ECEN 5488: Geometric Control Theory Final Project Report

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***Abstract*—We provide a rigorous treatment of the relationships between the differential flatness and controllability properties of both linear and nonlinear systems. In particular, we show that flatness is sufficient for controllability and for linear systems, it is also necessary. Furthermore, we use the FlatVCP approach for optimal control of a specific differentially flat system: The planar manipulator with n links.**

I. Introduction

Differential flatness is a system property which has been used for motion planning [1, 2, 3] in recent years with great success. The study of this property can be traced back as far as the mid 80s in [4]. However, it wasn't until later that the term "flatness" was coined and the property was formally established in the works [5, 6, 7]. Notably, [7] contains a list of examples of differentially flat systems.

While the flatness property is well-studied, most theoretical treatments such as [5] rely on arguments from differential algebra which can be foreign to the initiate control theory student. On the other hand, most practical implementations leveraging this property such as [3] rely only on the most superficial aspects of differential flatness. In this work, we seek to study differential flatness, specifically its relationship with controllability, from a more familiar context: Geometric control. We will see that flatness is a sufficient condition for controllability. Furthermore, in the special case of linear systems, flatness is also necessary. To aid in the exposition, the theoretical results will be accompanied by a practical example: The planar manipulator with n links. As we will see, this system is differentially flat. We will leverage the approach presented in [3] to design safe trajectories for the planar manipulator with n links.

The rest of this work is organized as follows: Section II defines differential flatness, introduces the model of the planar manipulator with n links and summarizes the FlatVCP approach; Section III studies the relationship between flatness and controllability for general systems; Section IV analyzes these properties specifically for the planar manipulator with n links; Section V follows the FlatVCP approach to solving an optimal control problem for the planar manipulator with n links; Section VI presents a few examples of solving the optimal control problem for planar manipulators with 4 and 8 links; Finally, Section VII concludes the work.

II. Background

In this section, we define the differential flatness and controllability properties, we present a model for the planar manipulator with n links and introduce the FlatVCP approach for optimal control of differentially flat systems.

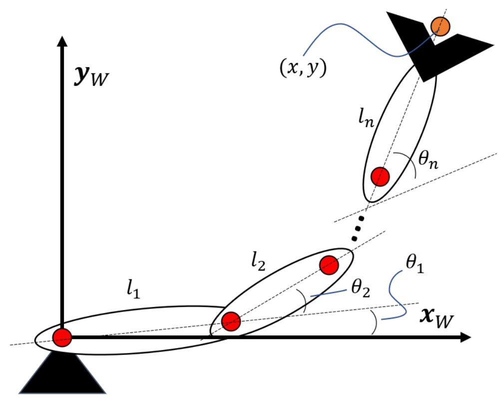

Fig. 1. Planar manipulator with n links.

A. Differential Flatness

Consider a nonlinear system:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \tag{1}$$

with state $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ and input $\mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^m$.

Definition 1. [8] *The system* (1) *is* differentially flat *if there exists a flat output* $\mathbf{y} = \xi(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \dots, \mathbf{u}^{(p)})$ *such that the elements of* $\mathbf{y} = (y_1, y_2, \dots, y_m)^T$ *are differentially independent, and the states and inputs can be expressed as algebraic functions of* $\mathbf{y}$ *and a finite number of its derivatives:*

$$\mathbf{x} = \Phi\big(\mathbf{y}, \dot{\mathbf{y}}, \dots, \mathbf{y}^{(q-1)}\big), \tag{2a}$$

$$\mathbf{u} = \Psi\big(\mathbf{y}, \dot{\mathbf{y}}, \dots, \mathbf{y}^{(q-1)}, \mathbf{y}^{(q)}\big), \tag{2b}$$

B. Controllability

Suppose (1) is a control system on a manifold $\mathcal{X}$ embedded in $\mathbb{R}^n$. Then, controllability is defined as follows

Definition 2. *The system* (1) *is controllable on* $\mathcal{X}$ *if for any* $\mathbf{x}, \mathbf{x'} \in \mathcal{X}$*, there is a control law that can steer the system from* $\mathbf{x}$ *to* $\mathbf{x}'$ *in finite time.*

C. Planar Manipulator with n links

Manipulators are commonly used in industry to automate manufacturing processes. Let the state and input vectors be, respectively

$$\mathbf{x} = \begin{bmatrix} x & y & \boldsymbol{\theta}^\top \end{bmatrix}^\top, \quad \mathbf{u} = \dot{\boldsymbol{\theta}},$$

where $(x, y)^\top$ is the position of the end effector and $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ are the angles of the n joints (see Figure 1).

The planar manipulator with n links is control-linear:

$$\dot{\mathbf{x}}(t) = g\big(\mathbf{x}(t)\big)\mathbf{u}(t), \tag{3}$$

where

$$\begin{aligned} g(\mathbf{x}) &= \begin{bmatrix} g_x(\boldsymbol{\theta})^\top & g_y(\boldsymbol{\theta})^\top & I_n \end{bmatrix}^\top, \\ g_x(\boldsymbol{\theta}) &\triangleq -\begin{bmatrix} \sum_{k=1}^n l_{y,k}(\boldsymbol{\theta}) & \sum_{k=2}^n l_{y,k}(\boldsymbol{\theta}) & \cdots & l_{y,n}(\boldsymbol{\theta}) \end{bmatrix}, \\ g_y(\boldsymbol{\theta}) &\triangleq \begin{bmatrix} \sum_{k=1}^n l_{x,k}(\boldsymbol{\theta}) & \sum_{k=2}^n l_{x,k}(\boldsymbol{\theta}) & \cdots & l_{x,n}(\boldsymbol{\theta}) \end{bmatrix}, \end{aligned}$$

with

$$l_{x,k}(\boldsymbol{\theta}) \triangleq l_k \cos\left(\sum_{i=1}^k \theta_i\right), \quad l_{y,k}(\boldsymbol{\theta}) \triangleq l_k \sin\left(\sum_{i=1}^k \theta_i\right),$$

where $l_j > 0$, $j = 1, \ldots, n$ is the length of the j-th link. The quantities $l_{x,k}$ and $l_{y,k}$ represent, the length of the k-th link projected onto the $\mathbf{x}_W$ and $\mathbf{y}_W$ axes given state $\mathbf{x}$, respectively.

D. FlatVCP

FlatVCP [3] is a specialized, pseudospectral, direct method to solving optimal control problems for differentially flat systems. The approach relies on B-spline basis functions to find sufficiently smooth flat output trajectories for flat systems. A d-th degree *B-spline basis* $\lambda_{i,d}(t)$ with $d \in \mathbb{Z}_{>0}$ is defined over a given *knot vector* $\boldsymbol{\tau} = (\tau_0, \ldots, \tau_\eta)^T$ satisfying $\tau_i \leq \tau_{i+1}$ for $i = 0, \ldots, \eta - 1$ and is computed recursively by the Cox-de Boor recursion formula [9]. Additionally, we consider the *clamped, uniform B-spline basis*, which is defined over knot vectors satisfying:

$$\text{(clamped)} \quad \tau_0 = \ldots = \tau_d, \quad \tau_{\eta-d} = \ldots = \tau_\eta, \tag{4a}$$
$$\text{(uniform)} \quad \tau_{d+1} - \tau_d = \ldots = \tau_{N+1} - \tau_N, \tag{4b}$$

where $N = \eta - d - 1$. A d-th degree *B-spline curve* $\mathbf{s}(t)$ is a m-dimensional parametric curve built by linearly combining *control points* $\mathbf{p}_i \in \mathbb{R}^m (i = 0, \ldots, N)$ and B-spline bases of the same degree. Given that $\mathbf{s}(t) = P\boldsymbol{\Lambda}_d(t)$, we compute a B-spline curve's r-th order derivative with:

$$\frac{\mathrm{d}^r\mathbf{s}}{\mathrm{d}t^r}(t) = \sum_{i=0}^N \mathbf{p}_i \mathbf{b}_{r,i+1}^T \boldsymbol{\Lambda}_{d-r}(t) = PB_r\boldsymbol{\Lambda}_{d-r}(t), \tag{5}$$

where the control points are grouped into a matrix $P = (\mathbf{p}_0, \ldots, \mathbf{p}_N) \in \mathbb{R}^{m\times(N+1)}$, the basis functions are grouped into a vector $\boldsymbol{\Lambda}_{d-r}(t) = \big(\lambda_{0,d-r}(t), \ldots, \lambda_{N+r,d-r}(t)\big)^T \in \mathbb{R}^{N+r+1}$, and $\mathbf{b}_{r,j}^T$ is the j-th row of a time-invariant matrix $B_r \in \mathbb{R}^{(N+1)\times(N+r+1)}$ constructed as $B_r = M_{d,d-r}C_r$, where matrices $M_{d,d-r} \in \mathbb{R}^{(N+1)\times(N-r+1)}$ and $C_r \in \mathbb{R}^{(N-r+1)\times(N+r+1)}$ are defined in [10, 11].

Definition 3. [10] *The columns of $P^{(r)} \triangleq PB_r$ are called the* r-th order virtual control points *(VCPs) of $\mathbf{s}(t)$ and denoted as $\mathbf{p}_i^{(r)}$ where $i = 0, 1, \ldots, N + r$, i.e.,*

$$P^{(r)} = PB_r = \begin{bmatrix} \mathbf{p}_0^{(r)} & \ldots & \mathbf{p}_{N+r}^{(r)} \end{bmatrix}. \tag{6}$$

III. Flatness and Controllability

In this section, we study the relationship between the flatness and controllability properties for different classes of systems.

Theorem 1. *[5] If a system is differentially flat, then it is controllable.*

Proof. The original definition of a differentially flat system according to [5] goes as follows: A linearizable system by an endogenous, dynamic compensator is called differentially flat. Let us work with this definition to show that, given two states $\mathbf{x}_0, \mathbf{x}_f \in \mathcal{X}$, we can always find two flat outputs $\mathbf{y}_0, \mathbf{y}_f \in \mathbb{R}^m$ and their $q-1$ derivatives $\dot{\mathbf{y}}_0, \ldots, \mathbf{y}_0^{(q-1)}, \dot{\mathbf{y}}_f, \ldots, \mathbf{y}_f^{(q-1)}$ such that $\Phi(\mathbf{y}_0, \dot{\mathbf{y}}_0, \ldots, \mathbf{y}_0^{(q-1)}) = \mathbf{x}_0$ and $\Phi(\mathbf{y}_f, \dot{\mathbf{y}}_f, \ldots, \mathbf{y}_f^{(q-1)}) = \mathbf{x}_f$.

A dynamic compensator is $\mathbf{u} = b(\mathbf{x}, \mathbf{z}, \mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^m$ is a fictitious input and $\mathbf{z} \in \mathbb{R}^p$ is a fictitious state that evolves as $\dot{\mathbf{z}} = a(\mathbf{x}, \mathbf{z}, \mathbf{v})$. It was shown in [12] that the feedback linearizability of the system also implies the existance of a diffeomorphism $\Xi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n+p}$ mapping the system

$$\begin{aligned} \dot{\mathbf{x}} &= f\big(\mathbf{x}, b(\mathbf{x}, \mathbf{z}, \mathbf{v})\big), \\ \dot{\mathbf{z}} &= a(\mathbf{x}, \mathbf{z}, \mathbf{v}), \end{aligned}$$

to a controllable, linear time invariant system

$$\frac{\mathrm{d}}{\mathrm{d}t}\Xi(\mathbf{x}, \mathbf{z}) = F\Xi(\mathbf{x}, \mathbf{z}) + G\mathbf{v}.$$

Let $\nu_1, \ldots, \nu_m$ be the controllability indices of the controllability matrix $\begin{bmatrix} G & FG & F^2G & \cdots & F^{n+p-1}G \end{bmatrix}$ (see, for example, [13]). Note that $\sum_{i=1}^m \nu_i = n+p$. Since the above linear system is controllable, there exists a nonsingular change of coordinates $P \in \mathbb{R}^{(n+p)\times(n+p)}$ such that [14]

$$P\Xi(\mathbf{x}, \mathbf{z}) = \begin{bmatrix} y_1 & \cdots & y_1^{(\nu_1-1)} & y_2 & \cdots & y_m^{(\nu_m-1)} \end{bmatrix}^\top \triangleq \tilde{\mathbf{y}},$$

and $P\frac{\mathrm{d}}{\mathrm{d}t}\Xi(\mathbf{x}, \mathbf{z})$ is such that $y_i^{(\nu_i)} = v_i$ for all $i = 1, \ldots, m$. This is called the Brunovsky canonical form. Furthermore, the index $q = \max(\nu_1, \nu_2, \ldots, \nu_m)$ is called the controllability index of (F, G) [13]. Note that the dynamic compensator being endogenous implies that the fictitious state is not a function of any independent (or exogenous) variables. That is, there exists a mapping from $(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(\rho)})$ to $\mathbf{z}$, for some ρ. Let $\mathbf{z}_0$ and $\mathbf{z}_f$ be obtained from $\mathbf{x}_0$ and $\mathbf{x}_f$, respectively, with arbitrary inputs $\mathbf{u}$ and their derivatives. Then, we have that $\tilde{\mathbf{y}}_0 = P\Xi(\mathbf{x}_0, \mathbf{z}_0)$ and $\tilde{\mathbf{y}}_f = P\Xi(\mathbf{x}_f, \mathbf{z}_f)$. Furthermore, since P is nonsingular and Ξ is a diffeomorphism, the map is invertible $(\mathbf{x}_0, \mathbf{z}_0) = \Xi^{-1}(P^{-1}\tilde{\mathbf{y}}_0)$ and $(\mathbf{x}_f, \mathbf{z}_f) = \Xi^{-1}(P^{-1}\tilde{\mathbf{y}}_f)$. In particular, the mapping $\Xi^{-1} \circ P^{-1}$ is an augmentation of Φ in (2).

We have found two flat outputs $\mathbf{y}_0$ and $\mathbf{y}_f$ as well as their derivatives up to$(q-1)$-order as we wanted. We now design a q-times differentiable curve in $\mathbb{R}^m$, namely $\mathbf{y}(t) \in C^q$ such that $\mathbf{y}^{(r)}(0) = \mathbf{y}_0^{(r)}$ and $\mathbf{y}^{(r)}(t_f) = \mathbf{y}_f^{(r)}$, $t_f > 0$, for all $r = 0, 1, 2, \ldots, q-1$. Let $\mathbf{y}(t)$ be a $(q+1)$-degree, m-dimensional B-spline curve defined as in (5) with $N+1$ control points $\mathbf{p}_j \in \mathbb{R}^m$,

$j = 0,1,\dots,N$. Let the knot vector $(\tau_0, \tau_1, \dots, \tau_{N+q+2})$ be clamped and uniform (4). Then, it suffices to set

$$\mathbf{p}_r^{(r)} = \mathbf{y}_0^{(r)}, \quad \mathbf{p}_N^{(r)} = \mathbf{y}_f^{(r)}, \quad r = 0,1,2,\dots,q-1,$$

where $\mathbf{p}_j^{(r)}$ is the j-th, r-th order virtual control point of $\mathbf{y}(t)$ introduced in Definition (3). The resulting B-spline curve is sufficiently smooth and satisfies the terminal conditions. Furthermore, by virtue of being clamped and by the definition of the VCPs, the r-th order derivative of $\mathbf{y}(t)$ also satisfies the necessary terminal constraints. Finally, we can pass this B-spline curve and its derivatives at any time $t \in [0, t_f)$ through the map Ψ from (2) to obtain a valid control input signal $\mathbf{u}(t)$ that steers the system from $\mathbf{x}_0$ to $\mathbf{x}_f$ in the time interval $[0, t_f]$. Thus, it follows from Definition 2 that the system is controllable. □

Theorem 2. *[5] A linear system is differentially flat if and only if it is controllable.*

Proof. If the linear system is differentially flat, it follows by Theorem 1 that it is controllable. For the reverse implication, we will restrict ourselves to linear time-invariant (LTI) systems

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{u} \in \mathbb{R}^m$ is the input vector with $m \le n$. Suppose the pair (A, B) is controllable and note that the pair (A, I_n) is observable. Then, by Theorem 17.3 of [15], the LTI system is a minimal realization of some transfer function $\hat{G}(s)$. Let $D = 0_{n\times m}$ such that the system output is $I_n\mathbf{x}(t) = \mathbf{x}(t)$. Then, the transfer function $\hat{G}(s) = (sI_n - A)^{-1}B \in \mathbb{R}^{n\times m}$ is strictly proper. Denote the monic least common denominator of all the entries of $\hat{G}(s)$ by $d(s) = s^q + \alpha_1 s^{q-1} + \dots + \alpha_{q-1}s + \alpha_q$, where q is such that $qm = n$. We can now expand $\hat{G}(s) = \big(1/d(s)\big)(N_1 s^{q-1} + N_2 s^{q-2} + \dots + N_{q-1}s + N_q)$, where the $N_i \in \mathbb{R}^{n\times m}$, $i = 1,2,\dots,q$ are constant matrices. By Theorem 17.4 of [15], all minimal realizations of $\hat{G}(s)$ are algebraically equivalent. Therefore, there exists a change of coordinates $\bar{\mathbf{x}} \triangleq P\mathbf{x}$ where $P \in \mathbb{R}^{n\times n}$ is nonsingular such that the system $\dot{\bar{\mathbf{x}}}(t) = PAP^{-1}\bar{\mathbf{x}}(t) + PB\mathbf{u}(t)$ is in controllable canonical form with

$$PAP^{-1} = \begin{bmatrix} -\alpha_1 I_m & -\alpha_2 I_m & \cdots & -\alpha_{q-1} I_m & -\alpha_q I_m \\ I_m & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times m} \\ 0_{m\times m} & I_m & \cdots & 0_{m\times m} & 0_{m\times m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & \cdots & I_m & 0_{m\times m} \end{bmatrix},$$

and $PB = \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix}^\top$. Define a vector $\mathbf{y}$ as the last m elements of $\bar{\mathbf{x}}$. Equivalently, $\mathbf{y} = \begin{bmatrix} 0_{m\times(n-m)} & I_m \end{bmatrix} P\mathbf{x}$. It follows from the form of PAP^{-1} and PB that

$$\bar{\mathbf{x}} = \begin{bmatrix} y_1^{(q-1)} & \dots & y_m^{(q-1)} & y_1^{(q-2)} & \cdots & y_m \end{bmatrix}^\top \in \mathbb{R}^{qm} = \mathbb{R}^n.$$

Recovering the original coordinates,

$$\mathbf{x} = P^{-1}\begin{bmatrix} y_1^{(q-1)} & \dots & y_m^{(q-1)} & y_1^{(q-2)} & \cdots & y_m \end{bmatrix}^\top.$$

It also follows from the form of PAP^{-1} and PB that

$$u_i = y_i^{(q)} + \alpha_1 y_i^{(q-1)} + \dots + \alpha_q y_i, \quad i = 1,2,\dots,m.$$

We have expressed the system's state $\mathbf{x}$ and input $\mathbf{u}$ vectors as algebraic functions of differentially independent coordinates $\mathbf{y}$ and their derivatives up to order q. Therefore, the system is differentially flat with flat output $\mathbf{y}$ by Definition 1. □

Remark 1. *The choice of flat output $\mathbf{y}$ for controllable linear systems is not unique. Recall that in the proof of Theorem 2 we assumed $C = I_n$ and $D = 0_{n\times m}$. Any other choice of C such that the pair (A, C) is observable or any other choice of D will give rise to a different transfer function $\hat{G}(s)$ and, thus, to a different similarity transformation P or different coefficients α_i.*

IV. Analyzing the Planar Manipulator with n Links

In this section we show that the planar manipulator with n links is differentially flat. We also verify controllability via Chow-Rashevskii's Theorem.

Lemma 1. *The planar manipulator with n links* (3) *is differentially flat.*

Proof. Consider the candidate flat output vector $\boldsymbol{\theta}$. We can write the position of the end-effector (x, y) as follows

$$x = \sum_{k=1}^{n} l_k \cos\left(\sum_{i=1}^{k}\theta_i\right), \quad y = \sum_{k=1}^{n} l_k \sin\left(\sum_{i=1}^{k}\theta_i\right).$$

Thus, the state vector $\mathbf{x}$ is an algebraic function of $\boldsymbol{\theta}$ and the input vector $\mathbf{u} = \dot{\boldsymbol{\theta}}$ is identically the first-order derivative of $\boldsymbol{\theta}$. By Definition 1, the planar manipulator with n links (3) is differentially flat with flat output $\boldsymbol{\theta}$ and index $q = 1$. □

Let S^1 denote the unit circle $S^1 \triangleq \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x}^\top\mathbf{x} = 1\}$. Note that any point $\mathbf{x} \in S^1$ can be represented as $\mathbf{x} = (\cos\theta, \sin\theta)^\top$ for some $\theta \in [0, 2\pi)$. Let us denote by $T_\theta S^1$ the tangent space of S^1 at the point $\mathbf{x} = (\cos\theta, \sin\theta) \in S^1$. It can be shown that $T_\theta S^1 = \{\mathbf{v} \in \mathbb{R}^2 \mid v_1\cos\theta + v_2\sin\theta = 0\}$. Note that any point $\mathbf{v} \in T_\theta S^1$ can be represented as $\mathbf{v} = \alpha(-\sin\theta, \cos\theta)$ for some $\alpha \in \mathbb{R}$. With a slight abuse of notation, we will write $\theta \in S^1$ to denote, implicitly, $(\cos\theta, \sin\theta) \in S^1$ and $\alpha \in T_\theta S^1$ to denote, implicitly, $\alpha(-\sin\theta, \cos\theta) \in T_\theta S^1$. Let us denote the cartesian product of n unit circles by T^n. That is

$$T^n \triangleq \underbrace{S^1 \times \dots \times S^1}_{n \text{ times}}.$$

We chose symbol T because T^2 is a torus embedded in $\mathbb{R}^3$.

Lemma 2. *The planar manipulator with n links* (3) *is a control system on T^n embedded in $\mathbb{R}^{n+2}$.*

Proof. We begin by rewriting the planar manipulator with n links (3) as a linear combination of n controlled vector fields:

$$\dot{x}(t) = \sum_{i=1}^{n} g_i\big(\mathbf{x}(t)\big)u_i,$$

where each g_i for $i = 1,2,\dots,n$ is

$$g_i(\mathbf{x}) = \begin{bmatrix} -\sum_{k=i}^{n} l_{y,k}(\mathbf{x}) & \sum_{k=i}^{n} l_{x,k}(\mathbf{x}) & \mathbf{e}_i^\top \end{bmatrix}^\top, \qquad (7)$$

with $\mathbf{e}_i$ the i-th canonical basis vector of $\mathbb{R}^n$. Next we verify that each g_i is indeed a vector field on T^n. We can write the tangent space of the manifold T^n at a point $\mathbf{x} \in T^n$ as

$$T_{\mathbf{x}}T^n = \{(x, y, \boldsymbol{\alpha}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \mid \alpha_i \in T_{\theta_i}S^1, i = 1,\dots,n\}.$$

Note that, for any $\theta \in S^1$, any point $\alpha \in \mathbb{R}$ is also in $T_\theta S^1$. Thus, for any $\mathbf{x} \in T^n$, we have that $T_{\mathbf{x}} T^n \approx \mathbb{R}^{n+2}$. It follows that any function $f : T^n \to \mathbb{R}^{n+2}$ is a vector field on T^n. In particular, g_i, $i = 1, 2, \ldots, n$ are vector fields on T^n and the planar manipulator with n links (3) is a control system on manifold T^n. Since the state $\mathbf{x} \in \mathbb{R}^{n+2}$, we say that T^n is embedded on $\mathbb{R}^{n+2}$. □

Corollary 1. *The planar manipulator with n links* (3) *is controllable.*

Proof. Lemma 1 and Theorem 1 immediately provide this result. Nonetheless, we will prove the statement without resorting to flatness.

Define the distribution

$$D^0 \triangleq \text{span}\{g_1, g_2, \ldots, g_n\},$$

where the g_i, $i = 1, 2, \ldots, n$ are the controlled vector fields of the planar manipulator with n links given in (7). It is clear that, for any $\mathbf{x} \in M$, $\dim(D^0_{\mathbf{x}}) = n$. Thus, D^0 is regular of dimension n. Let us now define

$$D^1 \triangleq D^0 \cup \text{span}\{[g_i, g_j]\}, \quad i < j, \quad i, j \in \{1, 2, \ldots, n\},$$

where $[f, g]$ denotes the Lie bracket operator defined as

$$[f, g](\mathbf{x}) = \frac{\partial g}{\partial \mathbf{x}} f(\mathbf{x}) - \frac{\partial f}{\partial \mathbf{x}} g(\mathbf{x}).$$

To understand the dimension of D^1, let us compute $[g_i, g_j]$. We begin by computing

$$\frac{\partial l_{x,k}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 0 & -l_{y,k}(\mathbf{x})\mathbf{1}_n^\top \end{bmatrix}, \quad \frac{\partial l_{y,k}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 0 & l_{x,k}(\mathbf{x})\mathbf{1}_n^\top \end{bmatrix}.$$

With the above partial derivatives, we have the i-th Jacobian

$$\frac{\partial g_i}{\partial \mathbf{x}} = -\begin{bmatrix} 0_{1\times 2} & \sum_{k=i}^n l_{x,k}(\mathbf{x})\mathbf{1}_n^\top \\ 0_{1\times 2} & \sum_{k=i}^n l_{y,k}(\mathbf{x})\mathbf{1}_n^\top \\ 0_{n\times 2} & 0_{n\times n} \end{bmatrix}.$$

Then, for any $j \in \{1, 2, \ldots, n\}$,

$$\frac{\partial g_i}{\partial \mathbf{x}} g_j(\mathbf{x}) = -\begin{bmatrix} \sum_{k=i}^n l_{x,k}(\mathbf{x}) & \sum_{k=i}^n l_{y,k}(\mathbf{x}) & 0_{1\times n} \end{bmatrix}^\top.$$

Finally, for any $i < j$, with $i, j \in \{1, 2, \ldots, n\}$, the Lie bracket

$$[g_i, g_j](\mathbf{x}) = \begin{bmatrix} \sum_{k=i}^j l_{x,k}(\mathbf{x}) & \sum_{k=i}^j l_{y,k}(\mathbf{x}) & 0_{1\times n} \end{bmatrix}^\top.$$

We can interpret the 1-st and 2-nd entries of $[g_i, g_j](\mathbf{x})$ as the sum of the projections of $l_i, l_{i+1}, \ldots, l_j$, given state $\mathbf{x}$, onto the $\mathbf{x}_W$ and $\mathbf{y}_W$ axes, respectively. For most $\mathbf{x} \in M$, we have that $\dim(D^1_{\mathbf{x}}) = n+2$. However, this is not always the case. Consider, for example, $\mathbf{x} = (\sum_{k=1}^n l_k, 0, 0_{1\times n})^\top$. In this case, $l_{y,k}(\mathbf{x}) = 0$ and $l_{x,k}(\mathbf{x}) = l_k > 0$, for any $k \in \{1, 2, \ldots, n\}$. Therefore, $\dim(D^1_{\mathbf{x}}) = n + 1$. Nonetheless, it can be shown that D^1 is regular of dimension $n + 1$. Let us continue by defining

$$D^2 \triangleq D^1 \cup \text{span}\{[g_1, [g_i, g_j]]\}, \quad i < j, \quad i, j \in \{1, 2, \ldots, n\},$$

To understand the dimension of D^2, let us compute $[g_1, [g_i, g_j]]$. We begin by computing

$$\frac{\partial [g_i, g_j]}{\partial \mathbf{x}} = \begin{bmatrix} 0_{1\times 2} & -\sum_{k=i}^j l_{y,k}(\mathbf{x})\mathbf{1}_n^\top \\ 0_{1\times 2} & \sum_{k=i}^j l_{x,k}(\mathbf{x})\mathbf{1}_n^\top \\ 0_{n\times 2} & 0_{n\times n} \end{bmatrix}.$$

Note that

$$\frac{\partial [g_i, g_j]}{\partial \mathbf{x}} g_l(\mathbf{x}) = \begin{bmatrix} -\sum_{k=i}^j l_{y,k}(\mathbf{x}) & \sum_{k=i}^j l_{x,k}(\mathbf{x}) & 0_{1\times n} \end{bmatrix}^\top,$$

for any $l \in \{1, 2, \ldots, n\}$. Furthermore,

$$\frac{\partial g_l}{\partial \mathbf{x}} [g_i, g_j](\mathbf{x}) = 0_{n\times 1}, \quad i, j, l \in \{1, 2, \ldots, n\}.$$

Therefore, $[g_1, [g_i, g_j]] = [g_2, [g_i, g_j]] = \cdots = [g_n, [g_i, g_j]]$ and this explains why the choice of $[g_1, [g_i, g_j]]$ in the definition of D^2 was made without loss of generality. It follows that, for $i < j$,

$$[g_1, [g_i, g_j]](\mathbf{x}) = \begin{bmatrix} -\sum_{k=i}^j l_{y,k}(\mathbf{x}) & \sum_{k=i}^j l_{x,k}(\mathbf{x}) & 0_{1\times n} \end{bmatrix}^\top.$$

Examining $[g_i, g_j]$ and $[g_1, [g_i, g_j]]$, we can determine that D^2 is regular of dimension $n + 2$. To see why this is true, suppose for a contradiction that there exists a $\mathbf{x} \in T^n$ such that $\dim(D^2_{\mathbf{x}}) = n + 1$. It follows by definition of D^2 that $\dim(D^1_{\mathbf{x}}) = n + 1$. This can happen in one of two cases:

1) Either $l_{x,k}(\mathbf{x}) = 0$, or $l_{y,k}(\mathbf{x}) = 0$ for all $k = 1, 2, \ldots, n$. Without loss of generality, suppose $l_{x,k}(\mathbf{x}) = 0$. Since $l_k > 0$, it follows that $\cos\left(\sum_{i=1}^k \theta_i\right) = 0$. However, this implies that $\sin\left(\sum_{i=1}^k \theta_i\right) = \pm 1$. Thus, $[g_i, g_j](\mathbf{x})$ and $[g_1, [g_i, g_j]](\mathbf{x})$ are linearly independent. Furthermore, $[g_1, [g_i, g_j]](\mathbf{x})$ and $g_k(\mathbf{x})$, for any $k \in \{1, 2, \ldots, n\}$ are also linearly independent. Thus, $\dim(D^2_{\mathbf{x}}) = n + 2$, which is a contradiction.

2) $l_{x,k}(\mathbf{x}) \propto l_{y,k}(\mathbf{x})$ for all $k = 1, 2, \ldots, n$. For an example $\mathbf{x} \in T^n$ where this is the case, consider the point

$$\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \sum_{k=1}^n l_k & \frac{1}{\sqrt{2}} \sum_{k=1}^n l_k & \frac{\pi}{4} & 0_{1\times n-1} \end{bmatrix}^\top.$$

Note that in this case, $[g_i, g_j](\mathbf{x})$ and $[g_1, [g_i, g_j]](\mathbf{x})$ are still linearly independent regardless of the proportionality of $l_{x,k}(\mathbf{x})$ and $l_{y,k}(\mathbf{x})$. Furthermore, $[g_1, [g_i, g_j]](\mathbf{x})$ and $g_k(\mathbf{x})$, for any $k \in \{1, 2, \ldots, n\}$ are also linearly independent. Thus, $\dim(D^2_{\mathbf{x}}) = n + 2$, which is a contradiction.

Therefore, D^2 is regular of dimension $n + 2$. It follows that for any $\mathbf{x} \in T^n$, $D^2_{\mathbf{x}} = T_{\mathbf{x}} T^n$ Thus, the system is controllable by Chow-Rashevskii's Theorem. □

Remark 2. *Note that the controllability we established does not mean controllability over $\mathbb{R}^{n+2}$. In fact, we instead established controllability on the manifold T^n. That is, given any initial joint configuration $\boldsymbol{\theta}_0 \in T^n$, and any final joint configuration $\boldsymbol{\theta}_1 \in T^n$, there exists a control input signal that takes the system from $\mathbf{x}_0 = (x_0, y_0, \boldsymbol{\theta}_0^\top)^\top$ to $\mathbf{x}_1 = (x_1, y_1, \boldsymbol{\theta}_1^\top)^\top$ in finite time, where*

$$x_0 = \sum_{k=1}^n l_{x,k}(\boldsymbol{\theta}_0), \quad x_1 = \sum_{k=1}^n l_{x,k}(\boldsymbol{\theta}_1),$$
$$y_0 = \sum_{k=1}^n l_{y,k}(\boldsymbol{\theta}_0), \quad y_1 = \sum_{k=1}^n l_{y,k}(\boldsymbol{\theta}_1).$$

V. OPTIMAL CONTROL VIA FLATVCP

In this section, we formulate an optimal control problem for the planar manipulator with n links. Then, we use the FlatVCP approach [3] to approximate the solution. We begin by defining the safe state and input sets, respectively as

$$\mathcal{S}_x \triangleq \{\mathbf{x} \in T^n \mid \underline{\boldsymbol{\theta}} \leq \boldsymbol{\theta} \leq \overline{\boldsymbol{\theta}}\},$$
$$\mathcal{S}_u \triangleq \{\mathbf{u} \in \mathbb{R}^n \mid \|\mathbf{u}\|_\infty \leq \overline{u}\},$$

where $\underline{\boldsymbol{\theta}}$ and $\overline{\boldsymbol{\theta}}$ represent joint limits and $\overline{u}$ denotes the actuation effort limit for all joints. Consider the optimal control problem

$$\begin{aligned} \text{minimize} \quad & \int_0^T \big(\dot{x}^2(t) + \dot{y}^2(t)\big)\mathrm{d}t \qquad \text{(OCP)} \\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = g\big(\mathbf{x}(t)\big)\mathbf{u}(t), \\ & \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(T) = \mathbf{x}_f, \\ & \mathbf{x}(t) \in \mathcal{S}_x, \quad \mathbf{u}(t) \in \mathcal{S}_u, \end{aligned}$$

over trajectories $\mathbf{x}(t)$, $\mathbf{u}(t)$ and with given initial and final states $\mathbf{x}_0, \mathbf{x}_f \in T^n$. Note that the objective is exactly the length of the curve traced by the end-effector.

From Lemma 1, recall that the planar manipulator with n links is differentially flat. Thus, we can flatten (OCP) as follows [3]

$$\begin{aligned} \text{minimize} \quad & \int_0^T \left\| \begin{bmatrix} g_x\big(\boldsymbol{\theta}(t)\big) \\ g_y\big(\boldsymbol{\theta}(t)\big) \end{bmatrix} \dot{\boldsymbol{\theta}}(t) \right\|_2^2 \mathrm{d}t \qquad \text{(FLAT-OCP)} \\ \text{subject to} \quad & \boldsymbol{\theta}(0) = \boldsymbol{\theta}_0, \quad \boldsymbol{\theta}(T) = \boldsymbol{\theta}_f, \\ & \underline{\boldsymbol{\theta}} \leq \boldsymbol{\theta}(t) \leq \overline{\boldsymbol{\theta}}, \quad \dot{\boldsymbol{\theta}}(t) \in \mathcal{S}_u, \quad t \in \{0, T\}, \end{aligned}$$

over functions $\boldsymbol{\theta} \in C^1 : [0, T] \to \mathbb{R}^n$ and with initial and final conditions $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_f$ extracted from $\mathbf{x}_0$ and $\mathbf{x}_f$, respectively. Note that (OCP) and (FLAT-OCP) are equivalent [16].

Now, we let $\boldsymbol{\theta}(t)$ be a clamped, uniform, n-dimensional, 2-nd degree B-spline curve with $N+1$ control points $\mathbf{p}_j \in \mathbb{R}^n$, $j = 0, 1, 2, \ldots, N$. We will examine the objective and constraints of (FLAT-OCP) and reformulate them in terms of the control points $\mathbf{p}_j$. We begin examining the objective function of (FLAT-OCP). Since g_x and g_y make it nonconvex with respect to $\boldsymbol{\theta}(t)$, we seek a related objective which is convex. Note that for any $\boldsymbol{\theta}$,

$$\left\| \begin{bmatrix} g_x(\boldsymbol{\theta}) \\ g_y(\boldsymbol{\theta}) \end{bmatrix} \dot{\boldsymbol{\theta}} \right\|_2^2 = \dot{\boldsymbol{\theta}}^\top \begin{bmatrix} g_x(\boldsymbol{\theta}) \\ g_y(\boldsymbol{\theta}) \end{bmatrix}^\top \begin{bmatrix} g_x(\boldsymbol{\theta}) \\ g_y(\boldsymbol{\theta}) \end{bmatrix} \dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}} Q(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}},$$

is a positive semidefinite function of $\dot{\boldsymbol{\theta}}(t)$ because $Q(\boldsymbol{\theta}) \succeq 0$. Thus, we will insted consider minimizing

$$\int_0^T \|\dot{\boldsymbol{\theta}}(t)\|_2^2 \mathrm{d}t \approx \sum_{k=0}^{N_J} \sum_{i=0}^{N} \frac{T}{N_J} \big(\mathbf{b}_{1,i+1}^\top \boldsymbol{\Lambda}_1(kT/N_J)\big)^2 \mathbf{p}_i^\top \mathbf{p}_i,$$

where $N_J > 0$ is the approximation resolution. Note that the above approximation is convex with respect to the control points $\mathbf{p}_j$. Next, by Proposition 1 of [10], we have that

$$\underline{\boldsymbol{\theta}} \leq \mathbf{p}_j \leq \overline{\boldsymbol{\theta}},\ j = 0, 1, \ldots, N \implies \underline{\boldsymbol{\theta}} \leq \boldsymbol{\theta}(t) \leq \overline{\boldsymbol{\theta}},\ \forall t \in [0, T),$$

and that

$$\mathbf{p}_j^{(1)} \in \mathcal{S}_u,\ j = 1, 2, \ldots, N \implies \dot{\boldsymbol{\theta}}(t) \in \mathcal{S}_u,\ \forall t \in [0, T).$$

Note from Definition 3 that $\mathbf{p}_j^{(1)}$ is a linear combination of the control points $\mathbf{p}_j$ as follows

$$\mathbf{p}_j^{(1)} = \sum_{i=0}^{N} b_{1,i+1,j+1} \mathbf{p}_i,$$

where $b_{1,i,j}$ is the (i,j)-th entry of B_1. Thus, the conditions above are convex with respect to the control points $\mathbf{p}_j$. We can now formulate the following quadratic program (QP)

$$\begin{aligned} \text{minimize} \quad & \sum_{k=0}^{N_J} \sum_{i=0}^{N} \frac{T}{N_J} \big(\mathbf{b}_{1,i+1}^\top \boldsymbol{\Lambda}_1(kT/N_J)\big)^2 \mathbf{p}_i^\top \mathbf{p}_i, \qquad \text{(FQP)} \\ \text{subject to} \quad & \mathbf{p}_0 = \boldsymbol{\theta}_0, \quad \mathbf{p}_N = \boldsymbol{\theta}_f, \\ & \underline{\boldsymbol{\theta}} \leq \mathbf{p}_j \leq \overline{\boldsymbol{\theta}}, \quad j = 0, 1, \ldots, N \\ & \sum_{i=0}^{N} b_{1,i+1,j+1} \mathbf{p}_i \in \mathcal{S}_u, \quad j = 1, 2, \ldots, N, \end{aligned}$$

over the control points $\mathbf{p}_j$, $j = 0, 1, 2, \ldots, N$ of $\boldsymbol{\theta}(t)$.

Corollary 2. *If $\mathbf{p}_j$, $j = 0, 1, \ldots, N$ is feasible in* (FQP). *Then, the state-space trajectory $\mathbf{x}(t)$, $\mathbf{u}(t)$ obtained by passing the B-spline curve $\boldsymbol{\theta}(t)$ with control points $\mathbf{p}_j$, $j = 0, 1, \ldots, N$ through the flat map* (2) *of the planar manipulator with n links is feasible in* (OCP).

VI. EXAMPLES

In this section, we solve (FQP) for two different planar manipulators. One with $n = 4$ links (Figure 2) and another with $n = 8$ links (Figure 3). For each manipulator, we consider two different sets of constraints, initial and final conditions. In each case, we provide a figure showing the pose of the manipulator at 5 time instances along the trajectory and the resulting end-effector trajectory. We ommit the joint angle $\boldsymbol{\theta}(t)$ and input $\dot{\boldsymbol{\theta}}(t)$ trajectories (although they are indeed safe) because they are uninteresting. The problems are solved in MATLAB using YALMIP [17] and MOSEK [18][1].

VII. CONCLUSION

We presented a theoretical treatment of the differential flatness property and its relationship with controllability for different classes of systems. Then, we analyzed the flatness and controllability of the planar manipulator with n links. Finally, we used the FlatVCP approach to solve the optimal control problem for the planar manipulator with n links. We also presented visualizations of different trajectory solutions obtained with this approach. It is worth noting that a much more complicated version of the optimal control problem considered is the finding of trajectories when only the end-effector position is specified. Another useful constraint worth considering are constraints to the position of the end-effector. These questions are left for future work.

[1] Examples code: https://github.com/vifremel/FlatVCP/tree/master/matlab/vcp_manip

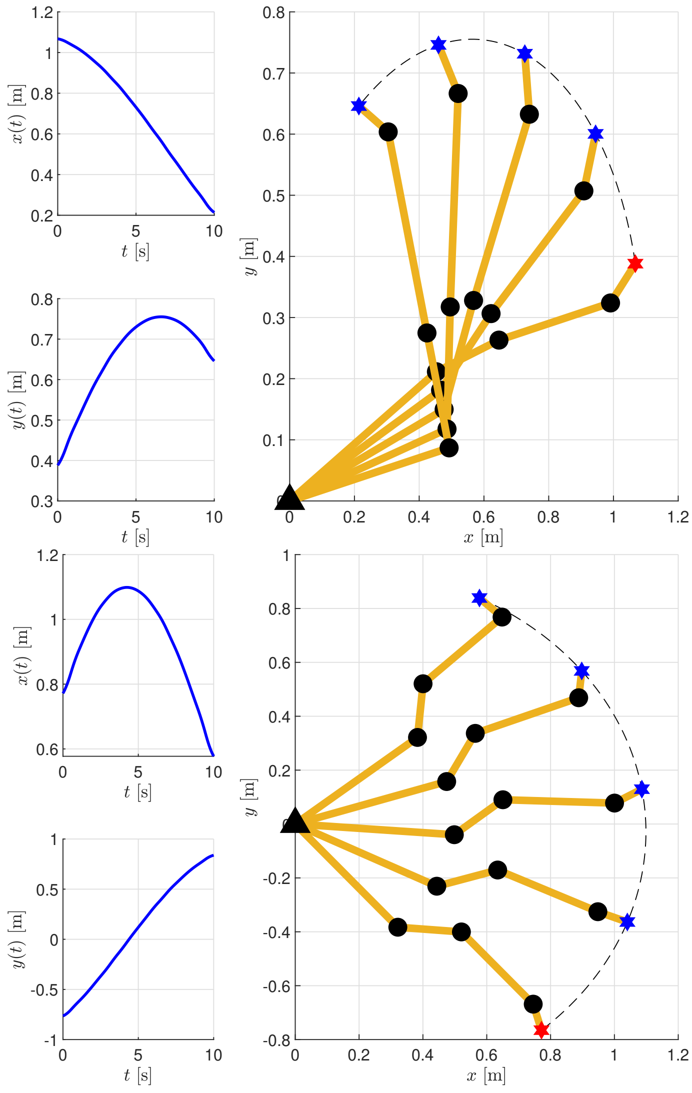

Fig. 2. Example trajectories for a planar manipulator with 4 links. The red hexagram marks the position of the end-effector when the manipulator is in the initial configuration. The lengths of each link are $l_1 = 0.5$, $l_2 = 0.2$, $l_3 = 0.35$ and $l_4 = 0.1$.

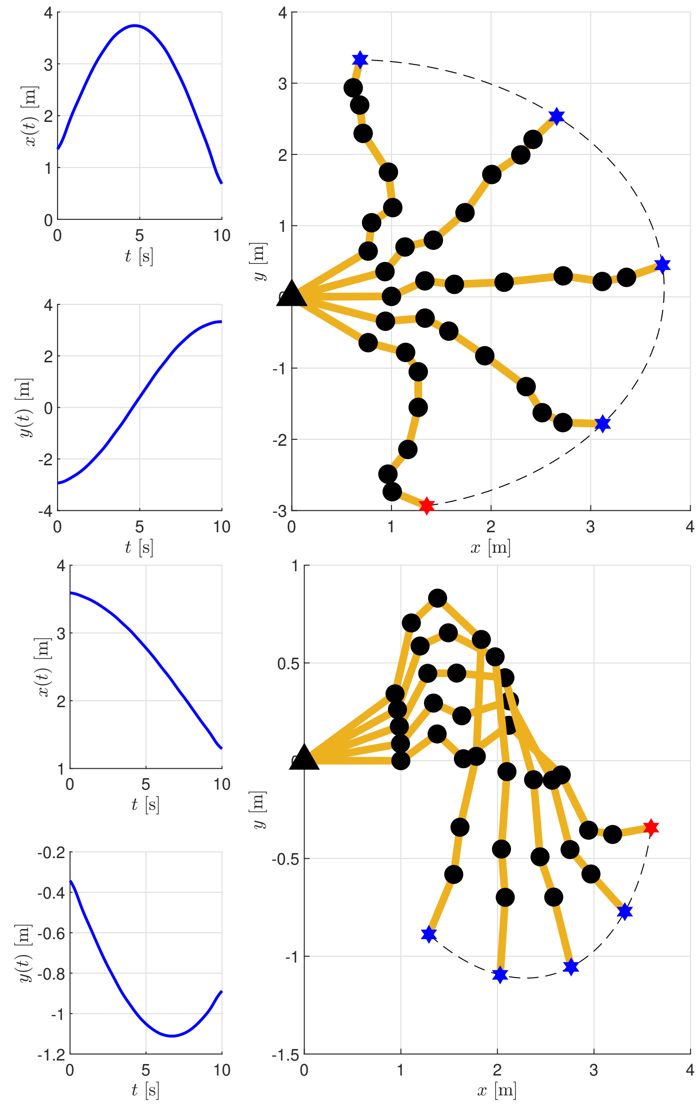

Fig. 3. Example trajectories for a planar manipulator with 8 links. The red hexagram marks the position of the end-effector when the manipulator is in the initial configuration. The lengths of each link are $l_1 = 1$, $l_2 = 0.4$, $l_3 = 0.3$, $l_4 = 0.5$ $l_5 = 0.6$, $l_6 = 0.4$, $l_7 = 0.25$ and $l_8 = 0.4$.

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