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THE SPATIAL REPRESENTATION OF SURFACE ROUGHNESS BY MEANS OF THE STRUCTURE FUNCTION: A PRACTICAL ALTERNATIVE TO CORRELATION

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Summary

The structure function, which is related simply to the autocorrelation function but is without some of its disadvantages, is proposed as a means of quantifying variations in surface texture. Both profile and surface structure functions can be computed; the latter conveniently shows the effect of anisotropy on roughness. Using the structure function, the plastic compliance of rough surfaces can be related to contact spot size. Finally it is shown that the size of recurrent features representing damage, *e.g.* lands, can be estimated from the structure function.

1. Introduction

The use of stochastic techniques to describe rough surfaces, pioneered more than thirty years ago [1], is now well established [2 - 5]. Common to all work on the subject is the need to define the "spatial" variation of surface geometry, *i.e.* the rapidity with which the height varies with horizontal distance. This variation is sometimes referred to as the texture of the surface as distinct from the actual roughness [6].

The autocovariance function or its normalized form the autocorrelation function (ACF) has been the most popular way of representing spatial variation. Longuet-Higgins [7], dealing with ocean waves, followed Rice [8] in specifying this by the derivatives of the ACF at the origin. Whitehouse and Archard [3] derive a significance from the distance over which it decays. Peklenik [2] uses the form of the decay to classify surfaces, although this can be misleading as many different structures can produce the same ACF. A general technique using the complete function is offered by the application of Gram–Charlier series [9]. Tellas and Dukler [10] have applied this to determine a mean wave shape of a falling liquid film.

Although the ACF has been widely applied to surface description it suffers from a number of disadvantages; some are historical and conceptual.

The mathematical development of the ACF was largely in the hands of communications engineers, with the result that its terminology is unfamiliar to the mechanical engineer or the production engineer. Again, the ACF is a sum of terms each of which is the product of two amplitudes. This is an uncomfortable concept to someone not trained in field theory, particularly when the amplitudes represent not voltages but heights. More serious are the mathematical disadvantages. The ACF will not cope well with a non-stationary mean, which is a common feature of engineering surfaces [11]. Similarly many engineering surfaces are anisotropic, *i.e.* they have a pronounced lay, and the autocovariance function, which should have a variance at the origin common to all profiles, can only be obtained if all profiles are measured from the same mean plane. If they are not, singularities are created at the origin of the function.

In this paper an alternative representation is examined which avoids some of these difficulties, and we show how this may be used to approach some interesting problems of surface characterization which have not hitherto been readily amenable to analysis.

2. The structure function

Consider a signal $z(x)$ whose statistical properties for a delay are to be investigated. The ACF sums the delayed products. Suppose instead the expected value $S(\tau)$ of the sums of the squares of the delayed amplitude differences is considered:

$$S(\tau) = \mathrm{E}\{[z(x) - z(x+\tau)]^2\} \tag{1}$$

where E{ } denotes an expectation. This has been termed the variance function or structure function (SF) [12 - 14]. It is independent of the mean plane; thus any profile structure function, irrespective of its mean line, is a section of the surface structure function — a property not shared by the ACF.

To demonstrate the structure function's independence of the mean and to obtain its relationship to the ACF, consider the profile in Fig. 1(a):

$$\begin{aligned} S(\tau) &= \mathrm{E}\{[\,z(x) + h \;-\; \{z(x+\tau) + h\}]^2\} \\ &= \mathrm{E}\{[z(x) + h]^2\} + \mathrm{E}\{[z(x+\tau) + h]^2\} - \\ &\quad -2\mathrm{E}\{[z(x) + h]\,[z(x+\tau) + h]\} \end{aligned}$$

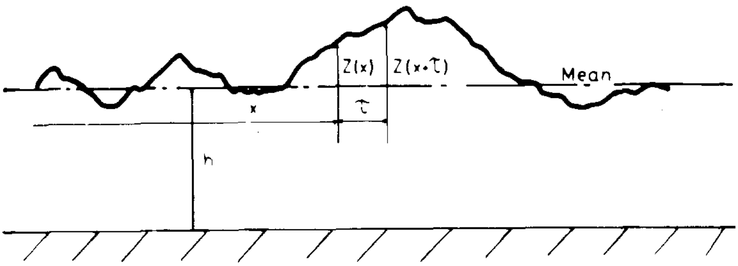
On multiplying out, the terms containing h cancel leaving

$$S(\tau) = \mathrm{E}\{z^2(x+\tau)\} + \mathrm{E}\{z^2(x)\} - 2\mathrm{E}\{z(x)z(x+\tau)\}$$

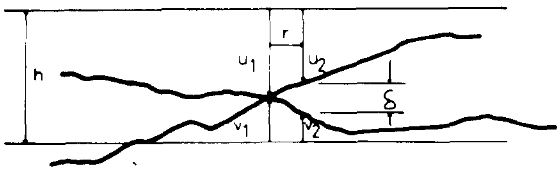
The product term represents the covariance function $\psi(\tau)$ and for stationary data $\mathrm{E}\{z^2(x)\} = \mathrm{E}\{z^2(x+\tau)\} = \sigma^2$. Thus

$$S(\tau) = 2\{\sigma^2 - \psi(\tau)\} \tag{2}$$

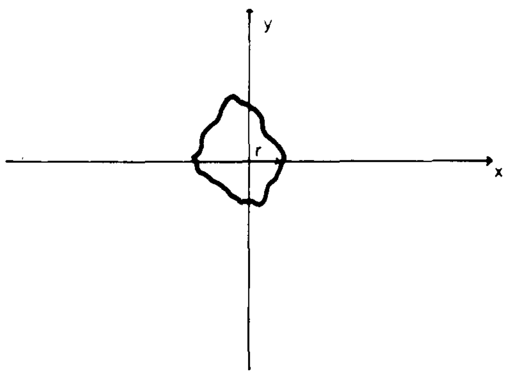
or in terms of the ACF $R(\tau) = \psi(\tau)/\sigma^2$



(a)



(b)



(c)

Fig. 1. (a) Profile of a randomly rough surface, indicating the nomenclature; (b), (c) nomenclature associated with the contact contour shape and size.

$$S(\tau) = 2\sigma^2\{1 - R(\tau)\} \tag{3}$$

2.1. The profile structure function

A spatial property of significance in surface studies is the correlation length [3], which is defined as the length over which the ACF decays to a value of 0.1. The SF is useful in examining the effects of high-pass filtering on such properties. Figure 2 represents a typical profile SF; the maximum value it achieves over the profile length involved is $2\sigma_1^2$. For a correlation of 0.1 the correlation length l_1 will occur at $S(\tau) = 0.9 \times 2\sigma_1^2$.

Now consider high-pass filtering, which often has to be performed to define any sort of correlation length. The effect required is to reduce or

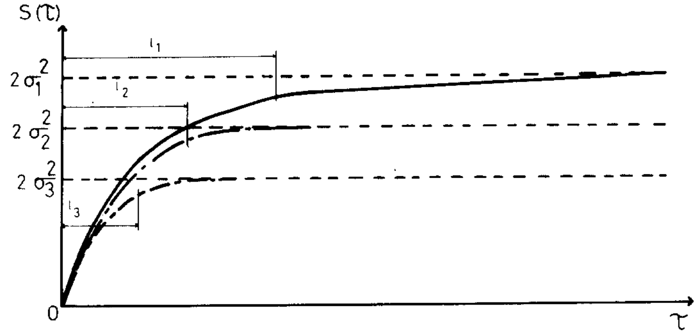


Fig. 2. Structure function of a typical random surface profile, showing the effects of high-pass filtering on the apparent roughness and correlation length l at various cut-offs.

remove variations in height occurring over long spatial lengths; *i.e.*, if necessary, $S(\tau)$ is reduced to a constant of $2\sigma^2$ when τ exceeds some significant length (usually expressed in terms of a cut-off wavelength). The result of this is shown by the horizontal broken line marked $2\sigma_2^2$ in Fig. 2. Being now a stationary function it can be converted into an ACF if required and will have a correlation length of l_2. Continuing the filtering process to smaller lengths (a shorter wavelength cut-off) will obviously produce smaller correlation lengths. The effect is demonstrated for several cut-offs in Fig. 2. It becomes clear from the above discussion that only when the surface is stationary, *i.e.* a natural asymptote $2\sigma^2$ exists, will a unique correlation length exist.

2.2. *The surface structure function* S(r, θ)

$$S(r, \theta) = \mathrm{E}\{[z(x, y) - z\{(x, y) + r(\theta)\}]^2\} \tag{4}$$

where $z(x, y)$ is the surface height at coordinates (x, y) in the plane of the surface and $z\{(x, y) + r(\theta)\}$ is the surface height at a radial distance r from (x, y) in a direction θ.

The majority of machined surfaces are more or less anisotropic, *i.e.* their finish has a pronounced lay or directional character; the statistical properties of a profile through such a surface will depend on its orientation with respect to the lay. This is especially true of ground surfaces. Surface grinding in particular produces a finish composed of a very large number of superimposed parallel scratches, each of which may be from 10 to 100 times as long as it is broad. Clearly such a surface will possess highly anisotropic properties.

There are considerable difficulties in characterizing an anisotropic surface by a correlation function. Its roughness, *i.e.* the standard deviation of its height distribution, will vary with direction. Profiles parallel to the lay will contain much less power at some wavelengths than will profiles at right angles to the lay and hence if they are measured at the same cut-off they will appear much smoother. If one attempts therefore to plot a three-dimensional autocovariance function it will be multivalued at the origin. If it is sought to avoid this by normalizing individual autocovariance functions by

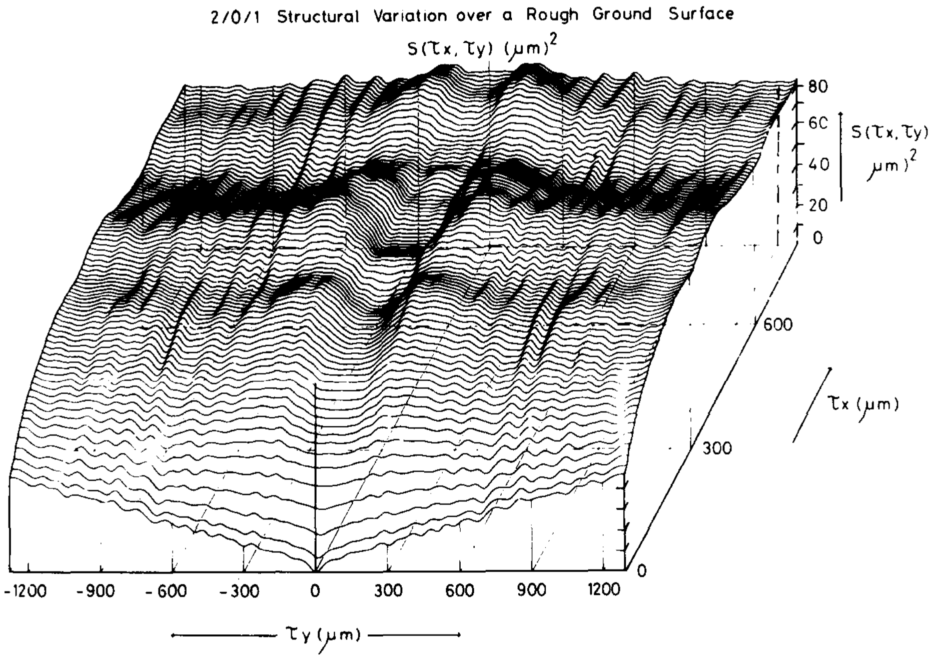


Fig. 3. A quadrant of the three-dimensional structure function of a ground surface reflected about the τ_y = 0 axis. The lay direction is parallel to the τ_y axis.

their respective variances, the resulting three-dimensional autocorrelation function will give a totally misleading impression of the directional variation of roughness.

The structure function does not suffer from problems of this kind. Figure 3 shows an isometric projection of half of a complete three-dimensional structure function of a ground surface. The τ_y axis represents the direction of the lay and the τ_x axis, which is at right angles to it, has been expanded by a factor of 2 for convenience of presentation. The variation of roughness with direction can now be clearly demonstrated. For instance, take the values of the structure function parallel and perpendicular to the lay at a spatial distance of 600 μm. Here $S(0,600) = 6\ \mu m^2$ while $S(600,0) = 12\ \mu m^2$ and the corresponding roughnesses are 1.7 μm and 2.4 μm respectively.

3. Identification of surface features

For the structure function of Fig. 3 the computation was carried out over a quadrant of the total structure field and the plot shown represents this quadrant reflected about the τ_y = 0 axis. The lay direction is τ_y and the direction of profile measurement τ_x was across the lay. Computation was achieved using an ensemble of ten pairs of profiles to create each row of data in the τ_x direction.

No filtering or attenuation other than that imposed by sampling is used; thus the smoothness of the function is significant, particularly at relatively large distances from the origin where a more random appearance might have been expected. This smooth appearance makes the proposition of identifying certain features attractive. For example the longer wavelength corrugation seen running parallel to the lay direction τ_y can be explained as representing the average grain spacing of the grinding wheel [15]. The shorter wavelength effect seen parallel to the τ_x direction is more difficult to explain; at first sight it might be thought to represent a slight error created by computation over only a limited ensemble of ten traverses. However, closer inspection reveals that the crests of the features tend to merge quite smoothly throughout the surface of the function. This would not occur with the averaging errors suggested.

The most surprising and distinctive feature apparent is the trough at $\tau_y = 0$, running across the lay (τ_x direction). It is not evident at the origin but quickly forms as τ_x is increased and at certain values it is a distinct flat region. Mathematically the flats represent, for a small distance in the lay direction, a constant difference in surface height over relatively large values of separation τ_x parallel to the lay. Thus over short lengths in the lay direction profiles are parallel. This effect could be related to the effective cutting length of individual abrasive grains.

3.1. *Average contact size and shape*

Consider the region of contact of two surfaces just touching (Fig. 1(b)). The surfaces may be of any form or configuration for the following analysis to apply, provided that the height distribution and, in the case of anisotropic surfaces, the relative orientation of structure in the x–y plane are known. For, say, two ground surfaces this will be the relative orientation of the lay directions.

Figure 1(c) indicates the projected contour representing constant separation δ. If perfectly plastic contact is considered with the assumption of no geometric change in asperity shape, this contour will represent the contact geometry at a mean plane separation $h - \delta$. The assumption of constant geometry is not as serious as it may at first seem in view of the work of Pullen and Williamson [16].

The expected or mean geometry of a contact can be related to the SF $S(r)$ in the following way:

$$\delta = h - u_2 - v_2$$

and if

$$h = u_1 + v_1$$

then

$$\delta = u_1 - u_2 + v_1 - v_2$$

$$\mathrm{E}\{\delta\} = \mathrm{E}\{u_1 - u_2 + v_1 - v_2\}$$

$$\mathrm{E}\{\delta\} = \mathrm{E}\{u_1 - u_2\} + \mathrm{E}\{v_1 - v_2\} \tag{5}$$

When the expectation is computed over the whole x–y plane the terms on the right of eqn. (5) will be zero. In the case of the contact described, these terms cannot be locally zero other than at the origin of the contact. Therefore considering only positive values of $u_1 - u_2$ and $v_1 - v_2$, which will be assumed to be normally distributed (*i.e.* the slope distributions are Gaussian), the expectation can be expressed in terms of the standard deviation thus:

$$\mathrm{E}\{z^2\} = \sigma^2$$

$$\mathrm{E}\{z^+\} = \mathrm{E}\{|z|\} = 2\int_0^\infty zp(z)\mathrm{d}z = \left(\frac{2}{\pi}\right)^{1/2}\sigma$$

Therefore

$$\mathrm{E}\{z^+\} = [(2/\pi)\mathrm{E}\{z^2\}]^{1/2} \tag{6}$$

Combining eqns. (5) and (6) gives

$$\mathrm{E}\{\delta\} = (2/\pi)^{1/2}([\mathrm{E}\{(u_1 - u_2)^2\}]^{1/2} + [\mathrm{E}\{(v_1 - v_2)^2\}]^{1/2})$$

The term $\mathrm{E}\{(u_1 - u_2)^2\}$ is the SF $S_u(r)$; therefore

$$\mathrm{E}\{\delta\} = (2/\pi)^{1/2}\{S_u^{1/2}(r) + S_v^{1/2}(r)\} \tag{7}$$

Thus for two surfaces having the same SF

$$\mathrm{E}\{\delta\} = 2(2/\pi)^{1/2}S^{1/2}(r) \tag{8}$$

or

$$\mathrm{E}\{\delta\} \approx 1.6S^{1/2}(r)$$

If the form of $S(r)$ is known the expected radius r can be expressed in terms of the mean plane displacement δ after initial contact.

For the Whitehouse and Archard exponential autocorrelation model and an isotropic surface, eqn. (8) becomes

$$\mathrm{E}\{\delta\} = 2(2/\pi)^{1/2}[2\sigma^2\{1 - \exp(-r/\beta^*)\}]^{1/2} \tag{9}$$

$$r = -\beta^* \ln\left[1 - \frac{\pi}{16}\left(\frac{\mathrm{E}\{\delta\}}{\sigma}\right)^2\right] \qquad 0 < \mathrm{E}\{\delta\} \leqslant 2.25\sigma$$

Equations (7) and (8) represent useful local compliance relations as they are based on profilometric rather than asperity models. As an asperity model this form of relationship, particularly when a computer analysis is involved, has advantages over the more conventional geometric shapes [17].

3.2. Identification of recurrent features

This example is concerned with identifying spatial surface features which are in some way similar but which repeat themselves in a random way. Many machined or formed surfaces have such characteristics, *e.g.* a bead-blasted surface can be considered as a random array of craters, differing in size but similar in shape. A question of interest is to what extent the shape is represented in the structure or correlation function and whether the function will yield information which will specify spatial surface properties.

To investigate the possibilities of such an approach the simple but useful example of a surface having undergone some form of truncation is considered (Fig. 4(a)). The characteristic that repeats itself in this case is a series of plateaux created on the surface. The object of the analysis is to determine the extent of changes that occur in the SF or ACF and whether these changes can be interpreted in terms of the spatial characteristics (such as land lengths) involved.

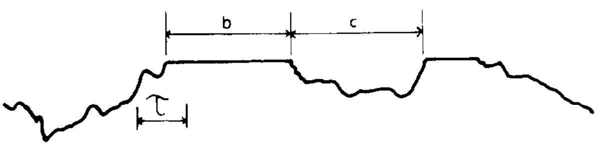


Fig. 4. Nomenclature associated with the identification of plateaux formed on a surface.

The technique employed is to consider the overall expected value as made up of n events. Thus the total expected value is given by the sum of their conditional expected values multiplied by the probability of each event [18]. Employing the SF

$$E\{[z(x)-z(x+\tau)]^2\}=\sum_{i=1}^{n} P_i E\{[z(x)-z(x+\tau)]^2 \mid A_i\} \tag{10}$$

where P_i is the probability of event A_i occurring and $E\{[z(x)-z(x+\tau)]^2 | A_i\}$ is the conditional expected value of event A_i, *i.e.* the value of E{ } when event A_i occurs.

The process of computing the total expectation for the example in question can be considered as the combination of only three events:

A_1 $z(x)$ and $z(x+\tau)$ lie in the unaffected region;

A_2 $z(x)$ or $z(x+\tau)$ lies in the unaffected region and the other does not;

A_3 $z(x)$ and $z(x+\tau)$ lie on a flat region.

It is convenient to use the spatial definition of a feature shown in Fig. 4(b) where b is the average flat size and c the average distance between flats. With such an arrangement the following approximate relations are shown in the Appendix to represent the conditional expectations and the probabilities of each event:

If $E_i = \mathrm{E}\{[z(x) - z(x+\tau)]^2 | \mathrm{A}_i\}$ then

$$\begin{aligned}
E_1 &= S_u(\tau) && \text{for all } \tau \\
E_2 &= S_u(\tau/2) && \tau < b \\
E_2 &= S_u(\tau - b/2) && \tau > b \\
E_3 &= 0 && \text{for all } \tau
\end{aligned} \tag{11}$$

$$\begin{aligned}
P_1 &= 1 - a - \tau q && \tau < b \\
P_1 &= 1 - 2a && b < \tau < c \\
P_1 &= (1-a)^2 && \tau > c \\
P_2 &= 2\tau q && \tau < b \\
P_2 &= 2a && b < \tau < c \\
P_2 &= 2a(1-a) && \tau > c \\
P_3 &= a - \tau q && \tau < b \\
P_3 &= 0 && b < \tau < c \\
P_3 &= a^2 && \tau > c
\end{aligned} \tag{12}$$

where $S_u(\tau)$ is the original surface SF, a is the ratio of the flat area to the nominal area and q is the density of flats. Combining eqns. (10), (11) and (12) yields

$$S_w(\tau) = (1 - a - \tau q)S_u(\tau) + 2\tau q S_u(\tau/2) \qquad \tau < b \tag{13}$$

$$S_w(\tau) = (1 - 2a)S_u(\tau) + 2aS_u(\tau - b/2) \qquad b < \tau < c \tag{14}$$

$$S_w(\tau) = (1 - a)S_u(\tau) + 2a(1-a)S_u(\tau - b/2) \qquad \tau > c \tag{15}$$

where $S_w(\tau)$ represents the total SF of the surface after the flat regions have been formed. The object of this analysis is not to apply the equations in a general sense but to use the differences occurring in the function to predict the parameters a, q and b. From eqn. (15) when $\tau \gg c$ for stationary data $S_u(\tau - b/2) \approx S_u(\tau)$. Thus

$$S_w(\tau)/S_u(\tau) = 1 - a^2 \tag{16}$$

Also for small τ $S_u(\tau/2) \approx S_u(\tau)/2$ and

$$S_w(\tau)/S_u(\tau) = 1 - a \tag{17}$$

A knowledge of a and b will also define the density q and average spacing c. To obtain b directly from the equations it is necessary to know the form of $S_u(\tau)$. However, for a hypothetical virgin structure function ($S_u(\tau)$ in Fig. 5) it can be seen that distinctive changes are predicted by eqns (13), (14) and (15) to occur at the spatial dimensions of most interest. The two broken curves S_{w1} and S_{w2} represent values of $a = 0.4$ and $a = 0.2$, respectively.

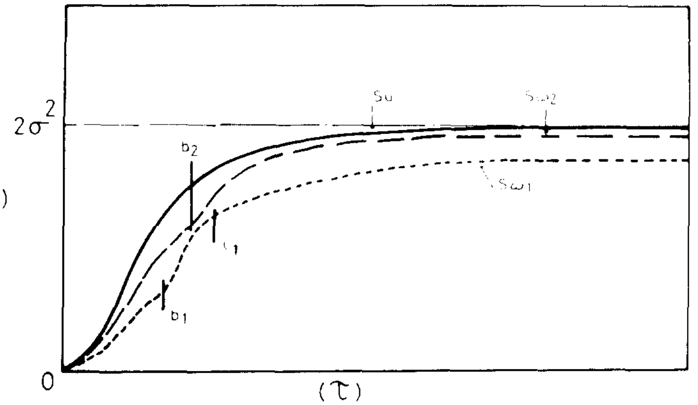


Fig. 5. Hypothetical structure function S_u of a virgin surface and two predicted functions S_{w1} and S_{w2} which could occur when plateaux are present at constant height. The inflections indicate the average flat size b and the average distance c between flats.

Curve S_{w1} shows inflections at the average flat size b and at the average spacing c.

The example used may well apply to some wear or machining operations but could also be interpreted as the ideal case of plastic deformation. A comparison of the results from a ground surface before and after deformation against a hardened steel flat is shown in Fig. 6. The broken line represents the predictions of the earlier analysis taking b as the point where the maximum difference occurs and estimating a from eqn. (16).

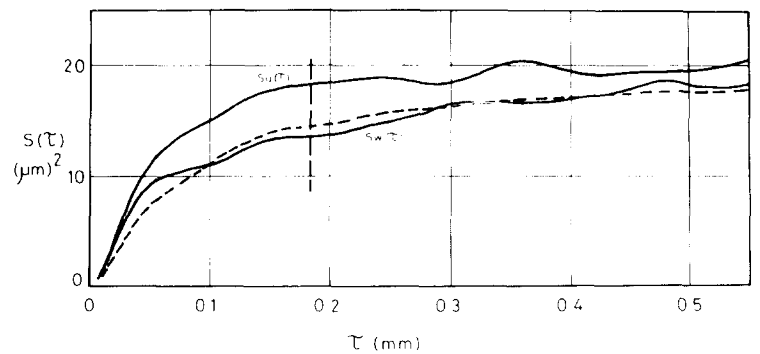


Fig. 6. Comparison of theory (broken line) with results from a ground surface before and after severe plastic deformation against a hardened steel flat. The vertical broken line shows the position of maximum difference, which was taken as the average flat size b.

Agreement is not good at small values of τ but this is interesting in itself as it is probably due to an asperity persistence effect which would be expected in this region. The interesting aspect is the extent over which it is seen to occur. The persistence of asperities discussed by Williamson and Hunt [19] has not been investigated as a spatially dependent property and this technique appears to offer a method of analysis.

4. Conclusions

It has been shown that the structure function discloses non-stationarity in a signal less ambiguously than the ACF and that the quantitative effect of high-pass filtering on roughness can be seen quickly by mere inspection of the SF. Proceeding from profile to surface properties, the relation between profile roughness and anisotropy is brought out more clearly by a surface SF, which will also pick out machining effects.

If the form of the SF is known and if pure plastic contact is assumed the average contact spot radius can be predicted for a given separation of surface mean planes. This relation is based on entirely profilometric measurements and makes no assumptions whatever about the geometry of individual asperities. As the total area of real contact would also be known from the load and hardness the number of contact spots and hence the contact resistance could be estimated without further information.

The main problem with identifying recurrent features from an ACF is that the ensemble averaging masks any changes which occur over only a limited number of the ensemble. The SF offers a means of overcoming this. It is independent of the mean and may be computed over only a proportion of the ensemble without loss of significance. For example, Sayles and Thomas [15] computed the function over only the positive regions of a grinding wheel profile. Thus the resulting function was significant to the peaks, which were the only features of interest in that particular application. The study of contact, friction and wear features occurring in only the higher regions of the surface is a further possible application of this technique. From a surface measurement point of view, the comparison of regional structure functions computed separately from peaks and valleys offers a means of studying the spatial asymmetry of surfaces. Many machined and formed surfaces fall into this category.

The actual computation of a SF presents no more difficulty than that of an ACF and one can readily be transformed into the other.

Acknowledgments

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Appendix

Derivation of eqns. (11) and (12)

Event A_1

$$\mathrm{E}\{[z(x) - z(x + \tau)]^2 | \mathrm{A}_1\} = S_u(\tau) \tag{A1}$$

When both ordinates are in an unaffected zone the expectation is simply that of the virgin surface $S_u(\tau)$. This is also valid in a contact situation if the uniform rise condition of ref. 16 is considered.

The probability

$$P_1 = P(\mathrm{A}_1) = 1 - P(\mathrm{A}_2) - P(\mathrm{A}_3) \tag{A2}$$

can be evaluated when $P(\mathrm{A}_2)$ and $P(\mathrm{A}_3)$ are known.

Event A_2

The expectation required for event A_2 depends upon the ordinate separation τ. It is convenient to consider the surface around one particular flat (Fig. 4(b)). When $\tau < b$ a reasonable approximation is

$$E\{[z(x) - z(x+\tau)]^2 | A_2\} \approx S_u(\tau/2) \quad \text{(A3)}$$

Similarly when $\tau > b$

$$E\{[z(x) - z(x+\tau)]^2 | A_2\} \approx S_u(\tau - b/2) \quad \text{(A4)}$$

From Fig. 4(b) if c is the average spatial distance between flats and b the average flat size, then the area ratio $A/A_n = a$ is simply

$$\frac{A}{A_n} = \frac{b}{b+c} = a \quad \text{(A5)}$$

In general $a \leqslant ½$ (in a contact situation); thus $b \leqslant c$. Also the density of flats will be

$$q = 1/(b+c) \quad \text{(A6)}$$

The spatial probability of events can be defined as

$$P(A_i) = \frac{\text{Length or area over which the event exists}}{\text{Total possible length or areas involved}}$$

Thus for $\tau < b$ event A_2 can occur over a length 2τ in the total event length $b + c$. For $\tau \leqslant b$

$$P(A_2) = 2\tau/(b+c) = 2q\tau \quad \text{(A7)}$$

Similarly for $b \leqslant \tau < c$

$$P(A_2) = 2b/(b+c) = 2a \quad \text{(A8)}$$

As τ tends to c the probability given by eqn. (A8) will reduce because variations in c will occur which cause one ordinate of the event to correspond to another contact and not the unaffected surface. When τ exceeds c the probability of this ordinate occurring on the unaffected surface will be a function of the area ratio a and for $\tau > c$ the probability of event A_2 becomes

$$P(A_2) = 2b(1-a)/(b+c) = 2a(1-a) \qquad \tau > c \quad \text{(A9)}$$

This result can be defined by conventional probability techniques, since the two possible ordinate occurrences can be assumed to be independent. Thus, P (one ordinate occurs on a flat) = $A/A_n = a$; also P (one ordinate does not occur on a flat) = $1 - A/A_n = 1 - a$. Therefore as the events are independent the joint probability is simply the product $a(1-a)$ and as there are two possibilities for each ordinate $P(A_2) = 2a(1-a)$, which corresponds to eqn. (A9).

Event A_3

The expected value of $[z(x) - z(x + \tau)]^2$ is zero if the operation creates flats. However, for plastic contact, if smaller scale asperities persist, then a value will probably exist but it will be small compared with the other events. This is particularly so as τ increases but for small τ the value may be significant if persistence occurs. Thus in general

$$E\{[z(x) - z(x + \tau)]^2 | A_3\} = 0$$

Using the same technique as in event A_2 the probabilities can be defined as follows:

$$P(A_3) = (b - \tau)/(b + c) = a - q\tau \qquad \tau \leqslant b \tag{A10}$$

$$P(A_3) = 0 \qquad b \leqslant \tau < c \tag{A11}$$

Again variations in b and c will create a small probability for the event given by eqn. (A11) but for this analysis it is sufficient to know that $P(A_3)$ will be relatively small. This argument also applies as τ begins to exceed c. When $\tau > c$ and each ordinate occurs on a different flat the events can be considered to be independent and thus

$$P(A_3) = (A/A_n)^2 = a^2 \qquad \tau > c \tag{A12}$$