

# Vectors

Derek, Vignesh

2020

## VECTORS [150 Marks]

1 Solve the following:

- a)  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ , where  $O$  is the origin. Given that lines  $OA$  and  $OB$  are parallel,  $|\mathbf{a}| = 2$  and  $\mathbf{a} \cdot \mathbf{b} = -2$ , express  $\mathbf{b}$  in terms of  $\mathbf{a}$ . [2]

Let  $\mathbf{b} = k\mathbf{a}$  for some  $k \in \mathbb{R}$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{a}) = k|\mathbf{a}|^2$$

$$-2 = k(2)^2 \Rightarrow k = -\frac{1}{2}$$

$$\therefore \mathbf{b} = -\frac{1}{2}\mathbf{a}$$

- b) A vector  $\mathbf{a}$  is such that  $\mathbf{a} = (\sqrt{2}\cos\alpha)\mathbf{i} - (\cos\alpha)\mathbf{j} + (\sqrt{2}\sin\alpha)\mathbf{k}$ , where  $0 \leq \alpha \leq 2\pi$  and  $|\mathbf{a}| = \sqrt{2}$ . Find the value(s) of  $\alpha$ . [2]

$$|\mathbf{a}| = \sqrt{2\cos^2\alpha + \cos^2\alpha + 2\sin^2\alpha}$$

$$\sqrt{2} = \sqrt{2 - \cos^2\alpha}$$

$$2 = 2 - \cos^2\alpha$$

$$\cos\alpha = 0$$

$$\alpha = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

- c) The points  $A, B$  and  $C$  with respect to the origin are represented by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  respectively. It is given that  $|\mathbf{b}| = 2$ ,  $\mathbf{a} \cdot \mathbf{b} = k$  and  $\mathbf{b} \cdot \mathbf{c} = 2$ . Given further

that point  $C$  divides the line  $AB$  such that  $AC : CB = 2 : 1$ , find  $k$ . [3]

$$\begin{aligned}\mathbf{c} &= \frac{\mathbf{a} + 2\mathbf{b}}{3} \\ \mathbf{b} \cdot \mathbf{c} &= \mathbf{b} \cdot \left( \frac{\mathbf{a} + 2\mathbf{b}}{3} \right) \\ 2 &= \frac{1}{3}(\mathbf{b} \cdot \mathbf{a} + 2|\mathbf{b}|^2) \\ 2 &= \frac{1}{3}(k + 8) \\ \therefore k &= -2\end{aligned}$$

d) Four vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  exist such that  $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$ . Show that  $\mathbf{b} \times (\mathbf{a} + \mathbf{c}) = \mathbf{d} \times \mathbf{b}$ . [2]

$$\begin{aligned}\mathbf{b} \times (\mathbf{a} + \mathbf{c}) &= \mathbf{b} \times (-\mathbf{b} - \mathbf{d}) \\ &= \mathbf{b} \times (-\mathbf{b}) - \mathbf{b} \times \mathbf{d} \\ &= \mathbf{d} \times \mathbf{b}\end{aligned}$$

e) Point  $A$  referred from the origin has vector  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ . The line  $OA$  makes an angle of  $\alpha$  with the  $y$ -axis and  $\beta$  with the  $z$ -axis, where  $\alpha, \beta < \pi$ . Show that

$$\alpha + \beta = \pi. \quad [3]$$

$$\cos \alpha = \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{(1)\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{3}$$

$$\cos \beta = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{(1)(3)} = -\frac{2}{3}$$

Since  $\cos \alpha = -\cos \beta$ , and  $\alpha, \beta < \pi$ ,

$$\cos \alpha = \cos(\pi - \beta)$$

$$\alpha = \pi - \beta$$

$$\alpha + \beta = \pi$$

**2** Referred to the origin  $O$ , points  $A$  and  $B$  have position vectors given by  $\mathbf{a}$  and  $\mathbf{b}$  respectively.  $C_0$  is the foot of perpendicular from  $A$  to  $OB$  with position vector  $\mathbf{c}_0$ . The angle between lines  $OA$  and  $OB$  is  $\alpha$ , where  $0 < \alpha < \frac{\pi}{2}$ .

**a)** By considering  $\cos \alpha$ , show that  $|\mathbf{c}_0| = \mathbf{a} \cdot \hat{\mathbf{b}}$ . [2]

$$\cos \alpha = \frac{|\mathbf{c}_0|}{|\mathbf{a}|}$$

$$\text{Also, } \cos \alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

$$\text{Hence, } \frac{|\mathbf{c}_0|}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

$$|\mathbf{c}_0| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \hat{\mathbf{b}}$$

**b)** The foot of perpendicular from  $C_0$  to  $OA$  is  $C_1$ . Show that  $|\mathbf{c}_1| = \mathbf{a} \cdot \hat{\mathbf{b}}(\cos \alpha)$ . [1]

$$\begin{aligned}\cos \alpha &= \frac{|\mathbf{c}_1|}{|\mathbf{c}_0|} \\ |\mathbf{c}_1| &= |\mathbf{c}_0| \cos \alpha \\ &= \mathbf{a} \cdot \hat{\mathbf{b}} \cos \alpha\end{aligned}$$

c)  $C_n$  is the  $n$ th foot of perpendicular. State  $|\mathbf{c}_n|$  in terms of  $a$ ,  $b$ ,  $n$  and  $\alpha$ . [1]

$$|\mathbf{c}_n| = \mathbf{a} \cdot \hat{\mathbf{b}} \cos^n \alpha$$

d) State the sum to infinity of scalar projections  $|\mathbf{c}_0| + |\mathbf{c}_1| + \dots + |\mathbf{c}_n| + \dots$  [1]

$$\begin{aligned}\text{Sum to infinity} &= \mathbf{a} \cdot \hat{\mathbf{b}} + \mathbf{a} \cdot \hat{\mathbf{b}} \cos \alpha + \dots + \mathbf{a} \cdot \hat{\mathbf{b}} \cos^n \alpha + \dots \\ &= \frac{\mathbf{a} \cdot \hat{\mathbf{b}}}{1 - \cos \alpha}\end{aligned}$$

**3** Referred to the origin  $O$ , points  $A$  and  $B$  have position vectors given by:  $\mathbf{a} = \mathbf{i} - p^2 \mathbf{k}$  and  $\mathbf{b} = \frac{2}{p} \mathbf{i} - \mathbf{j} + \mathbf{k}$  respectively, where  $p$  is to be found. Given that  $|\mathbf{a} \times \mathbf{b}|^2 = 4p^2 + 2$ , find

the value(s) that  $p$  can take.

[4]

$$\begin{aligned}
 |\mathbf{a} \times \mathbf{b}|^2 &= \left| \begin{pmatrix} 1 \\ 0 \\ -p^2 \end{pmatrix} \times \begin{pmatrix} 2p^{-1} \\ -1 \\ 1 \end{pmatrix} \right|^2 \\
 &= \left| \begin{pmatrix} -p^2 \\ -1 - 2p \\ -1 \end{pmatrix} \right|^2 \\
 &= (p^2)^2 + (1 + 2p)^2 + 1 \\
 &= p^4 + 4p^2 + 4p + 2 \\
 &= 4p^2 + 2 \\
 p^4 + 4p &= 0 \\
 p(p^3 + 4) &= 0 \\
 p &= -\sqrt[3]{4} \text{ since } p \neq 0
 \end{aligned}$$

- 4 The vector equation of  $l$  is given by  $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ ,  $\lambda \in \mathbb{R}$ . Point  $F$  is the foot of perpendicular from origin  $O$  to the line  $l$ . If  $|\mathbf{b}| = 1$  and  $\mathbf{a} \cdot \mathbf{b} = 1$ , express the position vector  $\overrightarrow{OF}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ . [3]

$$\text{Let } \overrightarrow{OF} = \mathbf{a} + k\mathbf{b} \text{ for some } k \in \mathbb{R}$$

$$\text{Since } OF \perp l, (\mathbf{a} + k\mathbf{b}) \cdot \mathbf{b} = 0$$

$$\mathbf{a} \cdot \mathbf{b} + k|\mathbf{b}|^2 = 0$$

$$1 + k(1)^2 = 0$$

$$k = -1$$

$$\therefore \overrightarrow{OF} = \mathbf{a} - \mathbf{b}$$

- 5 The equations of  $l$  and  $m$  are given by  $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ ,  $\lambda \in \mathbb{R}$  and  $m : \mathbf{r} = \mathbf{b} + \mu \mathbf{a}$ ,  $\mu \in \mathbb{R}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are co-planar vectors. State the conditions such that lines  $l$  and  $m$  are skew lines. [2]

Equating  $l$  and  $m$  and since skew lines are parallel and do not intersect,

$$\mathbf{a} + \lambda \mathbf{b} = \mu \mathbf{a} + \mathbf{b}$$

$$\lambda \neq 1, \mu \neq 1 \text{ and } \mathbf{a} \neq k\mathbf{b} \text{ for all } k \in \mathbb{R}$$

- 6** Points  $A$  and  $B$  have position vectors  $\mathbf{a}$  and  $\mathbf{b}$  with respect to the origin  $O$ . It is given that  $(\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) = 11$ . Find the perpendicular distance from point  $A$  to line  $OB$  if  $|\mathbf{b}| = 11$ . [4]

$$(\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) = \mathbf{a} \times \mathbf{a} + 7\mathbf{a} \times \mathbf{b} - 15\mathbf{b} \times \mathbf{a} - 21\mathbf{b} \times \mathbf{b}$$

$$11 = 22\mathbf{a} \times \mathbf{b}$$

$$\mathbf{a} \times \mathbf{b} = \frac{1}{2}$$

$$\begin{aligned} \text{Perpendicular distance from A to OB} &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \\ &= \frac{\left(\frac{1}{2}\right)}{11} \\ &= \frac{1}{22} \text{ units} \end{aligned}$$

- 7** Points  $A$  and  $B$  have position vectors  $\mathbf{a}$  and  $\mathbf{b}$  with respect to the origin  $O$ . It is given that  $|\mathbf{a}| = 3$ ,  $|\mathbf{b}| = 1$  and  $\mathbf{a} \cdot \mathbf{b} = 2$ .

- a) State the vector equation of line  $AB$ . [1]

$$l_{AB} : \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \lambda \in \mathbb{R}$$

- b) Find  $|\mathbf{b} - \mathbf{a}|$ . [2]

$$\begin{aligned} |\mathbf{b} - \mathbf{a}| &= \sqrt{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})} \\ &= \sqrt{|\mathbf{b}|^2 - 2\mathbf{b} \cdot \mathbf{a} + |\mathbf{a}|^2} \\ &= \sqrt{1 - 2(2) + 3^2} \\ &= \sqrt{6} \end{aligned}$$

- c) Find the position vector of  $F$ , the foot of perpendicular from  $O$  to  $AB$ , in terms of  $\mathbf{a}$  and  $\mathbf{b}$ . [3]

$$\text{Let } \overrightarrow{OF} = \mathbf{a} + k(\mathbf{b} - \mathbf{a}) \text{ for some } k \in \mathbb{R}$$

$$\text{Since } OF \perp AB, (\mathbf{a} + k(\mathbf{b} - \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\mathbf{a} \cdot \mathbf{b} - |\mathbf{a}|^2 + k|\mathbf{b} - \mathbf{a}|^2 = 0$$

$$2 - 3^2 + k(\sqrt{6})^2 = 0$$

$$\therefore k = \frac{7}{6}$$

$$\text{Substituting } k = \frac{7}{6} \text{ back into } \overrightarrow{OF},$$

$$\overrightarrow{OF} = \mathbf{a} + \frac{7}{6}(\mathbf{b} - \mathbf{a})$$

$$\overrightarrow{OF} = \frac{1}{6}(7\mathbf{b} - \mathbf{a})$$

- d) Find  $|7\mathbf{b} - \mathbf{a}|$ . Hence, find the exact area of triangle  $OAB$ . [3]

$$\begin{aligned} |7\mathbf{b} - \mathbf{a}| &= \sqrt{(7\mathbf{b} - \mathbf{a}) \cdot (7\mathbf{b} - \mathbf{a})} \\ &= \sqrt{49|\mathbf{b}|^2 - 14\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2} \\ &= \sqrt{49 - 14(2) + 3^2} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle OAB} &= \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OF}| \\ &= \frac{1}{2} |\mathbf{b} - \mathbf{a}| \left| \frac{1}{6}(7\mathbf{b} - \mathbf{a}) \right| \\ &= \frac{1}{2} (\sqrt{6}) \left( \frac{1}{6} \right) (\sqrt{30}) \\ &= \frac{\sqrt{5}}{2} \text{ units}^2 \end{aligned}$$

- 8 Referred to the origin  $O$ , points  $A$  and  $B$  have the position vectors  $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}$  and  $\overrightarrow{OB} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$  respectively.

a) Verify that  $P(3, -2, -6)$  lies on line  $AB$ .

[2]

$$\text{Line AB // } \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{Equation of line AB is } \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\text{Substitute } \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} :$$

$$\begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

There is a solution  $\lambda = -2$

Hence, P lies on AB.

b) Find the position vector of  $F$ , the foot of perpendicular from  $P$  to  $AB$ .

[3]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$



$$\begin{aligned}\overrightarrow{PF} &= \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -2 - k \\ -2 + k \\ 4 + 2k \end{pmatrix}\end{aligned}$$

c) Hence, find the equation of line  $PF$ .

[3]

Since  $PF \perp AB$ ,

$$\begin{pmatrix} -2 - k \\ -2 + k \\ 4 + 2k \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\therefore k = -\frac{4}{3}$$

Substituting  $k = -\frac{4}{3}$  back into  $\overrightarrow{PF}$ ,

$$\overrightarrow{PF} = \begin{pmatrix} -2 + \frac{8}{3} \\ -2 - \frac{4}{3} \\ 4 - \frac{8}{3} \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$$

$$\text{Equation of line } PF : \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$$

9 The equations of lines  $l_1$  and  $l_2$  are given by:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \frac{x+1}{9} = \frac{y}{7} = \frac{4-z}{3} \text{ respectively.}$$

Point  $A$  has coordinates  $(2, -1, 1)$  while the foot of perpendicular from  $A$  to  $l_2$  is  $F$ .

- a) Find the position vector of  $P$ , the point of intersection between  $l_1$  and  $l_2$ . [2]

$$\text{Vector equation of line } l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\text{Equating both lines, } \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}$$

$$\text{Using G.C., we obtain } \lambda = \frac{3}{2}, \mu = \frac{1}{2}.$$

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix}$$

- b) Find vector  $\overrightarrow{AF}$ . [3]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{AF} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 + 9k \\ 1 + 7k \\ 3 - 3k \end{pmatrix}$$

$$\text{Since } AF \perp l_2, \begin{pmatrix} -3+9k \\ 1+7k \\ 3-3k \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} = 0$$

$$\therefore k = \frac{29}{139}$$

$$\text{Substituting } k = \frac{29}{139} \text{ back into } \overrightarrow{AF}, \overrightarrow{AF} = \begin{pmatrix} -3+9\left(\frac{29}{139}\right) \\ 1+7\left(\frac{29}{139}\right) \\ 3-3\left(\frac{29}{139}\right) \end{pmatrix} = \frac{2}{139} \begin{pmatrix} -78 \\ 171 \\ 165 \end{pmatrix}$$

c) Hence, find the vector equation of  $l_3$ , the reflection of  $l_1$  in  $l_2$ . [3]

Let  $A'$  on  $l_3$  be the reflection of  $A$  in  $l_2$ .

$$\begin{aligned} \overrightarrow{AF} &= \overrightarrow{FA'} \\ &= \overrightarrow{OA'} - \overrightarrow{OF} \\ \overrightarrow{OA'} &= \overrightarrow{AF} + \overrightarrow{OF} \\ &= \frac{2}{139} \begin{pmatrix} -78 \\ 171 \\ 165 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \frac{29}{139} \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} \\ &= \frac{1}{139} \begin{pmatrix} -34 \\ 545 \\ 973 \end{pmatrix} \end{aligned}$$

$$l_3 // \frac{1}{139} \begin{pmatrix} -34 \\ 545 \\ 973 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{1041}{278} \\ \frac{117}{278} \\ \frac{9}{2} \end{pmatrix}$$

$$\therefore l_3 : \mathbf{r} = \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} -1041 \\ 117 \\ 1251 \end{pmatrix}, \alpha \in \mathbb{R}$$

**10** Points  $A$  and  $B$  with position vectors  $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $3\mathbf{i} + \mathbf{k}$  respectively both lie on  $l_1$ .

The line  $l_2$  has Cartesian equation  $l_2 : x = 7, y - 3 = z$ .

a) Show that  $l_1$  and  $l_2$  are skew lines.

[2]

$$l_1 // \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\text{Equating both lines, } \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{We have the following equations } \begin{cases} 2\lambda = 4 \\ -\lambda - \mu = 3 \\ \lambda - \mu = -1 \end{cases}$$

Using a G.C., there is no solution found.

Hence,  $l_1$  and  $l_2$  are skew lines.

- b) Find a vector that is perpendicular to both  $l_1$  and  $l_2$ . [1]

A vector that is perpendicular to both  $l_1$  and  $l_2$  //  $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$

Let the vector be  $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ .

- c) Hence, find the shortest distance between  $l_1$  and  $l_2$ . [3]

Let point  $C$  on  $l_2$  be  $C(7, 3, 0)$ .

Shortest distance = Projection of  $\overrightarrow{BC}$  onto  $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ .

$$= \frac{\left( \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right|}$$

$$= \frac{\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right|}$$

$$= \frac{8}{\sqrt{3}}$$

$$= \frac{8\sqrt{3}}{3} \text{ units}$$

- 11** Referred to an origin  $O$ , points  $A$  and  $B$  have coordinates  $(-1, 2, 2)$  and  $(0, 1, 2)$  respectively. The point  $P$  on  $OA$  is such that  $OP : PA = \lambda : 1$  and the point  $Q$  on  $OB$  is such that  $OQ : QB = \lambda : 1 - \lambda$ , where  $\lambda$  is a real constant to be determined.

**a)** Find the area of  $\triangle OAB$ .

[2]

$$\begin{aligned}
 \text{Area of } \triangle OAB &= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \\
 &= \frac{1}{2} \left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right| \\
 &= \frac{1}{2} \left| \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \right| \\
 &= \frac{1}{2} \sqrt{4 + 4 + 1} \\
 &= \frac{3}{2} \text{units}^2
 \end{aligned}$$

**b)** Express the ratio  $\frac{\text{Area of } \triangle OAB}{\text{Area of } \triangle OPQ}$  in terms of  $\lambda$ .

[3]

$$\begin{aligned}
 \text{Area of } \triangle OPQ &= \frac{1}{2} |\vec{OP} \times \vec{OQ}| \\
 &= \frac{1}{2} \left| \left( \frac{\lambda}{\lambda+1} \right) \mathbf{a} \times \left( \frac{\lambda}{\lambda+1-\lambda} \right) \mathbf{b} \right| \\
 &= \frac{1}{2} \left| \frac{\lambda^2}{\lambda+1} \mathbf{a} \times \mathbf{b} \right| \\
 &= \left( \frac{\lambda^2}{\lambda+1} \right) \left( \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \right), \text{ since } 0 < \lambda < 1 \text{ so } \frac{1}{\lambda+1} > 0 \\
 &= \left( \frac{\lambda^2}{\lambda+1} \right) (\text{Area of } \triangle OAB) \\
 \therefore \frac{\text{Area of } \triangle OAB}{\text{Area of } \triangle OPQ} &= \frac{\lambda+1}{\lambda^2}
 \end{aligned}$$

**c)** Deduce if  $PQ$  is ever parallel to  $AB$  for some value of  $\lambda$ .

[3]

$$\begin{aligned}
 \vec{PQ} &= \vec{OQ} - \vec{OP} \\
 &= \lambda \mathbf{b} - \left( \frac{\lambda}{\lambda+1} \right) \mathbf{a}
 \end{aligned}$$

Assuming  $PQ \parallel AB$ ,

$$\lambda \mathbf{b} - \left(\frac{\lambda}{\lambda+1}\right) \mathbf{a} = k(\mathbf{b} - \mathbf{a}) \text{ for some } k \in \mathbb{R}$$

Equating scalar multiples of  $\mathbf{b}$  and  $\mathbf{a}$ ,

$$\lambda = k \text{ and } \frac{\lambda}{\lambda+1} = k$$

$$\lambda = \frac{\lambda}{\lambda+1}$$

$$\lambda^2 = 0$$

However, clearly  $0 < \lambda < 1$ , so no value of  $k \in \mathbb{R}$  exists for  $PQ \parallel AB$ .

- 12** Line  $l$  has the equation  $-x = \frac{y-3}{2} = \frac{z+4}{2}$ . Line  $m$ , which is parallel to  $\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$  where  $c$  is some real constant, is obtained by rotating line  $l$   $45^\circ$  about the point  $A(0, 3, -4)$ . Find the possible vector equations of line  $m$ . [5]

$$\text{Equation of line } l : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\text{Equation of line } m : \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix} \right|} = \frac{2-c}{3\sqrt{c^2+1}}$$

$$\frac{1}{2} = \frac{(2-c)^2}{9(c^2+1)}$$

$$9c^2 + 9 = 2c^2 - 8c + 8$$

$$7c^2 + 8c + 1 = 0$$

$$c = -\frac{1}{7} \text{ or } -1$$

The two equations of line  $m$  are:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \mu \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

**13** Three points  $A, B$  and  $C$  referred from the origin  $O$  have position vectors given by:

$$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \mathbf{b} = -2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{c} = \frac{3}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} - 3\mathbf{k}.$$

a) Find the vector equations of lines  $AB$  and  $AC$ . [2]

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 2.5 \\ -3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Equations of lines  $AB$  and  $AC$  are:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \mu \in \mathbb{R} \text{ respectively.}$$

b) Find two vector equations of  $l$ , where  $l$  is the line representing the all the midpoints



of lines  $AB$  and  $AC$ .

[4]

$$\text{Unit vector of } AB, \mathbf{u}_1 = \frac{1}{\sqrt{4^2 + 1 + 3^2}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$

$$\text{Unit vector of } AC, \mathbf{u}_2 = \frac{1}{\sqrt{1 + 3^2 + 4^2}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Two possible midpoints have position vectors  $\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)$  and  $\frac{1}{2}(\mathbf{u}_1 - \mathbf{u}_2)$ .

$$\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2) = \frac{1}{2} \left( \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right) = \frac{1}{2\sqrt{26}} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$\frac{1}{2}(\mathbf{u}_1 - \mathbf{u}_2) = \frac{1}{2} \left( \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} - \frac{1}{\sqrt{26}} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right) = \frac{1}{2\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$$

Hence, two possible equations of  $l$  are:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R} \text{ and}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}, \quad t \in \mathbb{R}$$

- 14** Point  $A$  with position vector  $\mathbf{a}$  lies on plane  $\pi$  with normal parallel to vector  $\mathbf{n}$ . Given that  $|\mathbf{a} - \mathbf{n}|^2 = 3$  and  $|\mathbf{n}|^2 = 4 - |\mathbf{a}|^2$ , find the value of  $d$  if the equation of plane  $\pi$  is  $\mathbf{r} \cdot \mathbf{n} = d$ .

[4]

The equation of plane  $\pi$  is  $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ .

To find  $\mathbf{a} \cdot \mathbf{n}$ ,

$$|\mathbf{a} - \mathbf{n}|^2 = 3$$

$$(\mathbf{a} - \mathbf{n}) \cdot (\mathbf{a} - \mathbf{n}) = 3$$

$$|\mathbf{a}|^2 + |\mathbf{n}|^2 - 2\mathbf{a} \cdot \mathbf{n} = 3$$

$$4 - 2\mathbf{a} \cdot \mathbf{n} = 3$$

$$\mathbf{a} \cdot \mathbf{n} = \frac{1}{2}$$

$\therefore$  The equation of  $\pi$  is  $\mathbf{r} \cdot \mathbf{n} = \frac{1}{2}$ .

**15** The equations of parallel planes  $p$  and  $q$  are given by  $p : \mathbf{r} \cdot \mathbf{n} = d$  and  $q : \mathbf{r} \cdot \mathbf{n} = kd$ .

Line  $l$  given by equation  $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ ,  $\lambda \in \mathbb{R}$  intersects planes  $p$  and  $q$  at points  $A$  and  $B$  respectively.

a) Show that  $\overrightarrow{AB} = \mathbf{b} \left( \frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right)$ . [4]

Substitute equation of  $l$  into  $p$  and  $q$  to get  $\overrightarrow{OA}$  and  $\overrightarrow{OB}$  respectively.

For  $A$ ,  $(\mathbf{a} + \lambda \mathbf{b}) \cdot \mathbf{n} = d$

$$\lambda = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}$$

$$\overrightarrow{OA} = \mathbf{a} + \left( \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b}$$

For  $B$ ,  $(\mathbf{a} + \lambda \mathbf{b}) \cdot \mathbf{n} = kd$

$$\lambda = \frac{kd - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}$$

$$\begin{aligned}
\overrightarrow{OB} &= \mathbf{a} + \left( \frac{kd - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} \\
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= \mathbf{a} + \left( \frac{kd - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} - \left( \mathbf{a} + \left( \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} \right) \\
&= \left( \frac{kd - \mathbf{a} \cdot \mathbf{n} - d + \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \right) \mathbf{b} \\
&= \mathbf{b} \left( \frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right)
\end{aligned}$$

- b) Hence, or otherwise, show that the perpendicular distance between planes  $p$  and  $q$  is equal to  $\frac{d(k-1)}{|\mathbf{n}|}$  units. [2]

Perpendicular distance between planes  $p$  and  $q$  = Projection of  $\overrightarrow{AB}$  onto  $\mathbf{n}$

$$\begin{aligned}
&= \frac{\left( \mathbf{b} \left( \frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right) \right) \cdot \mathbf{n}}{|\mathbf{n}|} \\
&= \frac{\left( \frac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}} \right) (\mathbf{b} \cdot \mathbf{n})}{|\mathbf{n}|} \\
&= \frac{d(k-1)}{|\mathbf{n}|}
\end{aligned}$$

**16** The equations of plane  $\pi$  and  $l$  are given by:

$$\pi : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -1 \text{ and } l : \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix}, \lambda, k \in \mathbb{R} \text{ respectively.}$$

- a) Show that, for  $\pi$  and  $l$  to intersect,  $k \neq -\frac{7}{2}$ . [1]

If  $\pi$  and  $l$  do not intersect,  $l$  is perpendicular to the normal of  $\pi$ .

$$\text{i.e. } \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 2k + 7 = 0$$

$$k = -\frac{7}{2}$$

Hence, for intersection,  $k \neq -\frac{7}{2}$ .

For the rest of the question, assume  $k = 1$ .

- b)** Find the coordinates of point  $P$ , the point of intersection of  $\pi$  and  $l$ . [2]

Equating line and plane,

$$\left( \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -1$$

$$9s + 3 = -1$$

$$s = -\frac{4}{9}$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -\frac{1}{9} \begin{pmatrix} 21 \\ 8 \\ -14 \end{pmatrix}$$

- c)** Find the shortest distance from  $A(-1, 0, 2)$  to  $\pi$ . [3]

$$\overrightarrow{AP} = -\frac{1}{9} \begin{pmatrix} 21 \\ 8 \\ -14 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -\frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$$

Shortest distance = Projection of  $\overrightarrow{AP}$  onto normal of  $\pi$

$$\begin{aligned}
 & \left| -\frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| \\
 &= \frac{\left| -\frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|} \\
 &= \frac{4}{9} \left( \frac{23}{3} \right) \\
 &= \frac{92}{27} \text{units}
 \end{aligned}$$

d) Find the acute angle between  $\pi$  and  $l$ .

[2]

Let acute angle be  $\alpha$ .

$$\sin \alpha = \frac{\left( \frac{92}{27} \right)}{\left| \overrightarrow{AP} \right|} = \frac{\left( \frac{92}{27} \right)}{\frac{4}{9} \left| \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \right|} = \frac{\left( \frac{92}{27} \right)}{\frac{4}{9} \sqrt{77}}$$

$\therefore \alpha = 60.9^\circ$  (1 d.p.)

**17** Plane  $\pi$  has a normal parallel to  $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$  and has the equation:

$$\pi : \mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R},$$

where  $a$  is some real constant to be determined.

a) Find the value of  $a$ .

[2]

$$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} // \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ a-3 \\ 6 \end{pmatrix}$$

$$\text{Clearly, } \begin{pmatrix} -2 \\ a-3 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore a = 5$$

b) Find the scalar product equation of plane  $\pi$ .

[1]

$$\text{Equation is } \mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} = 6$$

c) Line  $l$  passes through  $\pi$ , the origin  $O$  and  $A(3, 2, 5)$ . Find the position vector of  $P$ , the point of intersection between line  $l$  and plane  $\pi$ .

[2]

$$\text{Equation of } l \text{ is } \mathbf{r} = \lambda \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \lambda \in \mathbb{R}.$$

$$\text{Substituting into equation of } \pi, k \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = 6 \text{ for some } k \in \mathbb{R}$$

$$14k = 6$$

$$k = \frac{4}{7}$$

$$\therefore \overrightarrow{OP} = \frac{4}{7} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

d) Find  $|\overrightarrow{PF}|$ , where  $F$  is the foot of perpendicular from  $A$  to plane  $\pi$ . [3]

$$\begin{aligned} |\overrightarrow{PF}| &= \frac{\left| \overrightarrow{PA} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|}{\left| \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|} \\ &= \frac{\left| \frac{3}{7} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|}{\sqrt{11}} \\ &= \frac{\frac{3}{7} \left| \begin{pmatrix} 1 \\ -14 \\ 5 \end{pmatrix} \right|}{\sqrt{11}} \\ &= 37\sqrt{1 + 14^2 + 5^2}\sqrt{11} \\ &= 37\sqrt{\frac{222}{11}} \text{ units} \end{aligned}$$

e) Hence find  $|\overrightarrow{PG}|$ , where  $G$  is the foot of perpendicular from  $O$  to plane  $\pi$ . [2]

$$\frac{|\overrightarrow{PG}|}{|\overrightarrow{PF}|} = \frac{|\overrightarrow{OP}|}{|\overrightarrow{PA}|} = \frac{4}{3} \text{ by similar triangles, so we have:}$$

$$\begin{aligned}
|\overrightarrow{PG}| &= \frac{4}{3} |\overrightarrow{PF}| \\
&= \frac{4}{3} \left( \frac{3}{7} \sqrt{\frac{222}{11}} \right) \\
&= \frac{4}{7} \sqrt{\frac{222}{11}} \text{ units}^2
\end{aligned}$$

**18** The equations of planes  $\pi_1$ ,  $\pi_2$  and  $\pi_3$  are such that:

$$\pi_1 : 2x + 3y + 4z = -1, \quad \pi_2 : -2x + y - z = 5 \quad \text{and} \quad \pi_3 : \mathbf{r} \cdot \begin{pmatrix} a \\ -5 \\ -a \end{pmatrix} = k.$$

**a)** Find the vector equation of  $l$ , the line of intersection between  $\pi_1$  and  $\pi_2$ . [3]

$$\text{We have the two equations, } \begin{cases} 2x + 3y + 4z = -1 \\ -2x + y - z = 5 \end{cases}$$

$$\text{Using a G.C., we obtain } x = -2 - \frac{7}{8}z, \quad y = 1 - \frac{3}{4}z, \quad z \in \mathbb{R}.$$

$$\text{Equation of } l \text{ is } \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

**b)** Given that  $a = 2$ , find the value of  $k$  such that  $\pi_3$  contains  $l$ . [2]

Assuming that  $\pi_3$  contains  $l$ , substituting equation of  $l$  into equation of  $\pi$ ,

$$\left( \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix} = k \text{ for all } \lambda \in \mathbb{R}$$

$$-4 + 14\lambda - 5 - 30\lambda + 16\lambda = k$$

$$k = -9$$



- c) Given that  $a = 1$ ,  $k = 3$ , find the point of intersection of  $\pi_1$ ,  $\pi_2$  and  $\pi_3$ . [2]

$$\begin{aligned} \text{We have the three equations, } & \begin{cases} 2x + 3y + 4z = -1 \\ -2x + y - z = 5 \\ x - 5y - z = 3 \end{cases} \\ \text{Using a G.C., we obtain } & x = -\frac{20}{3}, y = -3 \text{ and } z = \frac{16}{3}. \end{aligned}$$

- 19 An incident beam of light was reflected perfectly ( $\theta_1 = \theta_2$ ) on a round mirror.

Figure 1: Mirror

A student modelled the scenario such that the incident beam is  $l_1$ , the reflected beam is  $l_2$  and the mirror is  $\pi$ , where  $\pi$  contains the vectors  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ ,  $\mathbf{i} - \mathbf{j}$  and  $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ .

- a) Find the equation of the plane  $\pi$  in the form  $\mathbf{r} \cdot \mathbf{n} = d$ . [3]

$$\begin{aligned} \text{Normal of plane } \pi // & \left( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) \times \left( \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) \\ & = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ & = \begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix} \\ d & = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 5 \\ \therefore \text{Equation of } \pi \text{ is } & \mathbf{r} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 5 \end{aligned}$$

- b)** Show that  $P(1, 3, 2)$ , the point of intersection between  $l_1$  and  $l_2$  lies on  $\pi$ . [1]

$$\begin{aligned} \text{Substituting } \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \text{ into equation of } \pi, \\ \text{LHS} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 5 \\ = \text{RHS} \end{aligned}$$

- c)** State the vector equation of  $l_3$ , the axis of reflection between  $l_1$  and  $l_2$ . [1]

$$l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

- d)** Given that  $A(t, 1, 1)$  lies on  $l_1$ , where  $t > 0$ , find  $t$  such that  $\theta_1 = \theta_2 = \frac{\pi}{4}$ . [4]

$$\overrightarrow{PA} = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix}$$

Let  $\alpha$  be the angle between  $l_1$  and  $l_3$ .

$$\begin{aligned}\cos\alpha &= \frac{\begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix} \right| \cdot \left| \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|} \\ &= \frac{4t-4}{\sqrt{t^2-2t+6}\sqrt{21}}\end{aligned}$$

Since  $\theta_1 = \frac{\pi}{4}$ ,  $\alpha = \frac{3\pi}{4}$  or  $\frac{\pi}{4}$ .

Regardless,  $\cos^2\alpha = \frac{1}{2}$ .

$$\text{Squaring both sides, } \frac{1}{2} = \frac{16t^2 - 32t + 16}{21t^2 - 42t + 126}$$

$$11t^2 - 22t - 94 = 0$$

Using G.C.,  $t = 2.50$  or  $-3.41$  (3s.f.)

Since  $t > 0$ ,  $t = 2.50$

For the rest of the question, take  $t = 4$ .

e) Find the shortest distance from  $A$  to  $\pi$ .

[2]

$$\begin{aligned}
 \text{Shortest distance} &= \frac{\left| \overrightarrow{AP} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|} \\
 &= \frac{\left| \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{21}} \\
 &= \frac{\left| \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} \right|}{\sqrt{21}} \\
 &= \frac{5}{\sqrt{21}} \sqrt{1 + 2^2 + 1} \\
 &= \frac{5\sqrt{14}}{7} \text{ units}
 \end{aligned}$$

f) Find the coordinates of  $F$ , the foot of perpendicular from  $A$  to  $l_3$ . Hence, or otherwise, find the equation of  $l_2$ .

[6]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{AF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 + 4k \\ 2 - k \\ 1 + 2k \end{pmatrix}$$

Since  $\overrightarrow{AF} \perp l_3$ ,

$$\begin{pmatrix} -3 + 4k \\ 2 - k \\ 1 + 2k \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$-12 + 21k = 0$$

$$k = \frac{4}{7}$$

$$\begin{aligned} \overrightarrow{OF} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 23 \\ 17 \\ 22 \end{pmatrix} \\ \therefore F &\left( \frac{23}{7}, \frac{17}{7}, \frac{22}{7} \right) \end{aligned}$$

Let  $A'$  be the reflection of point  $A$  in  $l_3$ .

$$\begin{aligned}
\overrightarrow{OA'} &= \overrightarrow{OA} + 2\overrightarrow{AF} \\
&= \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + 2 \left( \frac{1}{7} \begin{pmatrix} 23 \\ 17 \\ 22 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \right) \\
&= \frac{1}{7} \begin{pmatrix} 18 \\ 27 \\ 37 \end{pmatrix} \\
\overrightarrow{A'P} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \frac{1}{7} \begin{pmatrix} 18 \\ 27 \\ 37 \end{pmatrix} \\
&= \frac{1}{7} \begin{pmatrix} -11 \\ -6 \\ -23 \end{pmatrix}
\end{aligned}$$

Since  $l_2$  is parallel to  $\overrightarrow{A'P}$  and contains  $P$ , equation of  $l_2$  is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 11 \\ 6 \\ 23 \end{pmatrix}, \mu \in \mathbb{R}.$$

20 A professional card stacker stacks two cards  $P$  and  $Q$  as follows:

Figure 2: Card Stack

Mr Poh models the scenario such that the two cards are planes  $P$  and  $Q$ , where the equation of plane  $P$  is  $P : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1$ , where line  $l_1$  is a normal to plane  $P$  that contains  $A(-3, 3, 2)$ . Line  $l_2$ , a normal to plane  $Q$ , also contains point  $A$  and is parallel to the vector  $\begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}$  where  $a < 0$ .  $l_3$  is the line of intersection between planes  $P$  and  $Q$ .

a) Find the coordinates of  $B$ , the point of intersection between  $l_1$  and plane  $P$ . [2]

$$\text{Equation of } l_1 \text{ is } \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Substitute equation of  $l_1$  into equation of  $P$  :

$$\left( \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1 \text{ for some } k \in \mathbb{R}$$

$$-10 + 11k = 1$$

$$k = 1$$

$$\overrightarrow{OB} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore B(0, 2, 3)$$

b) Given that  $l_2$  is obtained by rotating  $l_1$   $\cos^{-1} \frac{9}{11}$  about point  $A$ , find  $a$ ; Hence, find

the vector equation of  $l_2$ .

[4]

$$\pm \cos \left( \cos^{-1} \frac{9}{11} \right) = \frac{\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}}{\left\| \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix} \right\|}$$

$$\pm \frac{9}{11} = \frac{10+a}{\sqrt{11}\sqrt{10+a^2}}$$

$$\frac{81}{121} = \frac{100+a^2+20a}{110+11a^2}$$

$$8910 + 891a^2 = 12100 + 121a^2 + 2420a$$

$$770a^2 - 2420a - 3190 = 0$$

Using a G.C., we obtain  $a = -1$  or  $\frac{29}{7}$ . Since  $a < 0$ ,  $a = -1$ .

$$\therefore l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

- c) The point of intersection between  $l_2$  and plane  $Q$  is point  $B'$ . Given that  $\left| \overrightarrow{AB} \right| = \left| \overrightarrow{AB'} \right|$ , find the position vector of  $B'$  given that the  $x$ -coordinate of  $B' < 0$ . [3]

$$\begin{aligned} \left| \overrightarrow{AB'} \right| &= \left| \overrightarrow{AB} \right| \\ &= \left| \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{11} \end{aligned}$$



$$\text{Unit vector along } AB' = \frac{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right|} = \frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{OB'} = \overrightarrow{OA} \pm \frac{|\overrightarrow{AB'}|}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix}$$

$$\text{Since the } x \text{ - coordinate of } B' < 0, \overrightarrow{OB'} = \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix}$$

**d)** Find the equation of the plane  $Q$  in the form  $\mathbf{r} \cdot \mathbf{n} = d$ .

[2]

Since plane  $Q$  contains  $B'$ , equation of plane  $Q$  is

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 25$$

e) Find the vector equation of  $l_3$ .

[2]

We have the two equations, 
$$\begin{cases} 3x - y + z = 1 \\ 3x - y - z = -25 \end{cases}$$

Using a G.C., we obtain  $x = -4 + \frac{1}{3}y$ ,  $y \in \mathbb{R}$ ,  $z = 13$ .

Equation of  $l_3$  is  $\mathbf{r} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ ,  $\tau \in \mathbb{R}$ .

The equation of plane  $R$  is such that it reflects the image of plane  $P$  to form plane  $Q$ .

f) Find the vector  $\overrightarrow{AF}$ , where  $F$  is the foot of perpendicular from  $A$  to  $l_3$ .

Hence, find the equation of plane  $R$  in the form  $\mathbf{r} \cdot \mathbf{n} = d$ .

[5]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + k \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{AF} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + k \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+k \\ -3+3k \\ 11 \end{pmatrix}$$

Since  $\overrightarrow{AF} \perp l_3$ ,

$$\begin{pmatrix} -1+k \\ -3+3k \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 0$$

$$-10 + 10k = 0$$

$$k = 1$$

$$\therefore \overrightarrow{AF} = \begin{pmatrix} -1+1 \\ -3+3 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$$

Since plane  $R$  is parallel to both  $\overrightarrow{AF}$  and  $l_3$ ,

$$\text{Normal to plane } R // \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Equation of plane } R \text{ is } \mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = 12$$