STUDY AND CONTROL OF DYANMICS OF INVERTED PENDULUM SYSTEM

A PROJECT REPORT

Submitted by

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BONAFIDE CERTIFICATE

Certified that this project report **"STUDY AND CONTROL OF DYNAMICS OF INVERTED PENDULUM SYSTEM"** is the bonafide work of **RITIKA (312216105079), ROHITH KUMAR K (312216105080), SRIRAM S (312216105107), VIGNESH R (312216105120)** who carried out the project work under my supervision.

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VIVA VOICE EXAMINATION

The Viva-Voice examination for the project work " **STUDY AND CONTROL OF DYNAMICS OF INVERTED PENDULUM SYSTEM** " submitted **by RITIKA (312216105079), ROHITH KUMAR K (312216105080), SRIRAM S (312216105107), VIGNESH R (312216105120)** was held on.....................

INTERNAL EXAMINER **EXTERNAL EXAMINER**

ABSTRACT

An Inverted Pendulum is a pendulum that stands in an inverted position against its natural equilibrium position by using a control system to monitor the angle of the pendulum. The Inverted Pendulum is a classic problem in dynamics and control theory and is used as benchmark for testing control strategies.

The pendulum is mounted on a cart that can move horizontally with the help of a DC motor and when the pendulum starts to fall over, the cart is moved in the corresponding direction, thus, balancing the pendulum. The entire system is controlled by a STM32 microcontroller which is interfaced with encoders for feedback control and a motor driver circuit for speed and direction control of motor.

The inverted pendulum is a non-linear unstable system and hence by solving this problem, it provides insight into linearizing a non-linear system to determine the equations of motion and also, the design of a controller to stabilize the unstable system.

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CHAPTER 1
INTRODUCTION

1.1 GENERAL INTRODUCTION

An Inverted Pendulum is a pendulum that has its center of mass above its pivot point. It can be suspended stably in this inverted position by using a control system to monitor the angle of the pole and move the pivot point horizontally back under the center of mass when it starts to fall over, keeping it balanced. The inverted pendulum is a classic problem in dynamics and control theory and is used as a benchmark for testing control strategies.

The primary practical applications of inverted pendulum are orbital motion of satellite & Segway – self balancing scooters whose control mechanism is similar and it gives the primitive knowledge to design and control.

Figure 1.1 Application of Inverted Pendulum

1.2 PROBLEM STATEMENT

The Inverted Pendulum has to be controlled so that the pendulum arm stands at (i.e.,) 180^o against gravity and remains balanced. The controller should be designed in such a way that it should provide reliable performance or output, despite of external disturbance like wind etc. Another major problem is that the computation time required by microcontroller to run control algorithm. Excessive computation delay leads to more time for correction of the angle of the pendulum and leaves the system with steady state error and peak overshoots. The last problem statement is to make the microcontroller to communicate with the computer in order to plot the graph between the state variables of the system.

1.3 PROBLEM SOLVING

The goal of project is to investigate the use of digital control algorithm implemented on a microcontroller. The digital control algorithm uses proportional and derivative (PD), which provides effective control to attain the reference position (180^{o}) of the pendulum also to maintain the system stable. The current angle of the pendulum is measured by the rotary quadrature encoder. Over the period, the microcontroller provides versatile, quicker, cheaper and reliable output. The project is built with STM32F103C8T6, a powerful and cost-effective board. High torque DC gear motors are used to drive the inverted pendulum to balance at inverted position. Keeping track of angle is important because the information is fed back to microcontroller in a feedback loop from encoders, to minimize the error and to achieve equilibrium. The microcontroller maintains the inverted position of the pendulum by varying the cart position by varying the voltage given to the DC motor. To control the speed of the BLDC motor, electronic speed controllers (ESC) are used and pulse width modulated (PWM) pulses are applied to the ESC. By varying the duty cycle of the PWM pulse, the speed of the motor is varied.

1.4 OBJECTIVES

To design a controller for Inverted Pendulum using STM32F103C8T6.

- To balance the Pendulum at inverted position.
- To make the Pendulum robust to external disturbances.
- To plot the state variables of the system.

1.5 METHODOLOGY

The objective is fulfilled using the following method

1. Derive equations of motion based on theory of Lagrangian mechanics.
2. Form state space model for the system.
3. Find a controller that can control these two conditions.
4. Hardware construction and physical modeling.
5. Implementation of controller using STM32F103C8T6.
6. Testing the Final Design.

1.6 REPORT ORGANIZATION

Chapter 1 of the report deals with Introduction. It includes the problem statement, objectives and the scope of this project.

Chapter 2 of the report deals with Literature Review. It includes study about control techniques namely, PD, feedback linearization.

Chapter 3 of the report deals with Mathematical modeling of the system (plant). It includes the dynamic equations.

Chapter 4 of report deals with controller design namely, PID and PD controllers.

Chapter 5 of report deals with Hardware Description and Identification of System Parameters namely, b/I calculations, moment of inertia and center of mass. Also, it includes the study of Force Vs Voltage mapping for cart-pole pendulum system.

Chapter 6 of report deals with Pole placement technique for Inverted Pendulum.

Chapter 7 of this report deals with the response obtained from the Inverted Pendulum system and study of the system dynamics.

Chapter 8 of this project deals with the future extensions of this project.

CHAPTER 2

LITERATURE REVIEW

2.1 PENDULUM MODEL

To understand the system better, analysis of a simple pendulum is crucial. According to Figure 2.1, a mass, M and moment of inertia, I, is connected by a massless rod of length L to a frictionless pivot. The angular velocity, ω and the rate of change of angular velocity, $d\omega/dt$ are given by the equation 2.1 and 2.2,

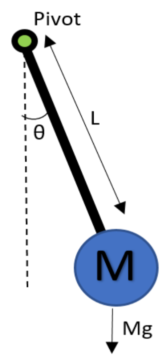

Figure 2.1: Simple Pendulum

$$\frac{d\theta}{dx} = \omega \qquad (2.1)$$

$$\frac{d\omega}{dt} = \frac{-\mathrm{MgL}\sin\theta}{J} \qquad (2.2)$$

2.2 MASS SPRING DAMPER SYSTEMS

The Inverted pendulum is a classical problem in control systems to explore the unstable dynamics. The common example to analyze it is by the Mass Spring Damper system as shown in figure 2.2.

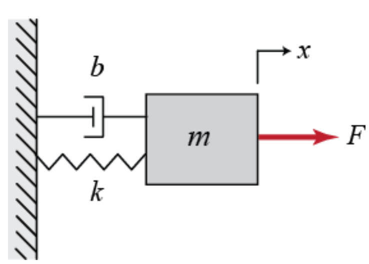

Figure 2.2 Mass Spring Damper system.

2.3 CONTROLLERS

To maintain the pendulum at the inverted position, commonly used controllers are Proportional Integral Derivative (PID) and the Linear Quadratic Regulator (LQR). Other research works have also explored the use of Linear - Gaussian Control (LQG), Fuzzy Logic and Pole-Placement method. In this project, the pendulum angle is controlled by PID controller.

2.3.1 PID

A Proportional–Integral–Derivative (PID) controller is a control loop feedback widely used in industrial control systems and in a wide range of applications requiring continuously modulated control. The controller, continuously calculates an error value, e(t), as the difference between a desired set point SP = r(t), measured process variable PV = y(t), and applies a correction based on proportional, integral, and derivative terms. The controller attempts to minimize the error over time by adjustment of a control variable u(t), determined by a weighted sum of the control terms. The Figure 2.3 shows the block diagram of the PID controller in a feedback loop.

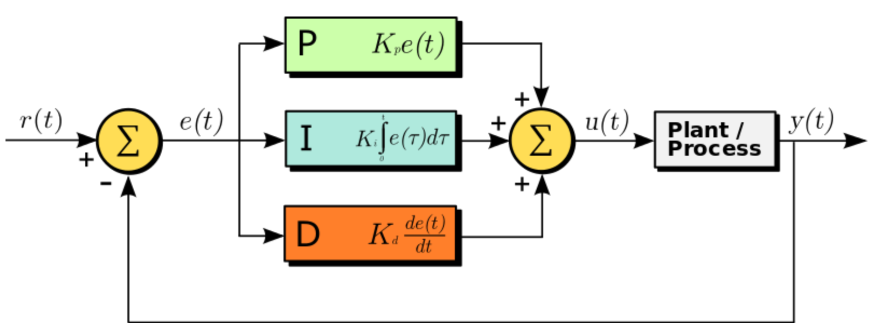

Figure 2.3 Block diagram of PID controller

- Term P is proportional to the current value of the error, e(t).
- Term I accounts for past values of the error and integrates them over time to produce I term. The integral term seeks to eliminate the residual error by adding a control effect due to the historic cumulative value of the error.
- Term D is the estimate of future trend of the error, based on its current rate of change. It is sometimes called anticipatory control, as it is effectively seeking to reduce the effect of the error by exerting a control influence generated by the rate of error change. The more rapid the change, the greater the controlling or dampening effect.

The overall control function can be expressed mathematically as,

$$\mathrm{u}(t) = K_p e(t) + K_i \int e(t)dt + K_\mathrm{d} \frac{d}{dt} e(t) \qquad (2.3)$$

Where, u(t) is the output of the controller and K_p, K_i and K_d are non-negative coefficients for proportional, integral and derivative terms respectively.

The proportional controller is effective on the errors since the output of the proportional controller is directly proportional to the error. But the proportional controller has a disadvantage of steady state error. The output will not settle at the set point. It will have error.

The derivative controller provides smoother response where the system will be having no oscillations. The output of the derivative controller is the function of the change in error. The damping factor decides the transient response of the system. This reduces the peak overshoot problem of the system.

The integral controller is used to eliminate the steady state error. The output of the integral controller is the function of the accumulated error. The past errors are summed, and the integral controller will create output proportional to the accumulated error.

2.3.1 LQR CONTROLLER

The linear quadratic regulator problem deals with minimization of the cost function.

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

The control law that minimizes the cost function is linear, given by,

$$u(t) = -R^{-1} B^T P x(t)$$

The control law guarantees that, $x(t)$ *and* $u(t) \to 0$ *as* $t \to \infty$ at a rate that is an optimal tradeoff between control effort and performance, demanded by Q and R.

The crucial and difficult task in the LQR controller design is a choice of the weighting matrices. We generally select weighting matrices Q and R to satisfy expected performance criterion. The different Q and R values give a different system response. The system will be more robust to disturbance and the settling time will be shorter if Q is larger (in a certain range). But there is no straightforward way to select these weighting matrices and it is usually done through an iterative simulation process.

For various values of Q and R matrix, the controller gain is calculated by solving Ricatti's Equation in MATLAB. An advantage of using the LQR optimal control scheme is that system designed will be stable and robust, except in the case where the system is not controllable.

CHAPTER 3

MATHEMATICAL MODEL

3.1 INVERTED PENDULUM MODEL

The pendulum's modeling is obtained from the schematic diagram shown in figure 3.1. The pendulum has 4 state variables x, $\dot{x}$, θ and $\dot{\theta}$ which represents the position of the cart, the velocity of the cart, angle and angular velocity of the pendulum respectively. The equations of motion of the system are derived using Lagrangian Mechanics and linearized using Jacobian linearization. The typical arrangement of the inverted pendulum is shown in fig 3.1. It is a freely hanging pendulum joined to a cart which is actuated by a motor. The linear motion of the cart creates a torque on the pendulum which helps balance the pendulum against its equilibrium position and prevents it from falling down.

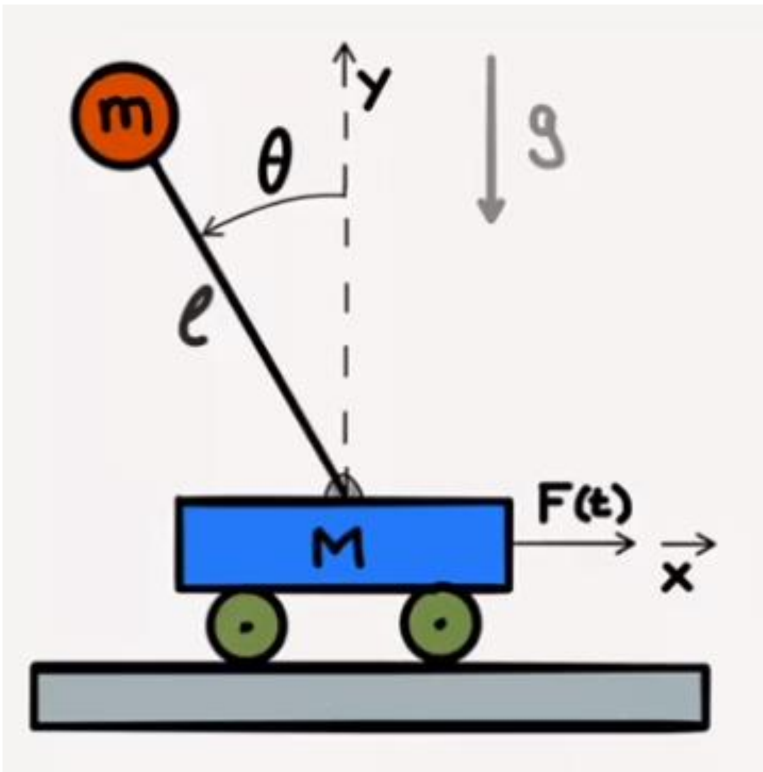

Figure 3.1 Schematic Diagram of Inverted Pendulum

3.2 SYSTEM PARAMETERS

The parameters that were used in the mathematical model are:

Symbol	**Units**	**Description**
g = 9.8	m/s^2	Gravitational acceleration at the surface of the earth
m = 0.05	kg	Mass of the rod
$M = 0.42$	kg	Mass of the Cart
l = 0.36	m	Length of the rod from axis of rotation
L = 0.396	m	Total length of the rod
I = 0.002547	kg m^2	Moment of inertia of the system
b =0.005	Nms/rad	Damping co-efficient of the pendulum
d =0.05	Ns/m	Cart Friction

Table 3.1 System parameters

3.3 SYSTEM DYNAMICS

Dynamics is concerned with the study of forces and torques and their effect on motion.

3.3.1 Lagrangian Mechanics:

Lagrangian mechanics is a fascinating and beautiful formulation of classical mechanics. Rather than the three laws of motion, Lagrangian mechanics is interested in a scalar quantity called the *action* which is a function of the state of the mechanical system. By applying a simple constraint known as the Principle of Least Action, you can get the equations of motion for any system in any coordinate system. To apply Lagrangian Mechanics to the cart-pole system, the Lagrangian of the system is determined which is defined as the difference between the potential and kinetic energies of the system as functions of the state.

The Lagrangian constant is given by $L = T - V$ (3.1)

Where

$T = \frac{1}{2}\sum_{k=1}^{N} m_k v_k^2$ is the total kinetic energy of the system (3.2)

$V = mgh$ is the potential energy of the system (3.3)

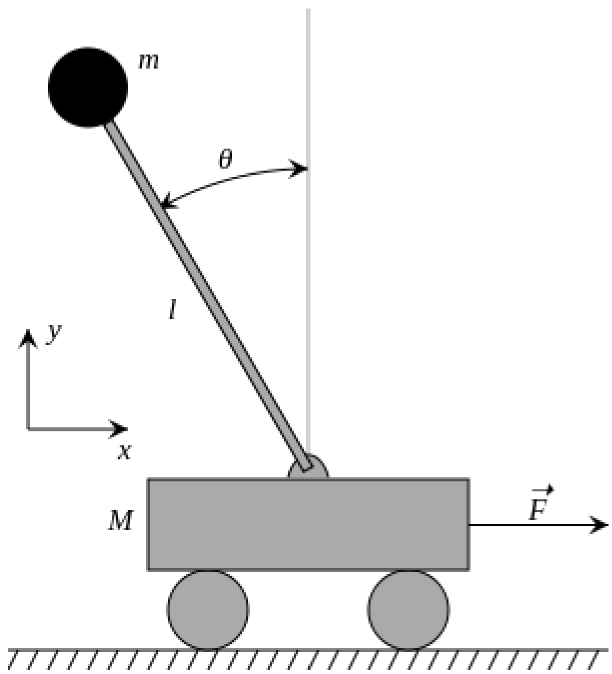

Figure 3.2 Free body diagram of the inverted pendulum system

The height is given by $h = l\cos\theta$

By substituting h in eq. (3.3),

Potential energy of the system is $V = mgl\cos\theta$ (3.4)

From eq. (3.2), total kinetic energy of the system is given by,

$T = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2$ (3.5)

where,

M is the mass of the cart

m is the mass of the pole

$v1$ is the velocity of the cart

$v2$ is the velocity of the point mass m

$v1$ and $v2$ can be expressed in terms of x and θ by writing the velocity as the first derivative of the position:

$$v_1^2 = \dot{x}^2 \tag{3.6}$$

$$v_2^2 = v_x^2 + v_y^2$$

By vectorizing,

$$v_x^2 = \left(\frac{d}{dt}(x - l\sin\theta)\right)^2$$

$$v_y^2 = \left(\frac{d}{dt}(l\cos\theta)\right)^2$$

Substituting v_x^2 and v_y^2 in v_2^2,

$$v_2^2 = \left(\frac{d}{dt}(x - l\sin\theta)\right)^2 + \left(\frac{d}{dt}(l\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to,

$$v_2^2 = \dot{x}^2 - 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2 \tag{3.7}$$

Substituting eq. (3.6) & (3.7) in eq. (3.5), the total kinetic energy of the system is given by,

$$T = \frac{1}{2}(M + m)\dot{x}^2 - ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 \tag{3.8}$$

The Lagrangian is now given by substituting eq. (3.3) & (3.8) in eq. (3.1),

$$L = \frac{1}{2}(M+m)\dot{x}^2 - ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{3.9}$$

In Lagrangian mechanics, as the system evolves over time, a quantity called *action* is defined as the integral of the Lagrangian.

$$S = \int_{t1}^{t2} L\,dt$$

According to the Principle of Least Action, the dynamics of the system evolves so that this quantity - the action - is minimized. Once we have the action, we can use the Euler-Lagrange equations to find the equations of motion of the system.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i$$

The q_i represent the state variables of the system and F_i generalized forces. By applying these equations once for each state variable, the complete equations of motion of the system is determined. In the case of the cart pole system, the state variables are the position of the cart and the angle between the cart and the pendulum. A force is applied only to the cart and no external force is applied to the pendulum. Therefore, the equations of motion are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F \tag{3.10}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{3.11}$$

To determine the equations of motion of the cart pole system, L (i.e.) eq. (3.9) is substituted in the above equations,

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - ml\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta \tag{3.12}$$

$$\frac{\partial L}{\partial x} = 0 \tag{3.13}$$

$$\frac{\partial L}{\partial \dot{\theta}} = -ml\dot{x}\cos\theta + ml^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -ml\ddot{x}\cos\theta + ml\dot{x}\dot{\theta}\sin\theta + ml^2\ddot{\theta} \tag{3.14}$$

$$\frac{\partial L}{\partial \theta} = ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta \tag{3.15}$$

Substituting eq. (3.12) & (3.13) in eq. (3.10) and eq. (3.14) & (3.15) in eq. (3.11),

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F \tag{3.16}$$

$$-ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta = 0 \tag{3.17}$$

The above equations describe the motion of the inverted pendulum.

These equations are nonlinear, but since the goal of a control system would be to keep the pendulum upright the equations can be linearized around $\theta \approx 0$.
Therefore, $\sin\theta \approx \theta, \cos\theta \approx 1, \dot{\theta} \approx 0$.

Also, introducing the cart friction, damping coefficient of pendulum and inertia of the system, the final equations of motion becomes,

$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\theta} = F \tag{3.18}$$

$$(I + ml^2)\ddot{\theta} - ml\ddot{x} + d\dot{\theta} - mgl\theta = 0 \tag{3.19}$$

The eq. (3.18) & (3.19) shows the model of the Inverted pendulum where the first equation provides the relationship between the input force produced by the motor actuating the cart and the relative movement of the cart. The second equation tells about the motion of the pendulum relative to the motion of the cart.

But the actual input we have is the voltage applied to the DC motor. Thus, the voltage given to the DC motor is converted to the force acting on the cart, which further generates a torque which acts on the pendulum and helps maintain the pendulum upright. Hence, the relationship between the voltage and force has to be calculated.

3.3.2. Transfer function Representation

To obtain the transfer functions of the linearized system equations, we must first take the Laplace transform of the system equations i.e., eq. (3.18) & (3.19) assuming zero initial conditions. The resulting Laplace transforms are shown below.

$$(I+ml^2)\theta(s)s^2 + d\theta(s)s - mgl\theta(s) = mlX(s)s^2$$

$$(M+m)X(s)s^2 + bX(s)s - ml\theta(s)s^2 = U(s)$$

A transfer function represents the relationship between a single input and a single output at a time. To find our first transfer function for the output $\theta(s)$ and an input of $U(s)$ we need to eliminate $X(s)$ from the above equations. Solve the first equation for $X(s)$.

$$X(s) = \left[\frac{I+ml^2}{ml} + \frac{d}{mls} - \frac{g}{s^2}\right]\theta(s)$$

Then substitute the above into the second equation.

$$(M+m)\left[\frac{I+ml^2}{ml} + \frac{d}{mls} - \frac{g}{s^2}\right]\theta(s)s^2 + b\left[\frac{I+ml^2}{ml} + \frac{d}{mls} - \frac{g}{s^2}\right]\theta(s)s - ml\theta(s)s^2 = U(s)$$

Rearranging, the transfer function is then the following,

$$\frac{\theta(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 + \frac{d(M+m)+b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl-bd}{q}s^2 - \frac{bmgl}{q}s}$$

where,

$$q = [(M+m)(I+ml^2) - (ml)^2]$$

From the transfer function above it can be seen that there is both a pole and a zero at the origin. These can be cancelled, and the transfer function becomes the following.

$$P_{pend}(s) = \frac{\theta(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{d(M+m)+b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl-bd}{q}s - \frac{bmgl}{q}} \qquad \left[\frac{rad}{N}\right] \tag{3.20}$$

Second, the transfer function with the cart position $X(s)$ as the output can be derived in a similar manner to arrive at the following.

$$P_{cart}(s) = \frac{X(s)}{U(s)} = \frac{\frac{(I+ml^2)s^2+ds-mgl}{q}}{s^4 + \frac{d(M+m)+b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl-bd}{q}s^2 - \frac{bmgl}{q}s} \qquad \left[\frac{m}{N}\right] \tag{3.21}$$

3.3.3. State Space Representation

The state space matrix is of the form $\dot{x} = Ax + Bu$ and $y = Cx + Du$

In this case,

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + Bu \tag{3.22}$$

Rearranging the equations of motion (i.e.) eq. (3.18) & (3.19), to get $\ddot{x}$ and $\ddot{\theta}$ in terms of the state variables $x, \dot{x}, \theta$ and $\dot{\theta}$,

$$\ddot{\theta} = \frac{ml\ddot{x} - d\dot{\theta} + mgl\theta}{I + ml^2}$$

$$\ddot{x} = \frac{F + ml\ddot{\theta} - b\dot{x}}{M + m}$$

Plugging $\ddot{\theta}$ in eq. (3.18) and $\ddot{x}$ in eq. (3.19),

$$(M + m)\ddot{x} + b\dot{x} - ml\left[\frac{ml\ddot{x} - d\dot{\theta} + mgl\theta}{I + ml^2}\right] = F$$

$$\ddot{x}[(I + ml^2)(M + m) - m^2l^2] = F(I + ml^2) - b\dot{x}(I + ml^2) - mld\dot{\theta} + m^2l^2g\theta$$

$$\ddot{x} = \frac{F(I+ml^2) - b\dot{x}(I+ml^2) - mld\dot{\theta} + m^2l^2g\theta}{q} \tag{3.23}$$

where, $q = I(M + m) + Mml^2$

$$(I + ml^2)\ddot{\theta} - ml\left[\frac{F + ml\ddot{\theta} - b\dot{x}}{M + m}\right] + d\dot{\theta} - mgl\theta = 0$$

$$\ddot{\theta}[(I + ml^2)(M + m) - m^2l^2] - mlF + mlb\dot{x} + (M + m)d\dot{\theta} - (M + m)(mgl\theta)$$

$$\ddot{\theta} = \frac{mlF - mlb\dot{x} + (M+m)(mgl\theta) + (M+m)d\dot{\theta}}{q} \tag{3.24}$$

where, $q = I(M + m) + Mml^2$

Comparing eq. (3.23) & (3.24) with the state space matrix given in eq. (3.22),

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b(I+ml^2)}{I(M+m)+Mml^2} & \frac{m^2l^2g}{I(M+m)+Mml^2} & \frac{mld}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \frac{(M+m)d}{I(M+m)+Mml^2} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

The above equations are the state space representation of the equations of motion of the inverted pendulum system, where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b(I+ml^2)}{I(M+m)+Mml^2} & \frac{m^2l^2g}{I(M+m)+Mml^2} & \frac{mld}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \frac{(M+m)d}{I(M+m)+Mml^2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The C matrix has 2 rows because both the cart's position and the pendulum's position are part of the output. Specifically, the cart's position is the first element of the output y and the pendulum's deviation from its equilibrium position is the second element of y.

The PD Controller is designed by solving this state space matrix and is discussed in the next chapter.

CHAPTER 4

CONTROLLER DESIGN

4.1 Open-Loop Response:

The open-loop transfer functions of the inverted pendulum system are,

from eq. (3.20) & (3.21),

$$P_{pend}(s) = \frac{\theta(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{d(M+m)+b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl-bd}{q}s - \frac{bmgl}{q}}$$

$$P_{cart}(s) = \frac{X(s)}{U(s)} = \frac{\frac{(I+ml^2)s^2+ds-mgl}{q}}{s^4 + \frac{d(M+m)+b(I+ml^2)}{q}s^3 - \frac{(M+m)mgl-bd}{q}s^2 - \frac{bmgl}{q}s}$$

The response of the pendulum to a 1N-sec impulse applied to the cart and 0.2m step command in cart position are examined using MATLAB.

4.1.1: Impulse Response:

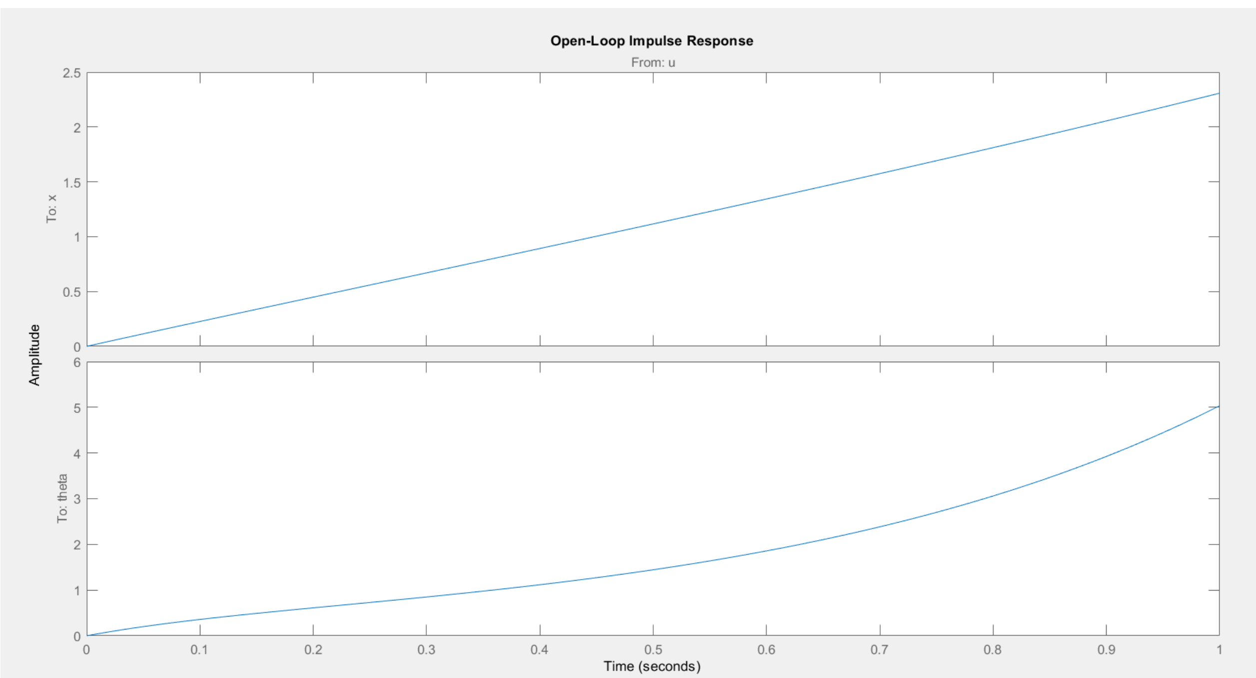

Figure 4.1 Open-Loop Impulse Response of the system

The figure 4.1 shows the cart and angle response for an impulse input given to the open-loop inverted pendulum system.

It can be seen from the plot that the system response is entirely unsatisfactory. In fact, it is not stable in open loop. Although the pendulum's position is shown to increase past increase past 5 radians, the model is only valid for small θ. Also, the cart’s position moves infinitely far towards one side, though there is no requirement on cart position for an impulsive force input.

The poles of a system can also tell about its time response. Since the system has two outputs and one input, it is described by two transfer functions. In general, all functions from each input to each output of a multi-input, multi-output (MIMO) system will have the same poles (but different zeros) unless there are pole-zero cancellations. The poles and zeros of the system are examined using MATLAB.

The zeros and poles of the system where the pendulum position is the output are as shown below,

Zeros:

- 0

Poles:

- -8.4898
- 2.4920
- -0.0106

The zeros and poles of the system where the cart position is the output are as shown below,

Zeros:

- -7.9859
- 2.4470

Poles:

- 0
- -8.4898
- 2.4920
- -0.0106

The poles for both transfer functions are identical. The pole at 2.4920 indicates that the system is unstable since the pole has a positive real part, that is, the pole is in the right half of the complex s-plane.

This agrees with the open-loop impulse plot of the system.

4.1.2: Step Response:

Since the system has a pole with positive real part, its response to a step input will also grow unbounded. This is verified using MATLAB.

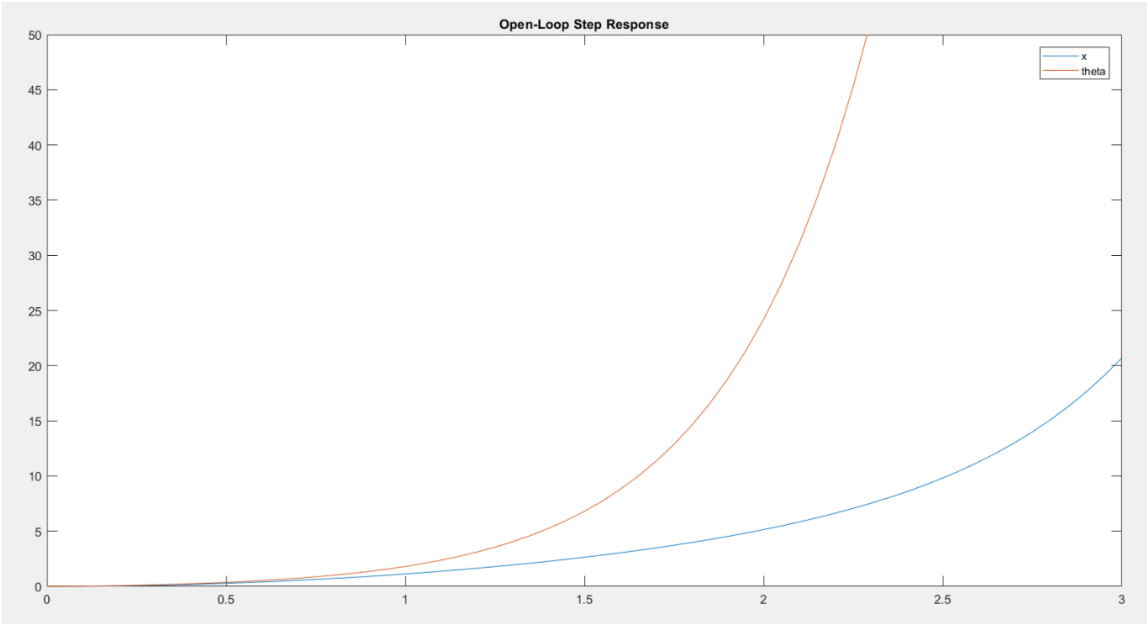

Figure 4.2 Open-Loop Step Response of the System

The figure 4.2 shows the cart and angle response for a step input given to the open-loop inverted pendulum system.

It can be seen from the plot that the system's response to a step input is unstable.

From the above plots, it can be inferred that some sort of control need to be designed to improve the response of the system. Here, a PD Controller is implemented to stabilize the inverted pendulum system.

4.2 PD Controller

A PD Controller is required to stabilize the system and at first, a single input single output system is assumed as described by the following transfer function from eq. (3.20). In this, the pendulum's angle is attempted to be controlled without regard to the cart's position.

$$P_{pend}(s) = \frac{\theta(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{d(M+m)+b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl-bd}{q}s - \frac{bmgl}{q}}$$

The controller will attempt to maintain the pendulum vertically upward when the cart is subjected to an impulse input.

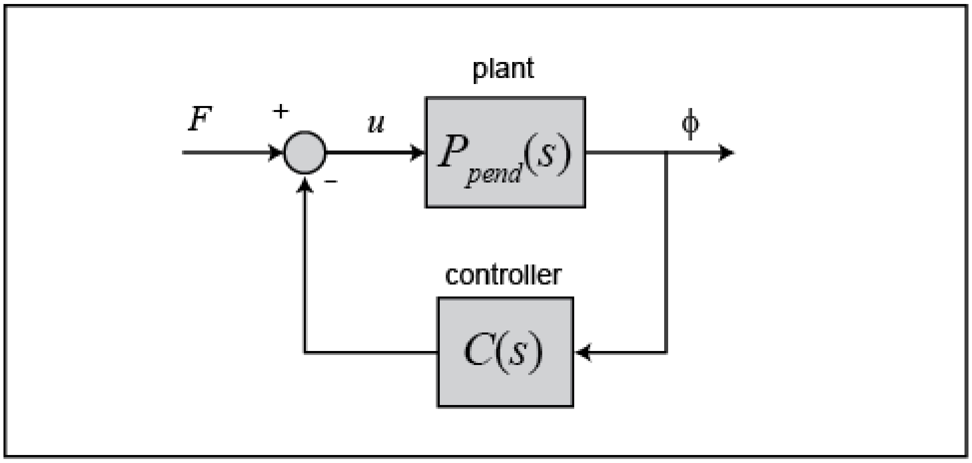

Figure 4.3 Inverted pendulum System with Controller

The figure 4.3 shows the structure of the inverted pendulum system with controller when only the pendulum's angle needs to be controlled.

The resulting transfer function of the closed loop system is given by,

$$T(s) = \frac{\theta(s)}{F(s)} = \frac{P_{pend}(s)}{1+C(s)P_{pend}(s)}$$

The response of the system for an impulse input is then examined using MATLAB for random PD values. (Initially, $K_p = 1$ and $K_d = 1$)

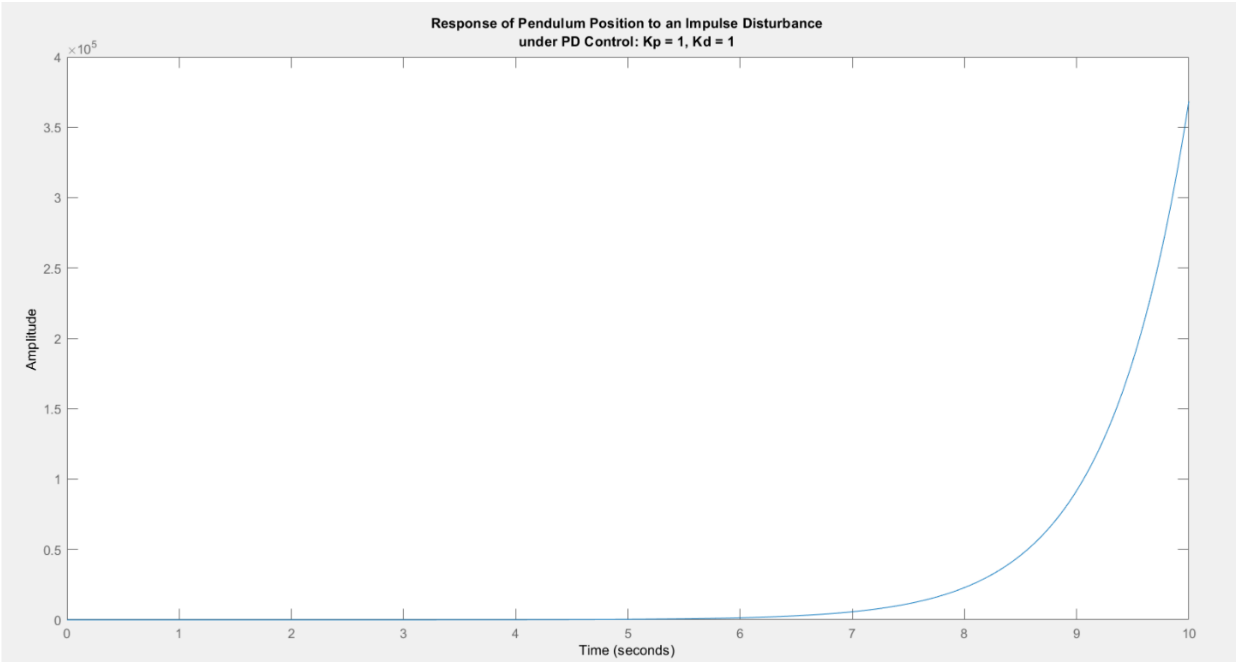

Figure 4.4 Response of System with Controller ($K_p = 1$ and $K_d = 1$)

The figure 4.4 shows the response of the system and is still not stable. So, the proportional gain is increased to 100 and the response is examined.

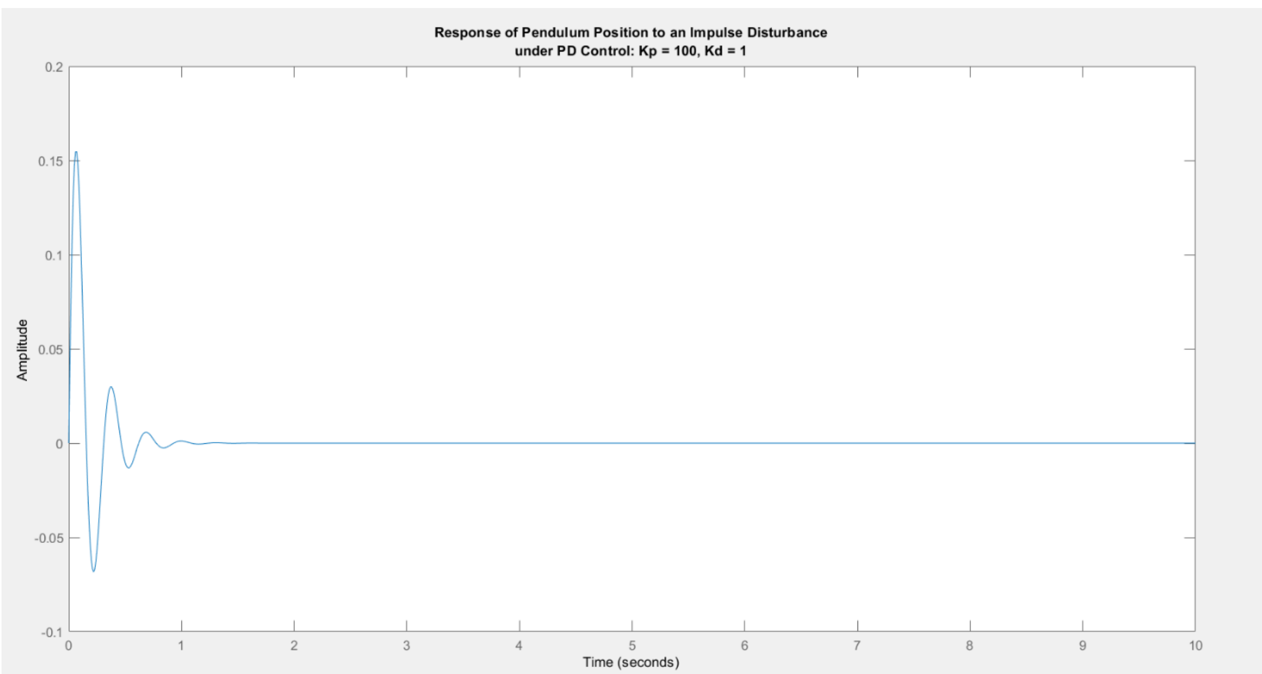

Figure 4.4 Response of system with controller ($K_p = 100$ and $K_d = 1$)

The figure 4.4 shows the response of the system and it shows that by tuning the PD values, the inverted pendulum system can be stabilized.

The cart's position should also be regarded during the pendulum's stabilization and is added in the system. The cart requires an independent controller to prevent it from moving too far towards one side. Hence, another PD Controller is implemented for the cart.

The PD gain values are determined using state feedback method.

4.3 PD Controller design using state feedback method:

We know that,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b(I+ml^2)}{I(M+m)+Mml^2} & \frac{m^2l^2g}{I(M+m)+Mml^2} & \frac{mld}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \frac{(M+m)d}{I(M+m)+Mml^2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix}$$

$K = [k_1\ k_2\ k_3\ k_4]$

where, K is the feedback matrix.

k_1 is the Proportional Gain for position

k_2 is the Derivative Gain for position

k_3 is the Proportional Gain for angle

k_4 is the Derivative Gain for angle

The control law states that,

$U = -Kx$

Then, the general state equation becomes,

$\dot{x} = Ax + B(-Kx)$

$\dot{x} = (A - BK)x$

Taking Laplace transform,

$Sx(s) = (A - BK)x(s)$

$$|SI - (A - BK)| = 0 \tag{4.4}$$

The eq. (4.4) gives the characteristic equation of the closed loop system.

$$BK = \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} [k_1\ k_2\ k_3\ k_4]$$

$$BK = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{(I+ml^2)k_1}{I(M+m)+Mml^2} & \frac{(I+ml^2)k_2}{I(M+m)+Mml^2} & \frac{(I+ml^2)k_3}{I(M+m)+Mml^2} & \frac{(I+ml^2)k_4}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 0 \\ \frac{mlk_1}{I(M+m)+Mml^2} & \frac{mlk_2}{I(M+m)+Mml^2} & \frac{mlk_3}{I(M+m)+Mml^2} & \frac{mlk_4}{I(M+m)+Mml^2} \end{bmatrix}$$

$$(A-BK)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b(I+ml^2)}{I(M+m)+Mml^2} & \frac{m^2l^2g}{I(M+m)+Mml^2} & \frac{mld}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \frac{(M+m)d}{I(M+m)+Mml^2} \end{bmatrix} -$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{(I+ml^2)k_1}{I(M+m)+Mml^2} & \frac{(I+ml^2)k_2}{I(M+m)+Mml^2} & \frac{(I+ml^2)k_3}{I(M+m)+Mml^2} & \frac{(I+ml^2)k_4}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 0 \\ \frac{mlk_1}{I(M+m)+Mml^2} & \frac{mlk_2}{I(M+m)+Mml^2} & \frac{mlk_3}{I(M+m)+Mml^2} & \frac{mlk_4}{I(M+m)+Mml^2} \end{bmatrix}$$

Simplifying, $(A-BK)$ is given by the matrix,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(I+ml^2)k_1}{I(M+m)+Mml^2} & \frac{-b(I+ml^2)}{I(M+m)+Mml^2}-\frac{(I+ml^2)k_2}{I(M+m)+Mml^2} & \frac{m^2l^2g}{I(M+m)+Mml^2}-\frac{(I+ml^2)k_3}{I(M+m)+Mml^2} & \frac{mld}{I(M+m)+Mml^2}-\frac{(I+ml^2)k_4}{I(M+m)+Mml^2} \\ 0 & 0 & 0 & 1 \\ \frac{-mlk_1}{I(M+m)+Mml^2} & \frac{-mlb}{I(M+m)+Mml^2}-\frac{mlk_2}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2}-\frac{mlk_3}{I(M+m)+Mml^2} & \frac{(M+m)d}{I(M+m)+Mml^2}-\frac{mlk_4}{I(M+m)+Mml^2} \end{bmatrix}$$

$$SI=\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

$$[SI-(A-BK)]=$$

$$\begin{bmatrix} s & -1 & 0 & 0 \\ \frac{(I+ml^2)k_1}{I(M+m)+Mml^2} & s+\frac{b(I+ml^2)}{I(M+m)+Mml^2}+\frac{(I+ml^2)k_2}{I(M+m)+Mml^2} & \frac{-m^2l^2g}{I(M+m)+Mml^2}+\frac{(I+ml^2)k_3}{I(M+m)+Mml^2} & \frac{-mld}{I(M+m)+Mml^2}+\frac{(I+ml^2)k_4}{I(M+m)+Mml^2} \\ 0 & 0 & s & -1 \\ \frac{mlk_1}{I(M+m)+Mml^2} & \frac{mlb}{I(M+m)+Mml^2}+\frac{mlk_2}{I(M+m)+Mml^2} & \frac{-mgl(M+m)}{I(M+m)+Mml^2}+\frac{mlk_3}{I(M+m)+Mml^2} & s-\frac{(M+m)d}{I(M+m)+Mml^2}+\frac{mlk_4}{I(M+m)+Mml^2} \end{bmatrix}$$

To determine the characteristic equation, the determinant of the above matrix is computed and is given by,

$$|SI-(A-BK)| = s^4 + s^3 \frac{\left[k_2\left(I+ml^2\right)+k_4 ml+bI-d(m+M)\right]}{q} + s^2 \frac{\left[k_1\left(I+ml^2\right)-d(k_2+b)+k_3 ml-mgl(M+m)\right]}{q} - s\frac{[k_1 d+mgl(k_2+b)]}{q} - \frac{k_1 mgl}{q}$$

Then, the closed loop characteristic equation of the pendulum becomes,

$$s^4 + s^3 \frac{[k_2(I+ml^2)+k_4 ml+bI-d(m+M)]}{q} + s^2 \frac{[k_1(I+ml^2)-d(k_2+b)+k_3 ml-mgl(M+m)]}{q} - s\frac{[k_1 d+mgl(k_2+b)]}{q} - \frac{k_1 mgl}{q} = 0 \tag{4.5}$$

where, $q = [(M+m)(I+ml^2)-(ml)^2]$

To determine the values of k_1, k_2, k_3, k_4, pole placement is done and is discussed in the upcoming chapters.

CHAPTER 5
HARDWARE STRUCTURE AND IDENTIFICATION OF SYSTEM PARAMETERS

5.1 HARDWARE MODEL

The Inverted Pendulum consists of linear rods with bearings that carry a cart. A pulley along with a timing belt is used to move the cart along the length of the rod. A DC motor and an encoder is placed at the ends of the timing belt to apply a force on the cart and to determine the position of the cart. A freely hanging rod is mounted on the cart and an encoder is used to determine the angle of the rod.

The rod is fixed with the encoder in such a way that the zero crossing detector of the encoder is triggered when the rod is upright, that is, at 180^0. This helps to overcome the issue of missing pulses during high frequency switching of the motor. When the pendulum is at 180^0, the z pulse count increases and the count value of A and B pulses is reset.

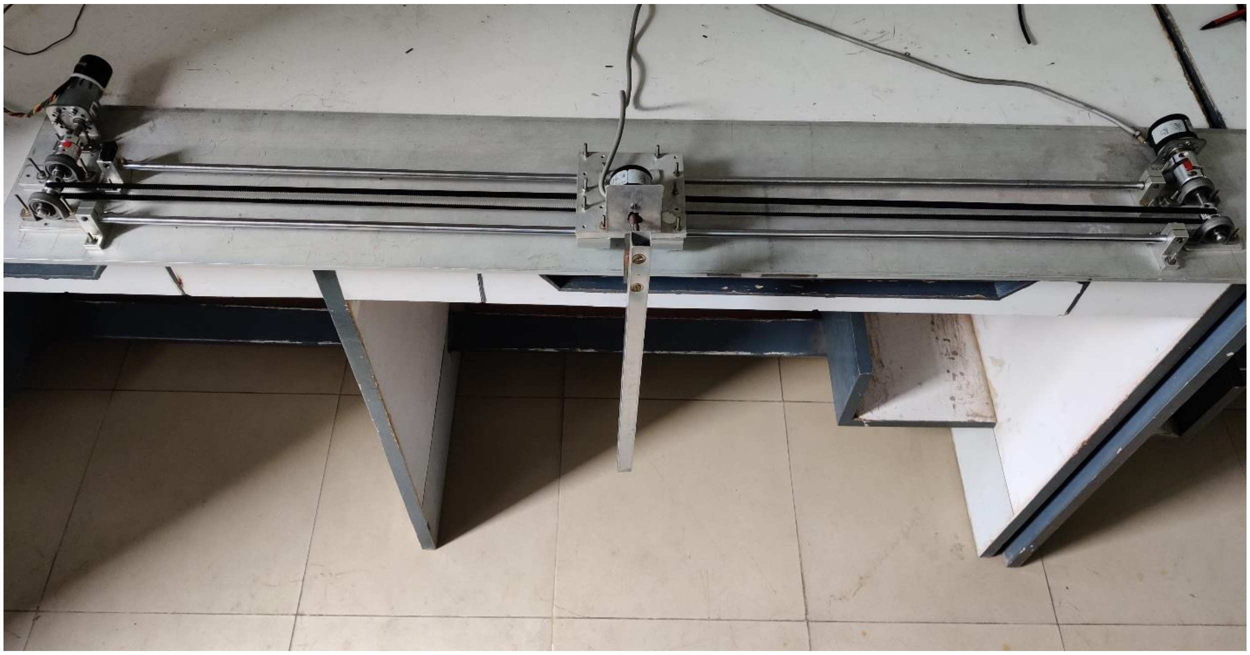

Figure 5.1 Hardware arrangement

A PCB is designed with motor driver to control the motor using the STM32F103C8T6 microcontroller and to interface the encoders as shown in figure 5.2.

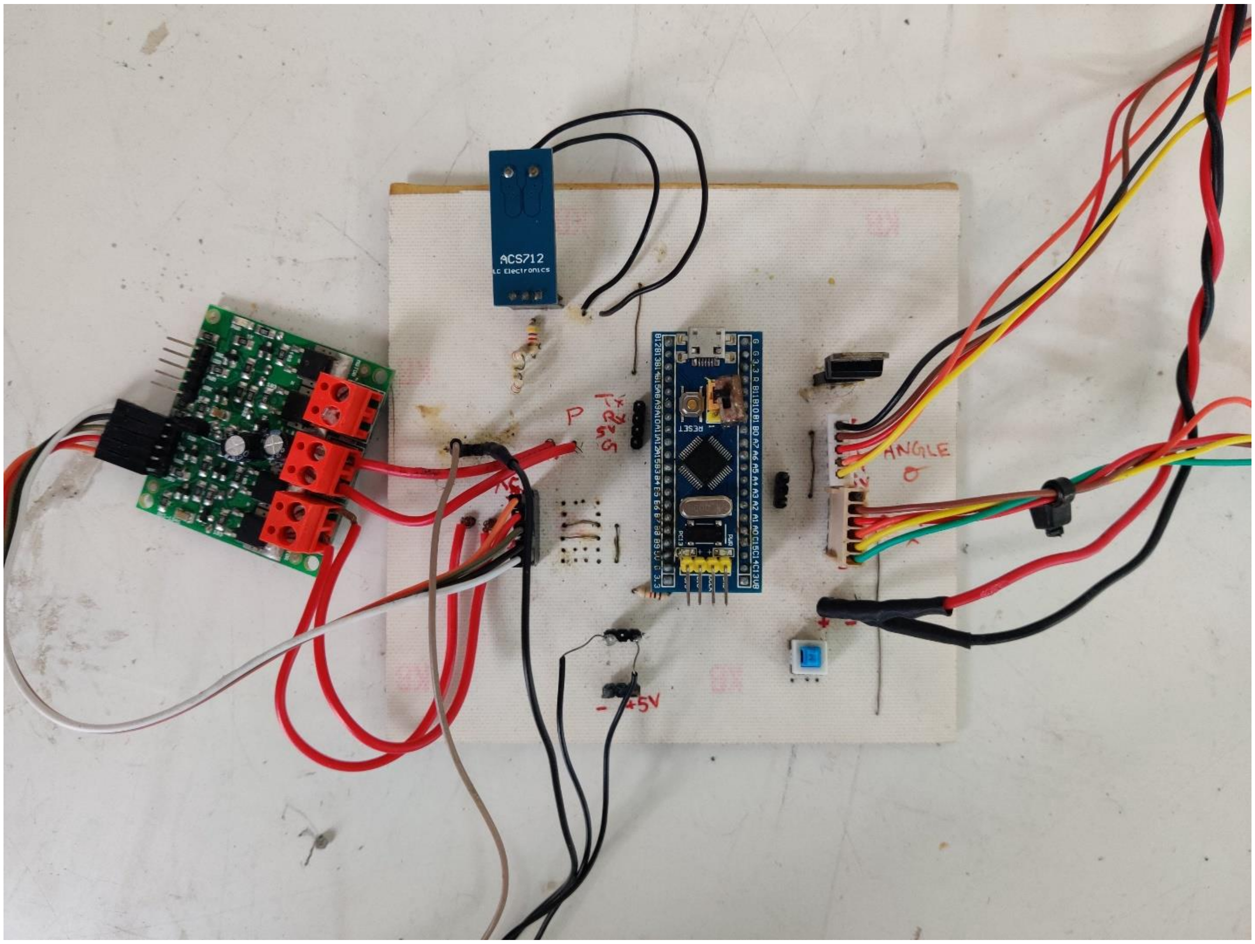

Figure 5.2 Microcontroller shield

5.2 IDENTIFICATION OF SYSTEM PARAMETERS

5.2.1 Total Mass calculation:

The total mass of the system is,

$$M = M_{rod} + M_{cart}$$

$$= 0.05 + 0.24$$

$$M = 0.29\text{kg}$$

5.2.2 Total Moment of inertia calculation:

$$I_{tot} = I_{rod} + I_{shaft}$$

5.2.2.1 Moment of inertia of rod:

We know that the moment of inertia of a rod,

$$I_{rod} = \frac{M_{rod}}{L_{rod}} \int_{l_1}^{l_2} x^2 dx$$

where, l_1 is the length of the rod up from axis of rotation and l_2 is the length of the rod down from axis of rotation.

Then,

$$I_{rod} = \frac{0.05}{0.396} \int_{-0.036}^{0.36} x^2 dx$$

$$= \frac{0.05}{0.396} [\frac{x^3}{3}]_{-0.036}^{0.36}$$

$$I_{rod} = 0.0019656\ kgm^2$$

5.2.2.2 Moment of inertia of shaft:

$$I_{shaft} = \frac{(M_{shaft} * R_{shaft}^2)}{2}$$

$$= \frac{(0.023089*0.003^2)}{2}$$

$$I_{shaft} = 1.039X\ 10^{-7}\ kgm^2$$

$$I_{tot}(or)\ I = 0.002547104 + 1.039X\ 10^{-7}$$

$$I_{tot}(or)\ I = 0.0025472079\ kgm^2$$

$$\frac{1}{I} = 392.5867221\ kg^{-1}m^{-2}$$

5.2.3 Damping co-efficient calculation (b):

The damping co-efficient of the pendulum is calculated by performing Free Oscillation test, which is responsible for the natural damping of the pendulum. The Pendulum is left to oscillate freely and the corresponding Time (t) and angle (theta) is noted and curve is plotted by taking time on X – axis and angle on Y –

axis using Microsoft Excel. The maxima of the curve are tracked and the decay rate is calculated by exponent curve fitting in Microsoft Excel.

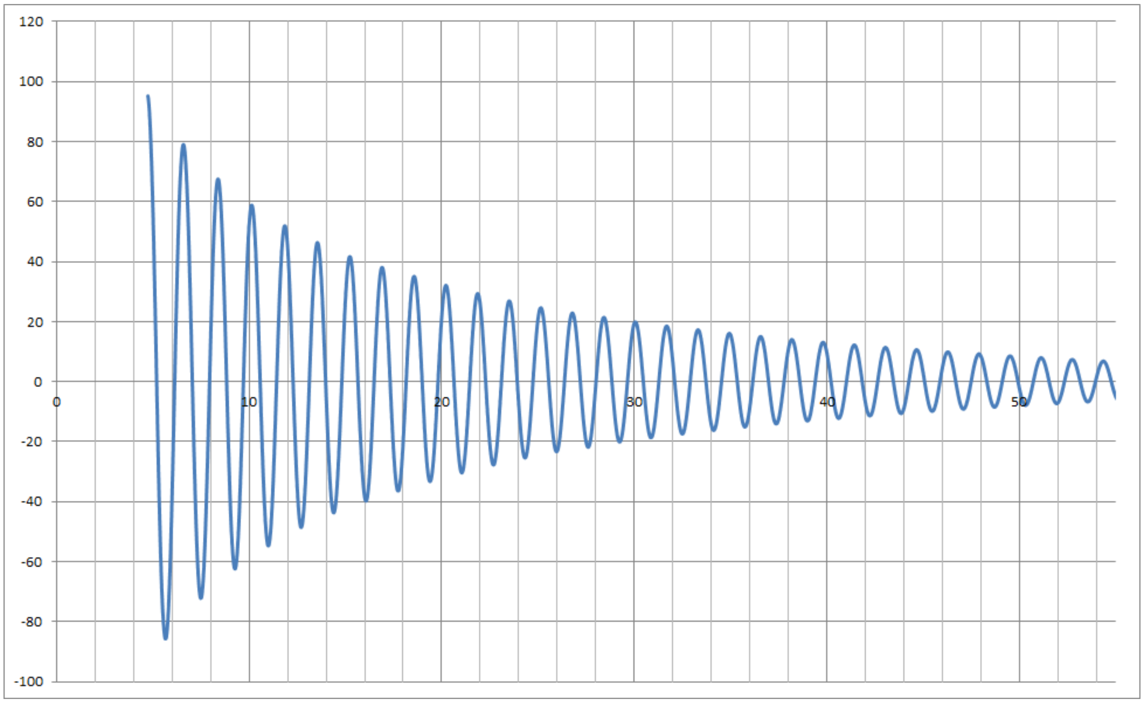

Figure 5.3 Free Oscillation Curve of the Pendulum

The exponent curve is shown in figure 5.4 and from the decay rate of the curve, the damping coefficient is calculated.

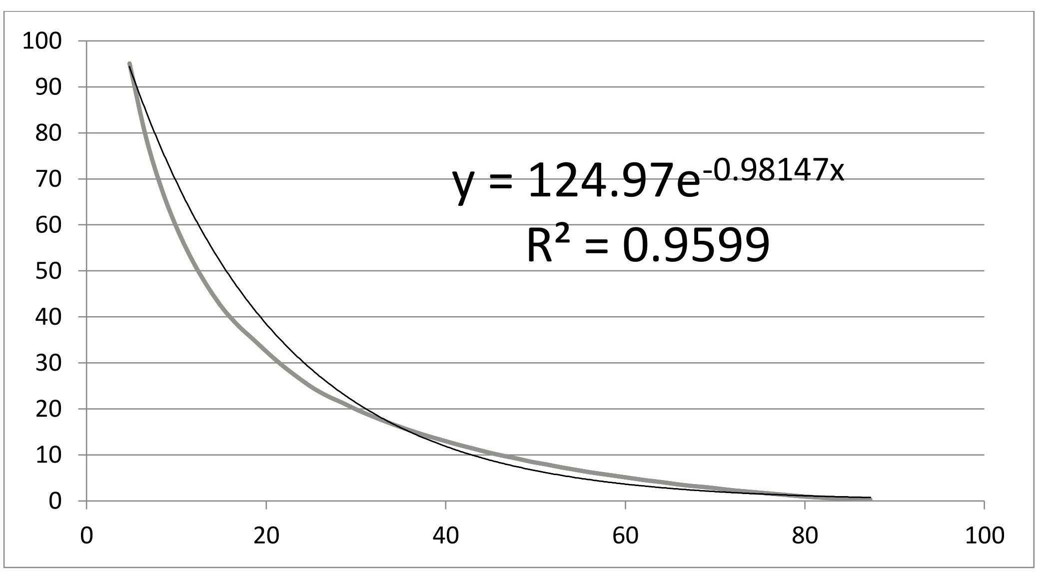

Figure 5.4 Exponent Curve

We know that the decay rate of the pendulum is, $Re^{-bt/_{2I}}$

Comparing with the decay rate obtained, we get,

$b/_{2I} = 0.98147$

$b/_{I} = 1.9629$

$b = 0.005$

5.2.4 Relation between Force and Voltage:

The relation between force and voltage has to be found since the model relates the angle of the pendulum which is maintained by linear movement of cart to the force applied on the cart, but the input given to the system is voltage. Hence, in order to relate the force on the cart with voltage, an experiment is conducted.

From the equations of motion, eq. (3.18) & (3.19),

$$(M+m)\ddot{x} + b\dot{x} - ml\left[\frac{ml\ddot{x} - d\dot{\theta} + mgl\theta}{I + ml^2}\right] = F$$

$$(I + ml^2)\ddot{\theta} - ml\left[\frac{F + ml\ddot{\theta} - b\dot{x}}{M+m}\right] + d\dot{\theta} - mgl\theta = 0$$

In this experiment, the motor torque is to be found for different PWM applied to the motor. A current sensor is connected in series to measure the current during the applied PWM and the speed of the motor is calculated using an encoder connected to the same belt as the motor. The torque is then given by,

$$T = \frac{60 * V * I * \eta}{1000 * 2\pi * Speed(rpm)} \quad Nm$$

where,

V is the voltage across the motor

I is the current drawn by the motor

η is the efficiency of the motor

The torque is calculated for different speed and is tabulated. The corresponding force is given by,

$$F = \frac{T}{r} \quad N$$

where,

r is the radius of the shaft of the motor

The force and the voltage values are then plotted in a graph as shown in figure 5.6 and the relationship between the two quantities is determined. This relation is then entered in the microcontroller.

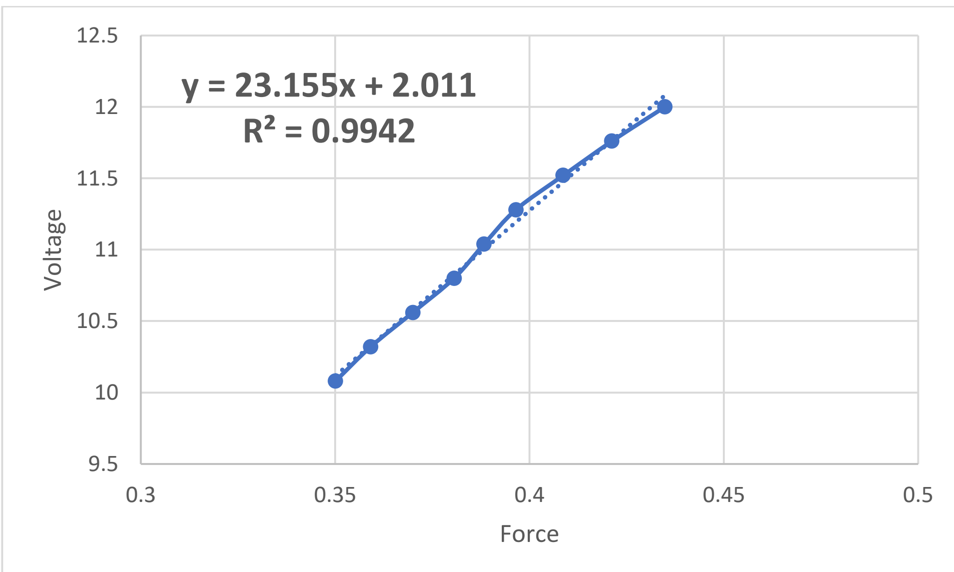

Figure 5.6 Force vs Voltage of Motor

CHAPTER 6
POLE PLACEMENT

6.1 Pole Placement Introduction:

Pole placement technique is the simple, robust and most widely used technique used to place the system poles at desired location. This technique requires the closed loop characteristic equation of the state feedback controller implemented system, which is a function of the feedback matrix K. By selecting the desired poles and from which the desired characteristic equation is formed. By comparing the desired characteristic equation with the actual system's closed loop equation, we will get the controller gain parameters.

The selection of the desired poles of the system is depended on various parameters such as, settling time, maximum peak overshoot, rise time etc.

The pendulum is designed to settle at 180^o angle and maintain that position even during occurrence of minor disturbances.

For an overdamped system, the settling time is approximately equal to,

$$T_s \cong \frac{1}{Dominant\ pole}$$

Where,

The dominant pole is the pole, which is placed near the zero, whose transient will decay after a long time. Hence, we can assume that the transient of the system is only dependent on the dominant pole response. So, the other poles should be far away from the dominant pole such that the transients of the other poles will decay quickly and that will not affect the system transient and it will be only depended on the dominant pole.

So, in order to get low settling time, the dominant poles have to be placed at -1 and -1.5. Also, the other poles have to be so big such that it should decay quickly, and the system transient is only depended on the dominant poles (-1 and -1.5).

6.2 Pole placement for PD controller:

The dominant pole of the system is placed at -1 and -1.5 and let us choose the other poles to be placed at -10 and -15, then the desired characteristic equation becomes,

$$(s+1)(s+1.5)(s+10)(s+15)=0$$

$$s^4+27.5s^3+214s^2+412.5s+225=0 \tag{6.1}$$

We obtained our closed loop characteristic equation of the state feedback controller (PD) implemented system from the eq. (4.5),

$$s^4+s^3\frac{[k_2(I+ml^2)+k_4ml+bI-d(m+M)]}{q}+s^2\frac{[k_1(I+ml^2)-d(k_2+b)+k_3ml-mgl(M+m)]}{q}-$$
$$s\frac{[k_1d+mgl(k_2+b)]}{q}-\frac{k_1mgl}{q}=0$$

where, $q=\ [(M+m)(I+ml^2)-(ml)^2]$

Comparing the above equations, we can determine the values of k_1, k_2, k_3 and k_4 when the system parameters is known.

Substituting, the values of the system parameters from Table 3.1, the state space matrix of the system is given by,

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -0.01152 & 0.8103 & 0.2297\\ 0 & 0 & 0 & 1\\ 0 & -0.02297 & 21.16 & 5.997\end{bmatrix}$$

$$B=\begin{bmatrix}0\\ 2.304\\ 0\\ 4.593\end{bmatrix}$$

The closed loop characteristic equation of the system is given by,

$$|SI-(A-BK)|=0$$

$$BK=\begin{bmatrix}0\\ 2.304\\ 0\\ 4.593\end{bmatrix}[k_1\ k_2\ k_3\ k_4]$$

$$BK = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2.304k_1 & 2.304k_2 & 2.304k_3 & 2.304k_4 \\ 0 & 0 & 0 & 0 \\ 4.593k_1 & 4.593k_2 & 4.593k_3 & 4.593k_4 \end{bmatrix}$$

$$(A - BK) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.01152 & 0.8103 & 0.2297 \\ 0 & 0 & 0 & 1 \\ 0 & -0.02297 & 21.16 & 5.997 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2.304k_1 & 2.304k_2 & 2.304k_3 & 2.304k_4 \\ 0 & 0 & 0 & 0 \\ 4.593k_1 & 4.593k_2 & 4.593k_3 & 4.593k_4 \end{bmatrix}$$

Simplifying, $(A - BK)$ is given by the matrix,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2.304k_1 & -0.01152 + 2.304k_2 & 0.8103 + 2.304k_3 & 0.2297 + 2.304k_4 \\ 0 & 0 & 0 & 1 \\ 4.593k_1 & -0.02297 + 4.593k_2 & 21.16 + 4.593k_3 & 5.997 + 4.593k_4 \end{bmatrix}$$

$$SI = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

$$[SI - (A - BK)] =$$

$$\begin{bmatrix} s & -1 & 0 & 0 \\ 2.304k_1 & s + 0.01152 + 2.304k_2 & -0.8103 + 2.304k_3 & -0.2297 + 2.304k_4 \\ 0 & 0 & s & -1 \\ 4.593k_1 & 0.02297 + 4.593k_2 & -21.16 + 4.593k_3 & s - 5.997 + 4.593k_4 \end{bmatrix}$$

To determine the characteristic equation, the determinant of the above matrix is computed and is given by,

$|SI-(A-BK)| = s^4 + s^3[2.304k_2 + 4.593k_4 - 5.98548] + s^2[2.304k_1 - 12.7621k_2 + 4.593k_3 - 21.2238] + s[-12.7621k_1 - 45.0309k_2 - 0.225151] - 45.0309k_1$

Then, the closed loop characteristic equation of the pendulum becomes,

$$s^4 + s^3[2.304k_2 + 4.593k_4 - 5.98548] + s^2[2.304k_1 - 12.7621k_2 + 4.593k_3 - 21.2238] + s[-12.7621k_1 - 45.0309k_2 - 0.225151] - 45.0309k_1 = 0 \quad (6.2)$$

Comparing eq. (6.1) & 6.2, we get the following system of equations,

$2.304k_2 + 4.593k_4 - 5.98548 = 27.5$

$2.304k_1 - 12.7621k_2 + 4.593k_3 - 21.2238 = 214$

$-12.7621k_1 - 45.0309k_2 - 0.225151 = 412.5$

$-45.0309k_1 = 225$

Solving the above equations,

$k_1 = -4.9983$

$k_2 = -7.7518$

$k_3 = 32.1825$

$k_4 = 11.1775$

These values are used in the PD Controller and the system response in simulation and real-time are observed. The responses are discussed in the next chapter.

CHAPTER 7

RESPONSE ANALYSIS OF INVERTED PENDULUM

The gain values of the controller have been determined and is used to obtain the response of the system in MATLAB.

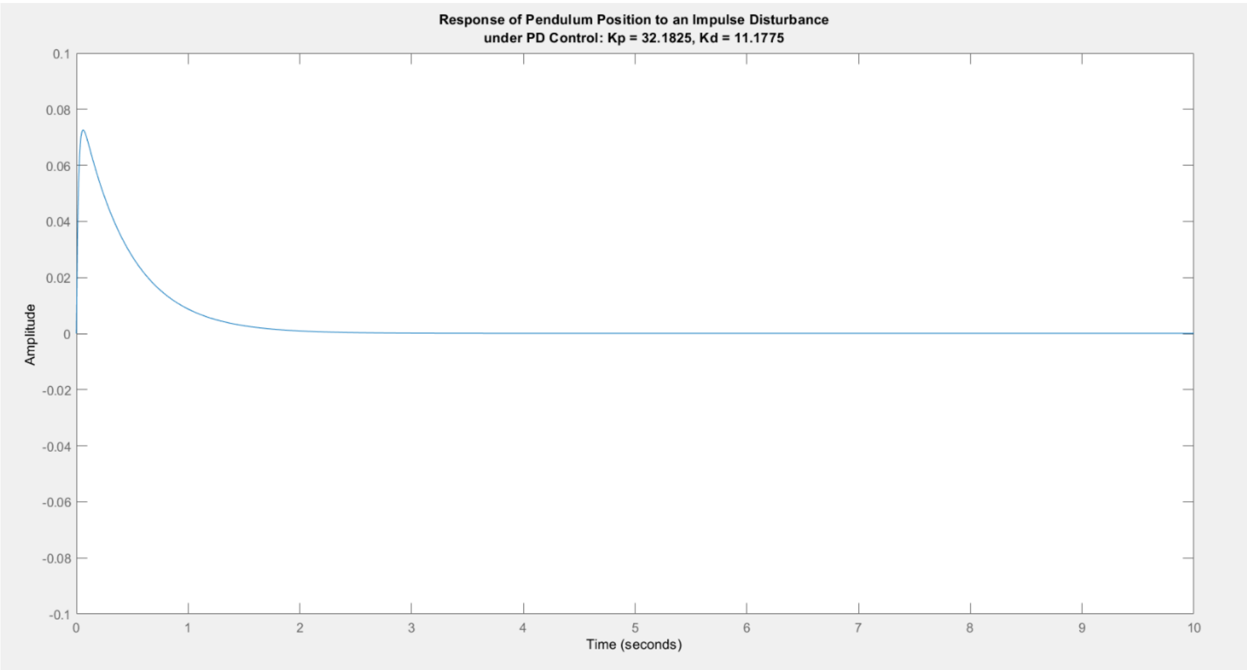

Figure 7.1 Response of system with controller ($K_p = 32.1825$ and $K_d = 11.1775$)

The figure 7.1 shows the response of the pendulum angle for an impulse disturbance with the controller gain values as calculated. It can be seen from the plot that the system is stable.

The zeros and poles of the system where the pendulum position is the output are as shown below,

Zeros:

- 0

Poles:

- -55.0486
- -2.3041
- 0

The poles are on the left half of the complex s-plane. This shows that the system is stable as described in the plot.

The real-time response of the system is as shown in figure 7.2.

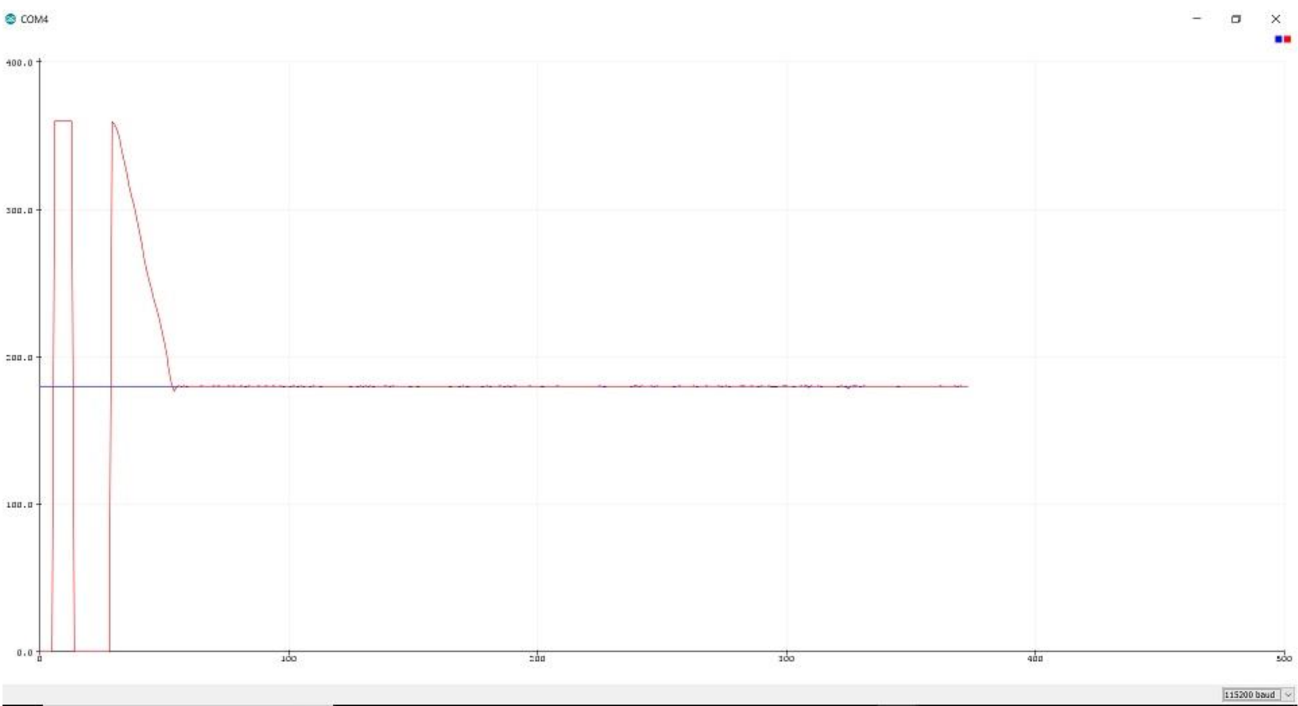

Figure 7.2 Real-time response of Inverted Pendulum System

In the above figure, red line indicates the angle of the pendulum and the blue line indicates the set angle, that is, 180^0. Initially, the pendulum is in a freely hanging position, or equilibrium position. The angle at that point is 0^0 or 360^0. Once the pendulum is lifted to the vertical position, it stays there and the oscillation of the pendulum around that angle can be seen from the plot.

CHAPTER 8

FUTURE IMPROVEMENTS

Further work will include increasing the level of autonomy of the pendulum by adding another link, thus allowing the pendulum to simulate human hand planner movement. Also, this project gives deeper understanding about the modelling and PD controller. Hence, this project can be extended to make a drone.

This system serves as a benchmark for testing controllers. So,

- Non-linear controllers can be designed and their performance can be tested.
- Instead of encoders, using gyroscope the angle can be measured which reduces the size of the system and would be helpful in case of dual link pendulums.
- This system can act as a model to perform comparative study on performance of PD controllers.

Also, stability analysis of the linearized control system on actual non-linear system can be quantitatively investigated.

APPENDIX

A1. Microcontroller Program:

```
unsigned long long previousmillis = 0,previousmillis1 = 0;
unsigned long long previousmillis2 = 0,previousmillis3 = 0;

double x=0.00,theta=0.00,x_dot=0.00,theta_dot=0.00;
float t1=0,t2=0,t3=0,t4=0;
double prev_theta=0.00, prev_x=0.00;
unsigned long pulses=0;
double x_ref=4.92, x_error=0.00, x_prev_error=0.00, x_dot_error=0.00,
x_tot_error=0.00, theta_tot_error=0.00, theta_ref=180.00, theta_error=0.00,
theta_prev_error=0.00, theta_dot_error=0.00, controller_out=0.00;

double K1=4.9983, K2=7.7518, K3=32.1825, K4=11.1775; // PD values derived
int flag=0,a=0,b=0,c=0;
int duty_cycle=0;
unsigned int duty_cycle1=0;
int m1=PB9; // motor direction pin
void setup() {
pinMode(m1,OUTPUT);
attachInterrupt(PA4,limit,CHANGE);  //limit switch interrupt
 // Datasheet page 392 Encoder Interface
  Serial.begin(115200);
  RCC_BASE->APB2ENR |= (1<<2); // Enable clock to Port A

  RCC_BASE->APB2ENR |= (1<<3); // Enable clock to Port B
```

```
  // PWM Pulse Generation// PB8 as Output PWM

  GPIOB_BASE->CRH |= ((1<<0)|(1<<1)|(1<<3));
  GPIOB_BASE->CRH &= ~((1<<2));
  TIMER4_BASE->CR1 |= 1;
  TIMER4_BASE->CCMR2 |=((1<<5)|(1<<6));
  TIMER4_BASE->CCMR2 &= ~((1<<0)|(1<<1)|(1<<3)|(1<<4));
  TIMER4_BASE->CCER |= (1<<8);
  TIMER4_BASE->PSC = 71;
  TIMER4_BASE->ARR = 50000;
  TIMER4_BASE->CNT = 0;
  TIMER4_BASE->CCR3 = 100;

// PA0,PA1,PA6,PA7  as INPUT_PULLUP
  GPIOA_BASE->CRL |= ( (1<<3)|(1<<7)|(1<<27)|(1<<31) );
  GPIOA_BASE->CRL &= ~(
1|(1<<1)|(1<<2)|(1<<4)|(1<<5)|(1<<6)|(1<<24)|(1<<25)|(1<<28)|(1<<29)|(1<<26)|
(1<<30));
  GPIOA_BASE->ODR |= 1|(1<<1)|(1<<7)|(1<<6);
  //TIMER2_BASE->CR1 = TIMER_CR1_CEN; // Enable Timer
  TIMER2_BASE->CR1 =1;
  TIMER2_BASE->CR2 = 0;
  TIMER2_BASE->SMCR |= 2; // Encoder Mode SMS = 011
  TIMER2_BASE->DIER = 0; // Disable Timer Interrupts
  TIMER2_BASE->EGR = 0;
  TIMER2_BASE->CCMR1 = 257; // Encoder Mode Enable
  TIMER2_BASE->CCMR2 = 0;
```

```
  TIMER2_BASE->CCER = 0;
  TIMER2_BASE->PSC = 0;
  TIMER2_BASE->ARR = 43382; // 43382 Pulse Per revolution for position
  TIMER2_BASE->DCR = 0;
  TIMER2_BASE->CNT = 0;

  TIMER3_BASE->CR1 =1;
  TIMER3_BASE->CR2 = 0;
  TIMER3_BASE->SMCR |= 3; // Encoder Mode SMS = 011
  TIMER3_BASE->DIER = 0; // Disable Timer Interrupts
  TIMER3_BASE->EGR = 0;
  TIMER3_BASE->CCMR1 = 257; // Encoder Mode Enable
  TIMER3_BASE->CCMR2 = 0;
  TIMER3_BASE->CCER = 0;
  TIMER3_BASE->PSC = 0;
  TIMER3_BASE->ARR = 3999; // 4000 Pulse Per revolution for angle
  TIMER3_BASE->DCR = 0;
  TIMER3_BASE->CNT = 0;
}

void loop() {
if(b==0){
  begin1();
  }
if(c==0 && b==1){
  begin2();
  TIMER3_BASE->CNT = 0;
```

```
  }

if(c==1){
//Serial.println(theta);
 theta = ((TIMER3_BASE->CNT)*0.09); // 4000 pulses per revolution
 x = ((TIMER2_BASE->CNT)*0.000223595); // 43382 pulses per revolution
 x_error=x_ref-x;
 theta_error= theta_ref-theta;

errorcal(1);

controller_out =  (x_error*K1)+(x_dot_error*K2*1000)+  (theta_error*K3) +
(theta_dot_error*K4*1000);
duty_cycle = (23.155*controller_out)+2.011; //from excel
if(duty_cycle>0&&duty_cycle<300){
  duty_cycle=300;
}
if(duty_cycle<0&&duty_cycle>-300){
  duty_cycle=-300;
}
duty_cycle1 = (abs(duty_cycle))*50000/360;
print1(100);
if(theta==180.00){
  flag=1;
}

if (flag==1){
```

```
  if(x<0.2){
    digitalWrite(m1,LOW);
    TIMER4_BASE->CCR3 = 25000;
  }
  else if(x>9){
    digitalWrite(m1,HIGH);
    TIMER4_BASE->CCR3 = 25000;
  }
  else if(duty_cycle>=0){
    digitalWrite(m1,HIGH);
    TIMER4_BASE->CCR3 = duty_cycle1;
   }
   else if(duty_cycle<0){
    digitalWrite(m1,LOW);
    TIMER4_BASE->CCR3 = duty_cycle1;
    }
}


}
}
void begin1(){
   digitalWrite(m1,HIGH);
    TIMER4_BASE->CCR3 = 30000;
    if(a==1){
      TIMER4_BASE->CCR3 = 0;
      delay(500);
      TIMER2_BASE->CNT = 0;
```

```
      b=1;
      }
  }
void begin2(){
      Serial.println(TIMER2_BASE->CNT);
      if(TIMER2_BASE->CNT < 22000){
      digitalWrite(m1,LOW);
      TIMER4_BASE->CCR3 = 40000;
      }
      else{
        TIMER4_BASE->CCR3 = 0;
        delay(10000);

        c=1;
        }
  }

void errorcal(unsigned int delayy){
  if ((millis()-previousmillis)>=delayy){
     theta_dot = int ((theta - prev_theta)*1000/delayy);
     x_dot = int (x-prev_x)*1000/(delayy);


    x_dot_error= x_error-x_prev_error;
    theta_dot_error=theta_error-theta_prev_error;
    x_tot_error+=x_error;
    theta_tot_error+=theta_error;
```

```
prev_theta = theta;
prev_x = x;
x_prev_error = x_error;
theta_prev_error = theta_error;
previousmillis = millis();


  }
}
void print1(unsigned int delayy1){
if ((millis()-previousmillis1)>=delayy1){
      Serial.print(duty_cycle);
      Serial.print("                        ");
      Serial.print(x);
      Serial.print("                        ");
      Serial.println(theta);
      previousmillis1 = millis();
}
  }
void limit(){
  a = 1;
}
```

A2. MATLAB Programs

A2.1 Open-loop impulse response

```
M = 0.42;
m = 0.05;
b = 0.005;
d = 0.05;
I = 0.002547;
g = 9.8;
l = 0.36;
q = (M+m)*(I+m*l^2)-(m*l)^2;
s = tf('s');

P_cart = (((I+m*l^2)/q)*s^2 +(d*s/q)- (m*g*l/q))/(s^4 + (d*(M+m))*s^3/q + (b*(I
+ m*l^2))*s^3/q - ((M + m)*m*g*l)*s^2/q + b*d*s^2/q - b*m*g*l*s/q);

P_pend = (m*l*s/q)/(s^3 + (d*(M+m))*s^2/q + (b*(I + m*l^2))*s^2/q - ((M +
m)*m*g*l)*s/q + b*d*s/q - b*m*g*l/q);

sys_tf = [P_cart ; P_pend];

inputs = {'u'};
outputs = {'x'; 'theta'};

set(sys_tf,'InputName',inputs)
set(sys_tf,'OutputName',outputs)
```

```
t=0:0.01:1;
impulse(sys_tf,t);
title('Open-Loop Impulse Response')
```

A2.2 Open-loop Step Response

```
t = 0:0.05:10;
u = ones(size(t));
[y,t] = lsim(sys_tf,u,t);
plot(t,y)
title('Open-Loop Step Response')
axis([0 3 0 50])
legend('x','theta')
```

A2.3 Impulse Response with PD Controller

```
M = 0.42;
m = 0.05;
b = 0.005;
d = 0.05;
I = 0.002547;
g = 9.8;
l = 0.36;
q = (M+m)*(I+m*l^2)-(m*l)^2;
s = tf('s');

P_pend = (m*l*s/q)/(s^3 + (d*(M+m))*s^2/q + (b*(I + m*l^2))*s^2/q - ((M +
m)*m*g*l)*s/q + b*d*s/q - b*m*g*l/q);

Kp = 32.1825;
```

```
Ki = 0;
Kd = 11.1775;

C = pid(Kp,Ki,Kd);
T = feedback(P_pend,C);
t=0:0.01:10;
impulse(T,t)
title({'Response of Pendulum Position to an Impulse Disturbance';'under PD Control: Kp = 32.1825, Kd = 11.1775'});
```

REFERENCES

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4. Equations of Motion for Inverted Pendulum (2DOF) using Lagrange's Equation. Link: https://www.youtube.com/watch?v=Fo7kuUAHj3s
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