Maclaurin's Series

Derek, Vignesh

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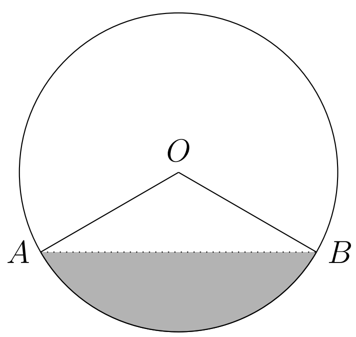
MACLAURIN'S SERIES [TBC Marks]

1 Find the first three non-zero terms in the expansion of $\mathrm{f}(x) = -\dfrac{x}{\sqrt{4-x^2}}$. [3]

We first express the denominator of $\mathrm{f}(x)$ in the form $(1+x)^n$ for Maclaurin expansion.

$$\begin{aligned}
\mathrm{f}(x) &= -\frac{x}{\sqrt{4-x^2}} \\
&= -x(4-x^2)^{-\frac{1}{2}} \\
&= -x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}} \cdot 4^{-\frac{1}{2}} \\
&= -\frac{1}{2}x\left(1+\frac{x^2}{8}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2+\cdots\right) \\
&= -\frac{x}{2}-\frac{x^3}{16}-\frac{3x^5}{256}-\cdots
\end{aligned}$$

2 Sector AOB of a circle centred at O with radius 4 cm is such that $\sphericalangle AOB = \frac{2\pi}{3}+\theta$ as shown in the diagram below:



Given that θ is sufficiently small for θ^4 and higher powers of θ to be neglected, find the approximate area of the shaded region in terms of powers of θ. [5]

$$\begin{aligned}
\text{Area of Segment } AOB &= \text{Area of Sector } AOB - \text{Area of } \Delta AOB \\
&= (4)\left(\frac{2\pi}{3}+\theta\right) - \frac{1}{2}\cdot 4^2 \sin\left(\frac{2\pi}{3}+\theta\right) \\
&= \frac{8\pi}{3} + 4\theta - 8\left(\sin\frac{2\pi}{3}\cos\theta + \cos\frac{2\pi}{3}\sin\theta\right) \\
&= \frac{8\pi}{3} + 4\theta - 8\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) \\
&= \frac{8\pi}{3} + 4\theta - 4\sqrt{3}(\cos\theta - \sin\theta) \\
&= \frac{8\pi}{3} + 4\theta - 4\sqrt{3}(\cos\theta - \sin\theta) \\
&= \frac{8\pi}{3} + 4\theta - 4\sqrt{3}\left(1 - \frac{\theta^2}{2!} + \cdots - \left(\theta - \frac{\theta^3}{3!} + \cdots\right)\right) \\
&= \frac{8\pi}{3} - 4\sqrt{3} + (4+4\sqrt{3})\theta + 2\sqrt{3}\,\theta^2 - \frac{2\sqrt{3}}{3}\theta^3 + \cdots
\end{aligned}$$

3 It is given that $y = \sqrt[3]{(1+x^2)(1+4x^2)}$.

(a) State the Maclaurin's expansion of y in ascending powers of x up to and including the x^4 term. [4]

We will express y as a product of two Maclaurin expansions and collect the x^0, x^2 and x^4 terms to simplify the expression.

$$\begin{aligned}
y &= \sqrt[3]{(1+x^2)(1+4x^2)} \\
&= (1+x^2)^{\frac{1}{3}}(1+4x^2)^{\frac{1}{3}} \\
&= \left(1 + \frac{1}{3}x^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^4 + \cdots\right)\left(1 + \frac{1}{3}(4x^2) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(16x^4) + \cdots\right) \\
&= \left(1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \cdots\right)\left(1 + \frac{4}{3}x^2 - \frac{16}{9}x^4 + \cdots\right) \\
&= 1 + \frac{5}{3}x^2 - \frac{13}{9}x^4 + \cdots
\end{aligned}$$

(b) Hence, show that $\frac{91487}{90000}$ is an accurate estimate for $\sqrt[3]{1.0504}$ up to 4 d.p.. [2]

We first evaluate $\sqrt[3]{1.0504}$ up to 4 decimal places using our G.C., getting a result of 1.0165. To obtain the fraction above, we use the substitution $x = 0.1$ i.e. $\sqrt[3]{(1+0.1^2)(1+4(0.1))^2} = \sqrt[3]{1.0504}$. Since x is sufficiently small, we use the Maclaurin expansion in **(a)** to estimate $\sqrt[3]{1.0504}$.

$$\begin{aligned}\sqrt[3]{1.0504} = y|_{x=0.1} &\approx 1 + \frac{5}{3}(0.1)^2 - \frac{13}{9}(0.1)^4 \\ &= \frac{91487}{90000} \\ &= 1.0165\dot{2} \\ &= 1.0165 \text{ (to 4 d.p.)}\end{aligned}$$

4 Given that $y = 2^{\sin^{-1} x}$, where $\sin^{-1} x$ denotes the principal value;

(a) Show that:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{x}{1-x^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}.$$

[4]

By repeated Differentiation w.r.t. x,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{(\ln 2)(\sin^{-1} x)} \\ &= \mathrm{e}^{(\ln 2)(\sin^{-1} x)}\left(\frac{\ln 2}{\sqrt{1-x^2}}\right) \\ &= y\left(\frac{\ln 2}{\sqrt{1-x^2}}\right) \\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\ln 2}{\sqrt{1-x^2}}\right) + y(\ln 2)\left(-\frac{1}{2}\right)(1-x^2)^{-\frac{3}{2}}(-2x) \\ &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right) + y\left(\frac{\ln 2}{\sqrt{1-x^2}}\right)\left(\frac{x}{1-x^2}\right) \\ &= \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{x}{1-x^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \text{ (shown)}\end{aligned}$$

(b) Hence, obtain the first three terms in the Maclaurin's series for y. [3]

The formula we want is $\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \cdots + \frac{x^n}{n!}\mathrm{f}^{(n)}(0) + \cdots$, which can be found in the MF26.

$$\begin{aligned} y\big|_{x=0} &= 2^0 = 1 \\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} &= (1)\left(\frac{\ln 2}{\sqrt{1-0^2}}\right) = \ln 2 \\ \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} &= \frac{1}{1}(\ln 2)^2 + 0 \\ &= (\ln 2)^2 \\ \therefore y &= 1 + x\ln 2 + \frac{x^2(\ln 2)^2}{2} + \cdots \end{aligned}$$

5 The graph of $y = \mathrm{f}(x)$ is given by:

$$\mathrm{f}(x) = \begin{cases} \mathrm{e}^{1/x^2}, & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

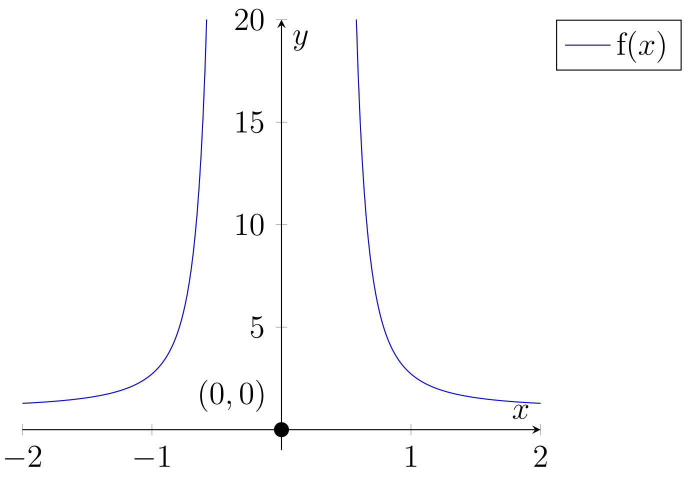
(a) Show that $\mathrm{f}(x)$ is not equal to the Maclaurin Series of e^{1/x^2}. [3]

We will first find the Maclaurin expansion of e^{1/x^2}, which is in the form $\mathrm{e}^{g(x)}$.

$$\mathrm{e}^{1/x^2} = 1 + \frac{1}{x^2} + \frac{1}{2x^4} + \cdots + \frac{1}{r!x^{2r}} + \cdots$$

As $x \to 0$ it is evident that the Maclaurin Series does not converge to 0, thus $f(x)$ is not equal to its Maclaurin Series

(b) Graph the function in **(a)**, labelling clearly the points at/around the origin. [3]



6 Given that $\tan 2x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$, by using the fact that $\sin 2x = \cos 2x \cdot \tan 2x$, obtain the expansion of $\tan 2x$ in ascending powers of x, up to and including the term in x^3. [4]

By expressing $\sin 2x$ and $\cos 2x$ in terms of their Maclaurin expansions, we can find each a_j for $j = 0, 1, 2, 3$ by comparing coefficients on the LHS and RHS of the equation.

$$\begin{aligned} \sin 2x &= \cos 2x \cdot \tan 2x \\ 2x - \frac{1}{3!}(2x)^3 + \cdots &= (1 - \frac{1}{2!}(2x)^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots) \\ 2x - \frac{4}{3}x^3 + \cdots &= a_0 + a_1 x + (a_2 - 2a_0)x^2 + (a_3 - 2a_1)x^3 + \cdots \end{aligned}$$

Comparing coefficients, we have $a_0 = 0$, $a_1 = 2$, $a_2 - 2a_0 = a_2 = 0$ and $a_3 - 2a_1 = a_3 - 4 = -\frac{4}{3} \implies a_3 = \frac{8}{3}$. Therefore, we have $\tan 2x = 2x + \frac{8}{3}x^3 + \cdots$.

7 It is given that $\mathrm{f}(x) = \dfrac{a-b}{(1-ax)(1-bx)}$, where $a, b > 0$.

(a) By expressing $\mathrm{f}(x)$ in terms of partial fractions and considering their Maclaurin expansions, show that:

$$\sum_{r=0}^{\infty} (a^{r+1} - b^{r+1}) x^r$$

[4]

$$\frac{a-b}{(1-ax)(1-bx)} \equiv \frac{A}{1-ax} + \frac{B}{1-bx} \text{ for some } A, B \in \mathbb{R}$$

$$a - b = A(1-bx) + B(1-ax)$$

By first substituting $x = \frac{1}{a}$ and then $x = \frac{1}{b}$, we obtain $A = a$ and $B = -b$ respectively. We now use the expansion of $(1+x)^n$ which can be found in the MF26.

$$\begin{aligned}
\mathrm{f}(x) &= \frac{a-b}{(1-ax)(1-bx)} \\
&= \frac{a}{1-ax} - \frac{b}{1-bx} \\
&= a(1-ax)^{-1} - b(1-bx)^{-1} \\
&= a(1 + ax + a^2x^2 + \cdots) - b(1 + ax + a^2x^2 + \cdots) \\
&= a\sum_{r=0}^{\infty} a^r x^r - b\sum_{r=0}^{\infty} b^r x^r \\
&= \sum_{r=0}^{\infty} (a^{r+1}x^r - b^{r+1}x^r) \\
&= \sum_{r=0}^{\infty} (a^{r+1} - b^{r+1})x^r
\end{aligned}$$

(b) Hence or otherwise, prove that, if x^3 and higher powers of x may be neglected, then $\frac{2}{(1-5x)(1-3x)} \approx 2 + 16x + 98x^2$. [2]

We will obtain the answer by setting $a = 5, b = 3$.

$$\begin{aligned}
\frac{2}{(1-5x)(1-3x)} &= \sum_{r=0}^{\infty} (5^{r+1} - 3^{r+1})x^r \\
&\approx \sum_{r=0}^{2} (5^{r+1} - 3^{r+1})x^r \\
&= (5-3) + (5^2 - 3^2)x + (5^3 - 3^3)x^2 \\
&= 2 + 16x + 98x^2 \text{ (shown)}
\end{aligned}$$

8 Find the Maclaurin's Series for e^{ix} up to and including the coefficient of x^7. [3]

Using the standard series provided in MF26:

$$e^{ix} = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \cdots$$

(a) Hence, prove that $e^{ix} = \cos x + i \sin x$. [3]

We notice that

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \cdots \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!} + i\sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!} = \cos x + i \sin x \quad \text{(shown)} \end{aligned}$$