

Vectors

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VECTORS [150 Marks]

1 Solve the following:

- a) $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin. Given that lines OA and OB are parallel, $|\mathbf{a}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = -2$, express \mathbf{b} in terms of \mathbf{a} . [2]

Let $\mathbf{b} = k\mathbf{a}$ for some $k \in \mathbb{R}$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{a}) = k|\mathbf{a}|^2$$

$$-2 = k(2)^2 \Rightarrow k = -\frac{1}{2}$$

$$\therefore \mathbf{b} = -\frac{1}{2}\mathbf{a}$$

- b) A vector \mathbf{a} is such that $\mathbf{a} = (\sqrt{2}\cos\alpha)\mathbf{i} - (\cos\alpha)\mathbf{j} + (\sqrt{2}\sin\alpha)\mathbf{k}$, where $0 \leq \alpha \leq 2\pi$ and $|\mathbf{a}| = \sqrt{2}$. Find the value(s) of α . [2]

$$|\mathbf{a}| = \sqrt{2\cos^2\alpha + \cos^2\alpha + 2\sin^2\alpha}$$

$$\sqrt{2} = \sqrt{2 - \cos^2\alpha}$$

$$2 = 2 - \cos^2\alpha$$

$$\cos\alpha = 0$$

$$\alpha = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

- c) The points A, B and C with respect to the origin are represented by the vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively. It is given that $|\mathbf{b}| = 2$, $\mathbf{a} \cdot \mathbf{b} = k$ and $\mathbf{b} \cdot \mathbf{c} = 2$. Given further

that point C divides the line AB such that $AC : CB = 2 : 1$, find k . [3]

$$\begin{aligned}\mathbf{c} &= \frac{\mathbf{a} + 2\mathbf{b}}{3} \\ \mathbf{b} \cdot \mathbf{c} &= \mathbf{b} \cdot \left(\frac{\mathbf{a} + 2\mathbf{b}}{3} \right) \\ 2 &= \frac{1}{3}(\mathbf{b} \cdot \mathbf{a} + 2|\mathbf{b}|^2) \\ 2 &= \frac{1}{3}(k + 8) \\ \therefore k &= -2\end{aligned}$$

d) Four vectors \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} exist such that $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$. Show that $\mathbf{b} \times (\mathbf{a} + \mathbf{c}) = \mathbf{d} \times \mathbf{b}$. [2]

$$\begin{aligned}\mathbf{b} \times (\mathbf{a} + \mathbf{c}) &= \mathbf{b} \times (-\mathbf{b} - \mathbf{d}) \\ &= \mathbf{b} \times (-\mathbf{b}) - \mathbf{b} \times \mathbf{d} \\ &= \mathbf{d} \times \mathbf{b}\end{aligned}$$

e) Point A referred from the origin has vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. The line OA makes an angle of α with the y -axis and β with the z -axis, where $\alpha, \beta < \pi$. Show that

$$\alpha + \beta = \pi. \quad [3]$$

$$\cos \alpha = \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{(1)\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{3}$$

$$\cos \beta = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{(1)(3)} = -\frac{2}{3}$$

Since $\cos \alpha = -\cos \beta$, and $\alpha, \beta < \pi$,

$$\cos \alpha = \cos(\pi - \beta)$$

$$\alpha = \pi - \beta$$

$$\alpha + \beta = \pi$$

2 Referred to the origin O , points A and B have position vectors given by \mathbf{a} and \mathbf{b} respectively. C_0 is the foot of perpendicular from A to OB with position vector \mathbf{c}_0 . The angle between lines OA and OB is α , where $0 < \alpha < \frac{\pi}{2}$.

a) By considering $\cos \alpha$, show that $|\mathbf{c}_0| = \mathbf{a} \cdot \hat{\mathbf{b}}$. [2]

$$\cos \alpha = \frac{|\mathbf{c}_0|}{|\mathbf{a}|}$$

$$\text{Also, } \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

$$\text{Hence, } \frac{|\mathbf{c}_0|}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

$$|\mathbf{c}_0| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \hat{\mathbf{b}}$$

b) The foot of perpendicular from C_0 to OA is C_1 . Show that $|\mathbf{c}_1| = \mathbf{a} \cdot \hat{\mathbf{b}}(\cos \alpha)$. [1]

$$\begin{aligned}\cos \alpha &= \frac{|\mathbf{c}_1|}{|\mathbf{c}_0|} \\ |\mathbf{c}_1| &= |\mathbf{c}_0| \cos \alpha \\ &= \mathbf{a} \cdot \hat{\mathbf{b}} \cos \alpha\end{aligned}$$

c) C_n is the n th foot of perpendicular. State $|\mathbf{c}_n|$ in terms of a , b , n and α . [1]

$$|\mathbf{c}_n| = \mathbf{a} \cdot \hat{\mathbf{b}} \cos^n \alpha$$

d) State the sum to infinity of scalar projections $|\mathbf{c}_0| + |\mathbf{c}_1| + \dots + |\mathbf{c}_n| + \dots$ [1]

$$\begin{aligned}\text{Sum to infinity} &= \mathbf{a} \cdot \hat{\mathbf{b}} + \mathbf{a} \cdot \hat{\mathbf{b}} \cos \alpha + \dots + \mathbf{a} \cdot \hat{\mathbf{b}} \cos^n \alpha + \dots \\ &= \frac{\mathbf{a} \cdot \hat{\mathbf{b}}}{1 - \cos \alpha}\end{aligned}$$

3 Referred to the origin O , points A and B have position vectors given by: $\mathbf{a} = \mathbf{i} - p^2 \mathbf{k}$ and $\mathbf{b} = \frac{2}{p} \mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively, where p is to be found. Given that $|\mathbf{a} \times \mathbf{b}|^2 = 4p^2 + 2$, find

the value(s) that p can take.

[4]

$$\begin{aligned}
 |\mathbf{a} \times \mathbf{b}|^2 &= \left| \begin{pmatrix} 1 \\ 0 \\ -p^2 \end{pmatrix} \times \begin{pmatrix} 2p^{-1} \\ -1 \\ 1 \end{pmatrix} \right|^2 \\
 &= \left| \begin{pmatrix} -p^2 \\ -1 - 2p \\ -1 \end{pmatrix} \right|^2 \\
 &= (p^2)^2 + (1 + 2p)^2 + 1 \\
 &= p^4 + 4p^2 + 4p + 2 \\
 &= 4p^2 + 2 \\
 p^4 + 4p &= 0 \\
 p(p^3 + 4) &= 0 \\
 p &= -\sqrt[3]{4} \text{ since } p \neq 0
 \end{aligned}$$

- 4 The vector equation of l is given by $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, $\lambda \in \mathbb{R}$. Point F is the foot of perpendicular from origin O to the line l . If $|\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 1$, express the position vector \overrightarrow{OF} in terms of \mathbf{a} and \mathbf{b} .

[3]

$$\text{Let } \overrightarrow{OF} = \mathbf{a} + k\mathbf{b} \text{ for some } k \in \mathbb{R}$$

$$\text{Since } OF \perp l, (\mathbf{a} + k\mathbf{b}) \cdot \mathbf{b} = 0$$

$$\mathbf{a} \cdot \mathbf{b} + k|\mathbf{b}|^2 = 0$$

$$1 + k(1)^2 = 0$$

$$k = -1$$

$$\therefore \overrightarrow{OF} = \mathbf{a} - \mathbf{b}$$

- 5 The equations of l and m are given by $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, $\lambda \in \mathbb{R}$ and $m : \mathbf{r} = \mathbf{b} + \mu \mathbf{a}$, $\mu \in \mathbb{R}$, where \mathbf{a} and \mathbf{b} are co-planar vectors. State the conditions such that lines l and m are skew lines.

[2]

Equating l and m and since skew lines are parallel and do not intersect,

$$\mathbf{a} + \lambda \mathbf{b} = \mu \mathbf{a} + \mathbf{b}$$

$$\lambda \neq 1, \mu \neq 1 \text{ and } \mathbf{a} \neq k\mathbf{b} \text{ for all } k \in \mathbb{R}$$

- 6** Points A and B have position vectors \mathbf{a} and \mathbf{b} with respect to the origin O . It is given that $(\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) = 11$. Find the perpendicular distance from point A to line OB if $|\mathbf{b}| = 11$. [4]

$$(\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) = \mathbf{a} \times \mathbf{a} + 7\mathbf{a} \times \mathbf{b} - 15\mathbf{b} \times \mathbf{a} - 21\mathbf{b} \times \mathbf{b}$$

$$11 = 22\mathbf{a} \times \mathbf{b}$$

$$\mathbf{a} \times \mathbf{b} = \frac{1}{2}$$

$$\begin{aligned} \text{Perpendicular distance from A to OB} &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \\ &= \frac{\left(\frac{1}{2}\right)}{11} \\ &= \frac{1}{22} \text{ units} \end{aligned}$$

- 7** Points A and B have position vectors \mathbf{a} and \mathbf{b} with respect to the origin O . It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 2$.

- a) State the vector equation of line AB . [1]

$$l_{AB} : \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \lambda \in \mathbb{R}$$

- b) Find $|\mathbf{b} - \mathbf{a}|$. [2]

$$\begin{aligned} |\mathbf{b} - \mathbf{a}| &= \sqrt{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})} \\ &= \sqrt{|\mathbf{b}|^2 - 2\mathbf{b} \cdot \mathbf{a} + |\mathbf{a}|^2} \\ &= \sqrt{1 - 2(2) + 3^2} \\ &= \sqrt{6} \end{aligned}$$

- c) Find the position vector of F , the foot of perpendicular from O to AB , in terms of \mathbf{a} and \mathbf{b} . [3]

$$\text{Let } \overrightarrow{OF} = \mathbf{a} + k(\mathbf{b} - \mathbf{a}) \text{ for some } k \in \mathbb{R}$$

$$\text{Since } OF \perp AB, (\mathbf{a} + k(\mathbf{b} - \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\mathbf{a} \cdot \mathbf{b} - |\mathbf{a}|^2 + k|\mathbf{b} - \mathbf{a}|^2 = 0$$

$$2 - 3^2 + k(\sqrt{6})^2 = 0$$

$$\therefore k = \frac{7}{6}$$

$$\text{Substituting } k = \frac{7}{6} \text{ back into } \overrightarrow{OF},$$

$$\overrightarrow{OF} = \mathbf{a} + \frac{7}{6}(\mathbf{b} - \mathbf{a})$$

$$\overrightarrow{OF} = \frac{1}{6}(7\mathbf{b} - \mathbf{a})$$

- d) Find $|7\mathbf{b} - \mathbf{a}|$. Hence, find the exact area of triangle OAB . [3]

$$\begin{aligned} |7\mathbf{b} - \mathbf{a}| &= \sqrt{(7\mathbf{b} - \mathbf{a}) \cdot (7\mathbf{b} - \mathbf{a})} \\ &= \sqrt{49|\mathbf{b}|^2 - 14\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2} \\ &= \sqrt{49 - 14(2) + 3^2} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle OAB} &= \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OF}| \\ &= \frac{1}{2} |\mathbf{b} - \mathbf{a}| \left| \frac{1}{6}(7\mathbf{b} - \mathbf{a}) \right| \\ &= \frac{1}{2} (\sqrt{6}) \left(\frac{1}{6} \right) (\sqrt{30}) \\ &= \frac{\sqrt{5}}{2} \text{ units}^2 \end{aligned}$$

- 8 Referred to the origin O , points A and B have the position vectors $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}$ and $\overrightarrow{OB} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively.

a) Verify that $P(3, -2, -6)$ lies on line AB .

[2]

$$\text{Line AB // } \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{Equation of line AB is } \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\text{Substitute } \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} :$$

$$\begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

There is a solution $\lambda = -2$

Hence, P lies on AB.

b) Find the position vector of F , the foot of perpendicular from P to AB .

[3]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\begin{aligned}\overrightarrow{PF} &= \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -2 - k \\ -2 + k \\ 4 + 2k \end{pmatrix}\end{aligned}$$

Since $PF \perp AB$,

$$\begin{pmatrix} -2 - k \\ -2 + k \\ 4 + 2k \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\therefore k = -\frac{4}{3}$$

$$\overrightarrow{OF} = \frac{1}{3} \begin{pmatrix} 7 \\ -4 \\ -14 \end{pmatrix}$$

c) Hence, find the equation of line PF .

[3]

Substituting $k = -\frac{4}{3}$ into \overrightarrow{PF} ,

$$\overrightarrow{PF} = \begin{pmatrix} -2 + \frac{8}{3} \\ -2 - \frac{4}{3} \\ 4 - \frac{8}{3} \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$$

$$\text{Equation of line } PF : \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$$

9 The equations of lines l_1 and l_2 are given by:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \frac{x+1}{9} = \frac{y}{7} = \frac{4-z}{3} \text{ respectively.}$$

Point A has coordinates $(2, -1, 1)$ while the foot of perpendicular from A to l_2 is F .

- a) Find the position vector of P , the point of intersection between l_1 and l_2 . [2]

$$\text{Vector equation of line } l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\text{Equating both lines, } \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}$$

Using G.C., we obtain $\lambda = \frac{3}{2}$, $\mu = \frac{1}{2}$.

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix}$$

b) Find vector \overrightarrow{AF} .

[3]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\begin{aligned} \overrightarrow{AF} &= \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -3 + 9k \\ 1 + 7k \\ 3 - 3k \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Since } AF \perp l_2, \begin{pmatrix} -3 + 9k \\ 1 + 7k \\ 3 - 3k \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} &= 0 \\ \therefore k &= \frac{29}{139} \end{aligned}$$

$$\text{Substituting } k = \frac{29}{139} \text{ back into } \overrightarrow{AF}, \overrightarrow{AF} = \begin{pmatrix} -3 + 9\left(\frac{29}{139}\right) \\ 1 + 7\left(\frac{29}{139}\right) \\ 3 - 3\left(\frac{29}{139}\right) \end{pmatrix} = \frac{2}{139} \begin{pmatrix} -78 \\ 171 \\ 165 \end{pmatrix}$$

c) Hence, find the vector equation of l_3 , the reflection of l_1 in l_2 .

[3]

Let A' on l_3 be the reflection of A in l_2 .

$$\begin{aligned}
 \overrightarrow{AF} &= \overrightarrow{FA'} \\
 &= \overrightarrow{OA'} - \overrightarrow{OF} \\
 \overrightarrow{OA'} &= \overrightarrow{AF} + \overrightarrow{OF} \\
 &= \frac{2}{139} \begin{pmatrix} -78 \\ 171 \\ 165 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \frac{29}{139} \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} \\
 &= \frac{1}{139} \begin{pmatrix} -34 \\ 545 \\ 799 \end{pmatrix}
 \end{aligned}$$

$$l_3 // \frac{1}{139} \begin{pmatrix} -34 \\ 545 \\ 799 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{1041}{278} \\ \frac{117}{278} \\ \frac{903}{278} \end{pmatrix}$$

$$\therefore l_3 : \mathbf{r} = \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} -1041 \\ 117 \\ 903 \end{pmatrix}, \alpha \in \mathbb{R}$$

10 Points A and B with position vectors $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + \mathbf{k}$ respectively both lie on l_1 .

The line l_2 has Cartesian equation $l_2 : x = 7, y - 3 = z$.

a) Show that l_1 and l_2 are skew lines.

[2]

$$l_1 // \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\text{Equating both lines, } \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{We have the following equations } \begin{cases} 2\lambda = 4 \\ -\lambda - \mu = 3 \\ \lambda - \mu = -1 \end{cases}$$

Using a G.C., there is no solution found.

Hence, l_1 and l_2 are skew lines.

b) Find a vector that is perpendicular to both l_1 and l_2 .

[1]

$$\text{A vector that is perpendicular to both } l_1 \text{ and } l_2 // \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

$$\text{Let the vector be } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

c) Hence, find the shortest distance between l_1 and l_2 .

[3]

Let point C on l_2 be $C(7, 3, 0)$.

$$\begin{aligned}
 \text{Shortest distance} &= \text{Projection of } \overrightarrow{BC} \text{ onto } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \\
 &= \frac{\left(\begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right|} \\
 &= \frac{\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right|} \\
 &= \frac{8}{\sqrt{3}} \\
 &= \frac{8\sqrt{3}}{3} \text{ units}
 \end{aligned}$$

- 11** Referred to an origin O , points A and B have coordinates $(-1, 2, 2)$ and $(0, 1, 2)$ respectively. The point P on OA is such that $OP : PA = \lambda : 1$ and the point Q on OB is such that $OQ : QB = \lambda : 1 - \lambda$, where λ is a real constant to be determined.

a) Find the area of ΔOAB .

[2]

$$\begin{aligned}
 \text{Area of } \Delta OAB &= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \\
 &= \frac{1}{2} \left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right| \\
 &= \frac{1}{2} \left| \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \right| \\
 &= \frac{1}{2} \sqrt{4 + 4 + 1} \\
 &= \frac{3}{2} \text{units}^2
 \end{aligned}$$

b) Express the ratio $\frac{\text{Area of } \Delta OAB}{\text{Area of } \Delta OPQ}$ in terms of λ .

[3]

$$\begin{aligned}
 \text{Area of } \Delta OPQ &= \frac{1}{2} \left| \overrightarrow{OP} \times \overrightarrow{OQ} \right| \\
 &= \frac{1}{2} \left| \left(\frac{\lambda}{\lambda+1} \right) \mathbf{a} \times \left(\frac{\lambda}{\lambda+1-\lambda} \right) \mathbf{b} \right| \\
 &= \frac{1}{2} \left| \frac{\lambda^2}{\lambda+1} \mathbf{a} \times \mathbf{b} \right| \\
 &= \left(\frac{\lambda^2}{\lambda+1} \right) \left(\frac{1}{2} |\mathbf{a} \times \mathbf{b}| \right), \text{ since } 0 < \lambda < 1 \text{ so } \frac{1}{\lambda+1} > 0 \\
 &= \left(\frac{\lambda^2}{\lambda+1} \right) (\text{Area of } \Delta OAB) \\
 \therefore \frac{\text{Area of } \Delta OAB}{\text{Area of } \Delta OPQ} &= \frac{\lambda+1}{\lambda^2}
 \end{aligned}$$

c) Deduce if PQ is ever parallel to AB for some value of λ .

[3]

$$\begin{aligned}
 \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\
 &= \lambda \mathbf{b} - \left(\frac{\lambda}{\lambda+1} \right) \mathbf{a}
 \end{aligned}$$

Assuming $PQ \parallel AB$,

$$\lambda \mathbf{b} - \left(\frac{\lambda}{\lambda+1}\right) \mathbf{a} = k(\mathbf{b} - \mathbf{a}) \text{ for some } k \in \mathbb{R}$$

Equating scalar multiples of \mathbf{b} and \mathbf{a} ,

$$\lambda = k \text{ and } \frac{\lambda}{\lambda+1} = k$$

$$\lambda = \frac{\lambda}{\lambda+1}$$

$$\lambda^2 = 0$$

However, clearly $0 < \lambda < 1$, so no value of $k \in \mathbb{R}$ exists for $PQ \parallel AB$.

- 12** Line l has the equation $-x = \frac{y-3}{2} = \frac{z+4}{2}$. Line m , which is parallel to $\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$ where c is some real constant, is obtained by rotating line l 45° about the point $A(0, 3, -4)$. Find the possible vector equations of line m . [5]

$$\text{Equation of line } l : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\text{Equation of line } m : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix} \right|} = \frac{2-c}{3\sqrt{c^2+1}}$$

$$\frac{1}{2} = \frac{(2-c)^2}{9(c^2+1)}$$

$$9c^2 + 9 = 2c^2 - 8c + 8$$

$$7c^2 + 8c + 1 = 0$$

$$c = -\frac{1}{7} \text{ or } -1$$

The two equations of line m are:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \mu \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

13 Three points A, B and C referred from the origin O have position vectors given by:

$$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \mathbf{b} = -2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{c} = \frac{3}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} - 3\mathbf{k}.$$

a) Find the vector equations of lines AB and AC . [2]

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 2.5 \\ -3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Equations of lines AB and AC are:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \mu \in \mathbb{R} \text{ respectively.}$$

b) Find two vector equations of l , where l is the line representing the all the midpoints

of lines AB and AC .

[4]

$$\text{Unit vector of } AB, \mathbf{u}_1 = \frac{1}{\sqrt{4^2 + 1 + 3^2}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$

$$\text{Unit vector of } AC, \mathbf{u}_2 = \frac{1}{\sqrt{1 + 3^2 + 4^2}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Two possible midpoints have position vectors $\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)$ and $\frac{1}{2}(\mathbf{u}_1 - \mathbf{u}_2)$.

$$\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2) = \frac{1}{2} \left(\frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right) = \frac{1}{2\sqrt{26}} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$\frac{1}{2}(\mathbf{u}_1 - \mathbf{u}_2) = \frac{1}{2} \left(\frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} - \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right) = \frac{1}{2\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

Hence, two possible equations of l are:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R} \text{ and}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}, \quad t \in \mathbb{R}$$

- 14** Point A with position vector \mathbf{a} lies on plane π with normal parallel to vector \mathbf{n} . Given that $|\mathbf{a} - \mathbf{n}|^2 = 3$ and $|\mathbf{n}|^2 = 4 - |\mathbf{a}|^2$, find the value of d if the equation of plane π is $\mathbf{r} \cdot \mathbf{n} = d$.

[4]

The equation of plane π is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

To find $\mathbf{a} \cdot \mathbf{n}$,

$$|\mathbf{a} - \mathbf{n}|^2 = 3$$

$$(\mathbf{a} - \mathbf{n}) \cdot (\mathbf{a} - \mathbf{n}) = 3$$

$$|\mathbf{a}|^2 + |\mathbf{n}|^2 - 2\mathbf{a} \cdot \mathbf{n} = 3$$

$$4 - 2\mathbf{a} \cdot \mathbf{n} = 3$$

$$\mathbf{a} \cdot \mathbf{n} = \frac{1}{2}$$

\therefore The equation of π is $\mathbf{r} \cdot \mathbf{n} = \frac{1}{2}$.

15 The equations of parallel planes p and q are given by $p : \mathbf{r} \cdot \mathbf{n} = d$ and $q : \mathbf{r} \cdot \mathbf{n} = kd$.

Line l given by equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, $\lambda \in \mathbb{R}$ intersects planes p and q at points A and B respectively.

a) Show that $\overrightarrow{AB} = \mathbf{b} \left(\frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right)$. [4]

Substitute equation of l into p and q to get \overrightarrow{OA} and \overrightarrow{OB} respectively.

For A , $(\mathbf{a} + \lambda \mathbf{b}) \cdot \mathbf{n} = d$

$$\lambda = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}$$

$$\overrightarrow{OA} = \mathbf{a} + \left(\frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b}$$

For B , $(\mathbf{a} + \lambda \mathbf{b}) \cdot \mathbf{n} = kd$

$$\lambda = \frac{kd - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}$$

$$\begin{aligned}
\overrightarrow{OB} &= \mathbf{a} + \left(\frac{kd - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} \\
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= \mathbf{a} + \left(\frac{kd - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} - \left(\mathbf{a} + \left(\frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} \right) \\
&= \left(\frac{kd - \mathbf{a} \cdot \mathbf{n} - d + \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} \\
&= \mathbf{b} \left(\frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right)
\end{aligned}$$

- b) Hence, or otherwise, show that the perpendicular distance between planes p and q is equal to $\frac{d(k-1)}{|\mathbf{n}|}$ units. [2]

Perpendicular distance between planes p and q = Projection of \overrightarrow{AB} onto \mathbf{n}

$$\begin{aligned}
&= \frac{\left(\mathbf{b} \left(\frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right) \right) \cdot \mathbf{n}}{|\mathbf{n}|} \\
&= \frac{\left(\frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right) (\mathbf{b} \cdot \mathbf{n})}{|\mathbf{n}|} \\
&= \frac{d(k-1)}{|\mathbf{n}|}
\end{aligned}$$

16 The equations of plane π and l are given by:

$$\pi : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -1 \text{ and } l : \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix}, \lambda, k \in \mathbb{R} \text{ respectively.}$$

- a) Show that, for π and l to intersect, $k \neq -\frac{7}{2}$. [1]

If π and l do not intersect, l is perpendicular to the normal of π .

$$\text{i.e. } \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 2k + 7 = 0$$

$$k = -\frac{7}{2}$$

Hence, for intersection, $k \neq -\frac{7}{2}$.

For the rest of the question, assume $k = 1$.

- b)** Find the coordinates of point P , the point of intersection of π and l . [2]

Equating line and plane,

$$\left(\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -1$$

$$9s + 3 = -1$$

$$s = -\frac{4}{9}$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -\frac{1}{9} \begin{pmatrix} 21 \\ 8 \\ -14 \end{pmatrix}$$

- c)** Find the shortest distance from $A(-1, 0, 2)$ to π . [3]

$$\overrightarrow{AP} = -\frac{1}{9} \begin{pmatrix} 21 \\ 8 \\ -14 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -\frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$$

Shortest distance = Projection of \overrightarrow{AP} onto normal of π

$$\begin{aligned}
 & \left| -\frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| \\
 &= \frac{\left| -\frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|} \\
 &= \frac{4}{9} \left(\frac{23}{3} \right) \\
 &= \frac{92}{27} \text{units}
 \end{aligned}$$

d) Find the acute angle between π and l .

[2]

Let acute angle be α .

$$\sin \alpha = \frac{\left(\frac{92}{27} \right)}{\left| \overrightarrow{AP} \right|} = \frac{\left(\frac{92}{27} \right)}{\frac{4}{9} \left| \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \right|} = \frac{\left(\frac{92}{27} \right)}{\frac{4}{9} \sqrt{77}}$$

$\therefore \alpha = 60.9^\circ$ (1 d.p.)

17 Plane π has a normal parallel to $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and has the equation:

$$\pi : \mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R},$$

where a is some real constant to be determined.

a) Find the value of a .

[2]

$$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} // \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ a-3 \\ 6 \end{pmatrix}$$

$$\text{Clearly, } \begin{pmatrix} -2 \\ a-3 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore a = 5$$

b) Find the scalar product equation of plane π .

[1]

$$\text{Equation is } \mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} = 6$$

c) Line l passes through π , the origin O and $A(3, 2, 5)$. Find the position vector of P , the point of intersection between line l and plane π .

[2]

$$\text{Equation of } l \text{ is } \mathbf{r} = \lambda \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \lambda \in \mathbb{R}.$$

$$\text{Substituting into equation of } \pi, k \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = 6 \text{ for some } k \in \mathbb{R}$$

$$14k = 6$$

$$k = \frac{3}{7}$$

$$\therefore \overrightarrow{OP} = \frac{3}{7} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

- d) Find $|\overrightarrow{PF}|$, where F is the foot of perpendicular from A to plane π . [3]

The easiest method is to use vector product projection of \overrightarrow{PA} onto the normal of π .

We can find \overrightarrow{PA} using the fact that O , P and A are collinear and $OP : PA = 3 : 4$.

$$\begin{aligned} |\overrightarrow{PF}| &= \frac{\left| \overrightarrow{PA} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|}{\left| \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|} \\ &= \frac{\left| \frac{4}{7} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|}{\sqrt{11}} \\ &= \frac{\frac{4}{7} \left| \begin{pmatrix} 1 \\ -14 \\ 5 \end{pmatrix} \right|}{\sqrt{11}} \\ &= \frac{4}{7} \frac{\sqrt{1 + 14^2 + 5^2}}{\sqrt{11}} \\ &= \frac{4}{7} \sqrt{\frac{222}{11}} \text{ units} \end{aligned}$$

- e) Hence find $|\overrightarrow{PG}|$, where G is the foot of perpendicular from O to plane π . [2]

$$\frac{|\overrightarrow{PG}|}{|\overrightarrow{PF}|} = \frac{|\overrightarrow{OP}|}{|\overrightarrow{PA}|} = \frac{4}{3} \text{ by similar triangles, so we have:}$$

$$\begin{aligned}
|\overrightarrow{PG}| &= \frac{4}{3} |\overrightarrow{PF}| \\
&= \frac{4}{3} \left(\frac{3}{7} \sqrt{\frac{222}{11}} \right) \\
&= \frac{4}{7} \sqrt{\frac{222}{11}} \text{ units}^2
\end{aligned}$$

18 The equations of planes π_1 , π_2 and π_3 are such that:

$$\pi_1 : 2x + 3y + 4z = -1, \quad \pi_2 : -2x + y - z = 5 \quad \text{and} \quad \pi_3 : \mathbf{r} \cdot \begin{pmatrix} a \\ -5 \\ -a \end{pmatrix} = k.$$

a) Find the vector equation of l , the line of intersection between π_1 and π_2 . [3]

$$\text{We have the two equations, } \begin{cases} 2x + 3y + 4z = -1 \\ -2x + y - z = 5 \end{cases}$$

$$\text{Using a G.C., we obtain } x = -2 - \frac{7}{8}z, \quad y = 1 - \frac{3}{4}z, \quad z \in \mathbb{R}.$$

$$\text{Equation of } l \text{ is } \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

b) Given that $a = 2$, find the value of k such that π_3 contains l . [2]

Assuming that π_3 contains l , substituting equation of l into equation of π ,

$$\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix} = k \text{ for all } \lambda \in \mathbb{R}$$

$$-4 + 14\lambda - 5 - 30\lambda + 16\lambda = k$$

$$k = -9$$

c) Given that $a = 1$, $k = 3$, find the point of intersection of π_1 , π_2 and π_3 . [2]

We have the three equations,
$$\begin{cases} 2x + 3y + 4z = -1 \\ -2x + y - z = 5 \\ x - 5y - z = 3 \end{cases}$$

Using a G.C., we obtain $x = -\frac{20}{3}$, $y = -3$ and $z = \frac{16}{3}$.

19 An incident beam of light was reflected perfectly ($\theta_1 = \theta_2$) on a round mirror.

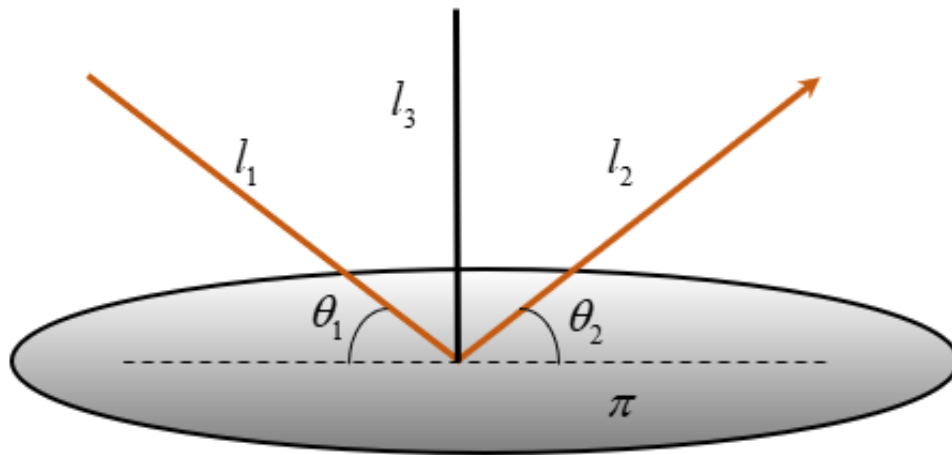


Figure 1: Mirror

A student modelled the scenario such that the incident beam is l_1 , the reflected beam is l_2 and the mirror is π , where π contains the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

a) Find the equation of the plane π in the form $\mathbf{r} \cdot \mathbf{n} = d$.

[3]

$$\begin{aligned}
 \text{Normal of plane } \pi // & \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) \times \left(\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix} \\
 d &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 5 \\
 \therefore \text{Equation of } \pi \text{ is } & \mathbf{r} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 5
 \end{aligned}$$

b) Show that $P(1, 3, 2)$, the point of intersection between l_1 and l_2 lies on π .

[1]

$$\begin{aligned}
 \text{Substituting } \mathbf{r} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \text{ into equation of } \pi, \\
 \text{LHS} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 5 \\
 &= \text{RHS}
 \end{aligned}$$

- c) State the vector equation of l_3 , the axis of reflection between l_1 and l_2 . [1]

$$l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

- d) Given that $A(t, 1, 1)$ lies on l_1 , where $t > 0$, find t such that $\theta_1 = \theta_2 = \frac{\pi}{4}$. [4]

$$\overrightarrow{PA} = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix}$$

Let α be the angle between l_1 and l_3 .

$$\begin{aligned} \cos \alpha &= \frac{\begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix} \right| \cdot \left| \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|} \\ &= \frac{4t-4}{\sqrt{t^2-2t+6}\sqrt{21}} \end{aligned}$$

Since $\theta_1 = \frac{\pi}{4}$, $\alpha = \frac{3\pi}{4}$ or $\frac{\pi}{4}$.

Regardless, $\cos^2 \alpha = \frac{1}{2}$.

$$\text{Squaring both sides, } \frac{1}{2} = \frac{16t^2 - 32t + 16}{21t^2 - 42t + 126}$$

$$11t^2 - 22t - 94 = 0$$

Using G.C., $t = 2.50$ or -3.41 (3s.f.)

Since $t > 0$, $t = 2.50$

For the rest of the question, take $t = 4$.

e) Find the shortest distance from A to π .

[2]

$$\begin{aligned}
 \text{Shortest distance} &= \frac{\left| \overrightarrow{AP} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|} \\
 &= \frac{\left| \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{21}} \\
 &= \frac{\left| \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} \right|}{\sqrt{21}} \\
 &= \frac{5}{\sqrt{21}} \sqrt{1 + 2^2 + 1} \\
 &= \frac{5\sqrt{14}}{7} \text{ units}
 \end{aligned}$$

f) Find the coordinates of F , the foot of perpendicular from A to l_3 . Hence, or otherwise, find the equation of l_2 .

[6]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{AF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 + 4k \\ 2 - k \\ 1 + 2k \end{pmatrix}$$

Since $\overrightarrow{AF} \perp l_3$,

$$\begin{pmatrix} -3 + 4k \\ 2 - k \\ 1 + 2k \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$-12 + 21k = 0$$

$$k = \frac{4}{7}$$

$$\begin{aligned} \overrightarrow{OF} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 23 \\ 17 \\ 22 \end{pmatrix} \\ \therefore F &\left(\frac{23}{7}, \frac{17}{7}, \frac{22}{7} \right) \end{aligned}$$

Let A' be the reflection of point A in l_3 .

$$\begin{aligned}
 \overrightarrow{OA'} &= \overrightarrow{OA} + 2\overrightarrow{AF} \\
 &= \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + 2 \left(\frac{1}{7} \begin{pmatrix} 23 \\ 17 \\ 22 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \right) \\
 &= \frac{1}{7} \begin{pmatrix} 18 \\ 27 \\ 37 \end{pmatrix} \\
 \overrightarrow{A'P} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \frac{1}{7} \begin{pmatrix} 18 \\ 27 \\ 37 \end{pmatrix} \\
 &= \frac{1}{7} \begin{pmatrix} -11 \\ -6 \\ -23 \end{pmatrix}
 \end{aligned}$$

Since l_2 is parallel to $\overrightarrow{A'P}$ and contains P , equation of l_2 is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 11 \\ 6 \\ 23 \end{pmatrix}, \mu \in \mathbb{R}.$$

20 A professional card stacker stacks two cards P and Q as follows:

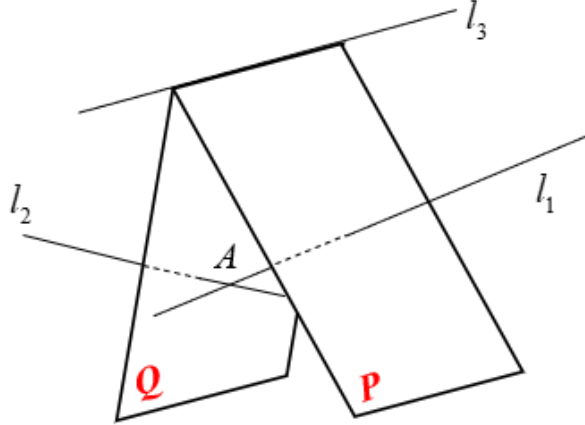


Figure 2: Card Stack

Mr Poh models the scenario such that the two cards are planes P and Q , where the equation of plane P is $P : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1$, where line l_1 is a normal to plane P that contains $A(-3, 3, 2)$. Line l_2 , a normal to plane Q , also contains point A and is parallel to the vector $\begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}$ where $a < 0$. l_3 is the line of intersection between planes P and Q .

- a) Find the coordinates of B , the point of intersection between l_1 and plane P . [2]

$$\text{Equation of } l_1 \text{ is } \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Substitute equation of l_1 into equation of P :

$$\left(\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1 \text{ for some } k \in \mathbb{R}$$

$$-10 + 11k = 1$$

$$k = 1$$

$$\overrightarrow{OB} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore B(0, 2, 3)$$

- b) Given that l_2 is obtained by rotating l_1 $\cos^{-1} \frac{9}{11}$ about point A , find a ; Hence, find the vector equation of l_2 . [4]

$$\pm \cos \left(\cos^{-1} \frac{9}{11} \right) = \frac{\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}}{\left\| \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix} \right\|}$$

$$\pm \frac{9}{11} = \frac{10+a}{\sqrt{11}\sqrt{10+a^2}}$$

$$\frac{81}{121} = \frac{100+a^2+20a}{110+11a^2}$$

$$8910 + 891a^2 = 12100 + 121a^2 + 2420a$$

$$770a^2 - 2420a - 3190 = 0$$

Using a G.C., we obtain $a = -1$ or $\frac{29}{7}$. Since $a < 0$, $a = -1$.

$$\therefore l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

- c) The point of intersection between l_2 and plane Q is point B' . Given that $\left| \overrightarrow{AB} \right| = \left| \overrightarrow{AB'} \right|$, find the position vector of B' given that the x -coordinate of $B' < 0$. [3]

$$\begin{aligned} \left| \overrightarrow{AB'} \right| &= \left| \overrightarrow{AB} \right| \\ &= \left| \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{11} \end{aligned}$$

$$\text{Unit vector along } AB' = \frac{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right|} = \frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{OB'} = \overrightarrow{OA} \pm \frac{|\overrightarrow{AB'}|}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix}$$

Since the x - coordinate of $B' < 0$, $\overrightarrow{OB'} = \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix}$

- d) Find the equation of the plane Q in the form $\mathbf{r} \cdot \mathbf{n} = d$. [2]

Since plane Q contains B' , equation of plane Q is

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 25$$

- e) Find the vector equation of l_3 . [2]

We have the two equations,
$$\begin{cases} 3x - y + z = 1 \\ 3x - y - z = -25 \end{cases}$$

Using a G.C., we obtain $x = -4 + \frac{1}{3}y$, $y \in \mathbb{R}$, $z = 13$.

Equation of l_3 is $\mathbf{r} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\tau \in \mathbb{R}$.

The equation of plane R is such that it reflects the image of plane P to form plane

Q .

f) Find the vector \overrightarrow{AF} , where F is the foot of perpendicular from A to l_3 .

Hence, find the equation of plane R in the form $\mathbf{r} \cdot \mathbf{n} = d$.

[5]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + k \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{AF} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + k \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+k \\ -3+3k \\ 11 \end{pmatrix}$$

Since $\overrightarrow{AF} \perp l_3$,

$$\begin{pmatrix} -1+k \\ -3+3k \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 0$$

$$-10 + 10k = 0$$

$$k = 1$$

$$\therefore \overrightarrow{AF} = \begin{pmatrix} -1+1 \\ -3+3 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$$

Since plane R is parallel to both \overrightarrow{AF} and l_3 ,

$$\text{Normal to plane } R // \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Equation of plane } R \text{ is } \mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = 12$$