Assignment 2
ICSE 2018 Class 12 Q.5(a)

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Outline

Question

Show that the function,

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

is continuous at $x = 1$ but not differentiable.

Solution

Given

$$f\left(x\right) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

We can say f is continuous at $x = 1$,iff

$$\lim_{x \to 1} f\left(x\right) = f\left(1\right) \tag{1}$$

In other words f should satisfy,

$$f\left(1^-\right) = f\left(1^+\right) = f\left(1\right) \tag{2}$$

where,

$$f\left(1^-\right) = \lim_{h \to 0} f\left(1 - h\right) \tag{3}$$

$$f\left(1^+\right) = \lim_{h \to 0} f\left(1 + h\right) \tag{4}$$

$$f\left(1\right) = 1 \tag{5}$$

Continuity at $x = 1$

Now,

$$f\left(1^-\right) = \lim_{h\to 0} f\left(1 - h\right) \tag{6}$$
$$= \lim_{h\to 0} \left(1 - h\right)^2 \tag{7}$$
$$\implies f\left(1^-\right) = 1 \tag{8}$$

And,

$$f\left(1^+\right) = \lim_{h\to 0} f\left(1 + h\right) \tag{9}$$
$$= \lim_{h\to 0} \frac{1}{\left(1 + h\right)} \tag{10}$$
$$\implies f\left(1^+\right) = 1 \tag{11}$$

Using eq 5 ,eq 8,eq 11,we can say that f is continuous at $x = 1$.

Differentiability at $x = 1$

We can say that f is differentiable at $x = 1$ iff the limit,

$$lim_{h\to 0}\frac{f\left(1+h\right)-f\left(1\right)}{h}$$

exists.
In that case f should satisfy,

$$\lim_{h\to 0}\frac{f\left(1+h\right)-f\left(1\right)}{h} = lim_{h\to 0}\frac{f\left(1\right)-f\left(1-h\right)}{h} \tag{12}$$

LHD

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\left(\frac{1}{(1+h)}\right)-1}{h} \tag{13}$$

$$= \lim_{h\to 0}\frac{(1-(1+h))}{h(1+h)} \tag{14}$$

$$= \lim_{h\to 0}\frac{-h}{h(1+h)} \tag{15}$$

$$= \lim_{h\to 0}\frac{-1}{(1+h)} \tag{16}$$

$$= -1 \tag{17}$$

$$\implies \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = -1. \tag{18}$$

RHD

RHS:

$$\lim_{h\to 0}\frac{f(1)-f(1-h)}{h} = \lim_{h\to 0}\frac{1-\left((1-h)^2\right)}{h} \tag{19}$$

$$= \lim_{h\to 0}\frac{-\left(2h-h^2\right)}{h} \tag{20}$$

$$= \lim_{h\to 0}\frac{h(2-h)}{h} \tag{21}$$

$$= \lim_{h\to 0} -(2-h) \tag{22}$$

$$= 2 \tag{23}$$

$$\implies \lim_{h\to 0}\frac{f(1)-f(1-h)}{h} = 2. \tag{24}$$

$$\therefore LHS \neq RHS$$

Hence, function $f(x)$ is not differentiable at $x = 1$.

Conclusion

Therefore, we proved that $f(x) = \begin{cases} x^2, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ is continuous at $x = 1$ but not differentiable.