Spectral Graph Theory

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1 Introduction

1.1 Eigenvalues and Optimization

Let M be a n dimensional symmetric matrix.

Definition 1.1: Rayleigh Quotient

The Rayleigh quotient for a matrix M is defined as

$$R(\boldsymbol{x}, M) := \frac{\boldsymbol{x}^T M \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$$

the matrix is ommitted if obvious from context.

Theorem 1.1

Let

$$\boldsymbol{x} \in \arg \max_{\boldsymbol{x} \in \mathbb{R}^n - \{0\}} R(\boldsymbol{x})$$

Such an $\boldsymbol{x}$ exists and is an eigenvector of M with the maximum eigenvalue μ_1. We can write a similar statement for the minimum eigenvalue by minimizing R.

Theorem 1.2: Spectral Theorem for Symmetric Matrices

There exist numbers $\mu_1, ..., \mu_n$ and orthonormal vectors $\boldsymbol{\psi}_1, ..., \boldsymbol{\psi}_n$ such that $M\boldsymbol{\psi}_i = \mu_i \boldsymbol{\psi}_i$ iff for $1 \leq i \leq n$

$$\boldsymbol{\psi}_i \in \arg \max_{\substack{\|\boldsymbol{x}\|=1 \\ \boldsymbol{x}^T \boldsymbol{\psi}_j = 0,\ j<i}} R(\boldsymbol{x})$$

or equivalently

$$\boldsymbol{\psi}_i \in \arg \min_{\substack{\|\boldsymbol{x}\|=1 \\ \boldsymbol{x}^T \boldsymbol{\psi}_j = 0,\ j>i}} R(\boldsymbol{x})$$

Theorem 1.3: Courant-Fischer Theorem

Let M have eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, then

$$\mu_k = \max_{\substack{S \subset \mathbb{R}^n \\ \dim(S)=k}} \min_{\substack{\boldsymbol{x} \in S \\ \boldsymbol{x} \neq 0}} R(\boldsymbol{x}) = \min_{\substack{T \subset \mathbb{R}^n \\ \dim(T)=n-k+1}} \max_{\substack{\boldsymbol{x} \in T \\ \boldsymbol{x} \neq 0}} R(\boldsymbol{x})$$

where S, T are subspaces of $\mathbb{R}^n$.

Theorem 1.4: Cauchy's Interlacing Theorem

Let A be a symmetric real matrix of dimension n. Let B be obtained by deleting the same row and column of A (N is a principal submatrix of dimension $n-1$). Let $\alpha_1 \geq \ldots \geq \alpha_n$ be the eigenvalues of A and $\beta_1 \geq \ldots \geq \beta_{n-1}$ be the eigenvalues of B. Then for $1 \leq k \leq n-1$

$$\alpha_k \geq \beta_k \geq \alpha_{k+1}$$

1.2 The Laplacian and Graph Drawing

Vectors on graphs are functions $V \to \mathbb{R}$, the vector $\mathbf{1}$ denotes the function $\mathbf{1}(a) = 1$. The degree function $\boldsymbol{d}$ is also a vector.

Matrices on graphs are functions $V \times V \to \mathbb{R}$ or can be viewed as linear operators on the space of vectors on graphs. Let G be a graph, then we use M_G to denote the adjacency matrix, D_G to denote the diagonal matrix of vertex degrees, and $L_G = D_G - M_G$ to denote the Laplacian matrix. Observe that $M_G\mathbf{1} = \boldsymbol{d}$ and $L_G\mathbf{1} = \mathbf{0}$.

The Laplacian is also a natural quadratic form on a graph

$$\boldsymbol{x}^T L_G \boldsymbol{x} = \sum_{(a,b) \in E(G)} w_{a,b}(\boldsymbol{x}(a) - \boldsymbol{x}(b))^2$$

Let $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of L_G with eigenvectors $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n$.

Lemma 1.5

G is connected iff $\lambda_2 \neq 0$.

We draw a graph in k dimensions by using the eigenvectors corresponding to $\lambda_2, \ldots, \lambda_{k+1}$ as the coordinates of vertices. These coordinates minimize the expression (excluding coordinates that lead to trivial drawings):

$$\sum_{(a,b) \in G} w_{a,b} \|x(a) - x(b)\|^2 = \sum_{i=1}^{k} \boldsymbol{x_i}^T L_G \boldsymbol{x_i}$$

where $x : V \to \mathbb{R}^k, x = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_k})$ is the coordinate function.

Theorem 1.6: Hall's Drawing Theorem

Let $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$ be orthonormal vectors that are orthogonal to $\mathbf{1}$, then

$$\sum_{i=1}^{k} \boldsymbol{x_i}^T L_G \boldsymbol{x_i} \geq \sum_{i=2}^{k+1} \lambda_1$$

where equality holds for $\boldsymbol{\psi}_j^T \boldsymbol{x_i} = 0$ for all i and $j > k+1$.

1.3 Adjacency Matrices

Let the eigenvalues of M_G be $\mu_1 \geq \ldots \geq \mu_n$.

Lemma 1.7

Let d_{avg} and d_{max} be the average and maximum degrees respectively, then

$$d_{avg} \leq \mu_1 \leq d_{max}$$

further, if H be a subgraph of G, then

$$d_{avg}(H) \leq \mu_1$$

Lemma 1.8

If G is connected and $\mu_1 = d_{max}$ then G is d_{max}-regular.

Theorem 1.9: Wilf's Theorem

Let $\chi(G)$ be the chromatic number of G, then

$$\chi(G) \leq \lfloor \mu_1 \rfloor + 1$$

Lemma 1.10

Let G be a connected weighted graph and $\boldsymbol{\phi}$ be a non negative eigenvector of M_G, then $\boldsymbol{\phi}$ is strictly positive.

Theorem 1.11: Perron-Frobenius Theorem

Let G be connected, then

1. μ_1 has a strictly positive eigenvector
2. $\mu_1 \geq -\mu_n$
3. $\mu_1 > \mu_2$

Lemma 1.12

If G is bipartite, then the eigenvalues of M_G are symmetric about 0.

Theorem 1.13

Let G be connected, $\mu_1 = -\mu_n$ iff G is bipartite.

1.4 Comparing Graphs

We introduce the partial order $\succeq$ on matrices as

$$A \succeq B \iff \forall \boldsymbol{x}, \boldsymbol{x}^T A \boldsymbol{x} \geq \boldsymbol{x}^T B \boldsymbol{x}$$

In particular $A \succeq 0$ means A is positive semidefinite. For graphs G, H on the same set of vertices, we write $G \succeq H$ iff $L_G \succeq L_H$. If H is a subset of G, we have

$$G \succeq H$$

For a graph H, define $c \cdot H$ to be the graph H with each edge weight multiplied by c. Let $\lambda_k(H)$ denote the kth smallest eigenvalue of L_H.

Lemma 1.14

If G, H are graphs

$$G \succeq c \cdot H \implies \lambda_k(G) \geq c\lambda_k(H)$$

Let $G_{a,b}$ be the graph with only the edge (a,b). The proof of the following lemmas follow trivially from the Cauchy Schwarz inequality applied to the Laplacian.

Lemma 1.15

Let P_n be the path graph on n vertices between vertex 1 and n.

$$G_{1,n} \preceq (n-1)P_n$$

Lemma 1.16: Extension to Weighted Paths

Let $P_{n,w}$ be the weighted path graph on n vertices with w_i the weight on the edge $(i, i+1)$.

$$G_{1,n} \preceq \left(\sum_{i=1}^{n-1} \frac{1}{w_i}\right) P_{n,w}$$

We use these lemmas to prove bounds on the eigenvalues of graphs.

Let K_n be the complete graph on n vertices, it is easy to see that $\lambda_i(K_n) = n$ for $i \geq 2$. We can also write that

$$L_{K_n} = \sum_{a<b} L_{G_{a,b}}$$

this allows us to prove the following

$$K_n = \sum_{a<b} G_{a,b} \preceq \sum_{a<b} (b-a)P_{a,b} \preceq \sum_{a<b} (b-a)P_n$$
$$\implies \lambda_2(K_n) = n \leq \lambda_2(P_n) \sum_{a<b} (b-a) = \frac{n(n+1)(n-1)}{6} \lambda_2(P_n)$$

Lemma 1.17: Bounding λ_2 of the Path Graph

$$\lambda_2(P_n) \geq \frac{6}{(n+1)(n-1)}$$

Let T_d be the complete binary tree of depth d, this will have $n = 2^{d+1} - 1$ vertices. Let $T_d^{a,b}$ be the shortest path between vertices a, b on T_d. Note that this path has size atmost $2d \leq 2\log_2 n$. Doing a similar comparision with K_n we get

$$K_n = \sum_{a<b} G_{a,b} \preceq \sum_{a<b} 2dT_d^{a,b} \preceq \sum_{a<b} 2\log_2 nT_d = \binom{n}{2} 2\log_2 nT_d$$
$$\implies \lambda_2(K_n) = n \leq \binom{n}{2} 2\log_2 n\lambda_2(T_d)$$

Lemma 1.18: Bounding λ_2 of Complete Binary Trees

$$\lambda_2(T_d) \geq \frac{1}{(n-1)\log_2 n}$$

2 Random Graphs

An Erdös-Rényi random graph is a graph in which each edge is present with probability p independent of other edges. We can write the adjacency matrix M of this graph as

$$M = p(J - I) + R$$

where J is the all ones matrix, I is the identity matrix and R is defined for off diagonal entries as

$$R(a, b) = \begin{cases} 1 - p & \text{with probability } p \\ -p & \text{with probability } 1 - p \end{cases}$$

clearly the expectation of M is $\rho(J - I)$, which means the expectation of R is the zero matrix. This can be easily verified from the definition of R. We will show that the eigenvalues of R are usually small.

2.1 Appendix

Theorem 2.1: Markov's Inequality

Let X be a non negative random variable with $a > 0$, then

$$P(X > aE(X)) \leq a^{-1}$$

Proof.

$$\begin{aligned} E(X) &= P(X \leq aE(X))E(X|X \leq aE(X)) + P(X > aE(X))E(X|X > aE(X)) \\ \implies P(X > aE(X)) &\leq \frac{E(X)}{E(X|X > aE(X))} \leq \frac{E(X)}{aE(X)} = a^{-1} \end{aligned}$$

□

Theorem 2.2: Sylvester's Theorem

Let A be an $m \times n$ matrix and B be an $n \times m$ matrix, then we can write

$$x^n \det(xI_m - AB) = x^m \det(xI_n - BA)$$

Proof. Define the following matrices

$$C = \begin{bmatrix} xI_m & A \\ B & I_n \end{bmatrix}, \quad D = \begin{bmatrix} I_m & 0 \\ -B & xI_n \end{bmatrix}$$

we can see that

$$CD = \begin{bmatrix} xI_m - AB & xA \\ 0 & xI_n \end{bmatrix}, \quad DC = \begin{bmatrix} xI_m & A \\ 0 & xI_n - BA \end{bmatrix}$$

using $\det(CD) = \det(DC)$, we get the required result. □

Lemma 2.3

Let A be an $m \times n$ matrix and B be an $n \times m$ matrix, then AB and BA have the same set of non zero eigenvalues.

Proof. This follows trivially from Sylvester's theorem or from the following proof. Let $\boldsymbol{x}$ be an eigenvector of AB with eigenvalue $\lambda \neq 0$, it is simple to check that $\boldsymbol{y} = B\boldsymbol{x}$ is an eigenvector of BA with eigenvalue λ. □

Lemma 2.4

For positive definite symmetric matrices A, $\sqrt{A}$ exists and is unique.

Theorem 2.5: Cauchy Interlacing Theorem for Rank One Updates

Let A be a symmetric matrix with eigenvalues $\alpha_1 \geq ... \geq \alpha_n$. Let $B = A + \boldsymbol{x}\boldsymbol{x}^T$ for some vector $\boldsymbol{x}$, let B have eigenvalues $\beta_1 \geq ... \geq \beta_n$, then

$$\beta_i \geq \alpha_i \geq \beta_{i+1}$$

Proof. Proving the first inequality is trivial using Courant Fischer theorem, the proof of the second inequality using this method is not known (to me). Therefore, we will make use of a different method. We will prove the result for positive definite symmetric matrices A by adding $(1 - \alpha_n)I$ to A.

Clearly $\sqrt{A}$ is a well defined, now define the $n \times (n+1)$ matrix C as

$$C := \begin{bmatrix} \sqrt{A} & \boldsymbol{x} \end{bmatrix}$$

then we can write

$$CC^T = B, \quad C^T C = \begin{bmatrix} A & \sqrt{A}x \\ x^T\sqrt{A} & x^T x \end{bmatrix}$$

By previous lemma, C^TC and CC^T share the same set of nonnegative eigenvalues. The eigenvalues of A interlace the eigenvalues of C^TC since it is a principal submatrix. The eigenvalues of C^TC are thus all positive except the smallest (which is 0). Since the eigenvalues of $CC^T = B$ are positive, the result follows. $\square$