Sequences & Series

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SEQUENCES & SERIES [55 Marks]

For all questions below, you may use the following standard results:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),\ \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

1 A geometric progression has first term $a = \frac{1}{2}$ and common ratio r. S_n denotes the sum of the first n terms of the progression.

(a) Given that the ratio $S_\infty : a = 3 : 2$, find r. [1]

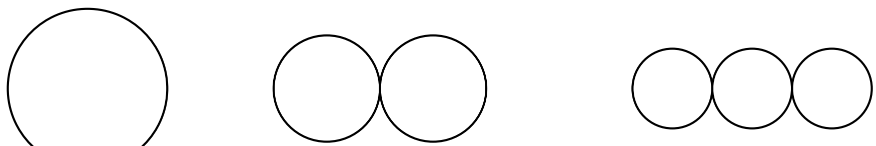
(b) Find the least value of k such that S_k is more than 99% of S_∞. [3]

2 The sum of the first n terms of a series is given by $S_n = 12 - 12\left(\frac{5}{6}\right)^n$.

(a) Find u_n. Hence, show that the series is geometric. [2]

(b) It is given that u_1 and u_2 are the third and fourth term respectively of an arithmetic progression $v_1, v_2, \ldots$. Find v_n in terms of n. [3]

3 A wire artist has a piece of wire of length L. With one piece of wire, he can choose to make a different number of circles of different radiuses as shown:

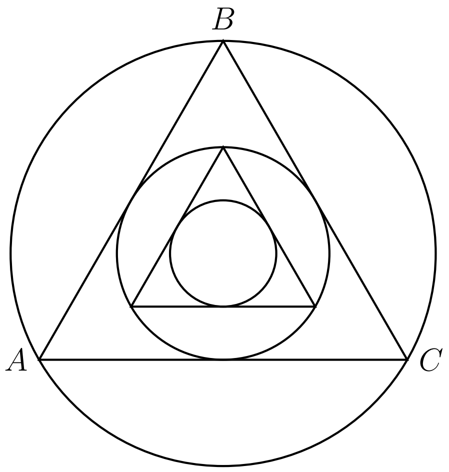


One Circle　　Two Circles　　Three Circles

(a) Show that the area of the circles A_n, where n denotes the number of circles formed with the wire, does not follow a geometric progression. [3]

(b) It is given that $B = \frac{1}{A}$. Describe the progression $B_1, B_2, \ldots, B_n, \ldots$. [2]

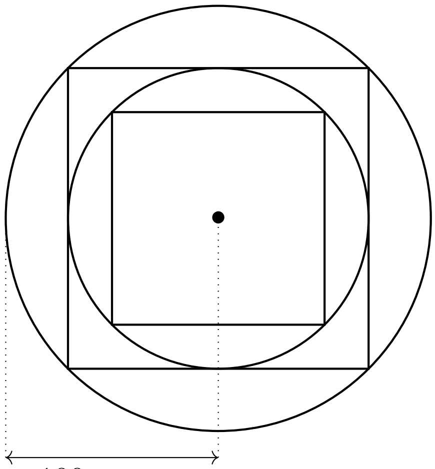
4 Equilateral triangle ABC is inscribed in circle ABC of radius r as shown in the diagram below. A circle is then inscribed in ΔABC, and the process continues indefinitely.

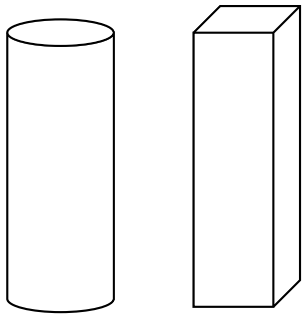


(a) Show that the areas of the equilateral triangles form a geometric progression. Hence, state the sum to infinity of this geometric progression. [5]

(b) An architect uses a similar design to build the tallest structure in the world constructed by cuboids stacked on top of cylinders stacked on top of cuboids and so on. Each cylindrical block is 20 storeys high and each cuboid block is 10 storeys high, where each storey is 3 metres in height. The first block, a cylinder, has a radius of

100 metres. Below is the bird's eye view of the building:





Due to building restrictions, the area of a block must be at least 60 m^2. Find the maximum height the building can be built to. [5]

5 On 1 January 2020, Jill has \$100,000 in her bank account. According to her savings plan, 2% of her savings will be credited to this same account at the end of each month.

(a) Given that she spends \$5000 at the start of every month including January, find the least value of n such that she cannot finance her expenses in the nth month. [3]

(b) Given instead that Jill spends \$ k at the start of every month including January, find the greatest value of \$ k such that she can afford a \$25,000 car at the end of 2020. Leave your answer to the nearest cent. [3]

6 On 1 January 2020, Jack has \$5,000 in his bank account. He is looking for a savings plan to maximise his savings. Given that he spends \$200 at the start of every month including January, what is the minimum value of r, where r % is the interest rate under the savings plan, such that Jack's balance will not go below \$4,000 by the end of 2020? [5]

7 It is given that $u_r = ar - r^2$, where $a > 0$. Given that $\frac{1}{2k(2k+1)} \sum_{r=1}^{2k} u_r = -\frac{13}{6}$ and $\sum_{r=k+1}^{5} u_r = -8$, solve for a and k [5]

8 It is given that $u_r = -\frac{1+2r}{2^{r+1}(r+1)!}$.

(a) Show that $u_r = \frac{1}{2^{r+1}(r+1)!} - \frac{1}{2^r(r)!}$. [2]

(b) Hence, evaluate $\sum_{r=1}^{n} u_r$. [3]

(c) Deduce the value of $\sum_{r=1}^{\infty} u_r$. [1]

9 **(a)** Show that $\cos(k+1)x - \cos(k-1)x = -2\sin kx \sin x$. [1]

(b) Find the sum of the series $\sin 2x + \sin 4x + \cdots + \sin 2nx$. [4]

10 Evaluate $\sum_{j=1}^{N} \mathrm{e}^{\left(\sum_{r=1}^{n} u_r\right)j}$ given that $u_r = \ln r$. [4]