_1 = 5$	$x_2 = 2.5,\ s_1 = 1.5$	B	Yes	Yes	7.5
$x_2 = s_1 = 0$	$2x_1 = 4$ $x_1 + s_2 = 5$	$x_1 = 2,\ s_2 = 3$	D	Yes	Yes	4
$x_2 = s_2 = 0$	$2x_1 + s_1 = 4$ $x_1 = 5$	$x_1 = 5,\ s_1 = -6$	E	Yes	No	–
$s_1 = s_2 = 0$	$2x_1 + x_2 = 4$ $x_1 + 2x_2 = 5$	$x_1 = 1,\ x_2 = 2$	C	Yes	Yes	8 (optimum)

Terminology – see table previous slide

Non-basic variables	The zero $n - m$ variables, i.e. the variables that you set equal to zero.
Basic variables	The remaining m variables.
Basic solution	The solution obtained by solving the m equations using the basic variables.

The simplex method



Common to start with setting the slack variables as basic variables.

-> Resulting corner point is the origin.

Then, aim to increase z. Should this happen in direction of x_1 or x_2? Choose x_2 as it has a higher rate of improvement for z.

Go to the next corner point and again ask if z can be increased by moving further, and so on …

Solving the Reddy Mikks example by simplex

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

s.t.

$$\begin{array}{rrrrrrl} 6x_1 & +4x_2 & +s_1 & & & & = 24 \\ x_1 & +2x_2 & & +s_2 & & & = 6 \\ -x_1 & +x_2 & & & +s_3 & & = 1 \\ & x_2 & & & & +s_4 & = 2 \end{array}$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

And we rewrite the objective function as

$$z - 5x_1 - 4x_2 = 0$$

The simplex tableau for the above model

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	1	–5	–4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	–1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

Current basic variables

Simplex feasibility condition: if one variable remains negative, the solution can still be improved.

Choose the variable with the most negative coefficient for fastest improvement of the solution (= **entering variable**).

Values of z and basic variables in the solution

Identifying the leaving variable by calculating ratios

As x_1 enters the solution, another variable must leave (= **leaving variable**). Calculate the ratios and choose the smallest positive ratio to determine the leaving variable.

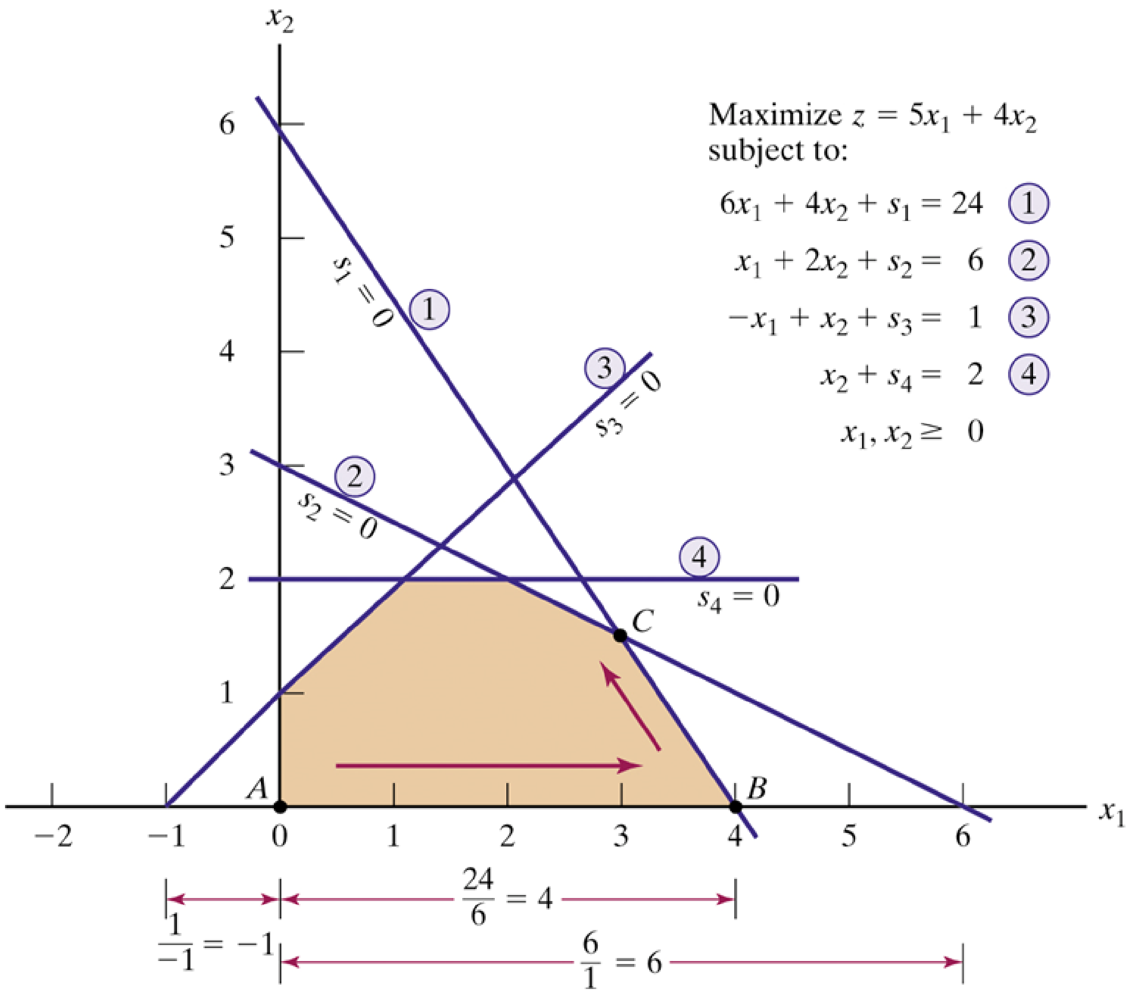
Basic	Entering x_1	Solution	Ratio (or intercept)
s_1	6	24	$x_1 = \frac{24}{6} = 4 \leftarrow$ minimum
s_2	1	6	$x_1 = \frac{6}{1} = 6$
s_3	−1	1	$x_1 = \frac{1}{-1} = -1$ (negative denominator, ignore)
s_4	0	2	$x_1 = \frac{2}{0} = \infty$ (zero denominator, ignore)
Conclusion: x_1 enters (at value 4) and s_1 leaves (at zero level)			

Graphical interpretation of the ratios

The ratios are the intercepts of the constraint lines with the entering variable x_1-axis.

Thus, choosing the smallest positive ratio graphically represents increasing x_1 to the next corner point. Increasing x_1 further leads to infeasible solutions.

The rule associated with the ratio computations is referred to as the **simplex feasibility condition**.



Swapping the variables

	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
			Enter ↓							
	z	1	–5	–4	0	0	0	0	0	
Leave ←	s_1	0	**6**	4	1	0	0	0	24	Pivot row
	s_2	0	1	2	0	1	0	0	6	
	s_3	0	–1	1	0	0	1	0	1	
	s_4	0	0	1	0	0	0	1	2	
			Pivot column							

Gauss-Jordan computations to produce the new basic solution

- The entering variable column is the **pivot column**.
- The leaving variable row is the **pivot row**.
- Their intersection is the **pivot element**.

1. Pivot row
   a) Replace the leaving variable in the basic column with the entering variable.
   b) New pivot row = current pivot row / pivot element.
2. All other rows, including z

   New row = (Current row) – (pivot column coefficient) * (new pivot row)

The new tableau becomes …

	Basic	z	x_1	x_2 ↓	s_1	s_2	s_3	s_4	Solution
	z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
	x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
←	s_2	0	0	$\mathbf{\frac{4}{3}}$	$-\frac{1}{6}$	1	0	0	2
	s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
	s_4	0	0	1	0	0	0	1	2

Constraint coefficients of the basic variable continue to form an identity matrix.

Setting the new non-basic variables x_2 and s_1 to zero, the solution column automatically gives the new basic solution ($x_1 = 4, s_2 = 2, s_3 = 5, s_4 = 2$).

Ratios for the next iteration

Basic	Entering x_2	Solution	Ratio
x_1	$\frac{2}{3}$	4	$x_2 = 4 \div \frac{2}{3} = 6$
s_2	$\frac{4}{3}$	2	$x_2 = 2 \div \frac{4}{3} = 1.5$ (minimum)
s_3	$\frac{5}{3}$	5	$x_2 = 5 \div \frac{5}{3} = 3$
s_4	1	2	$x_2 = 2 \div 1 = 2$

The final tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Interpretation of the final simplex tableau

Read straight from simplex tableau

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is $21,000

The status of the resources in the simplex tableau

Resource	Slack value	Status
Raw material, $M1$	$s_1 = 0$	Scarce
Raw material, $M2$	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Scarce : associated slack is 0. The activities of the model have used the resource completely.

Abundant : slack is positive.

Summary of the simplex method

Optimality condition: The entering variable in a maximization (minimization) problem is the non-basic variable with the most negative (positive) coefficient in the z-row. Ties are broken arbitrarily. The optimum is reached at the iteration where all z-row coefficients are non-negative (non-positive).

Feasibility condition: For both the maximization and the minimization problems, the leaving variable is the basic variable associated with the smallest non-negative ratio with strictly positive denominator. Ties are broken arbitrarily.

Gauss-Jordan row operations:

1. Pivot row
   a) Replace the leaving variable in the basic column with the entering variable.
   b) New pivot row = current pivot row / pivot element.
2. All other rows, including z

   New row = (Current row) – (pivot column coefficient) * (new pivot row)

Solve the following problem I

A company produced three products with amounts x_1, x_2, x_3. All products need to pass three production stages that have capacity constraints. The following linear program describes the optimal production. The three inequalities reflect the capacity constraints the three stages).

$$
\begin{array}{llrl}
 & 3x_1 + 4x_2 + 6x_3 & \rightarrow & \text{max.} \\
 & & & \\
\text{s.t.} & 2x_1 + x_2 + 2x_3 & \leq & 14 \\
 & x_1 + 2x_2 + x_3 & \leq & 21 \\
 & x_1 + x_2 + 2x_3 & \leq & 12 \\
 & x_1 , x_2 , x_3 & \geq & 0
\end{array}
$$

Solve the following problem II

a) Find the optimal production mix by using the simplex method.
b) What is the optimal revenue?
c) What products are not produced?
d) What product stages have scarce capacity, which ones are abundant?
e) How many units of abundant capacity remain available?

Artificial starting solutions

Ill-behaved LPs with equalities or larger-or-equal constraints have no slack variables that offer convenient starting solutions. Hence, if this case occurs, other starting solutions are necessary.

Special cases in the simplex method

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Non-existing (or infeasible) solutions

Degeneracy

$$\text{Maximize } z = 3x_1 + 9x_2$$

s.t.

$$\begin{aligned} x_1 + 4x_2 &\leq 8 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

When a tie for the minimum ratio occurs and is broken arbitrarily, at least one basic variable will be zero in the next iteration and the new solution is said to be degenerated.

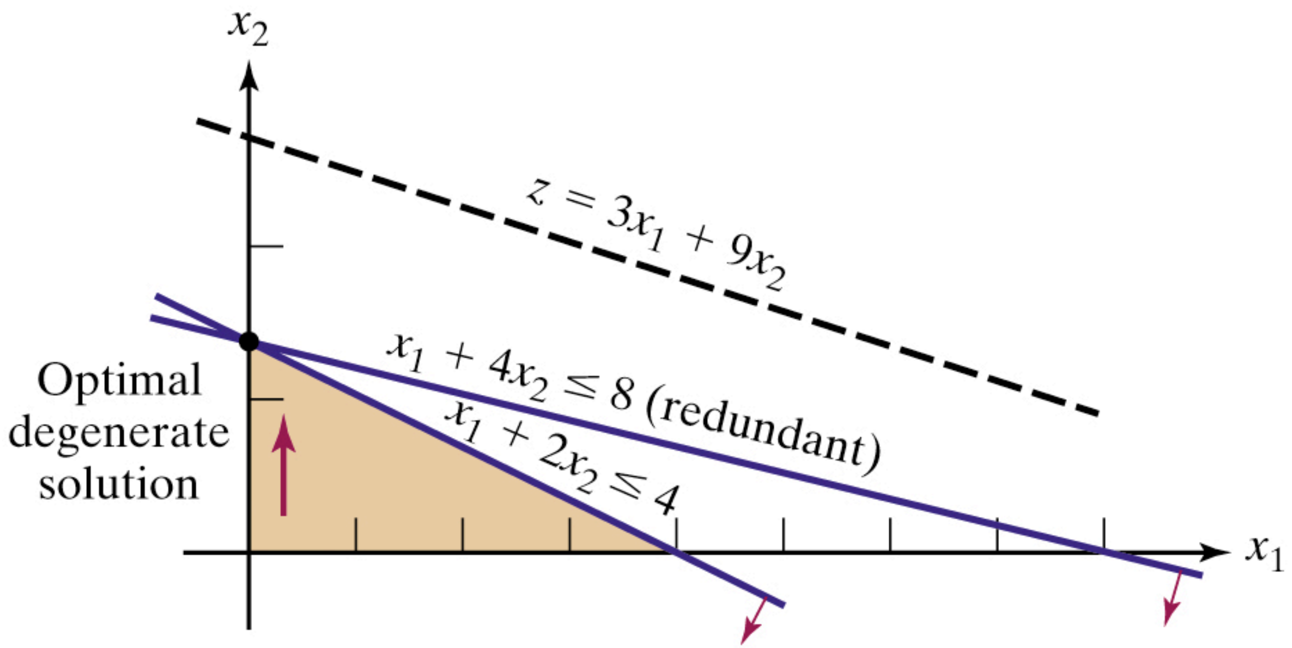
- The simplex iterations may cycle infinitely.
- There may be at least one redundant constraint.

Degeneracy

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	–3	–9	0	0	0
x_2 enters	x_3	1	4	1	0	8
x_3 leaves	x_4	1	2	0	1	4
1	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0
2	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
	x_1	1	0	–1	2	0

- In iteration 0, x_3 and x_4 tie for leaving.
- In iterations 1 and 2, the objective value does not improve and may drive simplex into cycling.

Degeneracy – redundant constraints



- In this case, we see that the problem is overdetermined, that is, one constraint is redundant.
- Redundancy may be created from a roundoff error during the course of solving an LP.

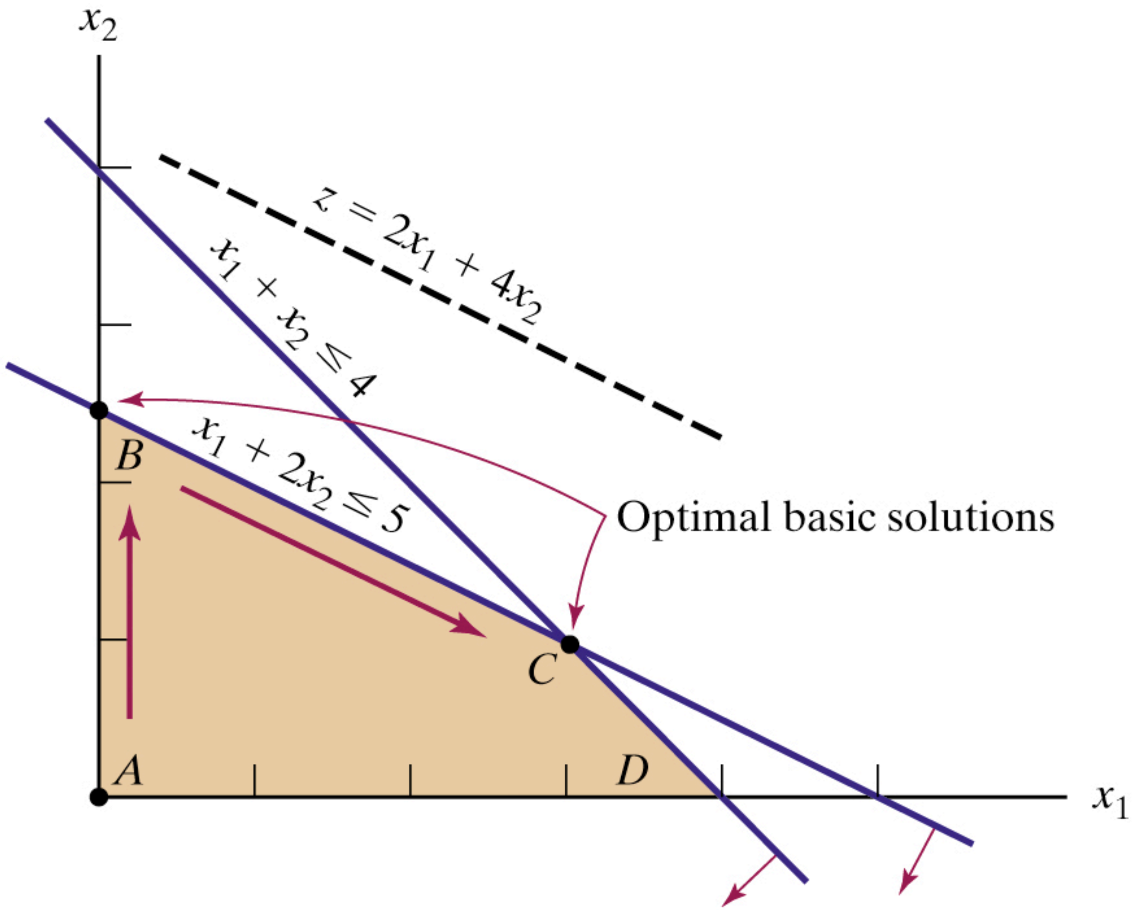
Alternative optima

An LP problem may have an infinite number of alternative optima, when the objective function is parallel to a non-redundant (=binding) constraint.

$$\text{Maximize } z = 2x_1 + 4x_2$$

s.t.

$$\begin{aligned} x_1 + 2x_2 &\leq 5 \\ x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Alternative optima

Detecting the existence of alternative optima:

- The zero coefficient of non-basic variable x_1 in iteration 1 indicates that x_1 can be made basic without changing the value of z.

Simplex only explores corner point solutions.

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	–2	–4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4
1 (optimum)	z	0	0	2	0	10
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
2	z	0	0	2	0	10
(alternative optimum)	x_2	0	1	1	–1	1
	x_1	1	0	–1	2	3

Unbounded solution

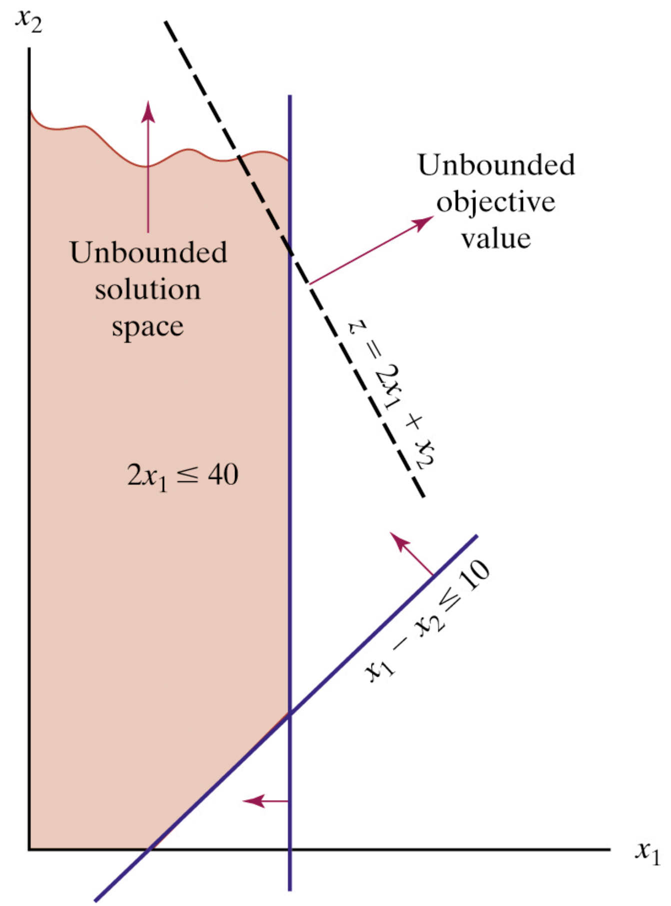
The solution space may be unbounded in at least one variable – meaning that the variables may be increased indefinitely without violating an of the constraints.

Typically, this indicates a modelling error such as missing a constraint.

$$\text{Maximize } z = 2x_1 + x_2$$

s.t.

$$\begin{array}{rcr} x_1 - x_2 & \leq & 10 \\ 2x_1 & \leq & 40 \\ x_1, x_2 & \geq & 0 \end{array}$$



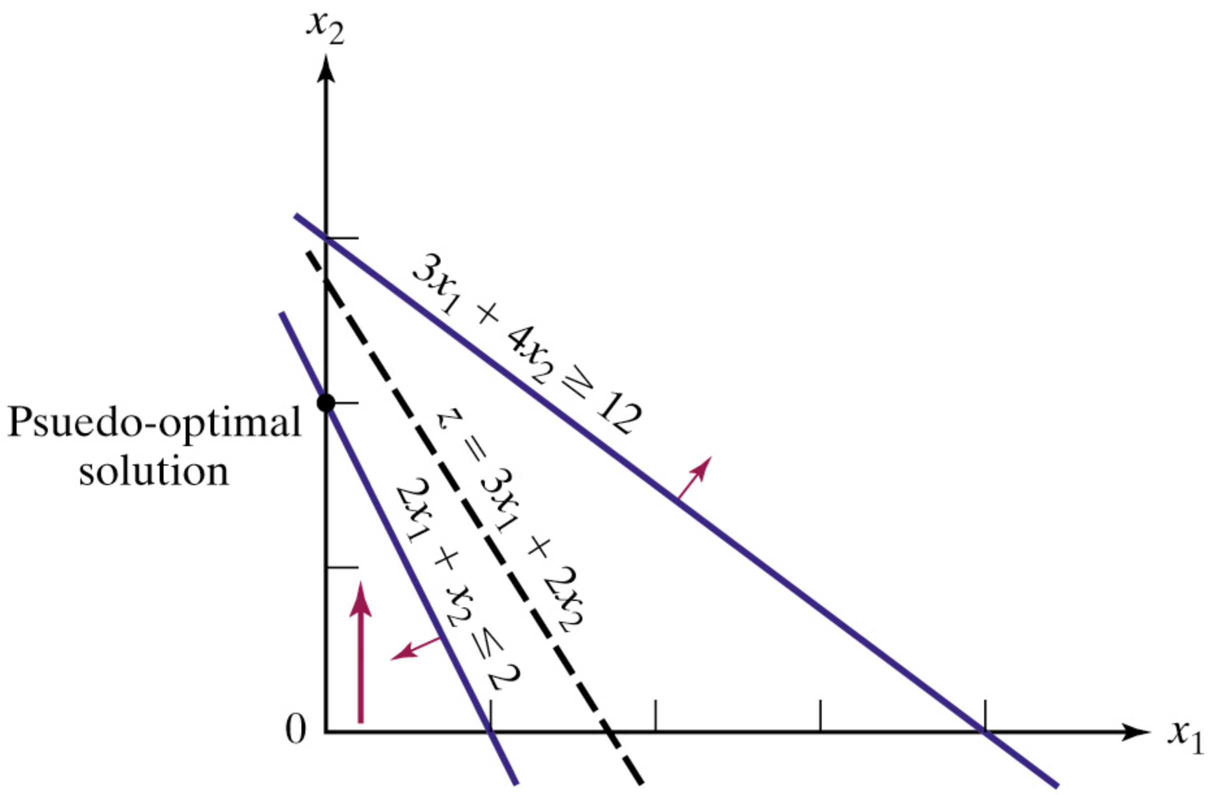
Unbounded solution

In the starting tableau, both x_1 and x_2 have negative coefficients in the z-equation.

Note that all constraint coefficients under x_2 are non-positive, indicating that x_2 can be increased indefinitely without violating any of them.

Basic	x_1	x_2	x_3	x_4	Solution
z	–2	–1	0	0	0
x_3	1	–1	1	0	10
x_4	2	0	0	1	40

Infeasible solution



LP models with inconsistent constraints have no feasible solution.

- Cannot occur if all constraints of $\leq$ type with non-negative rhs (the slacks provide a simple feasible solution).

Typically, an infeasible solution space points to the possibility that he model is not formulated correctly.

$$\text{Maximize } z = 3x_1 + 2x_2$$

s.t.

$$\begin{array}{rcc} 2x_1 + x_2 & \leq & 2 \\ 3x_1 + 4x_2 & \geq & 12 \\ x_1, x_2 & \geq & 0 \end{array}$$

Sensitivity analysis

Model parameters can change within certain limits without causing changes in the optimum. Exploring these limits is referred to as sensitivity analysis.

JOBCO example

JOBCO manufactures two products on two machines. A unit of product 1 requires 2 hrs on machine 1 and 1 hr on machine 2. For product 2, one unit requires 1 hr on machine 1 and 3 hrs on machine 2. The revenues per unit of products 1 and 2 are $30 and §20 respectively. The total daily processing time available for each machine is 8 hrs.

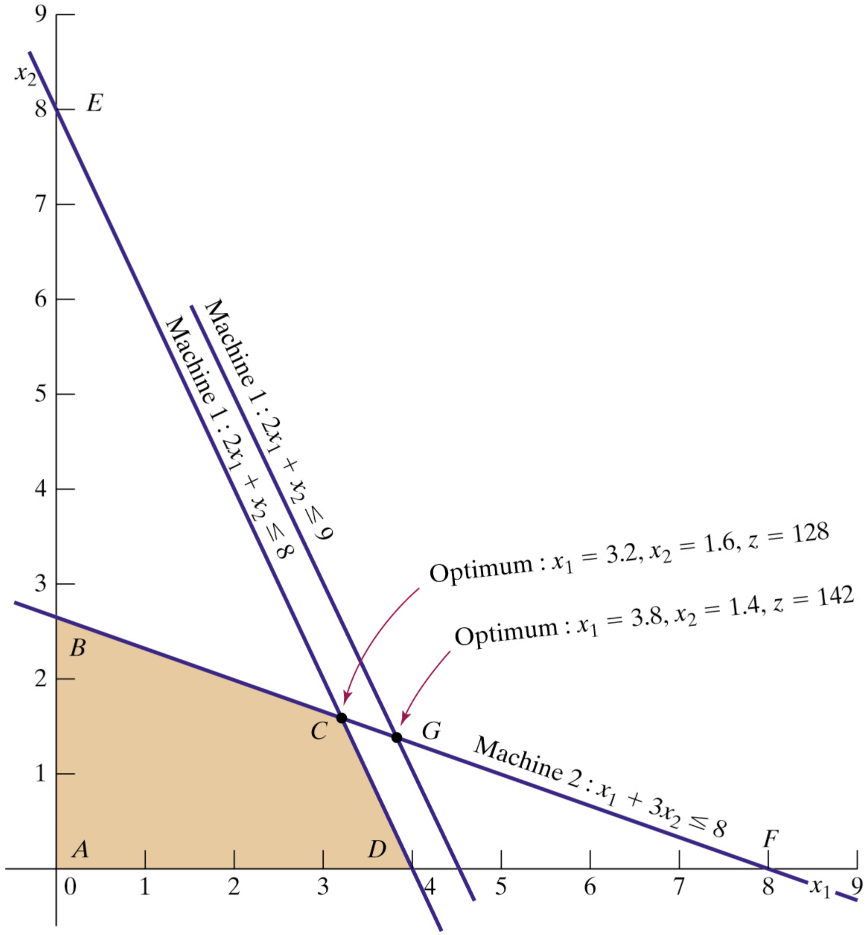
Letting x_1 and x_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as:

$$\text{Maximize } z = 30x_1 + 20x_2$$

s.t.

$$\begin{aligned} 2x_1 + x_2 &\leq 8 \\ x_1 + 3x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Sensitivity of the optimum solution to changes in the availability of resources



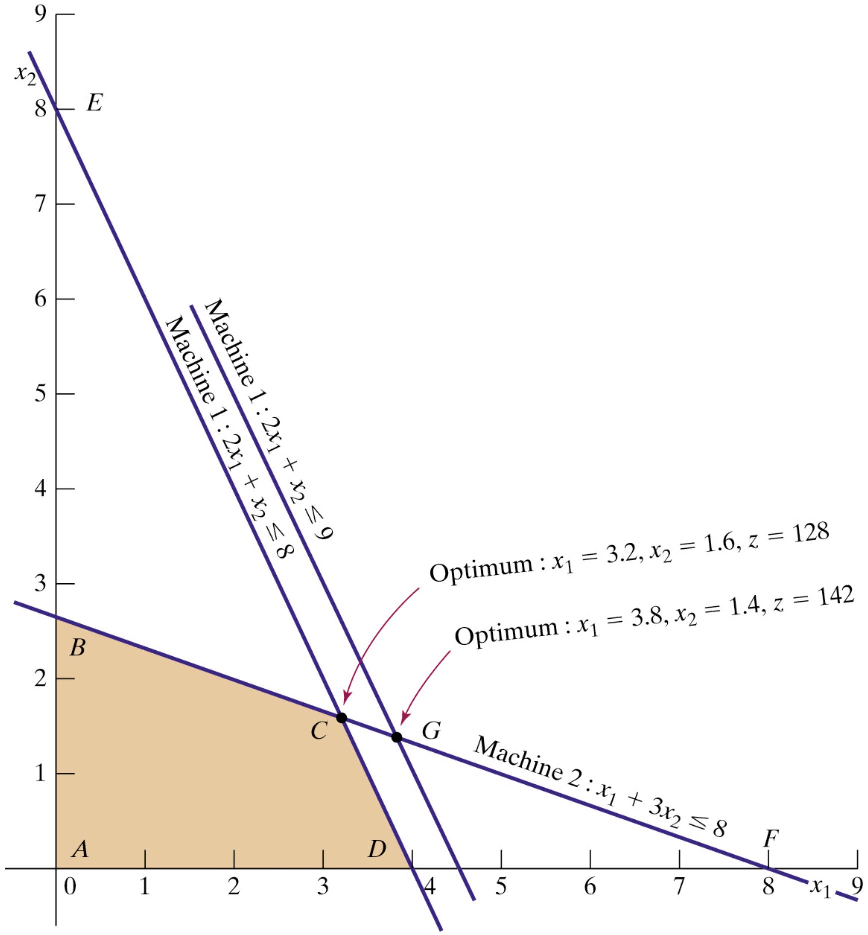
Rate of change from adding one extra hour to machine 1 (point C to point G)

$$\text{Rate} = \frac{z_G - z_C}{\text{capacity change}} = \frac{142-128}{9-1} = 14\,\frac{\$}{\text{hr}}$$

Thus, each extra hour (unit of input) is worth \$14 in output (=**unit worth of a resource**).

Traditionally, this is called the **shadow** (or **dual**) **price** of the resource. (and that's what it is oftentimes called in software packages).

Sensitivity of the optimum solution to changes in the availability of resources



Checking closely, the \$14/hr is only valid in the range from B to F, so only when

$$2.67 \text{ hrs} \leq \text{Machine 1 capacity} \leq 16 \text{ hrs}$$

Similarly, we find that an extra hour on machine 2 yields \$2.

$$4 \text{ hrs} \leq \text{Machine 2 capacity} \leq 24 \text{ hrs}$$

Sensitivity of the optimum solution to changes in the availability of resources

Potential questions to be addressed:

1. If JOBCO can increase the capacity of both machines, which machine should receive priority?
2. A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of $10/hr for each machine. Is this advisable?
3. If the capacity of machine 1 is increased from 8 to 13 hours, how will this increase impact the optimum revenue?
4. Suppose that the capacity of machine 1 is increased to 20 hrs, how will this increase affect the optimum revenue?

Sensitivity of the optimum solution to changes in the unit profit or unit cost

Changes in revenue units will change the slope of z.

The solution remains unchanged as long as the objective function lies between BF and DE.

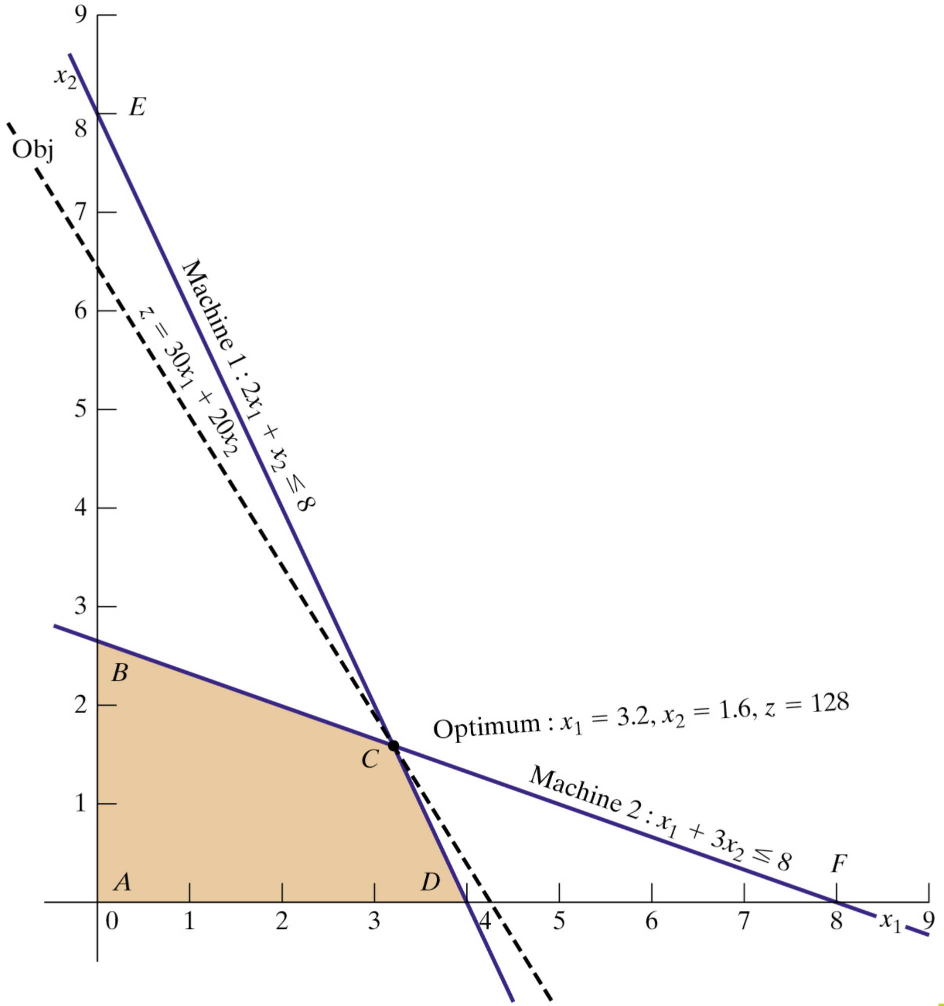
Finding the coefficients c_1 and c_2 such that the solution does not change:

Objective function in general form:

$$\text{Maximize } z = c_1x_1 + c_2x_2$$

The OF must not exceed either $2x_1 + x_2 = 8$ or $x_1 + 3x_2 = 8$. Thus,

$$\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$$



Sensitivity of the optimum solution to changes in the unit profit or unit cost

Potential questions to be addressed:

1. Suppose the unit revenues for products 1 and 2 are changed to \$35 and \$25 respectively. Will the current optimum reman the same?
2. Suppose the unit revenue of product 2 is fixed at its current value $c_2 = \$20$. What is the associated optimality range for the unit revenue for product 1, c_1, that will keep the optimum unchanged?

Algebraic Sensitivity Analysis

Follows essentially the same ideas.

1. Form teams of 2-3 three students and work through Chapter 3.6.2 and 3.6.3 (available in campUAS) by yourselves.
2. Then, create the TOYCO example in Excel and explore the sensitivity report provided in by the solver.

Take Away

- The simplex method allows to solve well-formulated LPs.
- You can connect the graphical solution to the simplex steps. I.e., you know not only how the method works, but also why.
- You can find a solution by using the simplex method.
- You can read and explain solutions provided in a simplex tableau.
- You can explain specific cases of LPs that challenge the simplex tableau.
- You can discuss the sensitivity of solution as to changes in the constraint levels and objective function costs both graphically and algebraically.