Assignment 2

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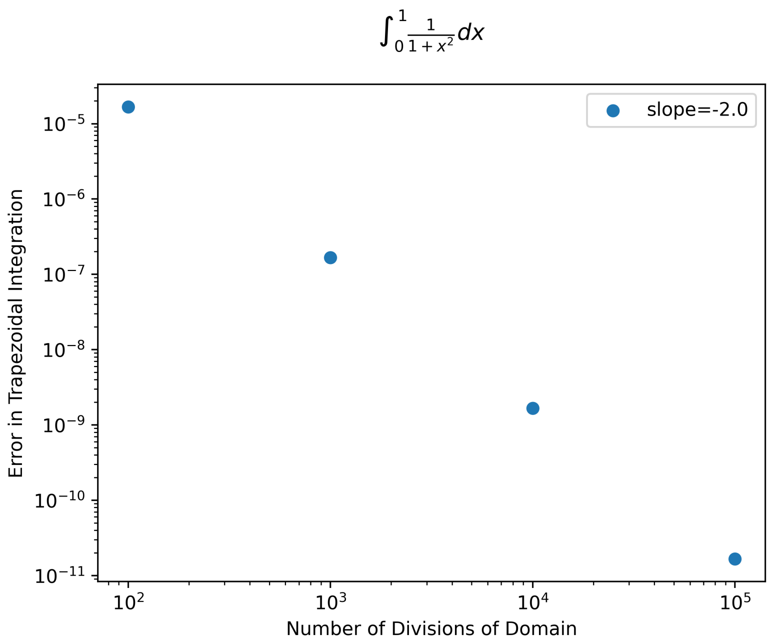
1. a)

$$\int_0^1 \frac{4}{1+x^2} dx = 3.1415919869231304$$

1. b)



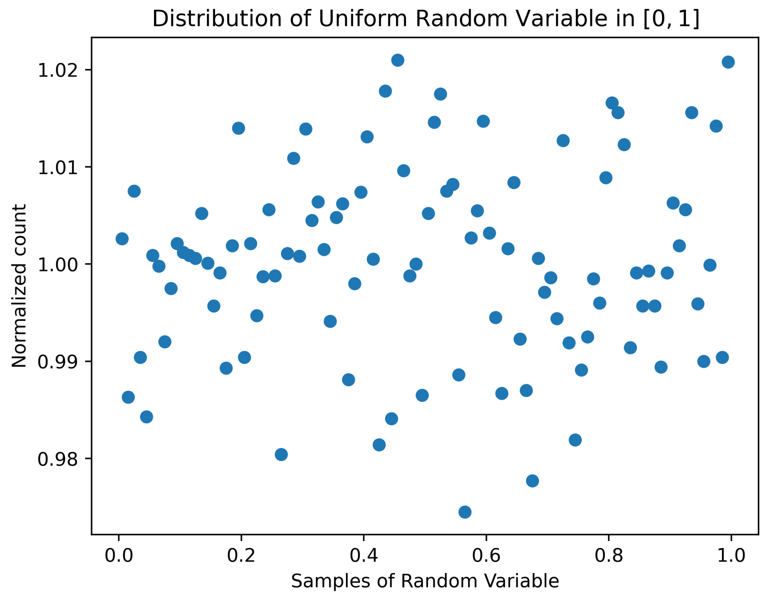
1. c)

$$\int_0^\pi \sin x dx = 1.9999934202594030$$

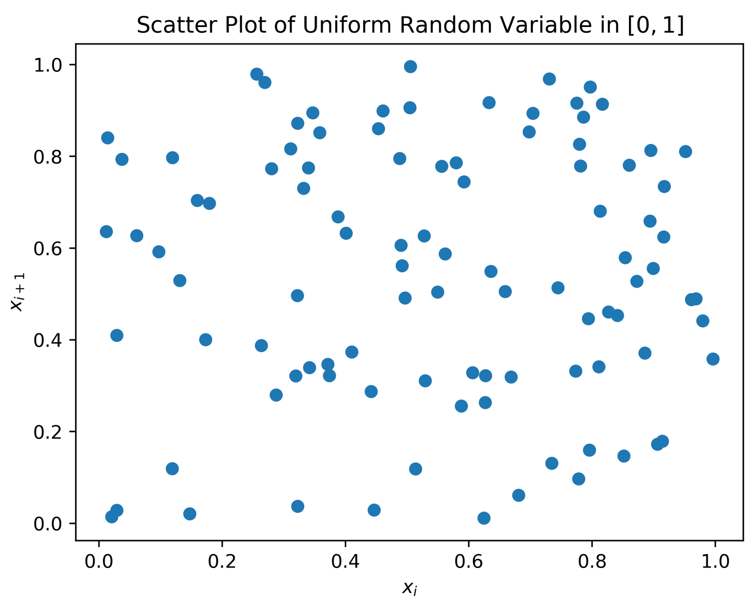
1. d)

$$\int_0^\pi \frac{1}{2\pi} \exp\left(-\frac{x^2}{2}\right) dx = 0.99729988484824805$$

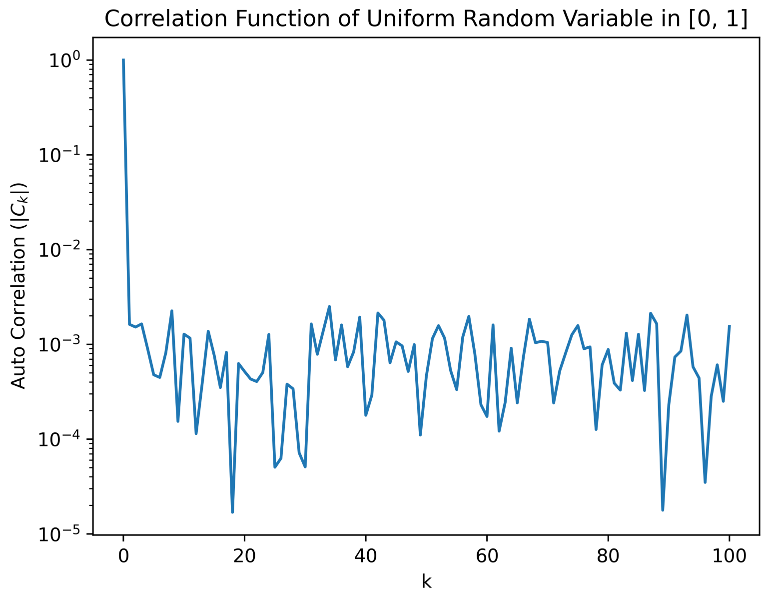
2. a)


Distribution of Uniform Random Variable in [0, 1]
Normalized count
Samples of Random Variable
1.02
1.01
1.00
0.99
0.98
0.0
0.2
0.4
0.6
0.8
1.0

2. b)

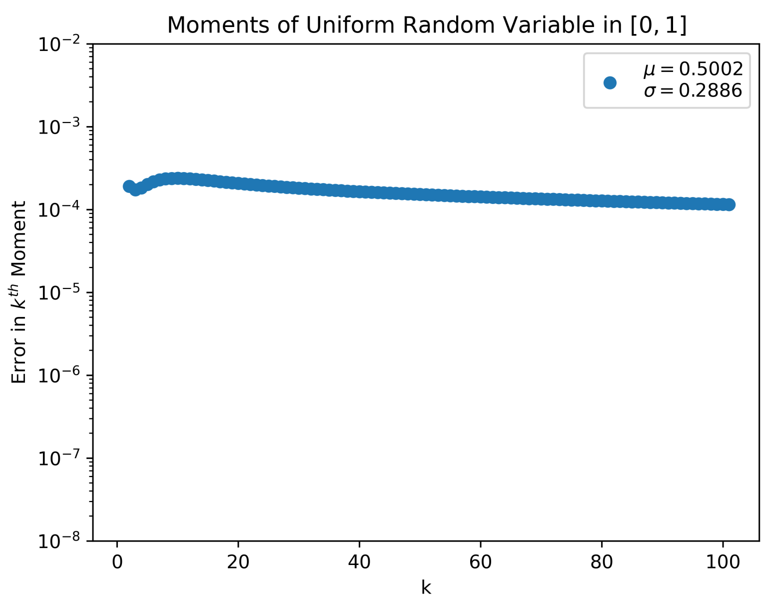

Scatter Plot of Uniform Random Variable in [0, 1]
x_{i+1}
x_i
0.0
0.2
0.4
0.6
0.8
1.0

2. c)



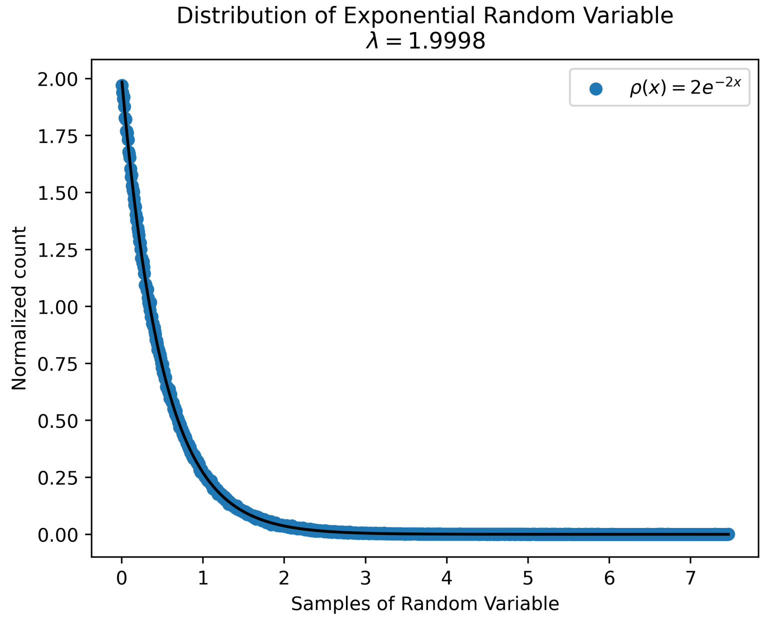
Correlation Function of Uniform Random Variable in [0, 1]
Auto Correlation ($|C_k|$)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0
20
40
60
80
100
k

2. d)



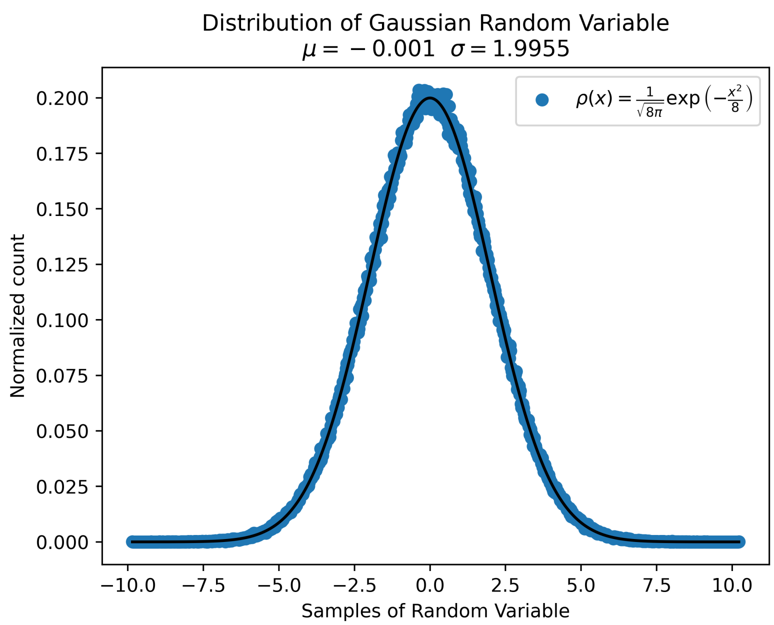
Moments of Uniform Random Variable in $[0, 1]$
$\mu = 0.5002$
$\sigma = 0.2886$
Error in k^{th} Moment
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
0
20
40
60
80
100
k

4. a)



Distribution of Exponential Random Variable
$\lambda = 1.9998$
$\rho(x) = 2e^{-2x}$
Normalized count
Samples of Random Variable
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
6
7

4. b)



Distribution of Gaussian Random Variable
$\mu = -0.001 \quad \sigma = 1.9955$
$\rho(x) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{8}\right)$
Normalized count
Samples of Random Variable
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

5. a)

We need to use brute force Monte Carlo in the interval S^6 where $S = [-a, a]$, let us get an estimate of the error of neglecting the region $\mathbb{R}^6 - S^6$.

$$\begin{aligned}
&g(\vec{x}, \vec{y}) \leq \exp(-\vec{x}^2 - \vec{y}^2) \\
\implies &\int_{\mathbb{R}^6 - S^6} g \leq \int_{\mathbb{R}^6 - S^6} \exp(-\vec{x}^2 - \vec{y}^2) dx^3 dy^3 \\
&\qquad = \int_{\mathbb{R}^6} \exp(-\vec{x}^2 - \vec{y}^2) dx^3 dy^3 - \int_{S^6} \exp(-\vec{x}^2 - \vec{y}^2) dx^3 dy^3 \\
&\qquad = \left(\int_{\mathbb{R}} \exp(-x^2) dx \right)^6 - \left(\int_{S} \exp(-x^2) dx \right)^6
\end{aligned}$$

Let $m = \int_S \exp(-x^2)dx$, $n = \int_{\mathbb{R}-S} \exp(-x^2)dx$, then we can rewrite the above expression as

$$\begin{aligned}
&= (m+n)^6 - m^6 \\
&= 6m^5 n + 15m^4 n^2 + 20m^3 n^3 + 15m^2 n^4 + 6mn^5 + n^6
\end{aligned}$$

Clearly we are only concerned with the case where $n \ll m$, which simplifies things to

$$\begin{aligned}
&\leq 6m^5 n + 57m^4 n^2 \\
&\leq 6(2a)^5 n + 57(2a)^4 n^2
\end{aligned}$$

we can also write

$$n = \sqrt{\pi} - \int_{-a}^{a} e^{-x^2} dx = \sqrt{\pi} \left(1 - \int_{-a\sqrt{2}}^{a\sqrt{2}} \mathcal{N}(x) dx \right)$$

where $\mathcal{N}$ is the normal distribution. For $a = 5$, $5\sqrt{2}$ corresponds to atleast 7σ deviations in the normal distribution which gives us

$$n \leq \sqrt{\pi}(1 - 0.999999999997440) \approx 4.537 \cdot 10^{-12}$$

Substituting back, we get

$$\int_{\mathbb{R}^6 - S^6} g \leq 2.222 \times 10^{-6}$$

which is a reasonable upper bound on the error.

Performing the integration with $N = 10^6$ (we keep this constant across all Monte Carlo integration methods), we get

$$I \approx 11.7771011$$

Error estimated by Monte Carlo is 1.2174044 while actual error is 0.8147269.

5. b)

Now we do importance sampling with the Gaussian distribution in all 6 dimensions because the form of g is similar to a Gaussian. Performing the integration, we get

$$I \approx 10.9596312$$

Error estimated by Monte Carlo is 0.0227881 while actual error is 0.0027431.

We can also integrate using brute force in a finite region if we do the change of variables $x_i = \tan y_i$ in each dimension, we get upon integration

$$I \approx 11.0263890$$

Error estimated by Monte Carlo is 0.0606193 while actual error is 0.0640148.