Random Walks and Electrical Networks

Goal of the talk: **Polya's Theorem about Random Walks** and an "application" to **Flatland**.

Reference: *Doyle and Snell. Random walks and electric networks, 2006*

Random Walks

- Definition of random walks on graphs using transition matrix
- Definition of the harmonic function problem on finite graphs and motivation from random walks
- Existence and uniqueness of harmonic functions

Electrical Networks

- Definition and notations: resistance, current, voltage
- Voltage is harmonic w.r.t. conductance
- Equivalence to the random walk problem: voltage is the hitting probability
- Motivation for probabilistic interpretation of current

Note: the probabilistic interpretations are Monte Carlo methods of solution to the harmonic function problem.

Effective Resistance

- Definition of effective resistance
- Escape probability definition: $p_{esc} = 1 - \sum_y P_{ay} v_y$
- $p_{esc} = \frac{C_{eff}}{C_a} = \frac{R_a}{R_{eff}}$
- The currents minimise the total energy, and the minimum energy is $i_a^2 R_{eff}$
- Rayleigh's Monotonicity Law: proof using energy
- Series and parallel resistors

Polya's Recurrence Problem

- Definition of recurrent and transient walks on $\mathbb{Z}^d$
- Polya's original definition is equivalent
- Spheres in $\mathbb{Z}^d$ and their boundary
- Formalizing escape probability by using limit of spheres
- Electrical formulation

- Shorting: setting a resistance to 0
- Cutting: setting a resistance to ∞
- Shorting decreases the effective resistance and cutting increases it by monotonicity

Shorting the Plane

- Short concentric squares which can now be treated as a single node

Effective Resistance of some Trees

- Resistance of full binary tree
- Their Hausdorff dimension is too much (infinite), but resistance is easy to calculate
- Number of vertices in ball of radius r is r^d and at the boundary is r^{d-1} in $\mathbb{Z}^d$

Embedding Trees

- Definition of NT_2 and NT_3
- "Embedding" NT_2
- Doing a similar construction in $\mathbb{Z}^d$ yields $NT_{\log_2 6}$