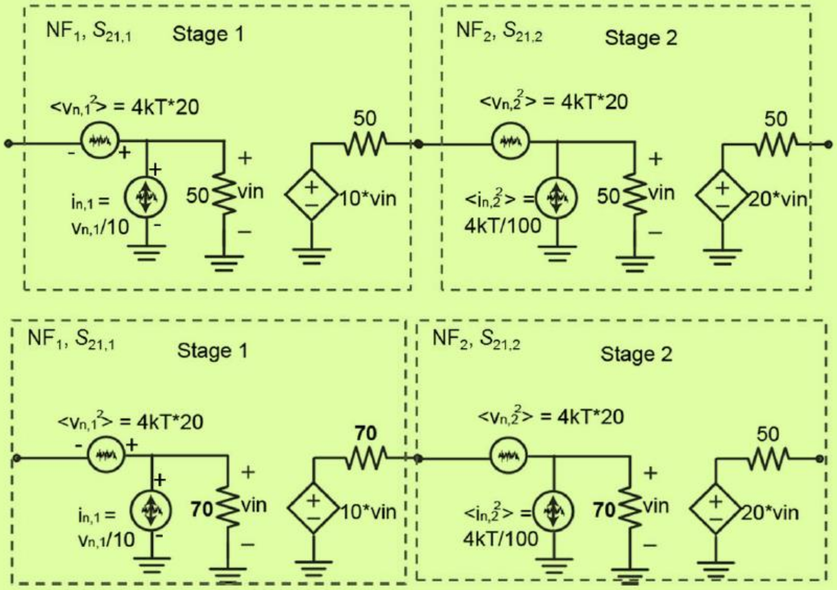
HW7 Solution

1. Noise Figure of Cascade Blocks and Lossy Transmission Line



(a) For the above two cascade circuits, calculate the power gains and noise figures for each stage (i.e., $S_{21,1}$, $S_{21,2}$, NF_1, NF_2) and the two-stage circuits ($S_{21,total}$, NF_{total}). The resistors are assumed to be noiseless.

First Circuit

$$S_{21,1} = \tfrac{10}{2} = 5$$

$$S_{21,2} = \tfrac{20}{2} = 10$$

$$S_{21,total} = \tfrac{10}{2} \times \tfrac{20}{2} = 50$$

$$NF_1 = \frac{SNR_i}{SNR_o} = \frac{N_o}{N_i|S_{21,1}|^2} = 1 + \frac{\overline{(V_{n,1}+\frac{V_{n,1}}{10}\times 50)^2}}{4kTR_s} = 1 + \frac{36\times 20}{50} = 15.4$$

$$NF_2 = \frac{SNR_i}{SNR_o} = \frac{N_o}{N_i|S_{21,1}|^2} = 1 + \frac{\overline{V_{n,2}^2}+50^2\overline{i_{n,2}^2}}{4kTR_s} = 1.9$$

$$NF_{total} = \frac{N_o}{N_i|S_{21,total}|^2} = 1 + \frac{\overline{(V_{n,1}+\frac{V_{n,1}}{10}\times 50)^2\times 5^2\times 10^2 + (\overline{V_{n,2}^2}+50^2\overline{i_{n,2}^2})\times 10^2}}{4kTR_s|S_{21,total}|^2} = 15.436$$

Second circuit:

$\Gamma_{in} = \frac{70-50}{70+50} = \frac{1}{6}$. So $S_{21,1} = \left(1+\frac{1}{6}\right) \times \frac{10}{70+50} \times 50 = \mathbf{4.861}$.

Similarly, $S_{21,2} = \left(1+\frac{1}{6}\right) \times \frac{20}{50+50} \times 50 = \mathbf{11.667}$.

$S_{21,total} = \left(1+\frac{1}{6}\right) \times \frac{10\times 70}{70+70} \times \frac{20\times 50}{50+50} = \mathbf{58.333} \neq S_{21,1} \times S_{21,2}$.

$NF_1 = 1 + \frac{\overline{v_{n,1}^2}(1+Y_cR_s)^2}{\overline{v_{R_s}^2}} = 1 + \frac{20(1+0.1\times 50)^2}{50} = \mathbf{15.4}$,

$NF_2 = 1 + \frac{\overline{v_{n,2}^2}+R_s^2\overline{i_{n,2}^2}}{\overline{v_{R_s}^2}} = 1 + \frac{20}{50} + 50 \times 0.01 = \mathbf{1.9}$.

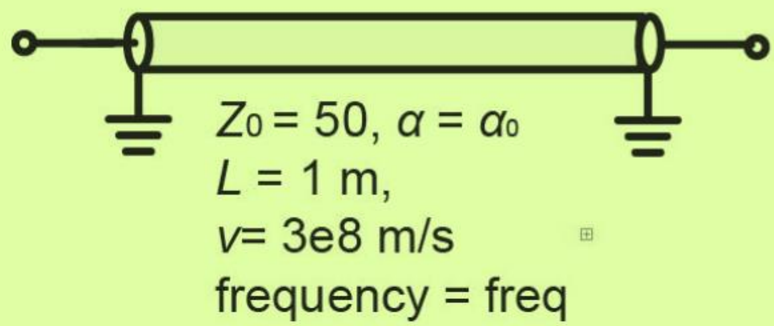
For the cascaded circuit, $\overline{v_{out}^2} = \left(\overline{v_{R_s}^2} + \overline{v_{n,1}^2}(1+Y_cR_s)^2\right)\left|S_{21,total}\right|^2 + \left(\overline{v_{n,2}^2} + 70^2\overline{i_{n,2}^2}\right) \times 100$. So $NF_{total} = \frac{\overline{v_{out}^2}}{\left|S_{21,total}\right|^2\overline{v_{R_s}^2}} = 1 + \frac{\overline{v_{n,1}^2}(1+Y_cR_s)^2}{\overline{v_{R_s}^2}} + \frac{\left(\overline{v_{n,2}^2}+70^2\overline{i_{n,2}^2}\right)\times 100}{\left|S_{21,total}\right|^2\overline{v_{R_s}^2}} = \mathbf{15.441} \neq NF_1 + \frac{NF_2-1}{\left|S_{21,1}\right|^2}$.

(b) Is the formula $NF_{total} = NF_1 + \frac{(NF_2-1)}{|S_{21,1}|^2}$ applicable?

1.b

For circuit above, it's applicable but for circuit below, it's not. Because in circuit above, all input and output impedances are matched and they are same as reference impedance Z_0, which is $50(\Omega)$. But in circuit below, some interfaces are not matched to $50(\Omega)$.

(c) For a lossy transmission line illustrated below, derive its noise figure.



c) Output noise $\overline{v_{out}^2} = 4kT \times 50 \times \delta f \times \left(\frac{50}{50+50}\right)^2$. Output noise due to source only $\overline{v_{out,source}^2} = 4kT \times 50 \times \delta f \times \left(\frac{50}{50+50}\right)^2 \times e^{-2\alpha L}$. Hence noise figure of a lossy transmission line is $NF_{lossT} = \frac{\overline{v_{out}^2}}{\overline{v_{out,source}^2}} = e^{2\alpha L} = \boldsymbol{e^{2\alpha_o}}$ (since L=1m). Also, $S_{21,lossT} = e^{-\alpha_o}$.

(d) If the T-line is used to connect the above two cascade circuits to the 50Ω source (e.g. antenna), what will be the new total noise figures?

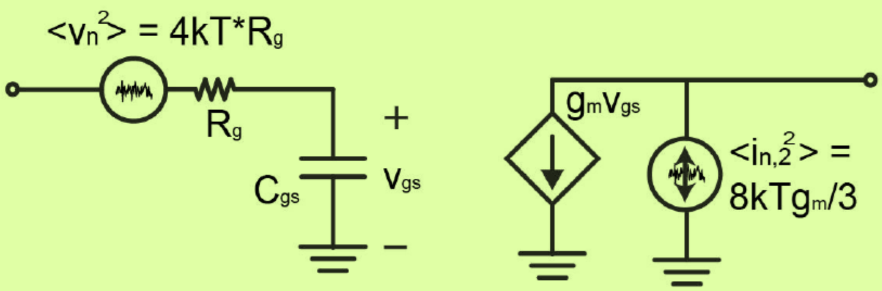
d) For the **first circuit**, $NF' = NF_{lossT} + \frac{NF_{total}-1}{|S_{21,lossT}|^2} = e^{2\alpha_o}(1 + 14.436) = \boldsymbol{15.436e^{2\alpha_o}}$.

For the **second circuit**, the formula cannot be applied directly since the lossy transmission line is not matched to the input of the cascaded stages. Now, $\overline{v_{out}^2} = \overline{v_{R_s}^2} + \left(\overline{v_{n,1}^2}(1 + Y_c R_s)^2\right)|S_{21,total}|^2 + \left(\overline{v_{n,2}^2} + 70^2\overline{i_{n,2}^2}\right) \times 100$, so $NF' = \boldsymbol{15.441e^{2\alpha_o}}$.

2. Matching for Low Noise versus Matching for High Gain

* In this problem, your answers should be functions of frequency.

(a) For a simplified common-source model shown below (with noise sources drawn), derive the input referred noise voltage and noise current.



a) Input referred noise voltage: Short the input terminal, and calculate noise current at output terminal.

Now, $\overline{i_{out,1}^2} = \overline{v_{n,1}^2}\frac{1}{1+\omega^2C_{gs}^2R_g^2}g_m^2 + \overline{i_{n,2}^2} = \overline{v_n^2}\frac{1}{1+\omega^2C_{gs}^2R_g^2}g_m^2$. Hence input referred noise voltage is $\sqrt{\overline{v_n^2}} = \sqrt{\overline{v_{n,1}^2} + \frac{\overline{i_{n,2}^2}(1+\omega^2C_{gs}^2R_g^2)}{g_m^2}}$.

Input referred noise current: Open the input terminal, and calculate noise current at output terminal.

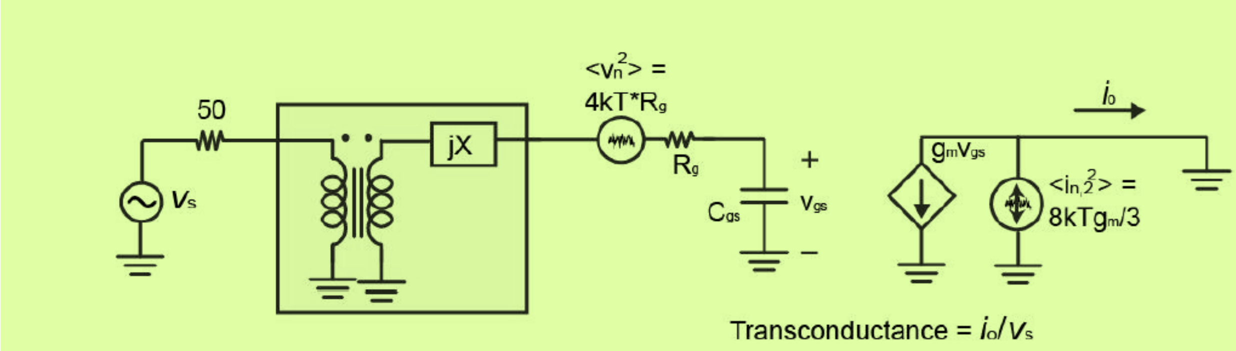
Now, $\overline{i_{out,2}^2} = \overline{i_{n,2}^2} = \overline{i_n^2}\frac{1}{\omega^2C_{gs}^2}g_m^2$. Hence input referred noise current is $\sqrt{\overline{i_n^2}} = \sqrt{\overline{i_{n,2}^2}}\frac{\omega C_{gs}}{g_m}$.

(b) Following part(a), what is the source impedance that optimizes the noise figure? What is the lowest noise figure?

b) Correlation between input referred noise voltage and current is $Z_c = \langle v_n, i_n \rangle = R_g + \frac{1}{j\omega C_{gs}}$.

Hence for minimum noise figure, $X_{s,opt} = -X_c = \frac{j}{\omega C_{gs}}$. Also, $R_c = R_u = R_g$, $G_n = \frac{2\omega^2C_{gs}^2}{3g_m}$. So $R_{s,opt} = \sqrt{\frac{R_u}{G_n} + R_c^2} = \sqrt{\frac{3R_gg_m}{2\omega^2C_{gs}^2} + R_g^2}$. The lowest noise figure is $\boldsymbol{NF_{min}} = 1 + 2R_cG_n + 2\sqrt{R_uG_n + R_c^2G_n^2} = 1 + \frac{4R_g\omega^2C_{gs}^2}{3g_m} + 2\sqrt{\frac{2R_g\omega^2C_{gs}^2}{3g_m} + \frac{4R_g^2\omega^4C_{gs}^4}{9g_m^2}}$.

(c) In practice, the source impedance is 50Ω (without any matching network). Design a input matching network to achieve the lowest noise figure. For your convenience, you can use an ideal transformer with arbitrary turns ratio and a series reactance to realize the matching network, as illustrated in the below figure.



c) Source impedance $R_s = 50\Omega$. If an ideal transformer of turns ratio $1{:}n$ is used to match to the common source model for minimum noise figure, then $n = \sqrt{\frac{R_{s,opt}}{50}}$. Also, $X = \frac{1}{\omega C_{gs}}$.

(d) Calculate the S_{11} and the achieved trans-conductance for your low-noise design.

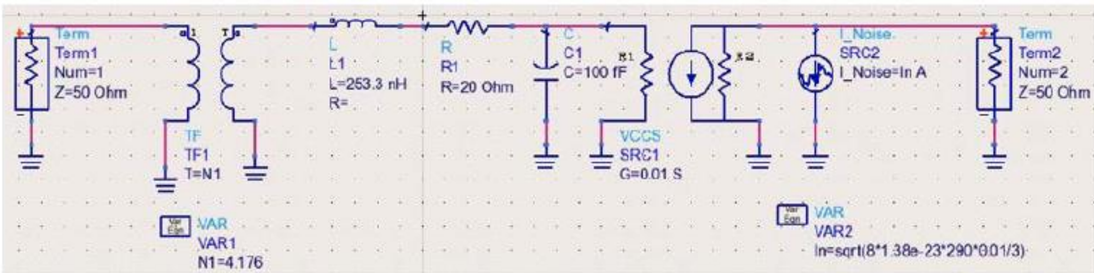
d) Since $S_{12} = 0$, $S_{11} = \Gamma_{in} = \frac{R_g - R_{s,opt}}{R_g + R_{s,opt}}$. The achieved transconductance is $\boldsymbol{G} = \frac{i_o}{v_s} = \frac{n \times \frac{1}{j\omega C_{gs}} \times g_m}{Z_{in} + Z_{s,opt}} = \frac{g_m\sqrt{\frac{R_{s,opt}}{50}}}{j\omega C_{gs}(R_{s,opt}+R_g)}$.

(e) Redesign the input matching network to achieve the maximum trans-conductance. Calculate the new trans-conductance and noise figure.

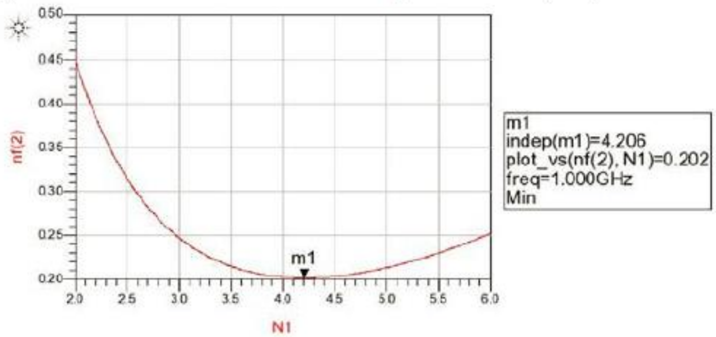
e) For maximum transconductance, $Z_{s,opt} = Z_{in}^*$ at resonant frequency ω_o. Hence, $jX = -\frac{1}{j\omega_o C_{gs}} = \frac{j}{\omega_o C_{gs}}$, i.e., $L = \frac{1}{\omega_o^2 C_{gs}}$ and $n = \sqrt{\frac{R_g}{50}}$. Then the maximum transconductance at resonance is $\boldsymbol{G}_{\omega_o} = \frac{i_o}{v_s} = \sqrt{\frac{R_g}{50}} \times Q \times g_m$ where $Q = \frac{1}{2\omega_o C_{gs} R_g}$. The output referred noise is $\overline{i_o^2} = 2\overline{v_{n,1}^2} \times Q^2 \times g_m^2 + \overline{i_{n,2}^2}$; output referred noise due to source only is $\overline{i_{o,source}^2} = \overline{v_{n,1}^2} \times Q^2 \times g_m^2$. Hence noise figure $NF_{\omega_o} = 2 + \frac{2}{3Q^2 g_m R_g}$.

(f) For a special case with frequency of 1 GHz, gm of 0.01, C_{gs} of 100 fF, and R_g of 20, verify the calculated results for the two designs in ADS.

Low noise figure design: $R_{s,opt} = 871.96\Omega$, $n \approx 4.176$.
Hence the calculated values are $NF_{min} \approx 1.047 = \mathbf{0.199dB}$, $S_{11} \approx \mathbf{-0.955}$, $|G| \approx \mathbf{0.075}$.
The schematic in ADS is shown below:



The plot of the noise figure vs turns ratio of the transformer shows the optimal value for minimum noise figure. The simulated values are approximately equal to the calculated values.



The simulated values of noise figure, S_{11}, and transconductance are given below.

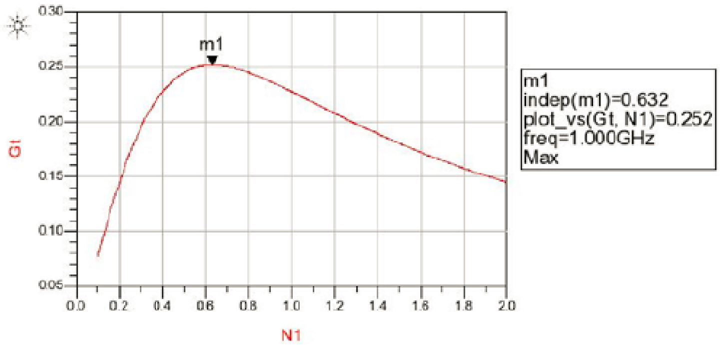
freq	nf(2)	NFmin	S(1,1)
1.000 GHz	0.202	0.202	0.955 / -179.9...

freq	Gt
1.000 GHz	0.075

Max trans-conductance design: $R_{s,opt} = 20\Omega, n \approx 0.632.$

Hence the calculated values are $|G_{max}| \approx \mathbf{0.252}$, $S_{11} \approx \mathbf{0}$, $NF \approx 2.002 = \mathbf{3.015dB}$.

The plot of trans-conductance vs turns ratio of the transformer shows the optimal value for maximum trans-conductance. The simulated values are close to the calculated values.



The simulated values of noise figure, S_{11}, and transconductance are given below.

freq	Gt
1.000 GHz	0.252

freq	nf(2)	NFmin	S(1,1)
1.000 GHz	3.079	0.202	8.576E-4 / -32...