NOTES FOR PROBABILITY THEORY

AARON NOTES SERIES

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1 Preface

My major in Electronic Engineering does not put enough focus on the students' mathematical basis. It's somewhat reasonable for some elementary courses like mathematical analysis and linear algebra but that's not accepatble for probability theory and stochastic process for they are the most important two courses for an EE researcher. Therefore, in order to compensate the weak basis of my probability theory and stochastic process, I decide to study comprehensively by myself. The following sections are mostly summarized from [1] and partly cited from other publications which will be presented in the specific position.

2 Basic Measure Theory

Definition 2.1 (σ-algebra) *A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a **σ-algebra** if it fulfils the following three conditions:*

- $\Omega \in \mathcal{A}$
- *$\mathcal{A}$ is closed under complements.*
- *$\mathcal{A}$ is closed under countable unions.*

Definition 2.2 (algebra) *A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called an **algebra** if it fulfils the following three conditions:*

- $\Omega \in \mathcal{A}$
- *$\mathcal{A}$ is $\setminus$-closed.*
- *$\mathcal{A}$ is $\cup$-closed.*

Theorem 2.1 *A class of sets $\mathcal{A} \subset 2^{\Omega}$ is an **algebra** iff. the following three conditions hold:*

- $\Omega \in \mathcal{A}$
- *$\mathcal{A}$ is closed under complements.*
- *$\mathcal{A}$ is closed under intersections.*

Definition 2.3 (ring) *A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a **ring** if it fulfils the following three conditions:*

- $\emptyset \in \mathcal{A}$
- *$\mathcal{A}$ is $\setminus$-closed.*
- *$\mathcal{A}$ is $\cup$-closed.*

A ring is called a σ-ring if it is also σ-$\cup$-closed.

Definition 2.4 (semiring) *A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a **semiring** if it fulfils the following three conditions:*

- $\emptyset \in \mathcal{A}$
- *for any two sets $A, B \in \mathcal{A}$ the difference set $B \backslash A$ is a finite union of mutually disjoint sets in $\mathcal{A}$.*
- *$\mathcal{A}$ is $\cap$-closed.*

Definition 2.5 (λ-system) *A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a **λ-system** (or Dynkin's λ-system) if*

- $\Omega \in \mathcal{A}$
- *for any two sets $A, B \in \mathcal{A}$ with $A \subset B$, the difference set $B \backslash A$ is in $\mathcal{A}$*
- $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ *for any choice of countably many pairwise disjoint sets $A_1, A_2, \dots \in \mathcal{A}$*

Definition 2.6 (liminf and limsup) *Let $A_1, A_2, \dots$ be subsets of Ω. The sets*

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \quad \textit{and} \quad \limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

*are called **limes inferior** and **limes superior**, respectively, of the sequence $(A_n)_{n\in\mathbb{N}}$.*

Definition 2.7 (Topology) *Let $\Omega \neq \emptyset$ be an arbitrary set. A class of sets $\tau \subset \Omega$ is called a* ***topology*** *on Ω if it has the following three properties:*

- $\emptyset, \Omega \in \tau$
- *τ is $\cap$-closed*
- *$\bigcup_{A \in \mathcal{F}} A \in \tau$ for any $\mathcal{F} \subset \tau$.*

The pair (Ω, τ) is called a topological space. The sets $A \in \tau$ are called open, and the sets $A \subset \Omega$ with $A^c \in \tau$ are called closed.

Definition 2.8 (Borel σ-algebra) *Let (Ω, τ) be a topological space. The σ-algebra*

$$\mathcal{B}(\Omega) := \mathcal{B}(\Omega, \tau) := \sigma(\tau)$$

that is generated by the open sets is called the ***Borel σ-algebra*** *on Ω. The elements $A \in \mathcal{B}(\Omega, \tau)$ are called* ***Borel sets*** *or* ***Borel measurable sets****.*

Definition 2.9 *Let $\mathcal{A} \subset 2^{\Omega}$ and let $\mu : \mathcal{A} \to [0, \infty]$ be a set function. We say that μ is*

1. ***monotone*** *if $\mu(A) \leqslant \mu(B)$ for any two sets $A, B \in \mathcal{A}$ with $A \subset B$,*
2. ***additive*** *if $\mu(\bigcup\limits_{i=1}^{n})A_i = \sum\limits_{i=1}^{n} \mu(A_i)$ for any choice of* finitely *many mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ with $A_i \in \mathcal{A}$,*
3. ***σ-additive*** *if $\mu(\bigcup\limits_{i=1}^{n})A_i = \sum\limits_{i=1}^{n} \mu(A_i)$ for any choice of* countably *many mutually disjoint sets $A_1, A_2, \dots \in \mathcal{A}$ with $A_i \in \mathcal{A}$,*
4. ***subadditive*** *if for any choice of* finitely *many sets $A, A_1, \ldots, A_n \in \mathcal{A}$ with $A \subset \bigcup\limits_{i=1}^{n} A_i$, we have $\mu(A) \leqslant \sum\limits_{i=1}^{n} A_i$, and*
5. ***σ-subadditive*** *if for any choice of* countably *many sets $A, A_1, A_2, \dots \in \mathcal{A}$ with $A \subset \bigcup\limits_{i=1}^{n} A_i$, we have $\mu(A) \leqslant \sum\limits_{i=1}^{\infty} \mu(A_i)$.*

Definition 2.10 *Let $\mathcal{A}$ be a semiring (as definition 2.4 indicates) and let $\mu : \mathcal{A} \to [0, \infty)$ be a set function with $\mu(\emptyset) = 0$. μ is called*

1. ***content*** *if μ is additive,*
2. ***premeasure*** *if μ is σ-additive,*
3. ***measure*** *if μ is a premeasure and $\mathcal{A}$ is a σ-algebra, and*
4. ***probability measure*** *if μ is a measure and $\mu(\Omega) = 1$.*

Definition 2.11 *Let $\mathcal{A}$ be a semiring. A content μ on $\mathcal{A}$ is called*

1. ***finite*** *if $\mu(A) < \infty$ for every $A \in \mathcal{A}$ and*
2. ***σ-finite*** *if there exists a sequence of sets $\Omega_1, \Omega_2, \dots \in \mathcal{A}$ such that $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$ and such that $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$.*

Definition 2.12 (weight function) *Let Ω be an (at most) countable nonempty set and let $\mathcal{A} = 2^{\Omega}$. Further, let $(p_\omega)_{\omega \in \Omega}$ be nonnegative numbers. Then $A \mapsto \mu(A) := \sum_{\omega \in A} p_\omega$ defines a σ-finite measure on 2^{Ω}. We call $p = (p_\omega)_{\omega \in \Omega}$ the weight function of μ. The number p_ω is called the weight of μ at point ω.*

Corollary 2.1 (probability vector) *If $\sum_{\omega \in \Omega} p_\omega = 1$, then μ is a probability measure. In this case, we interpret p_ω as the probability of the elementary event ω. The vector $p = (p_\omega)_{\omega \in \Omega}$ is called a* ***probability vector****.*

References

[1] A. KLENKE, *Probability theory: a comprehensive course*, Springer Science & Business Media, 2013.