EE 142 Problem Set 3

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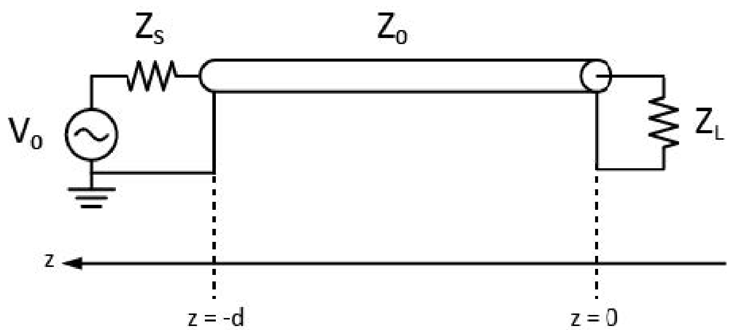
1 T-Lines at Steady State



Voltage source generates 10 Ghz sine with 10V amplitude.

Tline terminated with $Z_L = 80 - 40j\Omega$, and $Z_0 = 100\Omega$. $\epsilon_{eff} = 4$ and $d = 22.5$ mm.

1. Find the reflection coefficient at the load (z = 0) and at the source (z = -d)

   At the load:

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$
$$\rho_L = -0.0588 - 0.23539j$$
$$|\rho_L| = 0.242$$

   At any point on the line, we can derive an effective generalized $\rho(z)$ which represents the ratio of the backwards and forward traveling waves at a given point on the tline.

$$V(z) = V_0^+(e^{-j\beta z} + \rho_L e^{-j\beta z})$$
$$\rho(z) = \frac{V_0^+ \rho_L e^{j\beta z}}{V_0^+ e^{-j\beta z}}$$
$$\rho(z) = \rho_L e^{2j\beta z}$$

Notice that since $\beta = 2\pi/\lambda$, $\rho(z)$ repeats every $\lambda/2$ traversed along the tline back to the generator. We can find c_p and λ for this tline and frequency.

$$\begin{aligned}
c_p &= \frac{c_0}{\sqrt{\epsilon_{eff}}} \approx 1.5e8 \text{ m/s} \\
\lambda &= \frac{c_p}{f} = 0.015 \text{ m} \\
d/\lambda = 1.5 = 3 \cdot \frac{1}{2}\lambda
\end{aligned}$$

So, $\rho(z)$ at $z = -d$ is $rho_L = 0.242$.

2. Find the input impedance at the source (z = -d) and at z = 18.75mm.

   The general form is:

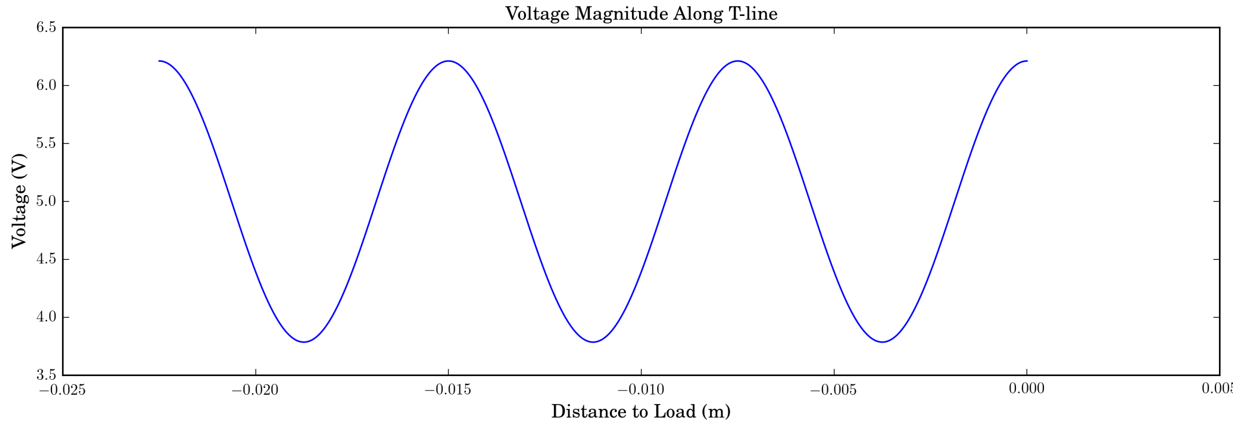
$$\begin{aligned}
Z_{in}(-l) &= Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \\
\beta &= \frac{2\pi}{\lambda} = 418.879 \\
Z_{in}(0) &= Z_L = 100\Omega \\
Z_{in}(-18.75 \text{ mm}) &= Z_{in}(\lambda + \lambda/4) = \frac{Z_0^2}{Z_L} = 100 + 50j
\end{aligned}$$

3. Plot the magnitude of the voltage along the line. Find voltage maximum, minimum, and SWR.

   We assume that $Z_S = Z_0$:

$$\begin{aligned}
SWR &= \frac{V_{max}}{V_{min}} = \frac{1 + |\rho_L|}{1 - |\rho_L|} \\
SWR &= 1.64 \\
V^+ &= \frac{Z_0}{Z_0 + Z_S} = 5 \text{ V} \\
V_{max} &= |V^+|(1 + |\rho_L|) = 6.2 \text{ V} \\
V_{min} &= |V^+|(1 - |\rho_L|) = 3.8 \text{ V}
\end{aligned}$$

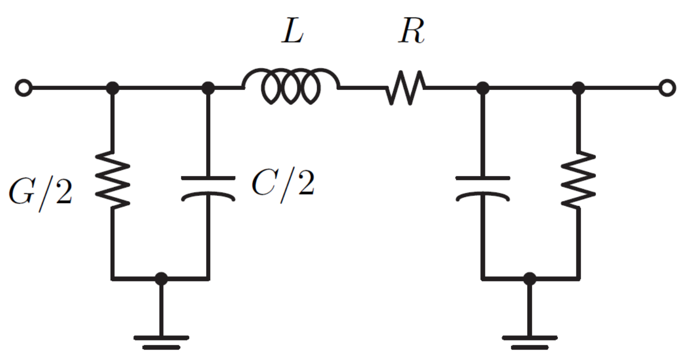
   Plot of voltage magnitude along line:



2 T-Line Modeling

We will derive an equivalent two-port circuit model for a short section of transmission line ($l << \lambda$) including loss.

1. For a "pi" equivalent circuit shown below, find the two-port Z matrix.



Let's call the current flowing *into* node 1 i_1 and the current flowing *into* node 2 i_2. The voltage applied across node is is v_1 and v_2 for node 2. We will call each section of the pi network Z_1, Z_2, Z_3 going left to right and $Z_1 = Z_3$.

$$Z_{11} = \frac{v_1}{i_1}\bigg|_{i_2=0} = (Z_1||(Z_2+Z_3)) = \frac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3}$$

$$Z_{22} = Z_{11} \text{ due to symmetry}$$

$$Z_{12} = \frac{v_1}{i_2}\bigg|_{i_1=0} = \frac{Z_1 Z_3}{Z_1+Z_2+Z_3}$$

$$Z_{21} = Z_{12} \text{ due to reciprocity}$$

$$Z_1 = Z_3 = \frac{2}{2j\omega C + G}$$
$$Z_2 = R + j\omega L$$

2. Consider a section of transmission line with loss. Find the two port Z matrix. Use the transmission line impedance equation (general expression with loss).

   We begin with the general form using ρ_L:

$$\begin{aligned}
Z_{in}(-l) &= \frac{V(-l)}{I(-l)} = Z_0 \frac{1 + \rho_L e^{-2\gamma l}}{1 - \rho_L e^{-2\gamma l}} \\
\rho_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\
Z_{in}(-l) &= Z_0 \frac{Z_L(1 + e^{-2\gamma l}) + Z_0(1 - e^{-2\gamma l})}{Z_0(1 + e^{-2\gamma l}) + Z_L(1 - e^{-2\gamma l})} \\
Z_{in}(-l) &= Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \\
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta
\end{aligned}$$

   Now we open and short the transmission line to measure its Z parameters.

$$\begin{aligned}
Z_{11} &= \left.\frac{v_1}{i_1}\right|_{i_2=0, Z_L=\infty} = Z_0 \frac{1}{\tanh(\gamma l)} \\
Z_{22} &= Z_{11} \text{ due to symmetry} \\
Z_{12} &= \left.\frac{v_1}{i_2}\right|_{i_1=0, Z_L=0} = Z_0 \frac{1}{\sinh(\gamma l)} \\
Z_{21} &= Z_{12} \text{ due to reciprocity}
\end{aligned}$$

   Keeping in mind that for a shorted tline, the voltage and current at a given point along the line are:

$$\begin{aligned}
v(z) &= V^+(e^{-\gamma z} + e^{\gamma z}) \\
i(z) &= \frac{V^+}{Z_0}(e^{-\gamma z} - e^{\gamma z})
\end{aligned}$$

3. Take the limit of a very short line and simplify the answer (Hint: use a Taylor series expansion and keep only the first few terms).

   Use the expansions:

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - ... \text{ for } |x| < \frac{\pi}{2}$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... \text{ for all x}$$

We plug the first two terms into the t-line's Z matrix for tanh and sinh. There isn't any other obvious simplification or limit taking to be done.

4. Using the previous results, now derive the values for L, R, C, and G for the equivalent circuit.

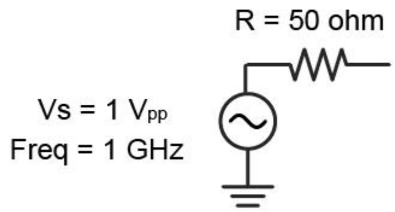
   I began this problem by equating the Z_{11} and Z_{12} parameters for the pi network and the transmission line. I also wrote down the equation relating γ and Z_0 with R,G,L,C.

```
%%
syms C w G R L
syms Z0 gamma l

Z1 = ((2/G)*(1/1i*w*C) / ((2/G) + (1/1i*w*C)));
%Z1 = 2 / (2*1i*w*C + G);
Z3 = Z1;
Z2 = R + 1i*w*L;

%%
Z_11 = (Z1*(Z2 + Z3)) / (Z1 + Z2 + Z3);
Z_12 = (Z1 * Z3) / (Z1 + Z2 + Z3);

Z_11_tline = Z0 / ((gamma*l - (gamma*l)^3 / 3 + 2*(gamma*l)^5 / 15));
Z_12_tline = Z0 / ((gamma*l + (gamma*l)^3 / 3*2 + (gamma*l)^5 / 5*4*3*2));

gamma_RLGC_relation = gamma == sqrt((R + 1i*w*L) * (G + 1i*w*C));
Z0_LC_relation = Z0 == sqrt((R + 1i*w*L)/(G + 1i*w*C));

sol = solve([Z_11 == Z_11_tline, Z_12 == Z_12_tline], [R, G, L, C]);
```

   I tried to give different permutations of these equations to the solver, but wasn't able to find any sensible solution for R, G, L, C in terms of Z_0, γ, and ω although I'm sure one exists.

3 Impedance Matching for Maximum Power Delivery

1. What is the maximum power that can be extracted from the source shown below? What is the optimal load impedance for the maximum power delivery to happen?

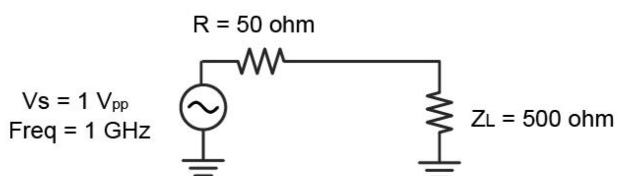


$$
\begin{aligned}
|I_s| &= \frac{|V_s|}{|R_s + R_L|} \\
I_{s,rms} &= \frac{1}{2}|I_s| \\
V_L &= I_{s,rms} R_L \\
P_L &= I_{s,rms} V_L = I_{s,rms}^2 R_L = 1/2(\frac{V_s}{R_s + R_L})^2 R_L \\
\frac{\partial P_L}{\partial R_L} &= (\frac{-R_s}{R_L})^2 + 1
\end{aligned}
$$

Setting the denominator of derivative to 0 and solving gives $R_L = \pm R_s \rightarrow R_L = R_s$. Indeed, this minimizes the denominator, and thus maximizes the power delivered to the load.

$$
P_{max} = \frac{V_s^2}{8R_L} = 2.5 \text{ mW}
$$

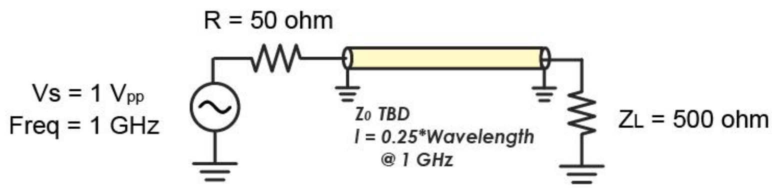
2. Use this source to drive a 500Ω load and we directly connect the load to the source, as illustrated by the figure below. What is the power delivered to the 500Ω load and the load voltage?



$$
\begin{aligned}
P_L &= 0.8 \text{ mW} \\
V_L = I_{s,rms} R_L &= 0.45 \text{ V}
\end{aligned}
$$

3. Let's try to achieve impedance matching by putting a quarter-wavelength transmission line between the load and source, as indicated by the below figure. Find the characteristic impedance

Z_0 that maximizes the power delivered to the load. What are the corresponding power and voltage at the load?



The general form of input impedance for a lossless tline is:

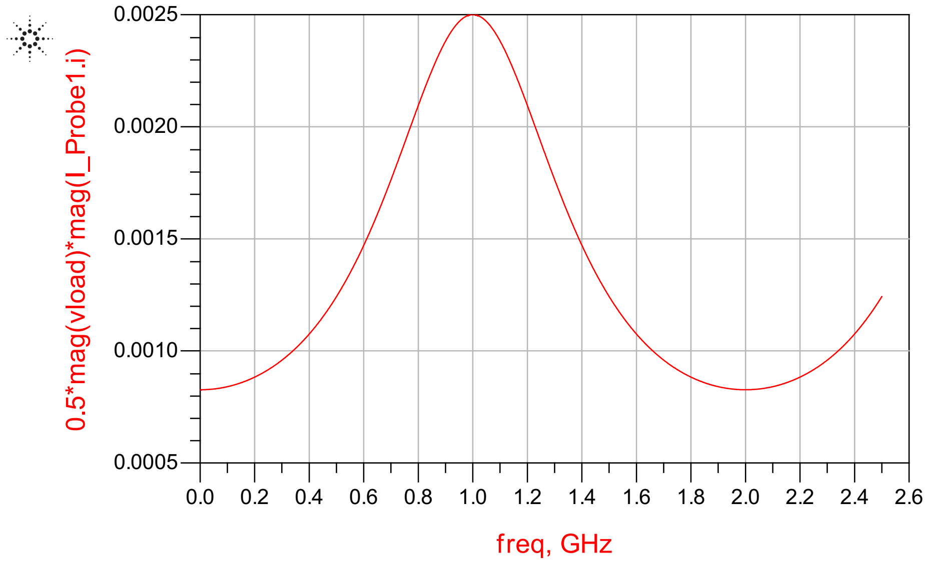
$$Z_{in}(-l) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$
$$Z_{in}(-\lambda/4) = \frac{Z_0^2}{Z_L} = 50 \rightarrow Z_0 = 158\Omega$$
$$P_L = P_{max} = 2.5mW \text{ since the tline is lossless}$$
$$V_L = \sqrt{P_L R_L} = 1.118 \text{ V}$$

4. Following part c) assume the source frequency can change, what is the frequency interval where the power delivered to the 500Ω load is less than 3 dB from the maximum value? Use ADS to verify.

   Assume that $\epsilon_{eff} = 4$. -3 dB power reduction is equivalent to the power halving. We fix the transmission line's length and Z_0 and sweep β to find where the effective resistance looking into the transmission line becomes sufficiently high to halve power.

$$P_L' = 1/2(\frac{1}{50 + R_L})^2 R_L = 2.5/2 \text{ mW}$$
$$\text{Solving: } R_L = 292\Omega$$
$$\beta = 76.1 \text{ to get an effective } R_L \text{ of } 292\Omega$$
$$\rightarrow f' = 1.8 \text{ Ghz}$$

   In ADS:



The simulation indicates a power halving at around 1.5 Ghz. I suspect my assumption about ϵ_{eff} was problematic in the hand calculation.