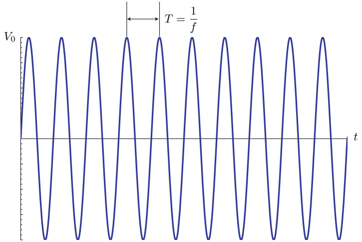
Sinusoidal Oscillators: Feedback Analysis

Prof. Ali M. Niknejad

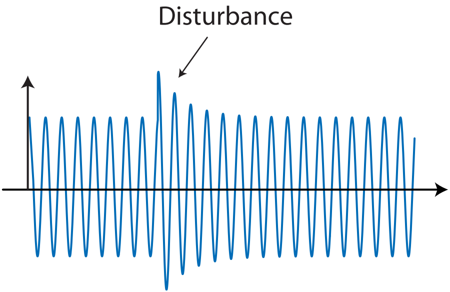
U.C. Berkeley

Oscillators



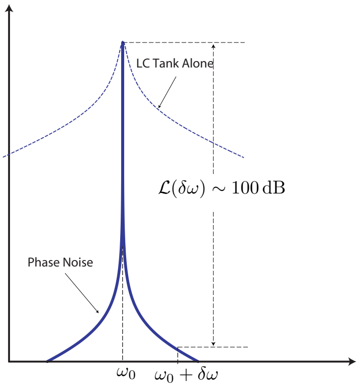
- An oscillator is an *autonomous* circuit that converts DC power into a periodic waveform. We will initially restrict our attention to a class of oscillators that generate a sinusoidal waveform.
- The period of oscillation is determined by a high-Q LC tank or a resonator (crystal, cavity, T-line, etc.). An oscillator is characterized by its oscillation amplitude (or power), frequency, "stability", phase noise, and tuning range.

Oscillators (cont)



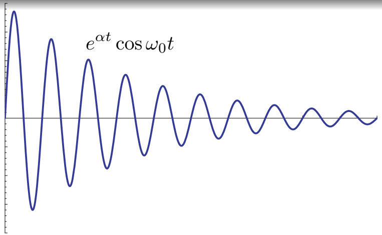
- Generically, a good oscillator is stable in that its frequency and amplitude of oscillation do not vary appreciably with temperature, process, power supply, and external disturbances.
- The amplitude of oscillation is particularly stable, always returning to the same value (even after a disturbance).

Phase Noise



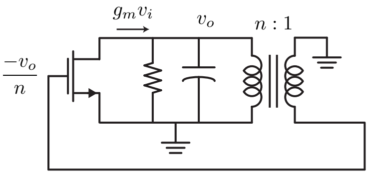
- Due to noise, a real oscillator does not have a delta-function power spectrum, but rather a very sharp peak at the oscillation frequency.
- The amplitude drops very quickly, though, as one moves away from the center frequency. E.g. a cell phone oscillator has a phase noise that is 100 dB down at an offset of only 0.01% from the carrier!

An LC Tank "Oscillator"



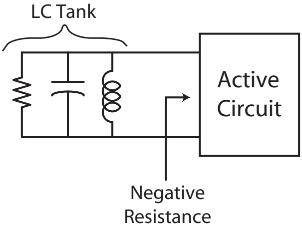
- Note that an LC tank alone is not a good oscillator. Due to loss, no matter how small, the amplitude of the oscillator decays.
- Even a very high Q oscillator can only sustain oscillations for about Q cycles. For instance, an LC tank at $1\mathrm{GHz}$ has a $Q \sim 20$, can only sustain oscillations for about $20\mathrm{ns}$.
- Even a resonator with high $Q \sim 10^6$, will only sustain oscillations for about $1\mathrm{ms}$.

Feedback Perspective



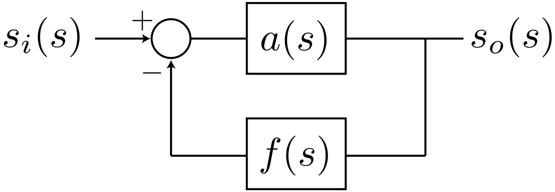
- Many oscillators can be viewed as feedback systems. The oscillation is sustained by feeding back a fraction of the output signal, using an amplifier to gain the signal, and then injecting the energy back into the tank. The transistor "pushes" the LC tank with just about enough energy to compensate for the loss.

Negative Resistance Perspective



- Another perspective is to view the active device as a negative resistance generator. In steady state, the losses in the tank due to conductance G are balanced by the power drawn from the active device through the negative conductance $-G$.

Feedback Approach



- Consider an ideal feedback system with forward gain $a(s)$ and feedback factor $f(s)$. The closed-loop transfer function is given by

$$H(s) = \frac{a(s)}{1 + a(s)f(s)}$$

Feedback Example

- As an example, consider a forward gain transfer function with three identical real negative poles with magnitude $|\omega_p| = 1/\tau$ and a frequency independent feedback factor f

$$a(s) = \frac{a_0}{(1 + s\tau)^3}$$

- Deriving the closed-loop gain, we have

$$H(s) = \frac{a_0}{(+s\tau)^3 + a_0 f} = \frac{K_1}{(1 - s/s_1)(1 - s/s_2)(1 - s/s_3)}$$

- where $s_{1,2,3}$ are the poles of the feedback amplifier.

Poles of Closed-Loop Gain

- Solving for the poles

$$(1+s\tau)^3 = -a_0 f$$

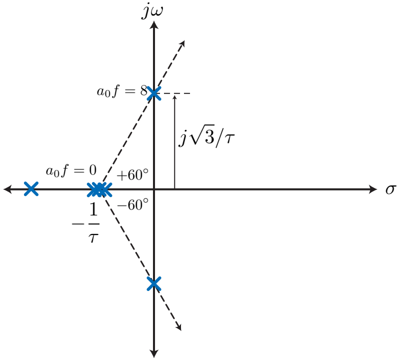
$$1+s\tau = (-a_0 f)^{\frac{1}{3}} = (a_0 f)^{\frac{1}{3}}(-1)^{\frac{1}{3}}$$

$$(-1)^{\frac{1}{3}} = -1,\ e^{j60^\circ},\ e^{-j60^\circ}$$

- The poles are therefore

$$s_1,\ s_2,\ s_3 = \frac{-1-(a_0 f)^{\frac{1}{3}}}{\tau},\ \frac{-1+(a_0 f)^{\frac{1}{3}} e^{\pm j60^\circ}}{\tau}$$

Root Locus



- If we plot the poles on the s-plane as a function of the DC loop gain $T_0 = a_0 f$ we generate a *root locus*
- For $a_0 f = 8$, the poles are on the $j\omega$-axis with value

$$s_1 = -3/\tau$$

$$s_{2,3} = \pm j\sqrt{3}/\tau$$

- For $a_0 f > 8$, the poles move into the right-half plane (RHP)

Natural Response

- Given a transfer function

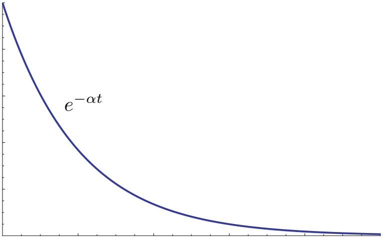
$$H(s) = \frac{K}{(s-s_1)(s-s_2)(s-s_3)} = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \frac{a_3}{s-s_3}$$

- The total response of the system can be partitioned into the *natural response* and the forced response

$$s_0(t) = f_1(a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}) + f_2(s_i(t))$$

- where $f_2(s_i(t))$ is the forced response whereas the first term $f_1()$ is the natural response of the system, even in the absence of the input signal. The natural response is determined by the initial conditions of the system.

Real LHP Poles



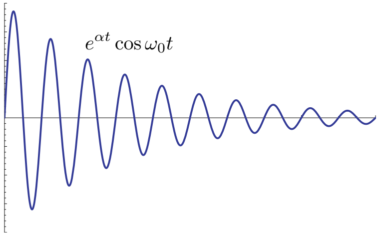
- Stable systems have all poles in the left-half plane (LHP).
- Consider the natural response when the pole is on the negative real axis, such as s_1 for our examples.
- The response is a decaying exponential that dies away with a time-constant determined by the pole magnitude.

Complex Conjugate LHP Poles

- Since $s_{2,3}$ are a complex conjugate pair

$$s_2,\ s_3 = \sigma \pm j\omega_0$$

- We can group these responses since $a_3 = \overline{a_2}$ into a single term

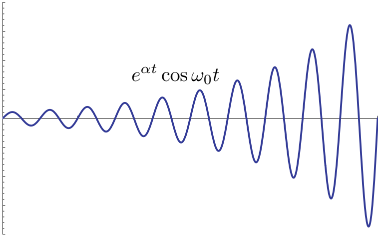


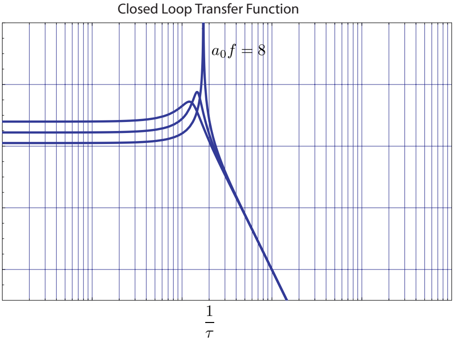
$$a_2 e^{s_2 t} + a_3 e^{s_3 t} = K_a e^{\sigma t}\cos\omega_0 t$$

- When the real part of the complex conjugate pair σ is negative, the response also decays exponentially.

Complex Conjugate Poles (RHP)

- When σ is positive (RHP), the response is an exponential growing oscillation at a frequency determined by the imaginary part ω_0
- Thus we see for any amplifier with three identical poles, if feedback is applied with loop gain $T_0 = a_0 f > 8$, the amplifier will oscillate.



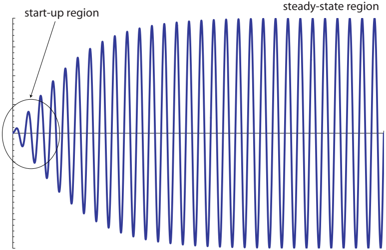


- In the frequency domain perspective, we see that a feedback amplifier has a transfer function

$$H(j\omega) = \frac{a(j\omega)}{1 + a(j\omega)f}$$

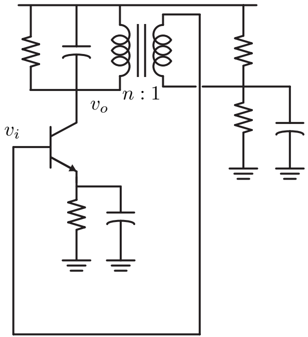
- If the loop gain $a_0 f = 8$, then we have with purely imaginary poles at a frequency $\omega_x = \sqrt{3}/\tau$ where the transfer function $a(j\omega_x)f = -1$ blows up. Apparently, the feedback amplifier has infinite gain at this frequency.

Oscillation Build Up



- In a real oscillator, the amplitude of oscillation initially grows exponentially as our linear system theory predicts. This is expected since the oscillator amplitude is initially very small and such theory is applicable. But as the oscillations become more vigorous, the non-linearity of the system comes into play.
- We will analyze the steady-state behavior, where the system is non-linear but periodically time-varying.

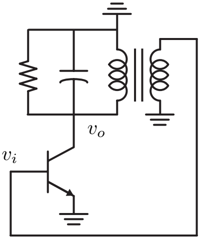
Example LC Oscillator

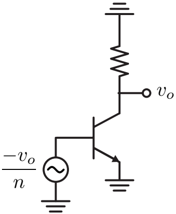


- The emitter resistor is bypassed by a large capacitor at AC frequencies.
- The base of the transistor is conveniently biased through the transformer windings.

- The LC oscillator uses a transformer for feedback. Since the amplifier has a phase shift of 180°, the feedback transformer needs to provide an additional phase shift of 180° to provide positive feedback.

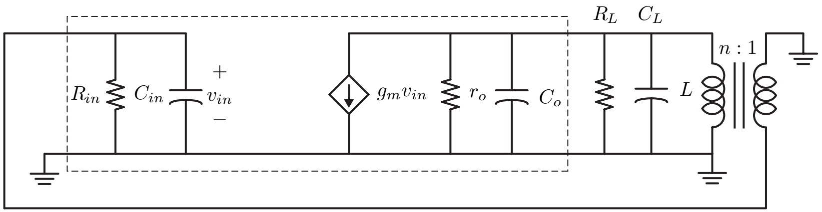
AC Equivalent Circuit





- At resonance, the AC equivalent circuit can be simplified. The transformer winding inductance L resonates with the total capacitance in the circuit. R_T is the equivalent tank impedance at resonance.

Small Signal Equivalent Circuit



- The forward gain is given by $a(s) = -g_m Z_T(s)$, where the tank impedance Z_T includes the loading effects from the input of the transistor

$$R = R_0 || R_L || n^2 R_i$$

$$C = C_L + \frac{C_i}{n^2}$$

Open-Loop Transfer Function

- The tank impedance is therefore

$$Z_T(s) = \frac{1}{sC + \frac{1}{R} + \frac{1}{Ls}} = \frac{Ls}{1 + s^2LC + sL/R}$$

- The loop gain is given by

$$af(s) = \frac{-g_m R}{n} \frac{\frac{L}{R}s}{1 + \frac{L}{R}s + s^2LC}$$

- The loop gain at resonance is the same as the DC loop gain

$$A_\ell = \frac{-g_m R}{n}$$

Closed-Loop Transfer Function

- The closed-loop transfer function is given by

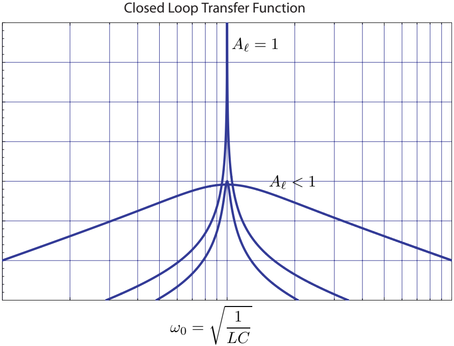
$$H(s) = \frac{-g_m R \frac{L}{R} s}{1 + s^2 LC + s\frac{L}{R}(1 - \frac{g_m R}{n})}$$

- Where the denominator can be written as a function of A_ℓ

$$H(s) = \frac{-g_m R \frac{L}{R} s}{1 + s^2 LC + s\frac{L}{R}(1 - A_\ell)}$$

- Note that as $n \to \infty$, the feedback loop is broken and we have a tuned amplifier. The pole locations are determined by the tank Q.

Oscillator Closed-Loop Gain vs A_ℓ



- If $A_\ell = 1$, then the denominator loss term cancels out and we have two complex conjugate imaginary axis poles

$$1 + s^2 LC = (1 + sj\sqrt{LC})(1 - sj\sqrt{LC})$$

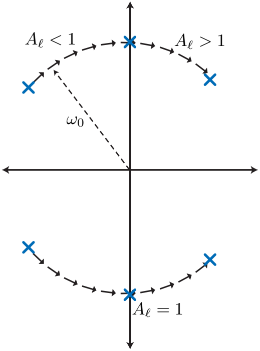
Root Locus for LC Oscillator

- For a second order transfer function, notice that the magnitude of the poles is constant, so they lie on a circle in the s-plane

$$s_1, s_2 = \frac{-a}{2b} \pm \frac{a}{2b}\sqrt{1 - \frac{4b}{a^2}} = \frac{-a}{2b} \pm j\frac{a}{2b}\sqrt{\frac{4b}{a^2} - 1}$$

$$|s_{1,2}| = \sqrt{\frac{a^2}{4b^2} + \frac{a^2}{4b^2}(\frac{4b}{a^2} + 1)} = \sqrt{\frac{1}{b}} = \omega_0$$

Root Locus (cont)

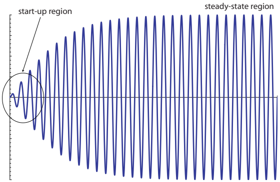


- We see that for $A_\ell = 0$, the poles are determined by the tank Q and lie in the LHP. As A_ℓ is increased, the action of the positive feedback is to boost the gain of the amplifier and to decrease the bandwidth. Eventually, as $A_\ell = 1$, the loop gain becomes infinite in magnitude.

Review: Role of Loop Gain

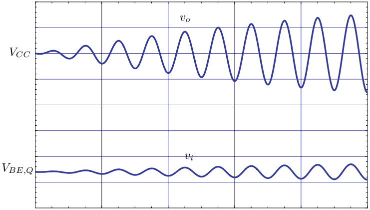
- The behavior of the circuit is determined largely by A_ℓ, the loop gain at DC and resonance. When $A_\ell = 1$, the poles of the system are on the $j\omega$ axis, corresponding to constant amplitude oscillation.
- When $A_\ell < 1$, the circuit oscillates but decays to a quiescent steady-state.
- When $A_\ell > 1$, the circuit begins oscillating with an amplitude which grows exponentially. Eventually, we find that the steady state amplitude is fixed.

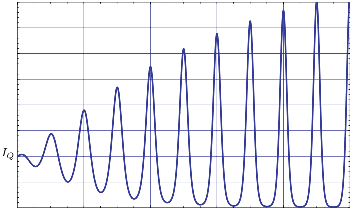
Steady-State Analysis



- To find the steady-state behavior of the circuit, we will make several simplifying assumptions. The most important assumption is the high tank Q assumption (say $Q > 10$), which implies the output waveform v_o is sinusoidal.
- Since the feedback network is linear, the input waveform $v_i = v_o/n$ is also sinusoidal.
- We may therefore apply the large-signal periodic steady-state analysis of the BJT to the oscillator.

Steady-State Waveforms





- The collector current is not sinusoidal, due to the large signal drive.
- The output voltage,though, is sinusoidal and given by

$$v_o \approx I_{\omega_1} Z_T(\omega_1) = G_m Z_T v_i$$

Steady State Equations

- But the input waveform is a scaled version of the output

$$v_o = G_m Z_T \frac{v_o}{n} = \frac{G_m Z_T}{n} v_o$$

- The above equation implies that

$$\frac{G_m Z_T}{n} \equiv 1$$

- Or that the loop gain in steady-state is unity and the phase of the loop gain is zero degrees (an exact multiple of 2π)

$$\left| \frac{G_m Z_T}{n} \right| \equiv 1 \qquad\qquad \angle \frac{G_m Z_T}{n} \equiv 0^\circ$$

Large Signal G_m

- Recall that the small-signal loop gain is given by

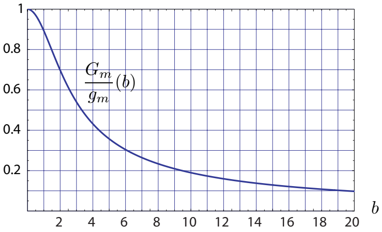
$$|A_\ell| = \left| \frac{g_m Z_T}{n} \right|$$

- Which implies a relation between the small-signal start-up transconductance and the steady-state large-signal transconductance

$$\left| \frac{g_m}{G_m} \right| = A_\ell$$

- Notice that g_m and A_ℓ are design parameters under our control, set by the choice of bias current and tank Q. The steady state G_m is therefore also fixed by initial start-up conditions.

Large Signal G_m (II)



- To find the oscillation amplitude we need to find the input voltage drive to produce G_m.
- For a BJT, we found that under the constraint that the bias current is fixed

$$I_{\omega_1} = \frac{2I_1(b)}{I_0(b)} I_Q = G_m v_i = G_m b \frac{kT}{q}$$

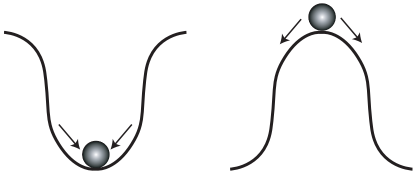
Large Signal G_m (III)

- Thus the large-signal G_m is given by

$$G_m = \frac{2I_1(b)}{bI_0(b)}\frac{qI_Q}{kT} = \frac{2I_1(b)}{bI_0(b)} g_m$$

$$\frac{G_m}{g_m} = \frac{2I_1(b)}{bI_0(b)}$$

Stability (Intuition)



- Here's an intuitive argument for how the oscillator reaches a stable oscillation amplitude. Assume that initially $A_l > 1$ and oscillations grow. As the amplitude of oscillation increases, though, the G_m of the transistor drops, and so effectively the loop gain drops.
- As the loop gain drops, the poles move closer to the $j\omega$ axis. This process continues until the poles hit the $j\omega$ axis, after which the oscillation ensues at a constant amplitude and $A_\ell = 1$.

Intuition (cont)

- To see how this is a stable point, consider what happens if somehow the loop gain changes. If the loop gain changes to $A_\ell + |\epsilon|$, then we already see that the system will roll back. If the loop gain drops below unity, $A_\ell - |\epsilon|$, then the poles move into the LHP and amplitude of oscillation will begin to decay.
- As the oscillation amplitude decays, the G_m increases and this causes the loop gain to grow. Thus the system also rolls back to the point where $A_\ell = 1$.

BJT Oscillator Design

- Say we desire an oscillation amplitude of $v_0 = 100\mathrm{mV}$ at a certain oscillation frequency ω_0.
- We begin by selecting a loop gain $A_\ell > 1$ with sufficient margin. Say $A_\ell = 3$.
- We also tune the LC tank to ω_0, being careful to include the loaded effects of the transistor (r_o, C_o, C_{in}, R_{in})
- We can estimate the required first harmonic current from

$$I_{\omega_0} = \frac{v_o}{R_T'}$$

Design (cont)

- This is an estimate because the exact value of R_T is not known until we specify the operating point of the transistor. But a good first order estimate is to neglect the loading and use R'_T
- We can now solve for the bias point from

$$I_{\omega_1} = \frac{2I_1(b)}{I_0(b)} I_Q$$

- b is known since it's the oscillation amplitude normalized to kT/q and divided by n. The above equation can be solved graphically or numerically.
- Once I_Q is known, we can now calculate R''_T and iterate until the bias current converges to the final value.