Integrated Circuits for Communication

Berkeley

Introduction to Transmission Lines

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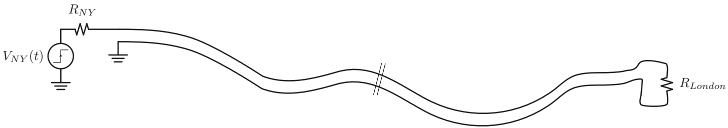
First Trans-Atlantic Cable

- Problem: A long cable – the trans-atlantic telephone cable – is laid out connecting NY to London. We would like analyze the electrical properties of this cable.
- For simplicity, assume the cable has a uniform cross-sectional configuration (shown as two wires here)

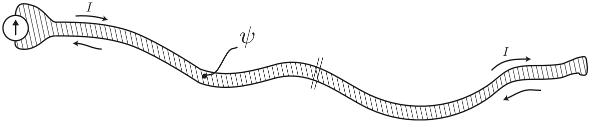


Trans-Atlantic Cable Analysis

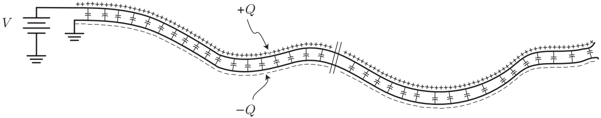
- Can we do it with circuit theory?
- Fundamental problem with circuit theory is that it assumes that the speed of light is infinite. So all signals are in phase: $V(z) = V(z + \ell)$
- Consequently, all variations in space are ignored: $\frac{\partial}{\partial z} \to 0$
- This allows the *lumped* circuit approximation.

Lumped Circuit Properties of Cable

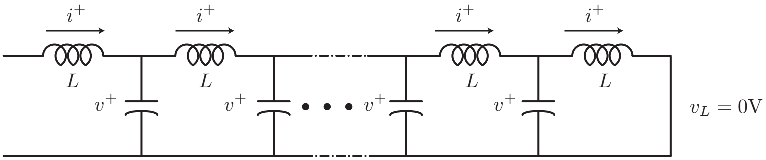
- Shorted Line: The long loop has *inductance* since the magnetic flux ψ is not negligible (long cable) ($\psi = LI$)



- Open Line: The cable also has substantial capacitance ($Q = CV$)



- So do we model the cable as an inductor or as a capacitor? Or both? How?
- Try a *distributed* model: Inductance and capacitance occur together. They are intermingled.

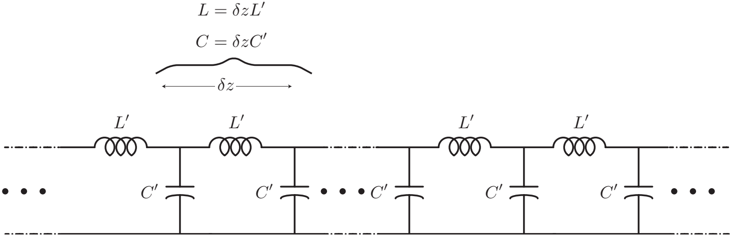


- Can add loss (series and shunt resistors) but let's keep it simple for now.
- Add more sections and solution should converge

Sectional Model (II)

- More sections $\rightarrow$ The equiv LC circuit represents a smaller and smaller section and therefore lumped circuit approximation is more valid
- This is an easy problem to solve with SPICE.
- But the people 1866 didn't have computers ... how did they analyze a problem with hundreds of inductors and capacitors?

Distributed Model



- Go to a fully distributed model by letting the number of sections go to infinity
- Define inductance and capacitance per unit length $L' = L/\ell$, $C' = C/\ell$
- For an infinitesimal section of the line, circuit theory applies since signals travel instantly over an infinitesimally small length

KCL and KVL for a small section

- KCL: $i(z) = \delta z C' \frac{\partial v(z)}{\partial t} + i(z + \delta z)$
- KVL: $v(z) = \delta z L' \frac{\partial i(z+\delta z)}{\partial t} + v(z + \delta z)$
- Take limit as $\delta z \to 0$

We arrive at "Telegrapher's Equatins"

$$\lim_{\delta z \to 0} \frac{i(z) - i(z + \delta z)}{\delta z} = -\frac{\partial i}{\partial z} = C' \frac{\partial v}{\partial t}$$

$$\lim_{\delta z \to 0} \frac{v(z) - v(z + \delta z)}{\delta z} = -\frac{\partial v}{\partial z} = L' \frac{\partial i}{\partial t}$$

Derivation of Wave Equations

- We have two coupled equations and two unkowns (i and v) ... can reduce it to two de-coupled equations:

$$\frac{\partial^2 i}{\partial t \partial z} = -C' \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial^2 v}{\partial z^2} = -L' \frac{\partial^2 i}{\partial z \partial t}$$

- note order of partials can be changed (at least in EE)

$$\frac{\partial^2 v}{\partial z^2} = L'C' \frac{\partial^2 v}{\partial t^2}$$

- Same equation can be derived for current:

$$\frac{\partial^2 i}{\partial z^2} = L'C' \frac{\partial^2 i}{\partial t^2}$$

The Wave Equation

The Wave Equation

We see that the currents and voltages on the transmission line satisfy the one-dimensional wave equation. This is a partial differential equation. The solution depends on boundary conditions and the initial condition.

$$\frac{\partial^2 i}{\partial z^2} = L'C'\frac{\partial^2 i}{\partial t^2}$$

Wave Equation Solution

Consider the function $f(z,t) = f(z \pm vt) = f(u)$:

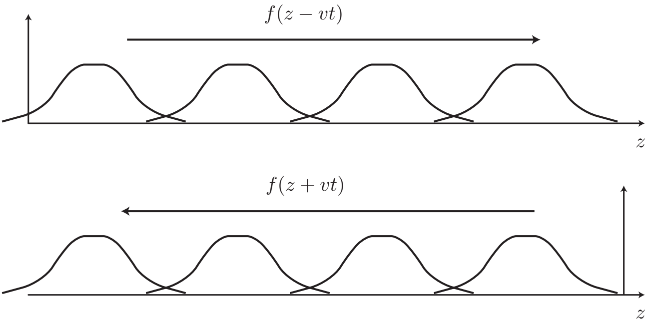
$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} = \frac{\partial f}{\partial u} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \pm v\frac{\partial f}{\partial u}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial u^2} \qquad \frac{\partial^2 f}{\partial t^2} = \pm v\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial t}\right) = v^2\frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

It satisfies the wave equation!

Wave Motion



- General voltage solution: $v(z,t) = f^+(z - vt) + f^-(z + vt)$
- Where $v = \sqrt{\frac{1}{L'C'}}$

Wave Speed

- Speed of motion can be deduced if we observe the speed of a point on the aveform

$$z \pm vt = \mathrm{constant}$$

- To follow this point as time elapses, we must move the z coordinate in step. This point moves with velocity

$$\frac{dz}{dt} \pm v = 0$$

- This is the speed at which we move with speed $\frac{dz}{dt} = \pm v$
- v is the velocity of wave propagation

"Ohm's Law" for T-Lines

Current / Voltage Relationship (I)

- Since the current also satisfies the wave equation

$$i(z,t) = g^+(z - vt) + g^-(z + vt)$$

- Recall that on a transmission line, current and voltage are related by

$$\frac{\partial i}{\partial z} = -C' \frac{\partial v}{\partial t}$$

- For the general function this gives

$$\frac{\partial g^+}{\partial u} + \frac{\partial g^-}{\partial u} = -C' \left(-v \frac{\partial f^+}{\partial u} + v \frac{\partial f^-}{\partial u} \right)$$

Current / Voltage Relationship (II)

- Since the forward waves are independent of the reverse waves

$$\frac{\partial g^+}{\partial u} = C' v \frac{\partial f^+}{\partial u}$$
$$\frac{\partial g^-}{\partial u} = -C' v \frac{\partial f^-}{\partial u}$$

- Within a constant we have

$$g^+ = \frac{f^+}{Z_0}$$
$$g^- = -\frac{f^-}{Z_0}$$

- Where $Z_0 = \sqrt{\frac{L'}{C'}}$ is the "Characteristic Impedance" of the line

A Side Note on Current

- Notice that the currents in the forward wave has the same sign

$$g^{+} = \frac{v^{+}}{Z_0}$$

- But the reverse wave has a negative sign

$$g^{-} = -\frac{v^{-}}{Z_0}$$

- This is related to the definition of current. If positive charges are moving left, then the corresponding current is negative.
- Clearly understand the definition of currents on a transmission line with respect to the two conductors. This is a “odd” mode current, since the top and bottom conductors carry equal and opposite currents. There's also an “even” mode current that we are neglecting for now.

Example: Step Into Infinite Line

- Excite a step function onto a transmission line
- The line is assumed uncharged: $Q(z,0)=0$, $\psi(z,0)=0$ or equivalently $v(z,0)=0$ and $i(z,0)=0$
- By physical intuition, we would only expect a forward traveling wave since the line is infinite in extent
- The general form of current and voltage on the line is given by

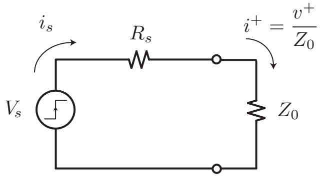
$$v(z,t) = v^{+}(z - vt)$$

$$i(z,t) = i^{+}(z - vt) = \frac{v^{+}(z - vt)}{Z_0}$$

- The T-line looks like a resistor of Z_0 ohms!

Example 1 (cont)

- We may therefore model the line with the following simple equivalent circuit



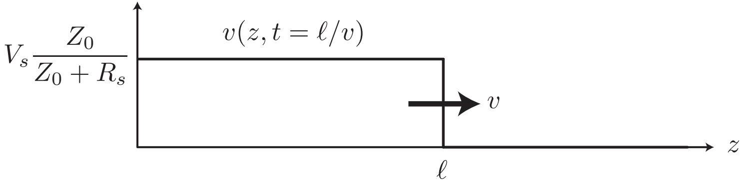
- Since $i_s = i^+$, the excited voltage wave has an amplitude of

$$v^+ = \frac{Z_0}{Z_0 + R_s} V_s$$

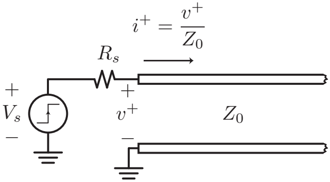
- It's surprising that the voltage on the line is not equal to the source voltage

Example 1 (cont)

- The voltage on the line is a delayed version of the source voltage



Energy to "Charge" Transmission Line



- The power flow into the line is given by

$$P^+_{line} = i^+(0,t)v^+(0,t) = \frac{\left(v^+(0,t)\right)^2}{Z_0}$$

- Or in terms of the source voltage

$$P^+_{line} = \left(\frac{Z_0}{Z_0 + R_s}\right)^2 \frac{V_s^2}{Z_0} = \frac{Z_0}{(Z_0 + R_s)^2} V_s^2$$

Energy Stored in Inds and Caps (I)

- But where is the power going? The line is lossless!
- Energy stored by a cap/ind is $\frac{1}{2}CV^2/\frac{1}{2}LI^2$
- At time t_d, a length of $\ell = vt_d$ has been "charged":

$$\frac{1}{2}CV^2 = \frac{1}{2}\ell C' \left(\frac{Z_0}{Z_0 + R_s}\right)^2 V_s^2$$

$$\frac{1}{2}LI^2 = \frac{1}{2}\ell L' \left(\frac{V_s}{Z_0 + R_s}\right)^2$$

- The total energy is thus

$$\frac{1}{2}LI^2 + \frac{1}{2}CV^2 = \frac{1}{2}\frac{\ell V_s^2}{(Z_0 + R_s)^2}\left(L' + C'Z_0^2\right)$$

Energy Stored (II)

- Recall that $Z_0 = \sqrt{L'/C'}$. The total energy stored on the line at time $t_d = \ell/v$:

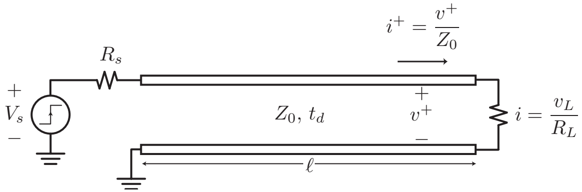
$$E_{line}(\ell/v) = \ell L' \frac{V_s^2}{(Z_0 + R_s)^2}$$

- And the power delivered onto the line in time t_d:

$$P_{line} \times \frac{\ell}{v} = \frac{\frac{l}{v} Z_0 V_s^2}{(Z_0 + R_s)^2} = \ell \sqrt{\frac{L'}{C'}} \sqrt{L'C'} \frac{V_s^2}{(Z_0 + R_s)^2}$$

- As expected, the results match (conservation of energy).

Transmission Line Termination



- Consider a finite transmission line with a termination resistance
- At the load we know that Ohm's law is valid: $I_L = V_L/R_L$
- So at time $t = \ell/v$, our pulse reaches the load. Since the current on the T-line is $i^+ = v^+/Z_0 = V_s/(Z_0 + R_s)$ and the current at the load is V_L/R_L, a discontinuity is produced at the load.

Reflections

- Thus a reflected wave is created at discontinuity

$$V_L(t) = v^+(\ell, t) + v^-(\ell, t)$$

$$I_L(t) = \frac{1}{Z_0} v^+(\ell, t) - \frac{1}{Z_0} v^-(\ell, t) = V_L(t)/R_L$$

- Solving for the forward and reflected waves

$$2v^+(\ell, t) = V_L(t)(1 + Z_0/R_L)$$

$$2v^-(\ell, t) = V_L(t)(1 - Z_0/R_L)$$

Reflection Coefficient

- And therefore the reflection from the load is given by

$$\Gamma_L = \frac{V^-(\ell, t)}{V^+(\ell, t)} = \frac{R_L - Z_0}{R_L + Z_0}$$

- The reflection coefficient is a very important concept for transmission lines: $-1 \leq \Gamma_L \leq 1$
- $\Gamma_L = -1$ for $R_L = 0$ (short)
- $\Gamma_L = +1$ for $R_L = \infty$ (open)
- $\Gamma_L = 0$ for $R_L = Z_0$ (match)
- Impedance match is the proper termination if we don't want any reflections

Propagation of Reflected Wave (I)

- If $\Gamma_L \neq 0$, a new reflected wave travels toward the source and unless $R_s = Z_0$, another reflection also occurs at source!
- To see this consider the wave arriving at the source. Recall that since the wave PDE is linear, a superposition of any number of solutins is also a solution.
- At the source end the boundary condition is as follows

$$V_s - I_s R_s = v_1^+ + v_1^- + v_2^+$$

- The new term v_2^+ is used to satisfy the boundary condition

Propagation of Reflected Wave (II)

- The current continuity requires $I_s = i_1^+ + i_1^- + i_2^+$

$$V_s = (v_1^+ - v_1^- + v_2^+)\frac{R_s}{Z_0} + v_1^+ + v_1^- + v_2^+$$

- Solve for v_2^+ in terms of known terms

$$V_s = \left(1 + \frac{R_s}{Z_0}\right)(v_1^+ + v_2^+) + \left(1 - \frac{R_s}{Z_0}\right) v_1^- +$$

- But $v_1^+ = \frac{Z_0}{R_s+Z_0} V_s$

$$V_s = \frac{R_s + Z_0}{Z_0}\frac{Z_0}{R_s + Z_0} V_s + \left(1 - \frac{R_s}{Z_0}\right) v_1^- + \left(1 + \frac{R_s}{Z_0}\right) v_2^+$$

Propagation of Reflected Wave (III)

- So the source terms cancel out and

$$v_2^+ = \frac{R_s - Z_0}{Z_0 + R_s} v_1^- = \Gamma_s v_1^-$$

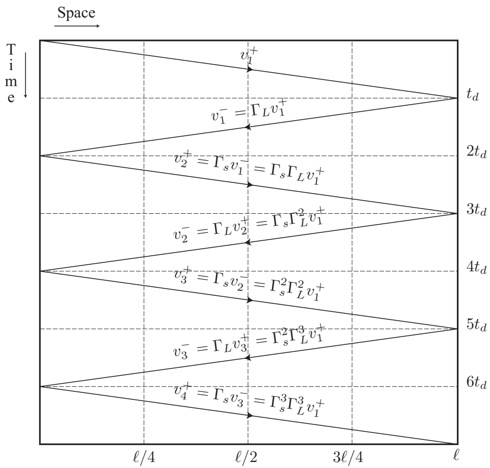
- The reflected wave bounces off the source impedance with a reflection coefficient given by the same equation as before

$$\Gamma(R) = \frac{R - Z_0}{R + Z_0}$$

- The source appears as a short for the incoming wave
- Invoke superposition! The term v_1^+ took care of the source boundary condition so our new v_2^+ only needed to compensate for the v_1^- wave ... the reflected wave is only a function of v_1^-

Bounce Diagram

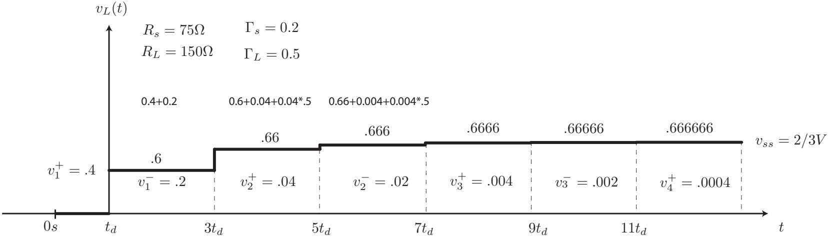
- We can track the multiple reflections with a "bounce diagram"



Freeze time

- If we freeze time and look at the line, using the bounce diagram we can figure out how many reflections have occurred
- For instance, at time $2.5t_d = 2.5\ell/v$ three waves have been excited (v_1^+,v_1^-, v_2^+), but v_2^+ has only travelled a distance of $\ell/2$
- To the left of $\ell/2$, the voltage is a summation of three components: $v = v_1^+ + v_1^- + v_2^+ = v_1^+(1 + \Gamma_L + \Gamma_L\Gamma_s)$.
- To the right of $\ell/2$, the voltage has only two components: $v = v_1^+ + v_1^- = v_1^+(1 + \Gamma_L)$.

- We can also pick at arbitrary point on the line and plot the evolution of voltage as a function of time
- For instance, at the load, assuming $R_L > Z_0$ and $R_S > Z_0$, so that $\Gamma_{s,L} > 0$, the voltage at the load will will increase with each new arrival of a reflection



Steady-State Voltage on Line (I)

- To find steady-state voltage on the line, we sum over all reflected waves:

$$v_{ss} = v_1^+ + v_1^- + v_2^+ + v_2^- + v_3^+ + v_3^- + v_4^+ + v_4^- + \cdots$$

- Or in terms of the first wave on the line

$$v_{ss} = v_1^+(1 + \Gamma_L + \Gamma_L\Gamma_s + \Gamma_L^2\Gamma_s + \Gamma_L^2\Gamma_s^2 + \Gamma_L^3\Gamma_s^2 + \Gamma_L^3\Gamma_s^3 + \cdots$$

- Notice geometric sums of terms like $\Gamma_L^k\Gamma_s^k$ and $\Gamma_L^{k+1}\Gamma_s^k$. Let $x = \Gamma_L\Gamma_s$:

$$v_{ss} = v_1^+(1 + x + x^2 + \cdots + \Gamma_L(1 + x + x^2 + \cdots))$$

- The sums converge since $x < 1$

$$v_{ss} = v_1^+ \left(\frac{1}{1 - \Gamma_L \Gamma_s} + \frac{\Gamma_L}{1 - \Gamma_L \Gamma_s} \right)$$

- Or more compactly

$$v_{ss} = v_1^+ \left(\frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_s} \right)$$

- Substituting for Γ_L and Γ_s gives

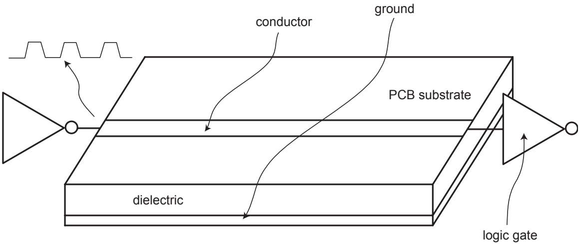
$$v_{ss} = V_s \frac{R_L}{R_L + R_s}$$

What Happend to the T-Line?

- For steady state, the equivalent circuit shows that the transmission line has disappeared.
- This happens because if we wait long enough, the effects of propagation delay do not matter
- Conversly, if the propagation speed were infinite, then the T-line would not matter
- But the presence of the T-line will be felt if we disconnect the source or load!
- That's because the T-line *stores* reactive energy in the capaciance and inductance
- Every real circuit behaves this way! Circuit theory is an abstraction

PCB Interconnect

- Suppose $\ell = 3\mathrm{cm}$, $v = 3 \times 10^8 \mathrm{m/s}$, so that $t_p = \ell / v = 10^{-10}\mathrm{s} = 100\mathrm{ps}$
- On a time scale $t < 100\mathrm{ps}$, the voltages on interconnect act like transmission lines!
  - Fast digital circuits need to consider T-line effects



Example: Open Line (I)

- Source impedance is $Z_0/4$, so $\Gamma_s = -0.6$, load is open so $\Gamma_L = 1$
- As before a positive going wave is launched $v_1^+ = Z_0/(Z_0 + Z_0/4)V_s = 0.8V_s$
- Upon reaching the load, a reflected wave of of equal amplitude is generated and the load voltage overshoots $v_L = v_1^+ + v_1^- = 1.6\mathrm{V}$
- Note that the current reflection is negative of the voltage

$$\Gamma_i = \frac{i^-}{i^+} = -\frac{v^-}{v^+} = -\Gamma_v$$

- This means that the sum of the currents at the load is zero (open)

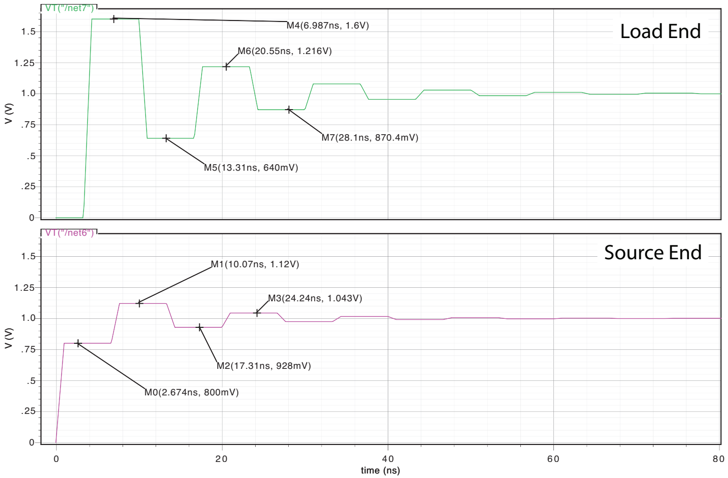
Example: Open Line (II)

- At the source a new reflection is created $v_2^+ = \Gamma_L \Gamma_s v_1^+$, and note $\Gamma_s < 0$, so $v_2^+ = -.6 \times 0.8 = -0.48$.
- At a time $3t_p$, the line charged initially to $v_1^+ + v_1^-$ drops in value

$$v_L = v_1^+ + v_1^- + v_2^+ + v_2^- = 1.6 - 2 \times .48 = .64$$

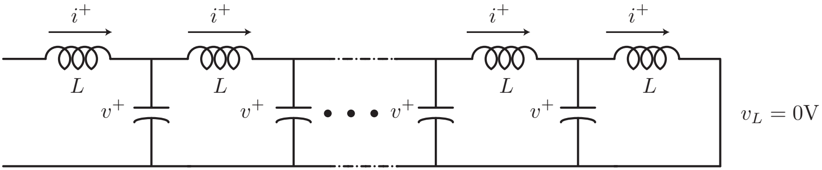
- So the voltage on the line undershoots (< 1 times V_s)
- And on the next cycle $5t_p$ the load voltage again overshoots
- We observe ringing with frequency $2t_p$

Example: Open Line Ringing



- Observed waveform as a function of time. Risetime due to SPICE $t_{step} = 20\text{ps}$.

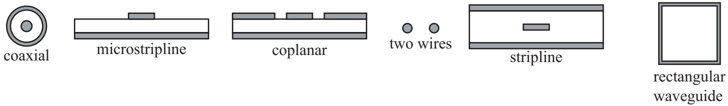
- The intitial step charges the "first" capacitor through the "first" inductor since the line is uncharged
- There is a delay since on the rising edge of the step, the inductor is an open
- Each successive capacitor is charged by "its" inductor in a uniform fashion ... this is the forward wave v_1^+



Physical Intuition: Shorted Line (II)

- The volage on the line goes up from left to right due to the delay in charging each inductor through the capacitors
- The last inductor, though, does not have a capacitor to charge
- Thus the last inductor is discharged ... the extra charge comes by discharging the last capacitor
- As this capacitor discharges, so does it's neighboring capacitor to the left
- Again there is a delay in discharging the caps due to the inductors
- This discharging represents the backward wave v_1^-

Transmission Line Menagerie

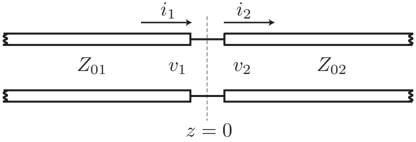


- T-Lines come in many shapes and sizes
- Coaxial usually 75Ω or 50Ω (cable TV, Internet)
- Microstrip lines are common on printed circuit boards (PCB) and integrated circuit (ICs)
- Coplanar also common on PCB and ICs
- Twisted pairs is almost a T-line, ubiquitous for phones/Ethernet

Waveguides and Transmission Lines

- The transmission lines we've been considering propagate the "TEM" mode or Transverse Electro-Magnetic. Later we'll see that they can also propagation other modes
- Waveguides cannot propagate TEM, but propage TM (Transverse Magnetic) and TE (Transverse Electric)
- In general, *any* set of more than one lossless conductors with uniform cross-section can transmit TEM waves. Low loss conductors are commonly approximated as lossless.

Cascade of T-Lines (I)



- Consider the junction between two transmission lines Z_{01} and Z_{02}
- At the interface $z = 0$, the boundary conditions are that the voltage/current has to be continuous

$$v_1^+ + v_1^- = v_2^+$$

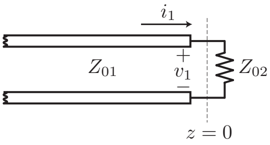
$$(v_1^+ - v_1^-)/Z_{01} = v_2^+/Z_{02}$$

Cascade of T-Lines (II)

- Solve these equations in terms of v_1^+
- The reflection coefficient has the same form (easy to remember)

$$\Gamma = \frac{v_1^-}{v_1^+} = \frac{Z_{02} - Z_{01}}{Z_{01} + Z_{02}}$$

- The second line looks like a load impedance of value Z_{02}



Transmission Coefficient

- The wave launched on the new transmission line at the interface is given by

$$v_2^+ = v_1^+ + v_1^- = v_1^+(1+\Gamma) = \tau v_1^+$$

- This "transmitted" wave has a coefficient

$$\tau = 1 + \Gamma = \frac{2Z_{02}}{Z_{01} + Z_{02}}$$

- Note the incoming wave carries a power

$$P_{in} = \frac{|v_1^+|^2}{2Z_{01}}$$

Conservation of Energy

- The reflected and transmitted waves likewise carry a power of

$$P_{ref} = \frac{|v_1^-|^2}{2Z_{01}} = |\Gamma|^2 \frac{|v_1^+|^2}{2Z_{01}}$$

$$P_{tran} = \frac{|v_2^+|^2}{2Z_{02}} = |\tau|^2 \frac{|v_1^+|^2}{2Z_{02}}$$

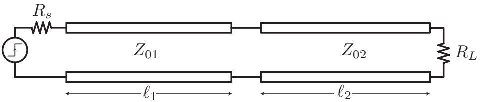
- By conservation of energy, it follows that

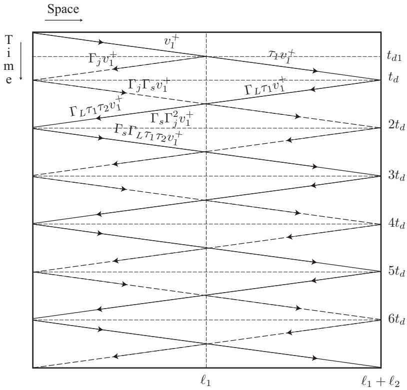
$$P_{in} = P_{ref} + P_{tran}$$

$$\frac{1}{Z_{02}}\tau^2 + \frac{1}{Z_{01}}\Gamma^2 = \frac{1}{Z_{01}}$$

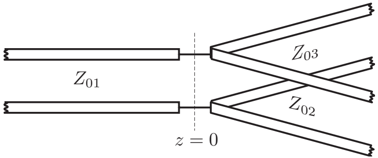
- You can verify that this relation holds!

- Consider the bounce diagram for the following arrangement





Junction of Parallel T-Lines



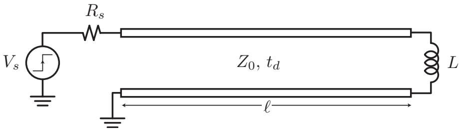
- Again invoke voltage/current continuity at the interface

$$v_1^+ + v_1^- = v_2^+ = v_3^+$$

$$\frac{v_1^+ - v_1^-}{Z_{01}} = \frac{v_2^+}{Z_{02}} + \frac{v_3^+}{Z_{02}}$$

- But $v_2^+ = v_3^+$, so the interface just looks like the case of two transmission lines Z_{01} and a new line with char. impedance $Z_{01}||Z_{02}$.

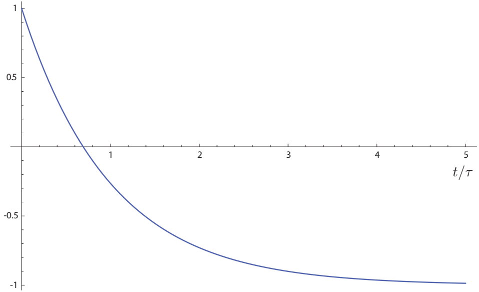
Reactive Terminations (I)



- Let's analyze the problem intuitively first
- When a pulse first "sees" the inductance at the load, it looks like an open so $\Gamma_0 = +1$
- As time progresses, the inductor looks more and more like a short! So $\Gamma_\infty = -1$

Reactive Terminations (II)

- So intuitively we might expect the reflection coefficient to look like this:



- The graph starts at $+1$ and ends at -1. In between we'll see that it goes through exponential decay (1st order ODE)

Reactive Terminations (III)

- Do equations confirm our intuition?

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}\left(\frac{v^+}{Z_0} - \frac{v^-}{Z_0}\right)$$

- And the voltage at the load is given by $v^+ + v^-$

$$v^- + \frac{L}{Z_0}\frac{dv^-}{dt} = \frac{L}{Z_0}\frac{dv^+}{dt} - v^+$$

- The right hand side is known, it's the incoming waveform

Solution for Reactive Term

- For the step response, the derivative term on the RHS is zero at the load

$$v^+ = \frac{Z_0}{Z_0 + R_s} V_s$$

- So we have a simpler case $\frac{dv^+}{dt} = 0$
- We must solve the following equation

$$v^- + \frac{L}{Z_0}\frac{dv^-}{dt} = -v^+$$

- For simplicity, assume at $t = 0$ the wave v^+ arrives at load

Laplace Domain Solution I

- In the Laplace domain

$$V^-(s) + \frac{sL}{Z_0}V^-(s) - \frac{L}{Z_0}v^-(0) = -v^+/s$$

- Solve for reflection $V^-(s)$

$$V^-(s) = \frac{v^-(0)L/Z_0}{1 + sL/Z_0} - \frac{v^+}{s(1 + sL/Z_0)}$$

- Break this into basic terms using partial fraction expansion

$$\frac{-1}{s(1 + sL/Z_0)} = \frac{-1}{1 + sL/Z_0} + \frac{L/Z_0}{1 + sL/Z_0}$$

Laplace Domain Solution (II)

- Invert the equations to get back to time domain $t > 0$

$$v^-(t) = (v^-(0) + v^+)e^{-t/\tau} - v^+$$

- Note that $v^-(0) = v^+$ since initially the inductor is an open
- So the reflection coefficient is

$$\Gamma(t) = 2e^{-t/\tau} - 1$$

- The reflection coefficient decays with time constant L/Z_0