AI1110 : Probability And Random Variables Assignment 2

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***Abstract*—This document provides solution of Assignment 2(ICSE 2018 12 Q.5(a))**

Question 5(a): Show that the function $f(x) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ is continuous at $x = 1$ but not differentiable.

Key Concept :

1) A function f is said to be continuous at $x = a$,iff the following three conditions satisfied.
   i The limit $\lim_{x \to a} f(x)$ should exist and it is finite.
   ii The functional value $f(a)$ should exist and it is finite.
   iii $\lim_{x \to a} f(x)$ = $f(a)$.
2) A function f is said to be differentiable at $x = a$ if and only if the limit,
$$lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
exists.

Solution : Given

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

We can say f is continuous at $x = 1$,iff

$$\lim_{x \to 1} f(x) = f(1) \tag{1}$$

In other words f should satisfy,

$$f\left(1^-\right) = f\left(1^+\right) = f(1) \tag{2}$$

where,

$$f\left(1^-\right) = \lim_{h \to 0} f(1-h) \tag{3}$$
$$f\left(1^+\right) = \lim_{h \to 0} f(1+h) \tag{4}$$

$$f(1) = 1 \tag{5}$$

Now,

$$f\left(1^-\right) = \lim_{h \to 0} f(1-h) \tag{6}$$
$$= \lim_{h \to 0} (1-h)^2 \tag{7}$$
$$\implies f\left(1^-\right) = 1 \tag{8}$$

And,

$$f\left(1^+\right) = \lim_{h \to 0} f(1+h) \tag{9}$$
$$= \lim_{h \to 0} \frac{1}{(1+h)} \tag{10}$$
$$\implies f\left(1^+\right) = 1 \tag{11}$$

Using (5) ,(8),(11),we can say that f is continuous at $x = 1$ and this can be seen in Fig 1.
Now from the concept of differentiability,we can say f is differentiable at $x = 1$ iff the limit,

$$lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

exists.
In that case f should satisfy,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = lim_{h \to 0} \frac{f(1) - f(1-h)}{h} \tag{12}$$

LHS:

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{(1+h)}\right) - 1}{h} \tag{13}$$
$$= \lim_{h \to 0} \frac{(1 - (1+h))}{h(1+h)} \tag{14}$$
$$= \lim_{h \to 0} \frac{-h}{h(1+h)} \tag{15}$$
$$= \lim_{h \to 0} \frac{-1}{(1+h)} \tag{16}$$
$$= -1 \tag{17}$$
$$\implies \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = -1. \tag{18}$$

RHS:

$$\lim_{h\to 0}\frac{f(1)-f(1-h)}{h} = \lim_{h\to 0}\frac{1-\left((1-h)^2\right)}{h} \quad (19)$$

$$= \lim_{h\to 0}\frac{-\left(2h-h^2\right)}{h} \quad (20)$$

$$= \lim_{h\to 0}\frac{h\left(2-h\right)}{h} \quad (21)$$

$$= \lim_{h\to 0} -\left(2-h\right) \quad (22)$$

$$= 2 \quad (23)$$

$$\implies \lim_{h\to 0}\frac{f(1)-f(1-h)}{h} = 2. \quad (24)$$

$$\therefore LHS \neq RHS$$

Hence, function $f(x)$ is not differentiable at $x = 1$.This can be seen in Fig 2.

Therefore, we proved that $f(x) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ is continuous at $x = 1$ but not differentiable.

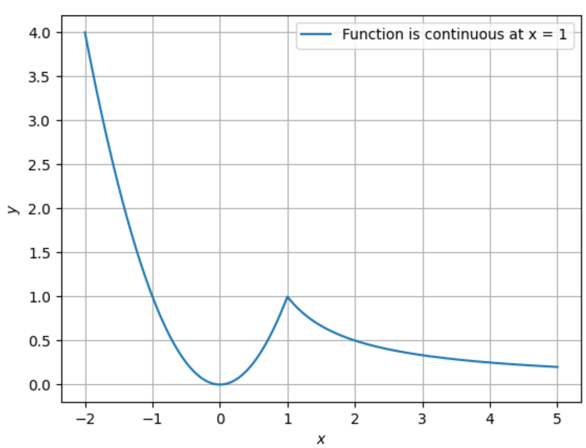

Fig. 1.

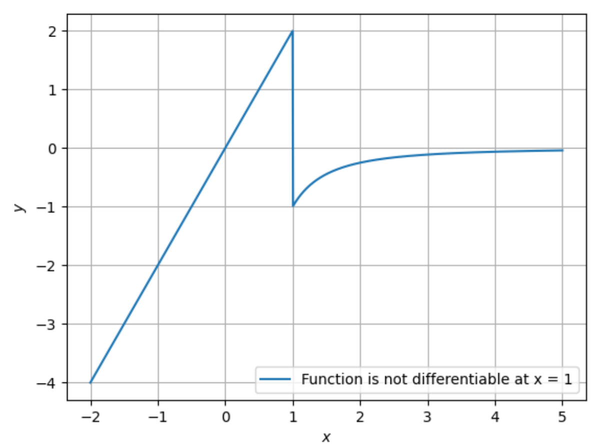

Fig. 2.