NOTES FOR STOCHASTIC PROCESS

AARON NOTES SERIES

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1 Conception

$\mathcal{F}$-measuable: (This is a reply from Stefan Hansen in StackExchange)

Let $(\Omega, \mathcal{F}, P)$ be a probability space, i.e. Ω is a non-empty set, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $P : \mathcal{F} \to [0, 1]$ is a probability measure on $\mathcal{F}$. Now, suppose we have a function $X : \Omega \to \mathbb{R}$ and we want to "measure" the probability of X belonging to some subset of $\mathbb{R}$. That is, we want to assign the probability to sets of the form

$$\{X \in A\} := X^{-1}(A) = \{\omega \in \Omega | X(\omega) \in A\} \tag{1.1}$$

for some Borel sets $A \in \mathcal{B}(\mathbb{R})$. For this to make sense, we need to make sure that $\{X \in A\} \in \mathcal{F}$ for all $A \in \mathcal{B}(\mathbb{R})$, otherwise we can't assign a probability to it (recall that P is only defined on $\mathcal{F}$). Whenever $X : \Omega \to R$ satisfies that $X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}(\mathbb{R})$ we say that X is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable or just $\mathcal{F}$-measurable when there is no chance of confusion. Thus, for a random variable X, it makes sense to assign the probability to any set of the form $\{X \in A\}$, and this defines the distribution of X:

$$P_X(A) := P(X \in A), \quad A \in \mathcal{B}(\mathbb{R}) \tag{1.2}$$

If $Y : \Omega \to \mathbb{R}$ is a random variable, then $\sigma(Y)$ is, by definition, given as

$$\sigma(Y) = \sigma(\{Y^{-1}(A) | A \in \mathcal{B}(\mathbb{R})\}), \tag{1.3}$$

i.e. the smallest σ-algebra containing all sets of the form $Y^{-1}(A)$. Another way of characterizing $\sigma(Y)$ is by saying that it is the smallest σ-algebra we can put on Ω that makes Y measurable. Similarly, if $Y : \Omega \to X \subset \mathbb{R}^T$ is measurable with respect to the cylinder σ-algebra $\sigma(\mathcal{F}_X)$ for X, then Y is called a stochastic process. The σ-algebra generated by Y is

$$\sigma(Y) := \{Y^{-1}(A) : A \in \sigma(\mathcal{F}_X)\} = \sigma(\{Y^{-1}(A) : A \in \mathcal{F}_X\}). \tag{1.4}$$

References