Tomlinson-Harashima Precoding in MIMO Systems: A Unified Approach to Transceiver Optimization Based on Multiplicative Schur-Convexity

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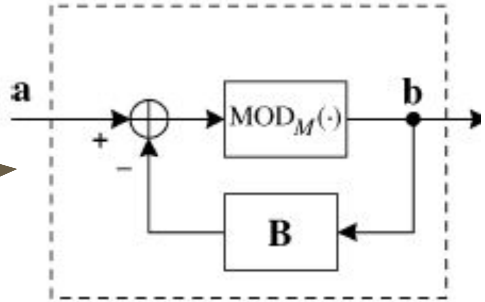
What is Tomlinson Harashima Precoding?

$$\mathrm{T(z) = S(z) - (A(z) - 1)T(z) - bN}$$

$$P(z) = T(z)\,A(z),$$

non-linear precoding unit



With matrices

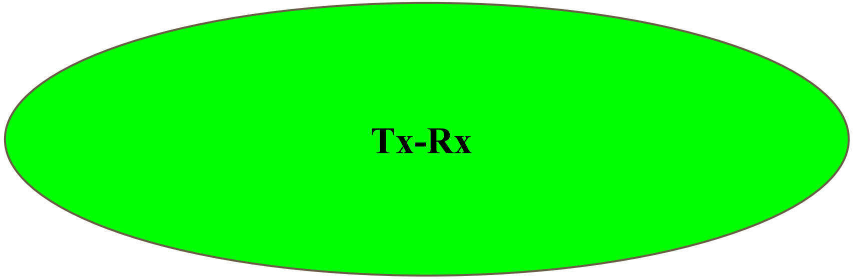
$$MOD_M(x) = x - 2\sqrt{M}\left\lfloor \frac{x+\sqrt{M}}{2\sqrt{M}} \right\rfloor$$

$$x = 0.99\sqrt{M};\, MOD_M(x) = 0.99\sqrt{M}$$

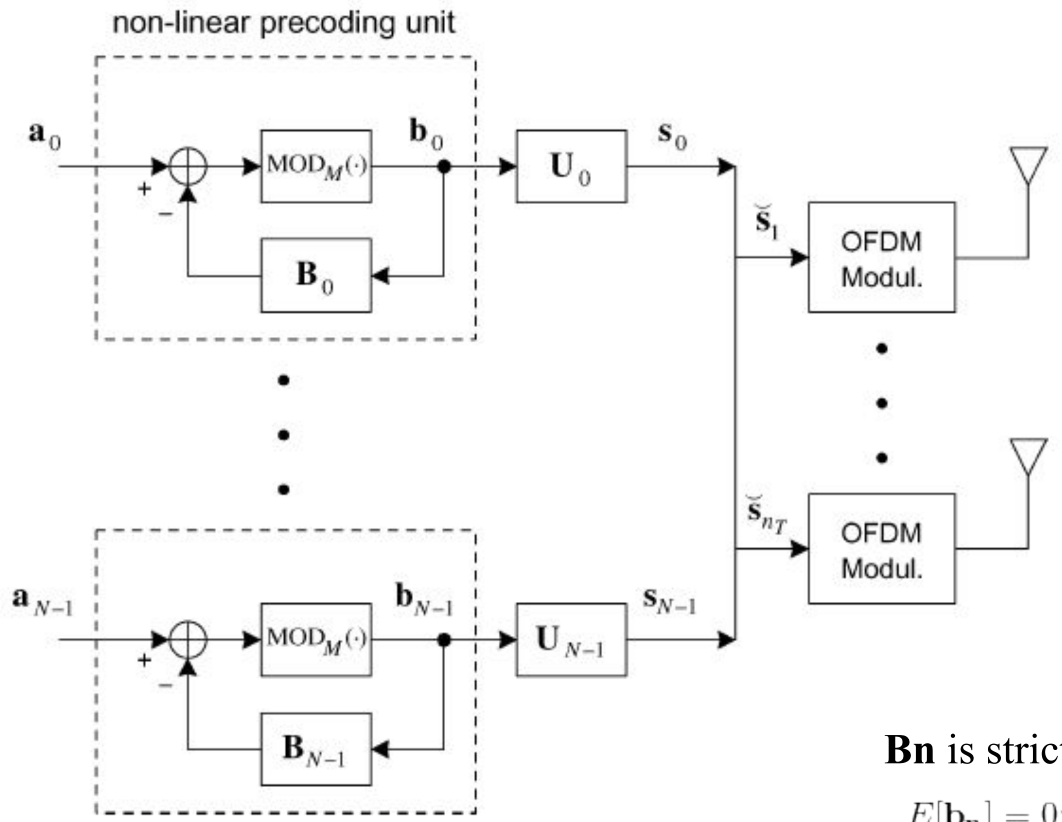
$$x = 1.99\sqrt{M};\, MOD_M(x) = -0.01\sqrt{M}$$

$$MOD_M(x)\ \epsilon\ \{-\sqrt{M}, \sqrt{M}\ \}$$

Why do Transceiver Optimization?



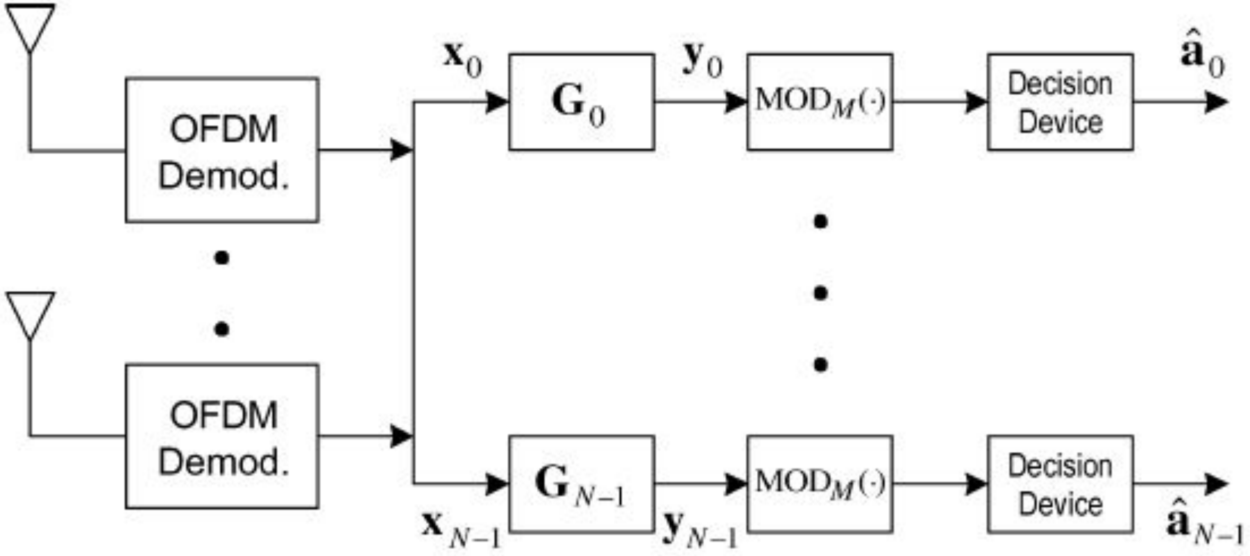
Non- Cooperative Scheme and Transmitter Structure



Bn is strictly lower triangular

$$E[\mathbf{b_n}] = 0; E[\mathbf{b_n} \mathbf{b_n^H}] = \sigma_a^2 \mathbf{I_K}$$

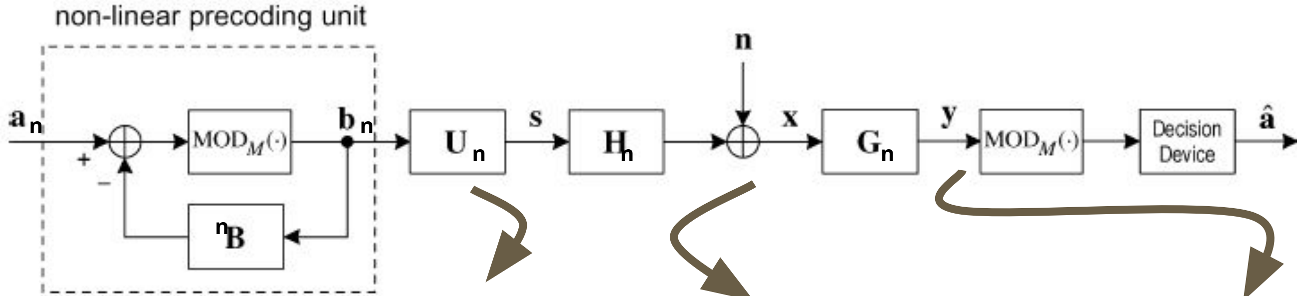
Receiver Structure



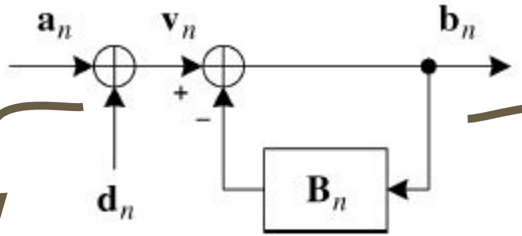
$$E[\mathbf{w_n}] = 0;\ E[\mathbf{w_n}\mathbf{w_n^H}] = \sigma_w^2 \mathbf{I_{n_R}}$$

$$\rho = \sigma_w^2 / \sigma_a^2$$

Signal Model



$$\mathbf{s_n} = \mathbf{U_n b_n} \qquad \mathbf{x_n} = \mathbf{H_n s_n} + \mathbf{w_n} \qquad \mathbf{y_n} = \mathbf{G_n x_n}$$



$$\mathbf{v_n} = \mathbf{a_n} + \mathbf{d_n}$$

$$\mathbf{v_n} = (\mathbf{I_k} + \mathbf{B_n})\mathbf{b_n} = \mathbf{C_n b_n}$$

$$\mathbf{y_n} = \mathbf{G_n H_n U_n b_n} + \mathbf{G_n w_n}$$

$$\mathrm{MSE}_{\mathbf{n,k}} = \mathrm{E}[|(\mathbf{y_n} - \mathbf{v_n})_{\mathbf{k}}|^2]$$

$$= \mathrm{E}[(\mathbf{y_n} - \mathbf{C_n b_n})(\mathbf{y_n} - \mathbf{C_n b_n})^{\mathbf{H}}]_{\mathbf{k,k}}$$

Optimization Problem and NC - OPA

Global Problem

$$\min_{P_n} \ \tilde{f}(P_0, P_1, \dots P_{N-1})$$

$$\text{s.t.} \quad \sum_{n=0}^{N-1} P_n = KN \ ; \ P_n \geq 0$$

where $\tilde{f}(P_0, P_1, \dots P_{N-1}) = f(\hat{\alpha_0}(P_0), \hat{\alpha_1}(P_1), \dots, \alpha_{\hat{N}-1}(P_{N-1}))$

$\hat{\alpha_n}(P_n)$ is the optimum solution to the below sub-problem

Sub - problem

$$\min_{\mathbf{C_n}, \mathbf{U_n}, \mathbf{G_n}} \ \alpha_n = f_n(MSE_{n,1}, \dots MSE_{n,K})$$

$$\text{s.t.} \quad Tr(\mathbf{U_n}\mathbf{U_n^H}) = P_n \ ; \ P_n \geq 0$$

Optimization of Gn

$$\mathbf{y_n} = \hat{\mathbf{v}}_\mathbf{n} = \mathbf{G_n x_n} \qquad \cong \quad \hat{\mathbf{x}} = \omega^{\mathbf{H}}\mathbf{y}$$

$$[(\mathbf{G_n})^{\mathbf{H}}_{\mathbf{opt}}]_{:,\mathbf{k}} = \mathbf{R}^{-1}_{\mathbf{x_n}}\mathbf{R}_{\mathbf{x_n v_n}} \qquad \because \omega_{\mathbf{opt}} = \mathbf{R}^{-1}_{\mathbf{y}}\mathbf{R}_{\mathbf{yx}}$$

$$[(\mathbf{G_n})^{\mathbf{H}}_{\mathbf{opt}}]_{:,\mathbf{k}} = (\mathbf{H_n U_n U_n^H H_n^H} + \rho \mathbf{I}_{\mathbf{n_R}})^{-1}[\mathbf{H_n U_n C_n^H}]_{:,\mathbf{k}}$$

$$\mathbf{LMMSE_{n,k}} = \mathbf{E}[|\tilde{\mathbf{x}}|^2] = \mathbf{E}[|(\tilde{\mathbf{v}}_\mathbf{n})_\mathbf{k}|^2]$$

$$\mathbf{LMMSE_{n,k}} = [\mathbf{R}_{\mathbf{v_n}}]_{\mathbf{k,k}} - [\mathbf{R}_{\mathbf{v_n x_n}}]_{\mathbf{k},:}[(\mathbf{G_n})^{\mathbf{H}}_{\mathbf{opt}}]_{:,\mathbf{k}} \qquad \because \mathbf{LMMSE} = \sigma^2_{\mathbf{x}} - \mathbf{R}_{\mathbf{xy}}\omega_{\mathbf{opt}}$$

$$\mathbf{MSE_{n,k}} = \mathbf{LMMSE_{n,k}} = \sigma^2_{\mathbf{w}}[\mathbf{C_n}(\mathbf{H_n U_n U_n^H H_n^H} + \rho \mathbf{I}_{\mathbf{n_R}})^{-1}\mathbf{C_n^H}]_{\mathbf{k,k}}$$

Optimization of Cn

$$\mathbf{MSE_{n,k}} = \sigma_{\mathbf{w}}^{2}([\mathbf{C_n^H}]_{:,\mathbf{k}})^{\mathbf{H}}(\mathbf{H_n U_n U_n^H H_n^H} + \rho \mathbf{I_{n_R}})^{-1}[\mathbf{C_n^H}]_{:,\mathbf{k}}$$

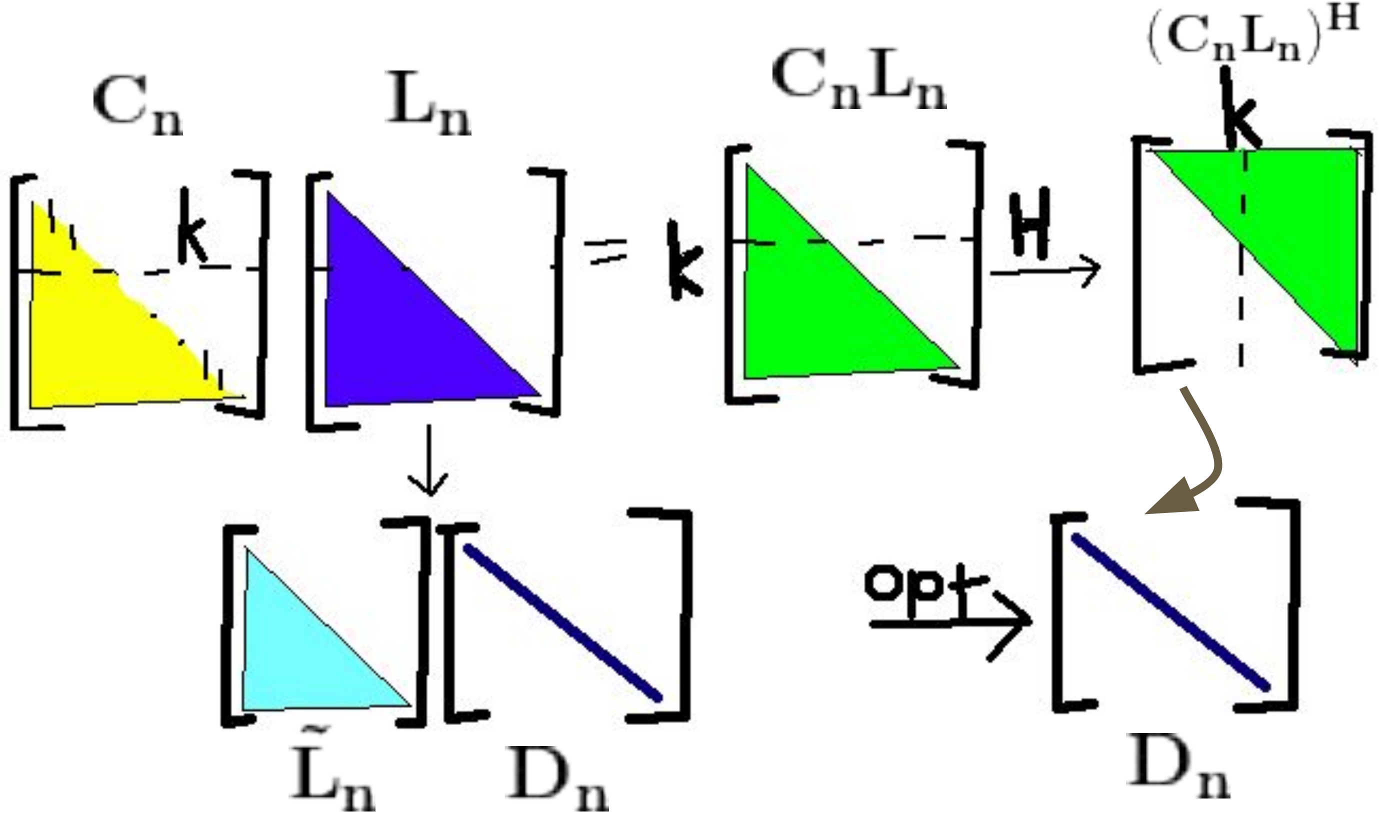
$$= \sigma_{\mathbf{w}}^{2}([\mathbf{C_n^H}]_{:,\mathbf{k}})^{\mathbf{H}}\mathbf{L_n L_n^H}[\mathbf{C_n^H}]_{:,\mathbf{k}} = \sigma_{\mathbf{w}}^{2}||[(\mathbf{C_n L_n})^{\mathbf{H}}]_{:,\mathbf{k}}||^{2}$$

$$= \sigma_{\mathbf{w}}^{2}\sum_{\mathbf{i}=0}^{\mathbf{k}-1}[\mathbf{L_n}]_{\mathbf{i},\mathbf{i}}^{2}\,|[(\mathbf{C_n \tilde{L}_n})^{\mathbf{H}}]_{\mathbf{i},\mathbf{k}}|^{2} + \sigma_{\mathbf{w}}^{2}[\mathbf{L_n}]_{\mathbf{k},\mathbf{k}}^{2}$$

where $\mathbf{L_n} = \mathbf{\tilde{L}_n D_n}\ ;\ \mathbf{D_n} = \mathrm{diag}\{[\mathbf{L_n}]_{1,1}, \ldots, [\mathbf{L_n}]_{\mathbf{K},\mathbf{K}}\}$

$\mathbf{C_n^{opt}} = \mathbf{\tilde{L}_n^{-1}} = \mathbf{D_n L_n^{-1}}$ $\qquad MSE_{n,k} = \sigma_w^2[\mathbf{L_n}]_{\mathbf{k},\mathbf{k}}^{2}$

Idea of Optimum Cn



Multiplicative Schur functions

Definition 1: Given a vector $\mathbf{x} = (x_1, x_2, \ldots, x_q)^T \in \mathbb{R}^q_{++}$, denote by $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[q]} > 0$ the elements of $\mathbf{x}$ in decreasing order.

Definition 2: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^q_{++}$ we say that $\mathbf{x}$ is multiplicatively majorized by $\mathbf{y}$ and we write $\mathbf{x} \prec_{\Pi} \mathbf{y}$ if

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]} \quad \text{for} \quad 1 \leq k \leq q-1$$

$$\prod_{i=1}^{q} x_{[i]} = \prod_{i=1}^{q} y_{[i]}.$$

Definition 3: A function $f : \mathbb{R}^q_{++} \to \mathbb{R}$ is said multiplicatively Schur-concave if for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^q_{++}$

$$\mathbf{x} \prec_{\Pi} \mathbf{y} \Rightarrow f(\mathbf{x}) \geq f(\mathbf{y}).$$

Similarly, f is said multiplicatively Schur-convex if

$$\mathbf{x} \prec_{\Pi} \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y}).$$

Optimization of Un for multiplicative Schur Concave ϕ

Let $\mathbf{U_n} = \mathbf{X}\ ;\ \mathbf{H_n^H H_n} = \mathbf{R}\ ;\ \mathbf{L_n} = \mathbf{L}\ ;\ T = P_n$

$$\mathbf{A} = (\mathbf{X^H R X} + \rho \mathbf{I_K})^{-1} = \mathbf{L L^H}$$

Where **A** is positive definite, **R** is positive semi-definite

$$\mathbf{d_L^2(X)} = ([\mathbf{L}]^2_{\mathbf{1,1}}, \ldots, [\mathbf{L}]^2_{\mathbf{q,q}})$$

$$\min_{\mathbf{X}} \phi(\mathbf{d_L^2(X)})$$

$$st\ Tr(\mathbf{XX^H}) = T$$

$$\mathbf{X_{opt}} = \mathbf{E\Gamma}$$

$$\Longrightarrow \quad \mathbf{U_n^{opt}} = \mathbf{E_n \Gamma_n}$$

$$\mathbf{C_n} = \mathbf{I_K}\ ;\ \mathbf{B_n} = \mathbf{0} \quad \because \mathbf{L_n} \sim \mathbf{diag}(..)$$

Eg :

$$f_n(MSE_{n,k}) = \prod_{k=1}^{K} MSE_{n,k}^{\beta_k}$$

$$s.t.\quad 0 \le \beta_1 \quad \le \quad \ldots.. \quad \le \beta_K$$

Outline of Proof

Obtain the condition on X (find lower bound)

$$\mathbf{d}^2_{\mathbf{L}}(\mathbf{X}) = ([\mathbf{L}]^2_{1,1}, ..., [\mathbf{L}]^2_{q,q})$$

$$\lambda_A(X) = (\lambda_1, ..., \lambda_q)$$

By Weyl's Theorem

$$\mathbf{d}^2_L(X) <_\pi \lambda_{\mathbf{A}}(\mathbf{X}) \Rightarrow \phi(\lambda_{\mathbf{A}}(\mathbf{X})) \le \phi(\mathbf{d}^2_{\mathbf{L}}(\mathbf{X}))$$

Equality is achieved when **A** is diagonal or $\mathrm{X^H R X}$ is diagonal

$$\tilde{\mathbf{X}} = \mathbf{X}\tilde{\mathbf{Q}}\ s.t\ Q : unitary$$

$$and\ \ \tilde{\mathbf{X}}^{\mathbf{H}}\mathbf{R}\tilde{\mathbf{X}}\ is\ diag$$

$$\phi(\mathbf{d}^2_{\mathbf{L}}(\tilde{\mathbf{X}})) = \phi(\lambda_{\mathbf{A}}(\mathbf{X})) \le \phi(\mathbf{d}^2_{\mathbf{L}}(\mathbf{X}))$$

Obtain closed form of X using the condition

If the conditions on the left are true then you can find

$$\tilde{\mathbf{X}}\ s.t.\ \tilde{\mathbf{X}}^{\mathbf{H}}\mathbf{R}\tilde{\mathbf{X}} = \mathbf{X}^{\mathbf{H}}\mathbf{R}\mathbf{X}$$

$$\tilde{\mathbf{X}} = \mathbf{E}\boldsymbol{\Sigma}\ and\ Tr(\tilde{\mathbf{X}}\tilde{\mathbf{X}}^{\mathbf{H}}) \le Tr(\mathbf{X}\mathbf{X}^{\mathbf{H}})$$

$$\mathbf{X}_{\mathbf{opt}} = \hat{\mathbf{X}} = \alpha\tilde{\mathbf{X}} = \mathbf{E}(\alpha\boldsymbol{\Sigma}) = \mathbf{E}\boldsymbol{\Gamma}$$

$$where\ \alpha = \sqrt{\frac{T}{Tr(\tilde{\mathbf{X}}\tilde{\mathbf{X}}^{\mathbf{H}})}} \ge 1$$

Optimization of Un for multiplicative Schur Convex ϕ

$$\mathbf{d}_{\mathbf{L}}^{\mathbf{2}}(\mathbf{X}) = ([\mathbf{L}]_{\mathbf{1,1}}^{\mathbf{2}}, ..., [\mathbf{L}]_{\mathbf{q,q}}^{\mathbf{2}})$$

$$\min_{\mathbf{X}} \phi(\mathbf{d}_{\mathbf{L}}^{\mathbf{2}}(\mathbf{X}))$$

$$st \; Tr(\mathbf{X}\mathbf{X}^{\mathbf{H}}) = T$$

$$\mathbf{X}_{\mathbf{opt}} = \mathbf{E}\mathbf{\Omega}\mathbf{F}$$

$$s.t. \; \mathbf{\Omega} = \mathbf{diag}(\omega_{\mathbf{1}}, ...\omega_{\mathbf{q}})$$

$$and \; \omega_i^2 = \left(\mu - \frac{\rho}{\eta_i}\right)_+ \quad \sum_{i=1}^{q} \omega_i^2 = T$$

F is an unitary matrix which makes all main diagonal elements of L equal

$$[\mathbf{d}_{\mathbf{L}}^{\mathbf{2}}(\mathbf{X}_{\mathbf{opt}})]_{\mathbf{k}} = \left(\prod_{\mathbf{i=1}}^{\mathbf{q}} \frac{\mathbf{1}}{\eta_{\mathbf{i}}\omega_{\mathbf{i}}^{\mathbf{2}} + \rho}\right)^{\frac{\mathbf{1}}{\mathbf{q}}}$$

$$MSE_n = \sigma_w^2 \left(\prod_{i=1}^{q} \frac{1}{\eta_i \omega_i^2 + \rho}\right)^{\frac{1}{q}}$$

$(\eta_1, ..., \eta_q)$ are the q largest eigen values of **R**

Eg: ARITH-MSE

$$\min_{\mathbf{Cn,Gn,Un}} \; f(MSE_{n,k}) = \sum_{n=0}^{N-1} \sum_{k=1}^{K} MSE_{n,k}$$

$$\text{s.t.} \quad \sum_{n=0}^{N-1} Tr(U_n U_n^H) = KN$$

$$\text{where} \quad \text{f}(\alpha_0, ..., \alpha_{N-1}) = \sum_{n=0}^{N-1} \alpha_n$$

$$\alpha_n = f_n(MSE_{n,1}, ..., MSE_{n,K}) = \sum_{k=1}^{K} MSE_{n,k}$$

$$\hat{\alpha_n}(P_n) = K(MSE_{n,k})$$

Outline of Proof

Obtain the condition on X (find lower bound)

$$\mathbf{x_{GM}} <_{\pi} \mathbf{x} \; where \; [\mathbf{x_{GM}}]_i = \left(\prod_{j=1}^{q} x_j\right)^{\frac{1}{q}}$$

$$\mathbf{F} \; : \; unitary \; s.t. \; \mathbf{F^H A F} = \mathbf{L L^H}$$
$$where \; [\mathbf{L}]_{\mathbf{k,k}} \; are \; all \; equal$$

$$\mathbf{F^H}(\mathbf{X^H R X} + \rho \mathbf{I_K})^{-1}\mathbf{F} = (\mathbf{F^H X^H R X F} + \rho \mathbf{I_K})^{-1}$$

$$\phi(\Delta^{1/q}, \ldots, \Delta^{1/q}) = \phi(\mathbf{d_L^2}(\mathbf{XF})) \leq \phi(\mathbf{d_L^2}(\mathbf{X}))$$

$$find \; \tilde{\mathbf{X}} \; which \; is \; the \; optimum \; of$$
$$\min_{\mathbf{X}} \; \Delta = det((\mathbf{X^H R X} + \rho \mathbf{I_K})^{-1})$$
$$s.t. \; Tr(\mathbf{X X^H}) = T$$

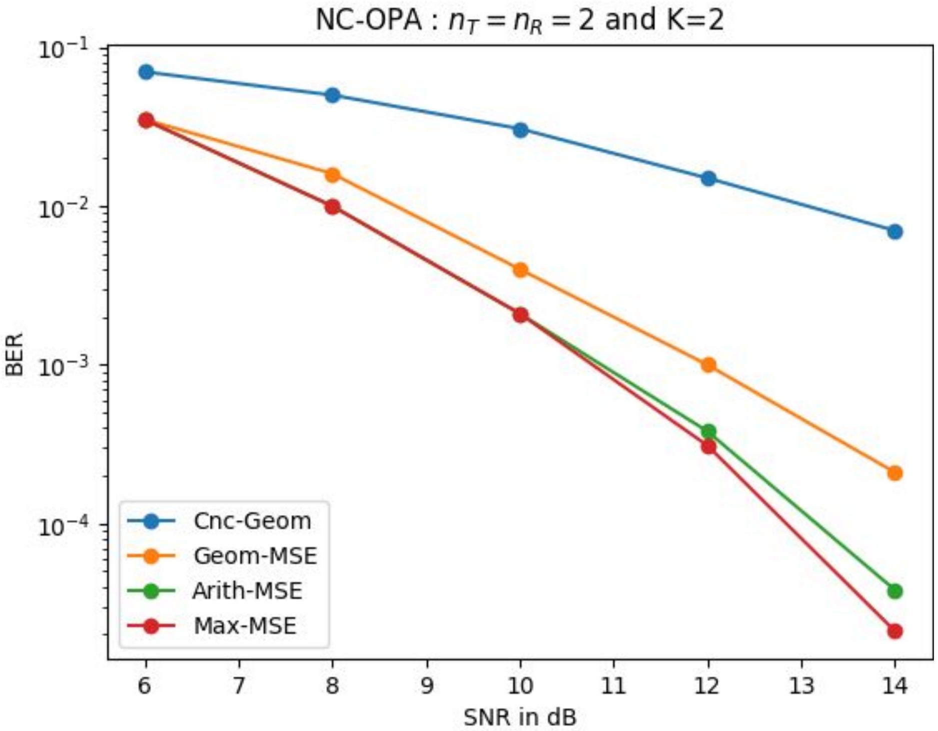
Obtain closed form of X using the condition

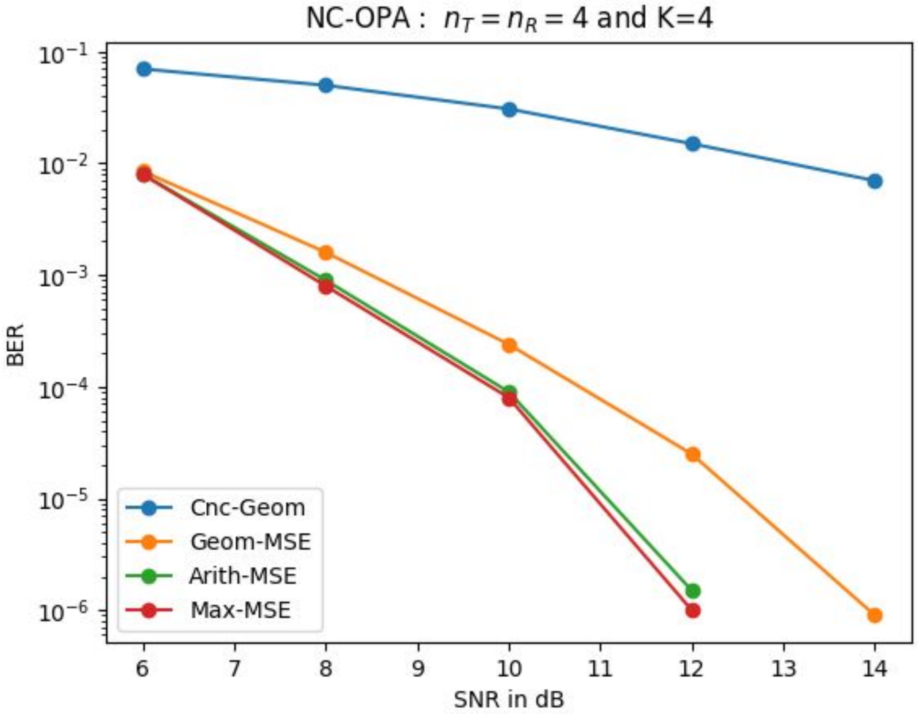
$$\hat{\mathbf{X}} = \mathbf{X_{opt}} = \tilde{\mathbf{X}}\mathbf{F}$$

If the conditions on the left are true then you can find

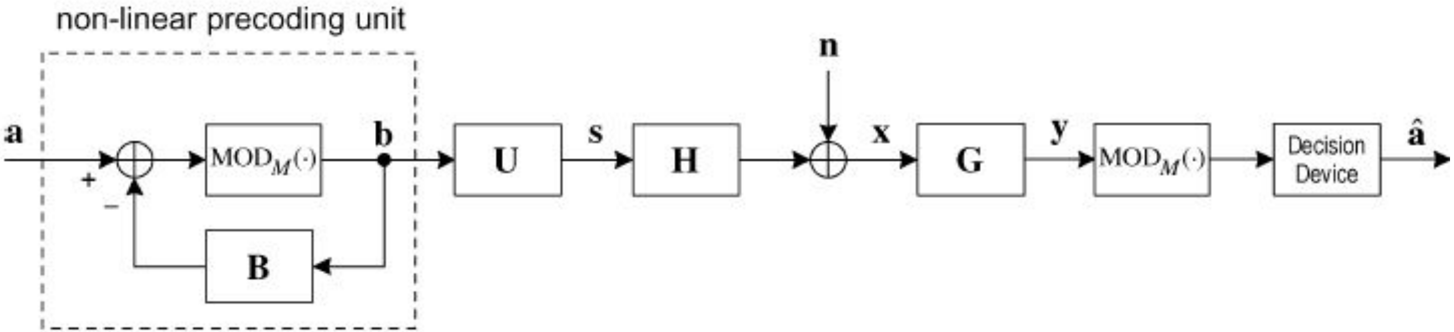
$$\mathbf{X_{opt}} = \mathbf{E \Omega F}$$
$$s.t. \; \mathbf{\Omega} = \mathbf{diag}(\omega_1, \ldots \omega_\mathbf{q})$$
$$and \; \omega_i^2 = \left(\mu - \frac{\rho}{\eta_i}\right)_+ \quad \sum_{i=1}^{q} \omega_i^2 = T$$

Simulation Results for NC-OPA





Carrier Cooperative Scheme



$$\mathbf{y} = \mathbf{GHUb} + \mathbf{Gw} \; where \; \mathbf{H} = \mathbf{diag}(\mathbf{H_0}, \ldots \mathbf{H_{N-1}})$$

From a mathematical point of view, the multicarrier cooperative model is equivalent to the single-carrier model. It is a more general scheme that is devised allowing cooperation among subcarriers (carrier-cooperative approach, [9]).

Takeaways

- THP is mainly used to remove interference
- In NC-OPA you can independently process and optimize for each sub-carrier
- Optimization of the Receive matrix is simply LMMSE estimation
- Optimization of the precoding matrix is norm minimization
- Optimization of Transmit matrix for Multiplicative Schur concave is aligning the vectors along the beam-forming directions (channel diagonalization)
- Optimization of Transmit matrix for Multiplicative Schur convex boils down to the water-filling problem
- Block processing of all carriers is done in carrier cooperative scheme, and its performance is better than Non-cooperative scheme

Thank You