EE 142 Problem Set 3

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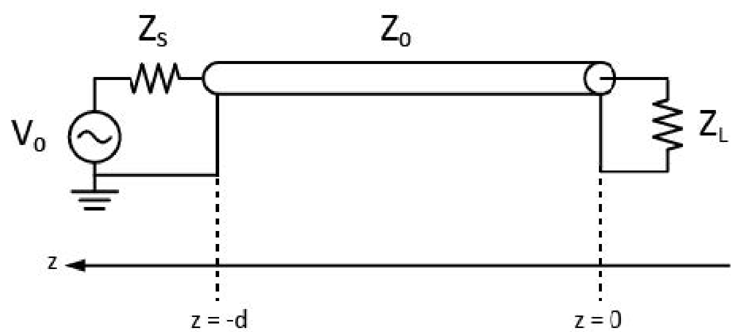
1 T-Lines at Steady State



Voltage source generates 10 Ghz sine with 10V amplitude.

Tline terminated with $Z_L = 80 - 40j\Omega$, and $Z_0 = 100\Omega$. $\epsilon_{eff} = 4$ and $d = 22.5$ mm.

1. Find the reflection coefficient at the load (z = 0) and at the source (z = -d)

   At the load:

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$
$$\rho_L = -0.0588 - 0.23539j$$
$$|\rho_L| = 0.242$$

   At any point on the line, we can derive an effective generalized $\rho(z)$ which represents the ratio of the backwards and forward traveling waves at a given point on the tline.

$$V(z) = V_0^+(e^{-j\beta z} + \rho_L e^{-j\beta z})$$
$$\rho(z) = \frac{V_0^+ \rho_L e^{j\beta z}}{V_0^+ e^{-j\beta z}}$$
$$\rho(z) = \rho_L e^{2j\beta z}$$

Notice that since $\beta = 2\pi/\lambda$, $\rho(z)$ repeats every $\lambda/2$ traversed along the tline back to the generator. We can find c_p and λ for this tline and frequency.

$$c_p = \frac{c_0}{\sqrt{\epsilon_{eff}}} \approx 1.5e8 \text{ m/s}$$
$$\lambda = \frac{c_p}{f} = 0.015 \text{ m}$$
$$d/\lambda = 1.5 = 3 \cdot \frac{1}{2}\lambda$$

So, $\rho(z)$ at $z = -d$ is $rho_L = 0.242$.

2. Find the input impedance at the source (z = -d) and at z = 18.75mm.

   The general form is:

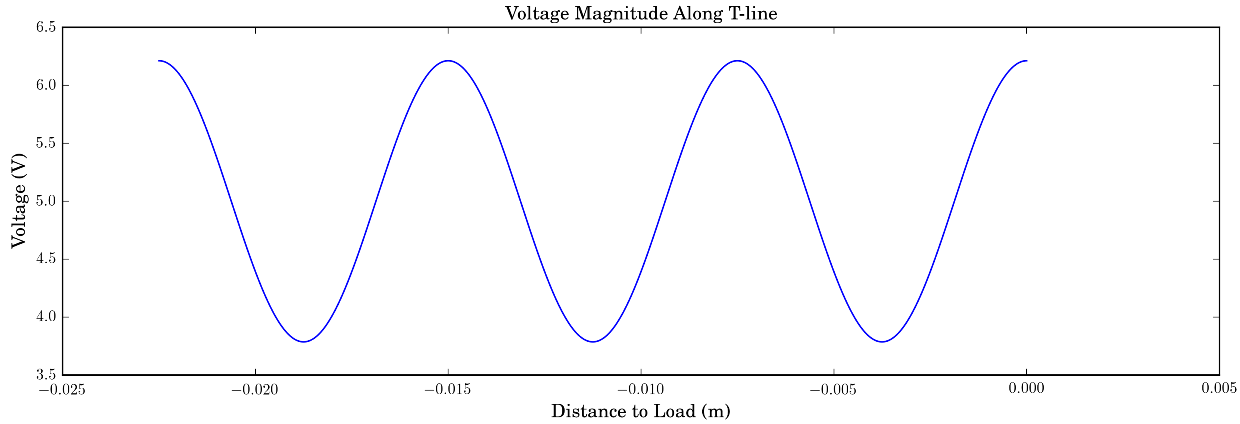
$$Z_{in}(-l) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$
$$\beta = \frac{2\pi}{\lambda} = 418.879$$
$$Z_{in}(0) = Z_L = 100\Omega$$
$$Z_{in}(-18.75 \text{ mm}) = Z_{in}(\lambda + \lambda/4) = \frac{Z_0^2}{Z_L} = 100 + 50j$$

3. Plot the magnitude of the voltage along the line. Find voltage maximum, minimum, and SWR.

   We assume that $Z_S = Z_0$:

$$SWR = \frac{V_{max}}{V_{min}} = \frac{1 + |\rho_L|}{1 - |\rho_L|}$$
$$SWR = 1.64$$
$$V^+ = \frac{Z_0}{Z_0 + Z_S} = 5 \text{ V}$$
$$V_{max} = |V^+|(1 + |\rho_L|) = 6.2 \text{ V}$$
$$V_{min} = |V^+|(1 - |\rho_L|) = 3.8 \text{ V}$$

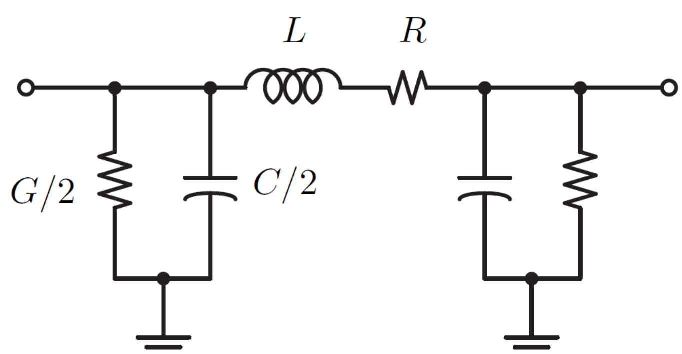
   Plot of voltage magnitude along line:



2 T-Line Modeling

We will derive an equivalent two-port circuit model for a short section of transmission line ($l << \lambda$) including loss.

1. For a "pi" equivalent circuit shown below, find the two-port Z matrix.



Let's call the current flowing *into* node 1 i_1 and the current flowing *into* node 2 i_2. The voltage applied across node is is v_1 and v_2 for node 2. We will call each section of the pi network Z_1, Z_2, Z_3 going left to right and $Z_1 = Z_3$.

$$
\begin{aligned}
Z_{11} &= \frac{v_1}{i_1}\bigg|_{i_2=0} = (Z_1||(Z_2+Z_3)) = \frac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3} \\
Z_{22} &= Z_{11} \text{ due to symmetry} \\
Z_{12} &= \frac{v_1}{i_2}\bigg|_{i_1=0} = \frac{Z_1 Z_3}{Z_1+Z_2+Z_3} \\
Z_{21} &= Z_{12} \text{ due to reciprocity}
\end{aligned}
$$

$$Z_1 = Z_3 = \frac{2}{2j\omega C + G}$$
$$Z_2 = R + j\omega L$$

2. Consider a section of transmission line with loss. Find the two port Z matrix. Use the transmission line impedance equation (general expression with loss).

   We begin with the general form using ρ_L:

$$\begin{aligned}
Z_{in}(-l) &= \frac{V(-l)}{I(-l)} = Z_0 \frac{1 + \rho_L e^{-2\gamma l}}{1 - \rho_L e^{-2\gamma l}} \\
\rho_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\
Z_{in}(-l) &= Z_0 \frac{Z_L(1 + e^{-2\gamma l}) + Z_0(1 - e^{-2\gamma l})}{Z_0(1 + e^{-2\gamma l}) + Z_L(1 - e^{-2\gamma l})} \\
Z_{in}(-l) &= Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \\
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta
\end{aligned}$$

   Now we open and short the transmission line to measure its Z parameters.

$$\begin{aligned}
Z_{11} &= \left.\frac{v_1}{i_1}\right|_{i_2=0, Z_L=\infty} = Z_0 \frac{1}{\tanh(\gamma l)} \\
Z_{22} &= Z_{11} \text{ due to symmetry} \\
Z_{12} &= \left.\frac{v_1}{i_2}\right|_{i_1=0, Z_L=0} = Z_0 \tanh(\gamma l) \\
Z_{21} &= Z_{12} \text{ due to reciprocity}
\end{aligned}$$

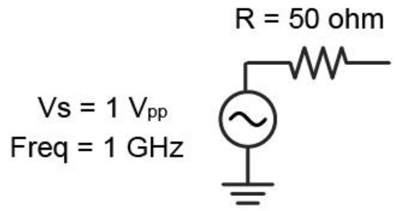
3. Take the limit of a very short line and simplify the answer (Hint: use a Taylor series expansion and keep only the first few terms).

   Use the expansion:

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots \text{ for } |x| < \frac{\pi}{2}$$

3 Impedance Matching for Maximum Power Delivery

1. What is the maximum power that can be extracted from the source shown below? What is the optimal load impedance for the maximum power delivery to happen?

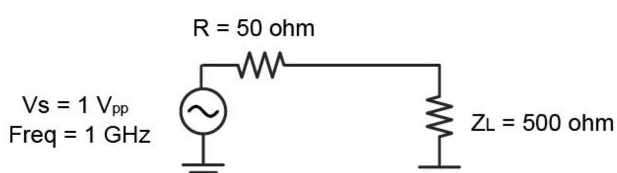


$$
\begin{aligned}
|I_s| &= \frac{|V_s|}{|R_s + R_L|} \\
I_{s,rms} &= \frac{1}{2}|I_s| \\
V_L &= I_{s,rms}R_L \\
P_L &= I_{s,rms}V_L = I_{s,rms}^2 R_L = 1/2(\frac{V_s}{R_s + R_L})^2 R_L \\
\frac{\partial P_L}{\partial R_L} &= (\frac{-R_s}{R_L})^2 + 1
\end{aligned}
$$

Setting the denominator of derivative to 0 and solving gives $R_L = \pm R_s \rightarrow R_L = R_s$. Indeed, this minimizes the denominator, and thus maximizes the power delivered to the load.

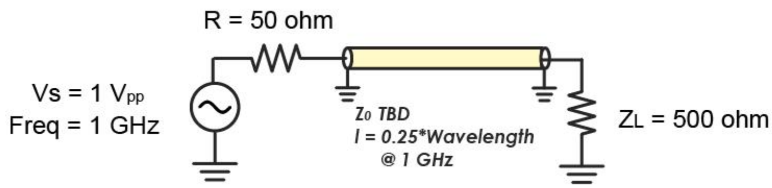
$$P_{max} = \frac{V_s^2}{8R_L} = 2.5 \text{ mW}$$

2. Use this source to drive a 500Ω load and we directly connect the load to the source, as illustrated by the figure below. What is the power delivered to the 500Ω load and the load voltage?



$$
\begin{aligned}
P_L &= 0.8 \text{ mW} \\
V_L = I_{s,rms}R_L &= 0.45 \text{ V}
\end{aligned}
$$

3. Let's try to achieve impedance matching by putting a quarter-wavelength transmission line between the load and source, as indicated by the below figure. Find the characteristic impedance Z_0 that maximizes the power delivered to the load. What are the corresponding power and voltage at the load?



The general form of input impedance for a lossless tline is:

$$Z_{in}(-l) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$