Notes for Game Theory - Auction Part

Aaron Notes Series

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1 The standard auciton type

Four basic types of auction are widely discussed and analyzed.

1. In the ascending auction, the price is successively raised until only one bidder remains, and that bidder wins the object at the final price.
2. The descending auction works in exactly the opposite way: the auctioneer starts at a very high price, and then lowers the price continuously. The first bidder who calls out that she will accept the current price wins the object at that price.
3. In the first-price sealed-bid auction each bidder independently submits a single bid, without seeing others' bids, and the object is sold to the bidder who makes the highest bid. The winner pays her bid (that is, the price is the highest or 'first' price bid).
4. In the second-price sealed-bid auction, also, each bidder independently submits a single bid, without seeing others' bids, and the object is sold to the bidder who makes the highest bid. However, the price she pays is the second-highest bidder's bid, or 'second price'.

2 The basic models of auciton

A key feature of auctions is the presence of asymmetric information.

In the basic *private-value* model each bidder knows how much she values the object(s) for sale, but her value is private information to herself. In the pure *common-value* model, by contrast, the actual value is the same for everyone, but bidders have different private information about what that value actually is. A general model encompassing both these as special cases assumes each bidder receives a private information signal, but allows each bidder's value to be a general function of all the signals.

3 Bidding

A key feature of bidding in auctions with common-values components is the *winner's curse*: each bidder must recognize that she wins the object only when she has the highest signal (in symmetric equilibrium).

4 Optimal auction design

We note that in this case, there is a strong assumption that the bidders' value estiamte are stochastically independent.

4.1 Basic definitions and assumptions

We assume there is one seller who has one single object to sell. He faces n bidders numbered $1, 2, \ldots, n$. We let $\mathcal{N}$ represent the set of bidders, so that

$$\mathcal{N} = \{1, \ldots, n\}. \tag{4.1.1}$$

For each bidder i, there is some quantity t_i which is i's *value estimate* for the object, which represents the maximum amount which i would be willing to pay for the object given his current information about it.

We shall assume the seller's uncertainty about the value estimate of bidder i can be described by a continuous probability distribution over a finite interval. Specifically, we let a_i represent the lowest possible value which i might assign to the object; we let b_i represent the highest possible value which i might assign to the object; and we let $f : [a_i, b_i] \to R$ be the probability density function for i's value estimate t_i. We assume that $-\infty < a_i < b_i < +\infty$; $f_i(t_i) > 0, \forall t_i \in [a_i, b_i]$; and $f_i(\cdot)$ is a continuous function on $[a_i, b_i]$. $F_i : [a_i, b_i] \to [0, 1]$ denotes the cumulative distribution function corresponding to the density $f_i(\cdot)$, so that

$$F_i(t_i) = \int_{a_i}^{t_i} f_i(s_i) ds_i. \tag{4.1.2}$$

Thus $F_i(t_i)$ is the seller's assessment of the probability that bidder i has a value estimate of t_i or less.

We use T to denote the set of all possible combinations of bidders' value estimate. That is,

$$T = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \tag{4.1.3}$$

Similarly,

$$T_{-i} = \bigtimes_{\substack{j \in \mathcal{N} \\ j \neq i}} [a_j, b_j]. \tag{4.1.4}$$

At first, we assume ill assume that the value estimates of the n bidders are stochastically independent random variables. That is,

$$f(t) = \prod_{j \in \mathcal{N}} f_j(t_j). \tag{4.1.5}$$

Of course, bidder i considers his own value estimate to be a known quantity, not a random variable. However, we assume that bidder i assesses the probability distributions for the other bidders' value estimates in the same way as the seller does. That is, both the seller and bidder i assess the joint density function on T_i for the vector $t_{-i} = (t_1, ., t_{i-1}, t_{i+1}, ..., t_n)$ of values for all bidders other than i to be

$$f_{-i}(t_{-i}) = \prod_{\substack{j \in \mathcal{N} \\ j \neq i}} f_j(t_j). \tag{4.1.6}$$

The seller's personal value estimate for the object, if he were to keep it and not sell it to any of the n bidders, will be denoted by t_0. We assume that the seller has no private information about the object, so that t_0 is known.

Remark 4.1.1 *There are two general reasons why one bidder's value estimates may be unknown to the seller and the other bidders. First, the bidder's personal preferences might be unknown to the other agents (for example, if the object is a painting, the others might not know how much he really enjoys looking at the painting). Second, the bidder might have some special information about the intrinsic quality of the object (he might know if the painting is an old master or a copy). We may refer to these two factors as* preference uncertainty *and* quality uncertainty. *This distinction is very important. If there are only preference uncertainties, then informing bidder i about bidder j's value estimate should not cause i to revise his valuation. (This does not mean that i might not revise his bidding strategy in an auction if he knew j's value estimate; this means only that i's honest preferences for having money versus having the object should not change.) However, if there are quality uncertainties, then bidder i might tend to revise his valuation of the object after learning about other bidders' value estimates. That is, if i learned that t_j was very low, suggesting that j had received discouraging information about the quality of the object, then i might honestly revise downward his assessment of how much he should be willing to pay for the object.*

We shall assume that there exist n revision effect functions $e_j : [a_i, b_i] \to \mathbb{R}$ such that, if another bidder i learned that t_j was j's value estimate for the object, then i would revise his own valuation by $e_j(t_j)$. Thus, if bidder i learned that

$t = (t_1, \ldots, t_n)$ was the vector of value estimates initially held by the n bidders, then i would revise his own valuation of the object to

$$v_i(t) = t_i + \sum_{-i} e_j(t_j). \tag{4.1.7}$$

Similarly, we shall assume that the seller would reassess his personal valuation of the object to

$$v_0(t) = t_0 + \sum_{j\in\mathcal{N}} e_j(t_j) \tag{4.1.8}$$

if he learned that t was the vector of value estimates initially held by the bidders. In the case of pure preference uncertainty, we would simply have $e_j(t_j) \equiv 0$.

4.2 Feasible auction mechanism

To begin, we shall restrict our attention to a special class of auction mechanisms: *the direct revelation mechanisms*.

In a direct revelation mechanism, the bidders simultaneously and confidentially announce their value estimates to the seller; and the seller then determines who gets the object and how much each bidder must pay, as some functions of the vector of announced value estimates $t = (t_1, \ldots, t_n)$. Thus, a direct revelation mechanism is described by a pair of outcome functions (p, x) (of the form $p : T \to \mathbb{R}^n$ and $x : T \to \mathbb{R}^n$) such that, if t is the vector of announced value estimates then $p_i(t)$ is the probability that i gets the object and $x_i(t)$ is the expected amount of money which bidder i must pay to the seller. (Notice that we allow for the possibility that a bidder might have to pay something even if he does not get the object.)

We shall assume that the seller and the bidders are *risk neutral and have additively separable utility functions* for money and the object being sold. Thus, if bidder i knows that his value estimate is t_i, then his expected utility from an auction mechanism described by (p, x) is

$$U_i(p, x, t_i) = \int_{T_{-i}} (v_i(t)p_i(t) - x_i(t)) f_{-i}(t_{-i}) dt_{-i} \tag{4.2.1}$$

where $dt_{-i} = dt_1 \ldots dt_{i-1} dt_{i+1} \ldots dt_n$.

Similarly, the expected utility for the seller from this auciton mechanism is

$$U_0(p, x) = \int_T \left(v_0(t) \left(1 - \sum_{j\in\mathcal{N}} p_j(t) \right) + \sum_{j\in\mathcal{N}} x_j(t) \right) f(t) dt \tag{4.2.2}$$

where $dt = dt_1 dt_2 \ldots dt_n$.

Not every pair of functions (p, x) represents a feasible auction mechanism however. There are three types of constraints which must be imposed on (p, x).

1. Since there is only one object to be allocated, the function p must satisfy the following probability conditions:
$$\sum_{j\in\mathcal{N}} p_j(t) \leqslant 1 \quad \text{and} \quad p_i(t) \geqslant 0, \quad \forall i \in \mathcal{N}, \quad \forall t \in T. \tag{4.2.3}$$
2. We assume that the seller cannot force a bidder to participate in an auction which offers him less expected utility then he could get on his own. If he did not participate in the auction, the bidder could not get the object, but also would not pay any money, so his utility payoff would be zero. Thus, to guarantee that the bidders will participate in the auction, the following *individual-rationality* conditions must be satisfied:
$$U_i(p, x, t_i) \geqslant 0, \quad \forall i \in \mathcal{N}, \quad \forall t_i \in [a_i, b_i] \tag{4.2.4}$$
3. We assume that the seller could not prevent any bidder from lying about his value estimate, if the bidder expected to gain from lying. Thus the revelation mechanism can be implemented only if no bidder ever expects to gain from lying. That is, honest responses must form a Nash equilibrium in the auction game. If bidder i claimed that s_i was his value estimate when t_i was his true value estimate, then his expected utility would be
$$\int_{T_{-i}} (v_i(t)p_i(t_{-i}, s_i) - x_i(t_{-i}, s_i)) f_{-i}(t_{-i}) dt_{-i},$$
where $(t_{-i}, s_i) = (t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)$. Thus, to guarantee that no bidder has any incentive to lie about his value estimate, the following *incentive-compatibility* conditions must be satisfied:
$$U_i(p, x, t_i) \geqslant \int_{T_{-i}} (v_i(t)p_i(t_{-i}, s_i) - x_i(t_{-i}, s_i)) f_{-i}(t_{-i}) dt_{-i}, \forall i \in \mathcal{N}, \forall t_i \in [a_i, b_i], \forall s_i \in [a_i, b_i] \tag{4.2.5}$$

We say that (p, x) is *feasible* (or that (p, x) represents a feasible auction mechanism) iff (4.2.3), (4.2.4), and (4.2.5) are all satisfied. That is, if the seller plans to allocate the object according to p and to demand monetary payments from bidders according to x, then the scheme can be implemented, with all bidders willing to participate honestly, if and only if (4.2.3)-(4.2.5) are satisfied.

Next we continue to discuss a general *auction game*. In a general auction game, each bidder has some set of strategy options Θ_i; and there are out functions

$$\hat{p} : \Theta_1 \times \cdots \times \Theta_n \to \mathbb{R}^n \quad \text{and} \quad \hat{x} : \Theta_1 \times \cdots \times \Theta_n \to \mathbb{R}^n, \tag{4.2.6}$$

which describes how the allocation of the object and the bidders' fees depend on the bidders' strategies.

An *auction mechanism* is any such auction game together with a description of the strategic plans which the bidders are expected to use in playing the game. Formally, a *strategic plan* can be represented by a function $\hat{\theta}_i : [a_i, b_i] \to \Theta_i$, such that $\hat{\theta}_i(t_i)$ is the strategy which i is expected to use in the auction game if his value estimate is t_i. In this general notation, our direct revelation mechanisms are simply those auction mechanisms in which $\Theta_i = [a_i, b_i]$ and $\hat{\theta}_i(t_i) = t_i$.

In this general framework, a feasible auction mechanism must satisfy constraints which generalize (4.2.3)-(4.2.5).

A vital insight: It might seem that problem of optimal auction design must be quite unmanageable, because there is no bound on the size or complexity of the strategy spaces Θ_i which the seller may use in constructing the auction game. The basic insight which enables us to solve auction design problems is that *there is really no loss of generality in considering only direct revelation mechanisms*.

Lemma 4.2.1 (The Revelation Principal) *Given any feasible auction mechanism, there exists an equivalent feasible direct revelation mechanism which gives to the seller and all bidders the same expected utilities as in the givn mechanism.*

The proof of a more general case is discussed as Theorem 2 in [1].

4.3 Analysis of the problem

Given an auction mechanism (p, x) we define

$$Q_i(p, t_i) = \int_{T_{-i}} p_i(t) f_{-i}(t_{-i}) dt_{-i} \tag{4.3.1}$$

for any bidder i and any value estimate t_i. So $Q_i(p, t_i)$ is the conditional probability that bidder i will get the object from the auction mechanism (p, x) given that his value estimate is t_i.

Lemma 4.3.1 *(p, x) is feasible iff the following conditions hold:*

$$s_i \leqslant t_i \Longrightarrow Q_i(p, s_i) \leqslant Q_i(p, t_i), \quad \forall i \in \mathcal{N}, \quad \forall s_i, t_i \in [a_i, b_i]; \tag{4.3.2a}$$

$$U_i(p, x, t_i) = U_i(p, x, a_i) + \int_{a_i}^{t_i} Q_i(P, s_i) ds_i, \quad \forall i \in \mathcal{N}, \quad \forall t \in T; \tag{4.3.2b}$$

$$U_i(p, x, a_i) \geqslant 0, \quad \forall i \in \mathcal{N}; \tag{4.3.2c}$$

$$\sum_{i \in \mathcal{N}} p_i(t) \leqslant 1 \quad \textit{and} \quad p_i(t) \geqslant 0, \quad \forall i \in \mathcal{N}, \quad \forall t \in T. \tag{4.3.2d}$$

The proof is provided as Lemma 2 in [2].

Lemma 4.3.2 *Suppose that $p : T \to \mathbb{R}^n$ maximizes*

$$\int_T \left(\sum_{i \in \mathcal{N}} \left(t_i - e_i(t_i) - \frac{1 - F_i(t_i)}{f_i(t_i)} - t_0 \right) p_i(t) \right) f(t) dt \tag{4.3.3}$$

subject to the constraints (4.3.2b) and (4.3.2d). Suppose also that

$$x_i(t) = p_i(t) v_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i) ds_i, \quad \forall i \in \mathcal{N}, \quad \forall t \in T. \tag{4.3.4}$$

Then (p, x) represents an optimal auction.

The proof is provided as Lemma 3 in [2].

Corollary 4.3.1 (The Revenue-Equivalence Theorem) *The seller's expected utility from a feasible auction mechanism is completely determined by the probability function p and the numbers $U_i(p, x, a_i)$ for all i.*

That is, once we know who gets the object in each possible situation (as specified by p) and how much expected utility each bidder would get if his value estimate were at its lowest possible level a_i, then the seller's expected utility from the auction does not depend on the payment function x. Thus, for example, the seller must get the same expected utility from any two auction mechanisms which have the properties that (1) the object always goes to the bidder with the highest value estimate above to and (2) every bidder would expect zero utility if his value estimate were at its lowest possible level.

4.4 Optimal Auction

The cumulative distribution function $F_i : [a_i, b_i] \to [0, 1]$ for bidder i is continuous and strictly increasing, since we assume that the density function f, is always strictly positive. Thus $F(\cdot)$ has an inverse $F_i^{-1} : [0, 1] \to [a_i, b_i]$, which is also continuous and strictly increasing.

For each bidder i, we now define four functions which have the unit interval $[0, 1]$ as their domain. First, for any q in $[0, 1]$,

$$\begin{aligned} h_i(q) &= F_i^{-1}(q) - e_i(F_i^{-1}(q)) - \frac{1-q}{f_i(F_i^{-1}(q))} \\ &= c_i(F_i^{-1}(q)), \end{aligned} \tag{4.4.1}$$

and let

$$H_i(q) = \int_0^q h_i(r)dr. \tag{4.4.2}$$

Next let $G_i : [0, 1] \to \mathbb{R}$ be the convex hull of the function $H(\cdot)$.

$$\begin{aligned} G_i(q) &= \text{conv} H_i(q) \\ &= \min\{\omega H_i(r_1) + (1-\omega)H_i(r_2) | \{\omega, r_1, r_2 \in [0, 1]\} \quad \text{and} \quad \omega r_1 + (1-\omega)r_2 = q\}. \end{aligned} \tag{4.4.3}$$

That is, $G_i(\cdot)$ is the highest convex function on $[0, 1]$ such that $G_i(q) < H_i(q)$ for every q.

As a convex function, G_i, is continuously differentiable except at countably many points, and its derivative is monotone increasing. We define $g_i : [0, 1] \to \mathbb{R}$ so that

$$g_i(q) = G_i'(q) \tag{4.4.4}$$

whenever this derivative is defined, and we extend $g_i(\cdot)$ to all of $[0, 1]$ by right-continuity.

We define $\bar{c}_i : [a_i, b_i] \to \mathbb{R}$ so that

$$\bar{c}_i(t_i) = g_i(F_i(t_i)). \tag{4.4.5}$$

Finally, for any or any vector of value estimates t, let $M(t)$ be the set of bidders for whom $\bar{c}(ti)$ is maximal among all bidders and is higher than t_0.

$$M(t) = \{i | t_0 \leqslant \bar{c}_i(t_i) = \max_{j \in \mathcal{N}} \bar{c}_j(t_j)\}. \tag{4.4.6}$$

We can now state our main result: that in an optimal auction, the object should always be sold to the bidder with the highest $\bar{c}_i(t_i)$, provided this is not less than t_0. Thus, we may think of $\bar{c}_i(t_i)$ as the priority level for bidder i when his value estimate is t_i, in the seller's optimal auction.

Theorem 4.4.1 *Let $\bar{p} : T \to \mathbb{R}^n$ and $\bar{x} : T \to \mathbb{R}^n$ satisfy*

$$\bar{p_i}(t) = \begin{cases} 1/|M(t)| & \text{if} \quad i \in M(t), \\ 0 & \text{if} \quad i \notin M(t) \end{cases} \tag{4.4.7}$$

and

$$\bar{x_i}(t) = \bar{p_i}(t)v_i(t) - \int_{a_i}^{t_i} \bar{p_i}(t_{-i}, s_i)ds_i \tag{4.4.8}$$

for all i in $\mathcal{N}$ and t in T. Then $(\bar{p}, \bar{x})$ represents an optimal auciton mechanism.

The proof is given as the theorem in [2].

References

[1] R. B. MYERSON, *Incentive compatibility and the bargaining problem*, Econometrica, 47 (1979), pp. 61–73.

[2] ——, *Optimal auction design*, Discussion Papers, 6 (1979), pp. 58–73.