Differentiation

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2020

DIFFERENTIATION [80 Marks]

1 Differentiate with respect to x the following:

(a) $\ln(\ln(x^2))$ [2]

(b) $4^{\cos 4x}$ [2]

(c) $\tan^{-1}(xy)$ [2]

(d) $\dfrac{x^2}{3} - \dfrac{y^2}{4} = 1$ [2]

2 Solve the following differential equations:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y,\ y = 1 \text{ when } x = 1$ [2]

(b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 4(x^2 + x + 1)$ [3]

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-\mathrm{e}^{-3x}}{3y}$ [3]

3 The parametric equations of curve C are:

$$x = t + 1,\ y = \frac{1}{t^2 + 2t + 2} \text{ where } -3 \leq t \leq 1.$$

(a) Find the Cartesian equation of C. [2]

(b) Find the equation of the tangent to the curve C when $t = 2$. [3]

4 Two variables x and y vary with time t in minutes and are connected by the equation $\sin(xy) = \frac{1}{2}$. Given that x increases at a rate of 3 units per minute, find the (maximum possible) rate of decrease of y when $y = 1$. [4]

5 The parametric equations of curve C are:

$$x = 1 + \mathrm{e}^{-at}, \ y = \mathrm{e}^{at} + \mathrm{e}^{-at} \quad \text{where } t \geq 0.$$

(a) Sketch C, including all stationary points and asymptotes where applicable. [2]

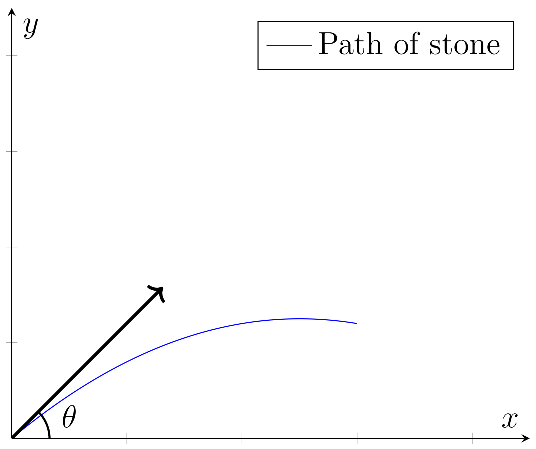
(b) Find the equation of the normal to the curve C at $t = \frac{\ln 2}{2a}$. [3]

(c) What can be said about the normals to the curve C as t gets large? [1]

6 A boy throws a rock in the air. It is launched at an angle θ to the horizontal ground, where $0 < \theta < \frac{\pi}{2}$. As shown in the diagram, its position in the air (x, y), where x is the horizontal position and y is the vertical position, is approximated by:

$$x = (20\cos\theta)t, \ y = (20\sin\theta)t - 5t^2,$$

where t is the time in seconds.



(a) Find, in terms of θ,

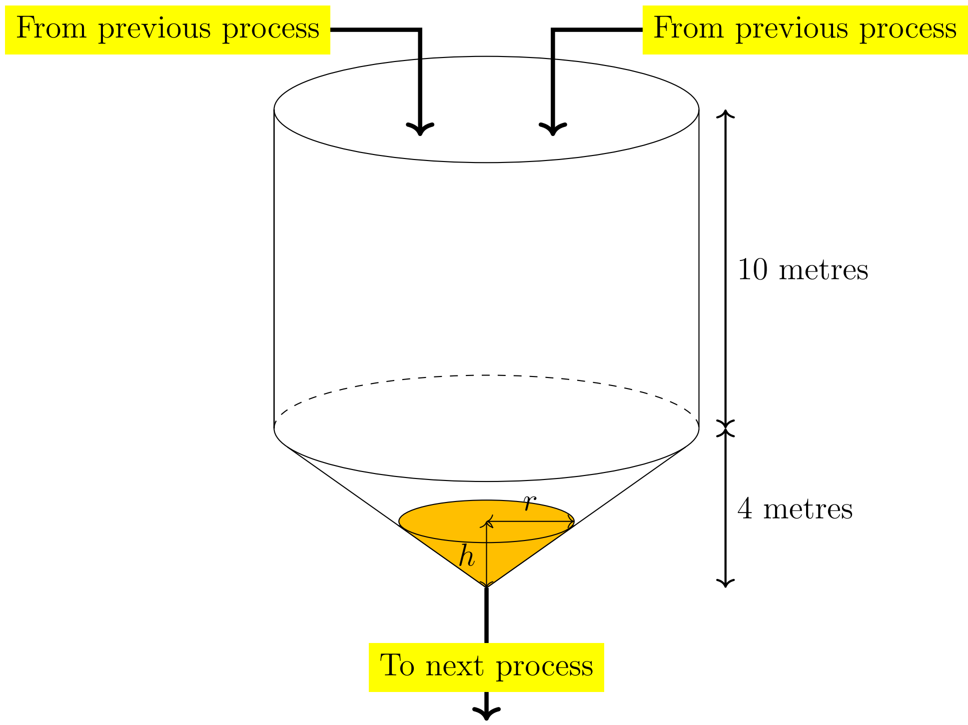
(i) The time taken for the toy plane to hit the ground; [2]

(ii) The maximum height reached by the toy plane; [2]

(iii) The horizontal distance travelled by the toy plane. [2]

(b) Find the angle θ at which the toy plane should be thrown to attain the maximum horizontal distance. [2]

7 A hopper in a butter processing factory is used to store pre-processed butter before it is discharged to the next stage of the manufacturing process. One such hopper is designed with the dimensions as follows:



The hopper can be said to be a cylinder attached to a cone, both of radius 3 metres. At t hours, the height of the butter in the cone is h and the radius of the liquid surface is r. At time $t = 0$ hours, the height of butter is 3 metres.

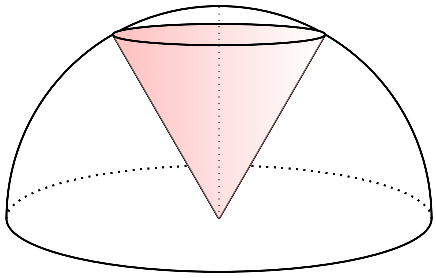
If the butter is allowed to flow out of the cone at -3 m^3 per hour,

(a) Show that $h^2 \dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{16}{3\pi}$. [3]

(b) Calculate the exact time it takes for the butter to completely drain out. [3]

(c) When the butter is completely drained out, the operator allows a net inflow of k m^3 per hour into the hopper. State the function $H(t)$, where H is the height of the butter in the hopper from time $t = 0$ to when the hopper is full. [7]

8 An art student conceptualises a design as shown below.

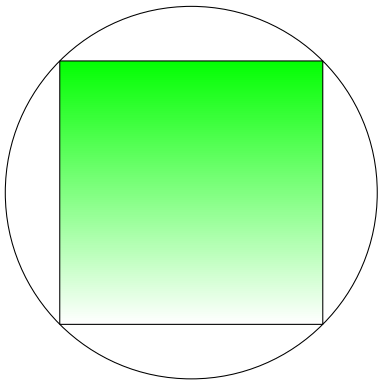


The design consists of an inverted glass cone of height h and radius r inscribed within a hemisphere of radius R.

(a) Show that the volume of the cone $V = \dfrac{1}{3}\pi(R^2 - h^2)h$. [2]

(b) Given that $R = \sqrt{3}$, find the maximum volume of the cone. [4]

9 An inflatable cube is inscribed inside an inflatable sphere such that the cross-section of the figure is as shown:

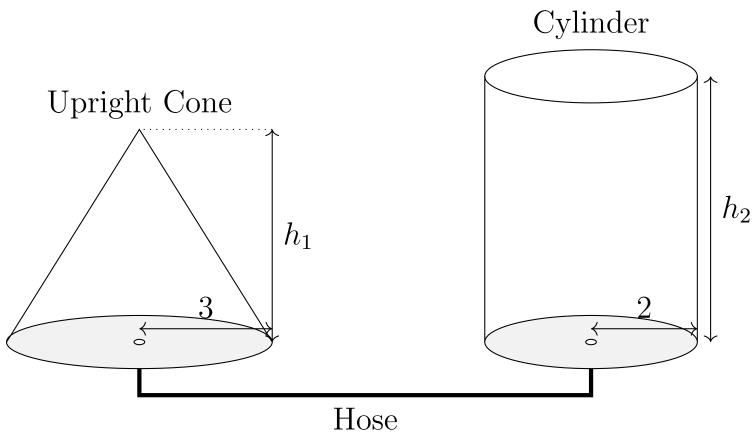


The cube has sides of length 1 unit initially. When the cube is allowed to inflate, the length of its side increase at a rate of $3t$, where t is the time in minutes. As the cube expands, the sphere expands uniformly and bursts when it has a volume of 27π units^3. (The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ where r is the radius of the sphere)

(a) Express the length of one of the cube's sides s as a function of time t. [2]

(b) Calculate the time taken for the sphere to burst correct to 3 significant figures. [3]

10 A student set up an experiment as shown below (not drawn to scale):



The inflatable upright cone and cylinder have fixed plates of radiuses 3 cm and 2 cm respectively such that only their heights are allowed to vary due to an increase or decrease in volume. The hose is such that gas can be transferred from the upright cone to the cylinder. The gas is incompressible.

(a) Show that $3\frac{\mathrm{d}h_1}{\mathrm{d}t} + 4\frac{\mathrm{d}h_2}{\mathrm{d}t} = 0$. [2]

(b) The gas is now being transferred continuously from the cone to the cylinder until just before the cone is empty. Show that the total surface area of the cone and the cylinder is increasing. [6]

11 In a city of Martians, the birth and death rates are high. The population is x Martians (in thousands) and t is the time in years since 2300. The birth rate is 2 times the population per year and the death rate at 1.5 times the square of the population per year.

(a) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}(4x - 3x^2)$ [2]

(b) Given that, in year 2300, the population was 1000, solve the differential equation in **(a)**. [5]

(c) Hence, find the population of Martians in the long run. [2]