EE142 Problem Set 4

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Problem 1

Calculate the scattering parameters of the following circuits:

(a) Find the input S_{11} for a general two-port terminated at port 2 with a load reflection coefficient of Γ_L.

We will call the wave going into port 1 V_1^+, the wave coming out of port 1 V_1^-, the wave *into* port 2 V_2^+ and the wave out of port 2 V_2^-.

We can then write the voltage waves in terms of the *two-port* S parameters.

$$V_1^- = S_{11}V_1^+ + S_{12}V_2^+$$
$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+$$

Now, if port two is terminated by a load which results in reflection coefficient Γ_L, then the network effectively becomes a 1 port network. We can write the one-port S_{11} in terms of the two-port S parameters.

Throughout this problem we will assume that the two-port S parameters, the one port S_{11}, and Γ_L are with respect to a reference of Z_0.

$$V_2^+ = V_2^-\Gamma_L$$
$$V_1^- = S_{11}V_1^+ + S_{12}V_2^-\Gamma_L \tag{1}$$
$$V_2^- = S_{21}V_1^+ + S_{22}V_2^-\Gamma_L \tag{2}$$

$$\text{Rewriting equ 2: } V_2^- = \frac{S_{21}V_1^+}{1 - S_{22}\Gamma_L}$$

$$\text{Plug into equ 1: } V_1^- = S_{11}V_1^+ + S_{12}\Gamma_L\frac{S_{21}V_1^+}{1 - S_{22}\Gamma_L}$$

$$\text{Finally: } \boxed{\frac{V_1^-}{V_1^+} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}}$$

$$\text{Notice: } \frac{V_1^-}{V_1^+} = S_{11,one-port}$$

(b) In the previous problem, what is the power that reaches the load in terms of the two-port scattering parameters and Γ_L? Suppose the input is driven with a matched source.

We derived the available power from the source for a 1-port network to be:

$$P_{avs} = \frac{|V_s|^2}{8Z_0}$$

where V_s is the amplitude at the generator. Assuming a perfect input match:

$$V_1^+ = V_s/2$$

The power seen by the load can be found as:

$$P_L = \frac{|V_2^-|^2 - |V_2^+|^2}{2Z_0} = \frac{|V_2^-|^2(1-|\Gamma_L|^2)}{2Z_0}$$

Recall from the previous part that V_2^- can be written in terms of S parameters:

$$V_2^- = \frac{S_{21}v_1^+}{1 - S_{22}\gamma_L}$$

Then, P_L can be written as such:

$$P_L = \frac{|S_{21}|^2|v_1^+|^2}{|1 - S_{22}\Gamma_L|^2} \cdot \frac{1-|\Gamma_L|^2}{2Z_0}$$

Substituting for V_1^+:

$$\boxed{P_L = \frac{|S_{21}|^2|v_s|^2}{4|1 - S_{22}\Gamma_L|^2} \cdot \frac{1-|\Gamma_L|^2}{2Z_0}}$$

(c) Derive the two-port scattering parameters of a three-port where port 3 is terminated in a load with reflection coefficient Γ_L.

We can write each voltage wave on each of the three ports in terms of the S parameters:

$$\begin{aligned} V_1^- &= S_{11}V_1^+ + S_{12}V_2^+ + S_{13}V_3^+ \\ V_2^- &= S_{21}V_1^+ + S_{22}V_2^+ + S_{23}V_3^+ \\ V_3^- &= S_{31}V_1^+ + S_{32}V_2^+ + S_{33}V_3^+ \end{aligned}$$

Next, by noticing $V_3^+ = V_3^- \cdot \Gamma_L$, the above equations can be written as:

$$\begin{aligned} V_1^- &= S_{11}V_1^+ + S_{12}V_2^+ + S_{13}V_3^+ \\ V_2^- &= S_{21}V_1^+ + S_{22}V_x^+ + S_{23}V_3^+ \\ 0 &= S_{31}V_1^+ + S_{32}V_2^+ + (\frac{S_{33}-1}{\Gamma_L})V_3^+ \end{aligned}$$

Finally, these equations can be combined to eliminate V_3^+:

$$V_1^- = S_{11}V_1^+ + S_{12}V_2^+ + \frac{S_{13}(S_{31}V_1^+ + S_{32}V_2^+)}{1/\Gamma_L - S_{33}}$$
$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+ + \frac{S_{23}(S_{31}V_1^+ + S_{32}V_2^+)}{1/\Gamma_L - S_{33}}$$

Now the new effective S_{11} can be read off as well as the other S parameters:

$$S_{11,new} = S_{11} + \frac{S_{13}S_{31}}{1/\Gamma_L - S_{33}}$$
$$S_{12,new} = S_{12} + \frac{S_{13}S_{32}}{1/\Gamma_L - S_{33}}$$
$$S_{21,new} = S_{21} + \frac{S_{23}S_{31}}{1/\Gamma_L - S_{33}}$$
$$S_{22,new} = S_{22} + \frac{S_{23}S_{32}}{1/\Gamma_L - S_{33}}$$

Problem 2

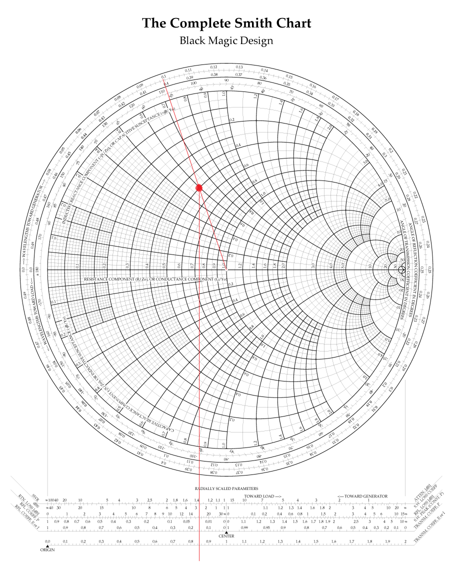
(a) Assume $Z_0 = 50\Omega$, use the Smith Chart to find ρ_L for a load impedance $Z_L = 25 + 30j\Omega$.

We first normalize the load impedance to Z_0: $Z_L' = 0.5 + 0.6j$. Then after plotting it on the Smith Chart, the magnitude and angle of the reflection coefficient can be read off.

We expect:

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} = -0.1494 + 0.45977j$$

The smith chart has been annotated (I'm not going to include any more charts):



We observe $|\rho_L| = 0.15$ and $phase(\rho_L) = 108.5\,\text{deg}$. Which translates to $\rho_L = -0.14158 + 0.42313j$. Pretty close.

(b) Assume $Z_0 = 50\Omega$, use the Smith Chart to find the load impedance for $\rho_L = 0.5 + 0.1j$.

To find Z_L we trace 0.5 units along the x axis and 0.1 units along the y axis of the Smith Chart. Then we just read off the impedance value. I found the value close to $Z_{L,norm} = 3.0 + 0.6j$, which translates to $Z_L = 150 + 30j$. The calculated value is $Z_L = 142 + 38j\Omega$, so the Smith Chart is once again pretty close.

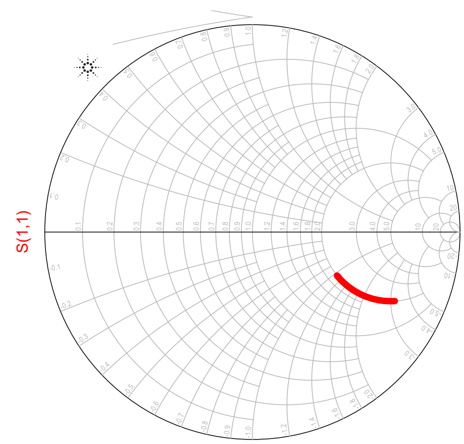
(c) Repeat (a) and (b) with Z_0 changed to 10Ω.

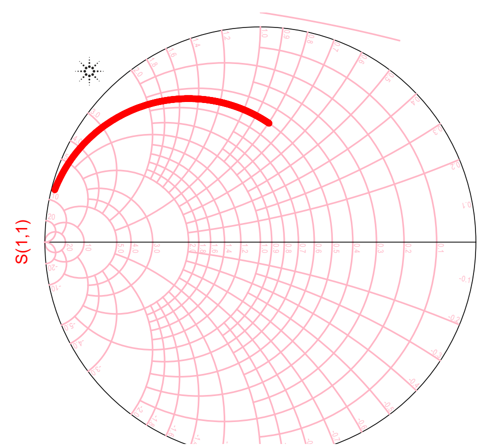
The same Smith Chart can't be used without modification. We have to normalize our chart to 10 instead of 50, but in addition to that, the scales on the reflection coefficient markers has to change. The Z_L measurement however is just scaled by 5.

$$\rho_L = 0.67 + 0.28j$$

$$Z_L = 29.4 + 7.7j\Omega$$

(d) For the following two circuits, trace ρ_L on a Smith Chart with $Z_0 = 50\Omega$.



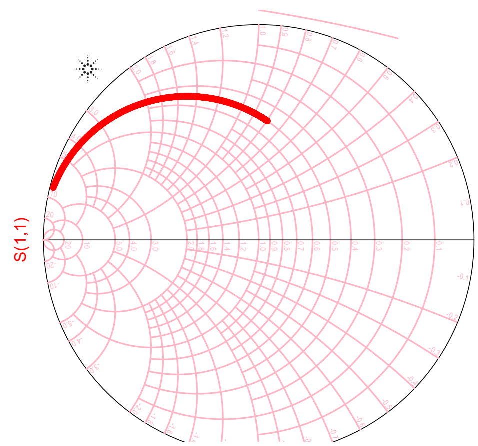


Problem 3

(a) Assume $Y_0 = 0.02S$, use the Admittance Smith Chart to find the ρ_L for a load impedance $Z_L = 25 + 30j\Omega$.

The method is similar to the impedance smith chart. We mark the load impedance on the admittance chart and then scale the value on the reflection axis to the measured phase. The result is identical $Z_L = -0.14 + 0.4j\Omega$.

(b) Trace ρ_L for the second circuit shown above with the parallel inductor, but on the admittance smith chart.



Problem 4

(a) What is the maximum power that can be extracted from the soure shown above? What is the load impedance for the max power delivery to happen?

The maximum power transfer occurs when the (real) load impedance matches the (real) source impedance. In that case, $Z_L = 50\Omega$ and the power delivered to the load is:

$$V_L = \frac{1}{2} V_s$$
$$I_L = \frac{V_s}{100}$$
$$P_{L,ac} = \frac{V_s^2}{200 \cdot 2} = 2.5 \text{ mW}$$

(b) Drive a 500Ω load. What is the power delivered and the load voltage?

$$V_L = \frac{Z_s}{Z_s + Z_L} V_s = 0.091$$
$$I_L = \frac{V_s}{500} = 2 \text{ mA}$$
$$P_L = 0.18 \text{ mW}$$

(c) Let's try to achieve impedance matching by putting a resistor in parallel with the load. What should the resistor value be and what is the actual power delivered to the load?

$$R_{match}||R_L = 50 \rightarrow R_{match} = \frac{50R_L}{R_L - 50} \rightarrow R_{match} = 55.5\Omega$$
$$V_{load,match} = \frac{1}{2} V_s$$
$$I_L = V_L/R_L = 1 \text{ mA}$$
$$P_L = 0.5 \text{ mW}$$

This approach doesn't work well since the maximum power is extracted from the load, but it is mostly wasted in the matching resistance.

(d) Design an impedance matching network using an ideal shunt capacitor and an ideal series inductor. What are the values?

We start on the real impedance axis at 10. We move on a constant conductance circle to the -0.3j constant suspectance curve. This yields $C = \frac{0.3/Z_0}{2\pi \cdot 1e9} = 0.95$ pF.

We then are on the constant resistance circle and can use a series inductor; we move on the 3j circle on the impedance chart. This yields $L = \frac{3 \cdot Z_0}{2\pi \cdot 1e9} = 23.9$ nH.

Simulation confirms that maximum power reaches the load.

(e) Calculate the load voltage and power.

We model each leg as an impedance, with the first leg being $Z_s + j\omega L$ and the second $\frac{Z_L}{1+j\omega C}$. Then V_L can be found as a voltage divider and the same current goes through both legs. We find $Re(V_L) = 0.509$ V, and $Re(I_L) = 10$ mA, thus total power is maximized to the load at $P_L = 2.5$ mW