Assignment 3

September 27, 2020

1 Exercise 4.1

1.1 Part 1

$X_t = 0.8X_{t-1} + \epsilon_t$

Backtracking gives,

$X_t = 0.8^2 X_{t-2} + 0.8\epsilon_{t-1} + \epsilon_t$

So backtracking k times gives,

$X_t = 0.8^k X_{t-k} + \sum_{j=0}^{k-1} 0.8^j \epsilon_{t-j}$

This gives the expression

$X_t = \sum_{j=0}^{\infty} 0.8^j \epsilon_{t-j}$

1.2 Part 2

$X_t = 1.25X_{t-1} + \epsilon_t$

$X_t = 0.8X_{t+1} - 0.8\epsilon_{t+1}$

Forward iterating gives,

$X_t = 0.8^2 X_{t+2} - 0.8^2 \epsilon_{t+2} - 0.8\epsilon_{t+1}$

So forward iterating k times gives,

$X_t = 0.8^k X_{t+k} - \sum_{j=0}^{k-1} 0.8^{j+1} \epsilon_{t+j+1}$

This gives the expression

$X_t = \sum_{j=0}^{\infty} 0.8^{j+1} \epsilon_{t+j+1}$

1.3 Part 3

1.3.1 Autocovariance of Part 1

$Cov(X_t, X_{t+k}) = Cov(\sum_{j=0}^{\infty} 0.8^j \epsilon_{t-j}, \sum_{j=0}^{\infty} 0.8^j \epsilon_{t+k-j})$

The first k - 1 terms of the term X_{t+k} do not have any correlated term in X_t. So mapping the remaining terms with each other gives covariance as

$\sum_{j=0}^{\infty} 0.8^{2j+k} Var(\epsilon_{t-j})$

$Cov(X_t, X_{t+k}) = 0.8^k \sum_{j=0}^{\infty} 0.8^{2j}$

$\implies Cov(X_t, X_{t+k}) = 0.8^k/0.36$

1.3.2 Autocovariance of Part 2

$Cov(X_t, X_{t+k}) = Cov(\sum_{j=0}^{\infty} 0.8^{j+1}\epsilon_{t+j+1}, \sum_{j=0}^{\infty} 0.8^{j+1}\epsilon_{t+k+j+1})$

The first k - 1 terms of the term X_t do not have any correlated term in X_{t+k}. So mapping the remaining terms with each other gives covariance as

$\sum_{j=0}^{\infty} 0.8^{2j+k+2} Var(\epsilon_{t+j+k+1})$

$Cov(X_t, X_{t+k}) = 0.8^{k+2} \sum_{j=0}^{\infty} 0.8^{2j}$

$\implies Cov(X_t, X_{t+k}) = 0.8^{k+2}/0.36$

2 Exercise 4.2

Let λ be a root of the characteristic equation of the given time series

So $1 - \sum_{j=1}^{p} \phi_j \lambda^j = 0$

$\implies \sum_{j=1}^{p} \phi_j \lambda^j = 1$

$\implies |\sum_{j=1}^{p} \phi_j \lambda^j| = 1$

Using triangle inequality,

$\sum_{j=1}^{p} |\phi_j \lambda^j| \geq |\sum_{j=1}^{p} \phi_j \lambda^j|$

$\implies \sum_{j=1}^{p} |\phi_j \lambda^j| \geq 1$ ——- condition (1)

But it is given that $\sum_{j=1}^{p} |\phi_j| < 1$

if $|\lambda| \leq 1$, $|\lambda^j| \leq 1$

So,

$|\phi_j \lambda^j| \leq |\phi_j| \; \forall j$

$\implies \sum_{j=1}^{p} |\phi_j \lambda^j| < 1$

But this contradicts condition (1)

So, absolute value of all the roots of the equation should be greater than 1 (They lie outside the circle and have a causal stationary solution)

3 Exercise 4.3

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$

Let λ_1 and λ_2 be the reciprocals of the roots of the characteristic equation of $1 - \phi_1 z - \phi_2 z^2$

i.e $(1 - \lambda_1 z)(1 - \lambda_2 z) = 0$

The roots lie outside the unit circle when the solution is causal

This gives the condition

$|\lambda_1| < 1$ and $|\lambda_2| < 1$

3.1 1st condition

From the equation the following condition is obtained

$\phi_1 = \lambda_1 + \lambda_2$

$\phi_2 = -\lambda_1 \lambda_2$

Since the absolute value of both λ_1 and λ_2 are less than 1, the absolute value of their products is less than 1.

So, $|\phi_2| < 1$

3.2 When both the roots are real

3.2.1 2nd condition

$\phi_1 + \phi_2 = \lambda_1 + \lambda_2 - \lambda_1 \lambda_2$

It is given that $\lambda_1 < 1$ since $|\lambda_1| < 1$

Since $\lambda_2 < 1$,The above equation gives the expression

$\lambda_1(1 - \lambda_2) < (1 - \lambda_2)$

Adding λ_2 on both sides,

$\implies \lambda_1(1 - \lambda_2) + \lambda_2 < 1$

$\implies \lambda_1 + \lambda_2 - \lambda_1 \lambda_2 < 1$

$\implies \phi_1 + \phi_2 < 1$

3.2.2 3rd condition

$\phi_2 - \phi_1 = -\lambda_1 - \lambda_2 - \lambda_1 \lambda_2$

It is given that $-\lambda_1 < 1$ since $|\lambda_1| < 1$

Since $\lambda_2 > -1$,The above equation gives the expression

$-\lambda_1(1 + \lambda_2) < (1 + \lambda_2)$

Subtracting λ_2 on both sides,

$\implies -\lambda_1(1 + \lambda_2) - \lambda_2 < 1$

$\implies -\lambda_1 - \lambda_2 - \lambda_1 \lambda_2 < 1$

$\implies \phi_2 - \phi_2 < 1$

3.3 When both the roots are not real

Let $\lambda_1 = re^{i\theta}$, $\lambda_2 = re^{-i\theta}$

$\phi_1 = 2rcos(\theta)$

$\phi_2 = r^2$

3.3.1 2nd condition

$\phi_1 + \phi_2 = 2rcos(\theta) - r^2$

Since $cos(\theta) \leq 1$

$\phi_1 + \phi_2 \leq 2r - r^2$

Since r < 1

$\phi_1 + \phi_2 < 1$

3.3.2 3rd condition

$\phi_2 - \phi_1 = -r^2 - 2rcos(\theta)$

Since -$cos(\theta) \leq 1$

$\phi_2 - \phi_1 \leq 2r - r^2$

Since r < 1

$\phi_2 - \phi_1 < 1$

4 Exercise 4.4

4.1 Part a

Let λ_1, λ_2 be the reciprocals of the roots of the given characteristic polynomial equation

This gives,

$\lambda_1 + \lambda_2 = \phi_1$

$\lambda_1\lambda_2 = -\phi_2$

Since the absolute value of the roots are greater than 1, λ_1 and λ_2 are less than 1

$|\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2|$

$\implies |\lambda_1 + \lambda_2| < 2$

$\implies |\phi_1| < 2$

$|\lambda_1\lambda_2| = |\lambda_1||\lambda_2|$

$\implies |\lambda_1\lambda_2| < 1$

$\implies |\phi_2| < 1$

$\implies |\phi_1| + |\phi_2| < 3$

$\implies |\phi_1| + |\phi_2| < 4$

4.2 Part b

Let λ_1, λ_2, λ_3, ……, λ_p be the reciprocals of the roots of the given characteristic polynomial equation

The following expressions are obtained from Vieta's formula

$|\lambda_1 \lambda_2 \lambda_3 .....\lambda_p| = |\phi_p|$

$|\sum_{j=1}^{p} 1/\lambda_j| = |\phi_{p-1}/\phi_p|$

This gives $|\phi_{p-1}|$ = Absolute value of sum of product of λs, (p-1) terms at a time

Using triangle inequality,

$|\phi_{p-1}| \leq$ Sum of absolute value of product of λs, (p-1) terms at a time

And continuing this gives

$|\phi_{p-k}|$ = Absolute value of sum of product of λs, (p-k) terms at a time

$|\phi_{p-k}| \leq$ Sum of absolute value of product of λs, (p-k) terms at a time

Since $|\phi_{p-k}|$ has ${}^{p}C_{p-k}$ terms and all the λs are less than 1,

$|\phi_{p-k}| \leq {}^{p}C_{p-k}$

$\sum_{k=1}^{p} |\phi_k| \leq \sum_{k=1}^{p} {}^{p}C_{p-k}$

$\implies \sum_{k=1}^{p} |\phi_k| \leq 2^p - 1$

$\implies \sum_{k=1}^{p} |\phi_k| \leq 2^p$

5 Exercise 4.5

5.1 Part a

$X_t = \frac{7}{3}\ X_{t-1} - \frac{2}{3}\ X_{t-2} + \epsilon_t$

The characteristic equation of the above expression is

$1 - \frac{7}{3}\ z + \frac{2}{3}\ z^2 = 0$

The roots are 1/2 and 3

Comparing with the expression $(1 - az)(1 - bz) = 0$

a = 2, b = 1/3

So, the series can be written as the sum of a causal and a non-causal time-series

Using the result (4.10) in the textbook, the following expression is obtained

$X_t = \frac{-3}{5} \sum_{j=0}^{\infty} (3^{-j-1}\epsilon_{t-j} + 2^{-j}\epsilon_{t+1+j})$

The above expression is non-causal, so there is no MA(∞) representation

5.2 Part b

$X_t = \frac{4\sqrt{3}}{5}\ X_{t-1} - \frac{4^2}{5^2}\ X_{t-2} + \epsilon_t$

The characteristic equation of the above expression is

1 - $\frac{4\sqrt{3}}{5}\ z + \frac{4^2}{5^2}\ z^2 = 0$

The roots are not real since the discriminant is negative

Comparing with the equation,

$1 - 2rcos(\theta) + r^2 = 0$

$r = 4/5$

Solving for θ,

$cos(\theta) = \frac{\sqrt{3}}{2}$

$\implies \theta = \pi/6$

Substituting the above values in equation (4.12) in the textbook,

$X_t = \frac{5}{4}\sum_{j=0}^{\infty} 2 * 0.8^{j+1} sin((j+1)\pi/6)\epsilon_{t-j}$

$\implies X_t = 2\sum_{j=0}^{\infty} 0.8^{j} sin((j+1)\pi/6)\epsilon_{t-j}$

The above expression is causal and $\sum_{j=0}^{\infty} |0.8^{j} sin((j+1)\pi/6)|$ is a converging series

So, this has an MA(∞) representation

5.3 Part c

$X_t = X_{t-1} - 4X_{t-2} + \epsilon_t$

The characteristic equation of the above expression is

$1 - z + 4z^2 = 0$

The roots are not real since the discriminant is negative

Comparing with the equation,

$1 - 2rcos(\theta) + r^2 = 0$

$r = 2$

Solving for θ,

$cos(\theta) = \frac{1}{4}$

$sin(\theta) = \frac{\sqrt{15}}{4}$

Substituting $b = re^{i\theta}$ and $a = re^{-i\theta}$ in the following equation,

$\frac{1}{(1-za)(1-zb)} = \frac{1}{(b-a)}(\frac{b}{(1-bz)} - \frac{a}{(1-az)})$

The solution of an AR(2) when both the roots are not real and lie inside the circle becomes

$X_t = \frac{1}{2rsin(\theta)}\sum_{j=0}^{\infty} 2r^{-j} sin(j\theta)\epsilon_{t+j+1}$

Substituting the values gives

$X_t = \frac{2}{\sqrt{15}} \sum_{j=0}^{\infty} r^{-j} sin(j\theta)\epsilon_{t+j+1}$

where $\theta = sin^{-1}(\sqrt{15}/4)$

The above expression is non-causal. So, there is no MA(∞) representation

6 Exercise 4.6

A causal stationary AR(2) process with pseudo-period 17 has the form $X_t = 2rcos(2\pi/17)X_{t-1} - r^2 X_{t-2} + \epsilon_t$ with 0 < r < 1

The process is plotted for r = 0.5 and r = 0.95. This gives,

$$X_t = cos(2\pi/17)X_{t-1} - 0.25X_{t-2} + \epsilon_t$$

and

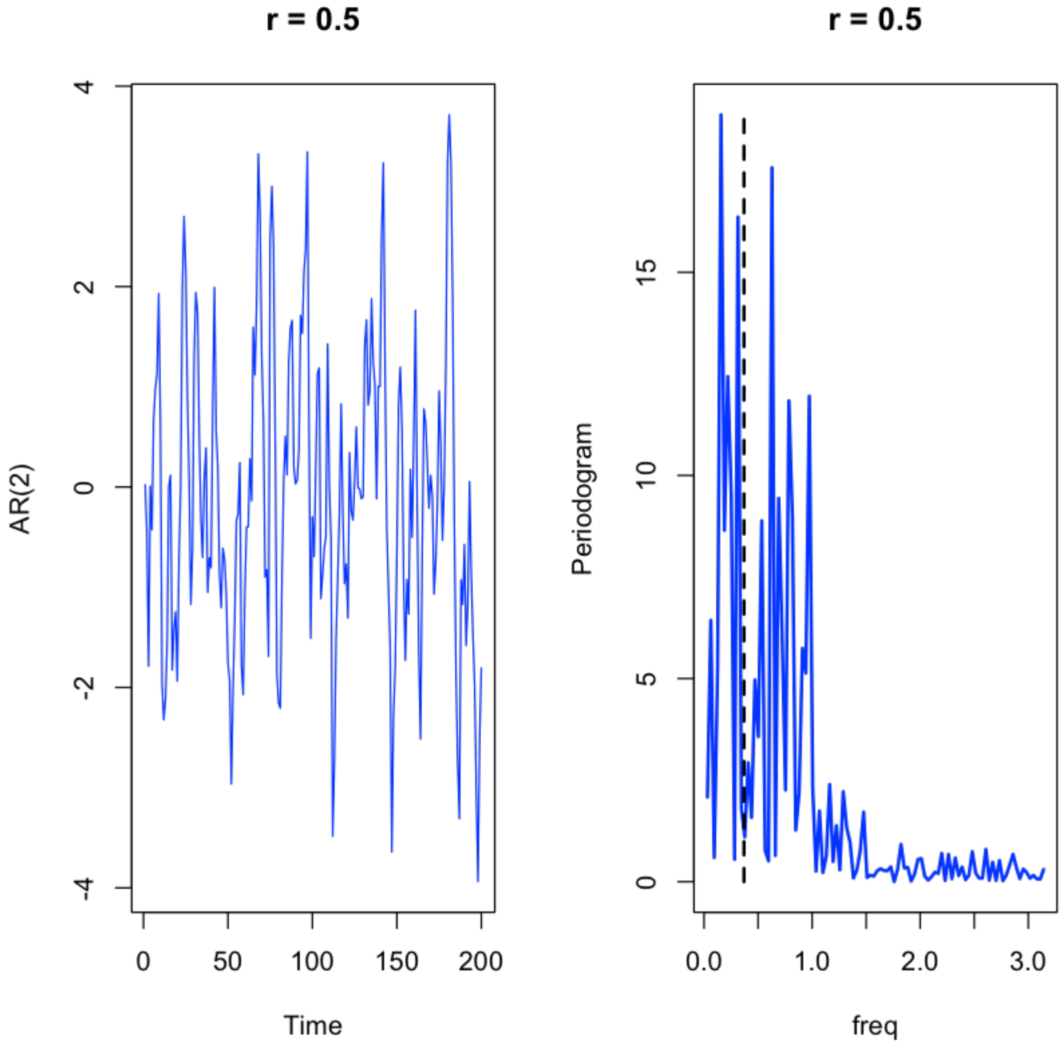
$$X_t = 1.9cos(2\pi/17)X_{t-1} - 0.9025X_{t-2} + \epsilon_t$$

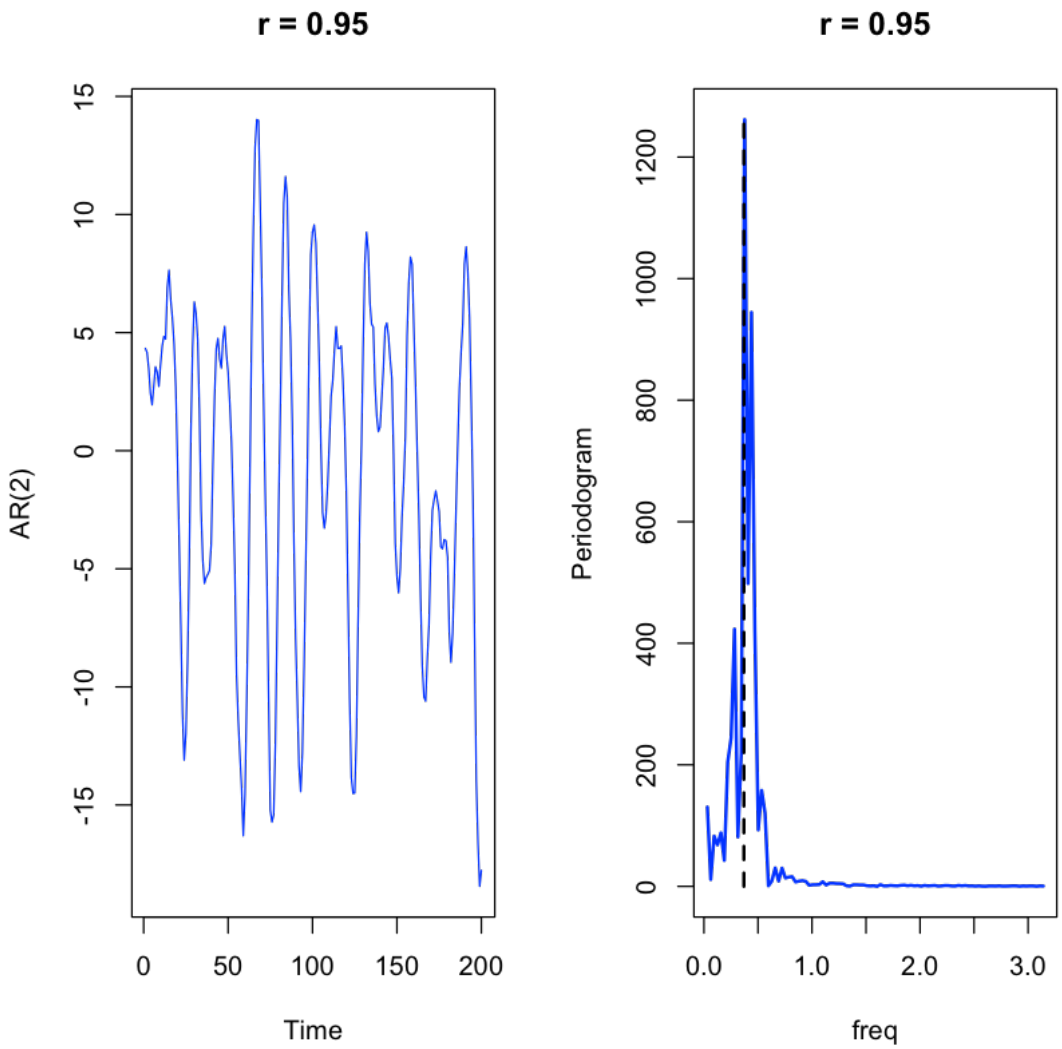
The above time series are plotted below and their periodograms are plotted to their right.

The periodogram's peak does not occur at 17 as 200/17 is not an integer.

But the peak is closer when r is increased closer to 1

[199]:
```
plot_function <- function (r, omega, n, label){
    ts.sim <- arima.sim(list(order = c(2,0,0), ar = c(2*r*cos(omega), -r*r)), n␣
 ↪= 200)
    F = abs(fft(ts.sim)/sqrt(n))**2
    F = F[c(1:(n/2))]
    freq = 2*pi*c(1:(n/2))/n
    par(mfrow=c(1,2))
    ts.plot(ts.sim, col = "blue", ylab = "AR(2)", main = label)
    plot(freq, F, lwd=2, col="blue", type="l", ylab="Periodogram", main = label)
    lines(c(omega, omega),c(0,max(F)),lwd=2,lty=2)
    return (n/which.max(F))
}

period_max1 <- plot_function(0.5, 2*pi/17, 200, "r = 0.5")
period_max2 <- plot_function(0.95, 2*pi/17, 200, "r = 0.95")
```


r = 0.5
AR(2)
Time
0
50
100
150
200
4
2
0
-2
-4
r = 0.5
Periodogram
freq
0.0
1.0
2.0
3.0
0
5
10
15




```
[208]: print("Period obtained from r = 0.5")
print(period_max1)
print("Period obtained from r = 0.95")
print(period_max2)
```

```
[1] "Period obtained from r = 0.5"
[1] 40
[1] "Period obtained from r = 0.95"
[1] 16.66667
```

7 Exercise 4.7

The noise used is a uniformly distributed uncorrelated noise between -1 and 1.

7.1 Part 1

This is a combination of a causal AR(2) and a non-causal AR(2) processes

Let $r_1e^{i\theta_1}$, $r_1e^{-i\theta_1}$, $r_2e^{i\theta_2}$, $r_2e^{-i\theta_2}$ be the reciprocals of the roots of the equation

$r_1 = 0.8$

$\theta_1 = 2\pi/13$

$r_2 = 1.5$

$\theta_1 = 2\pi/5$

So, the non-causal AR(2) process is simulated first with the characteristic equation

$1 - 3cos(2\pi/5)z + 2.25z^2 = 0$

Since this is non-causal, it is represented by the following backward recursion equation

$Y_t = \frac{1}{r^2}(-Y_{t+2} + 2rcos(\theta)Y_{t+1} + \epsilon_{t+2})$

where ϵ_t ~ Uniform(-1, 1)

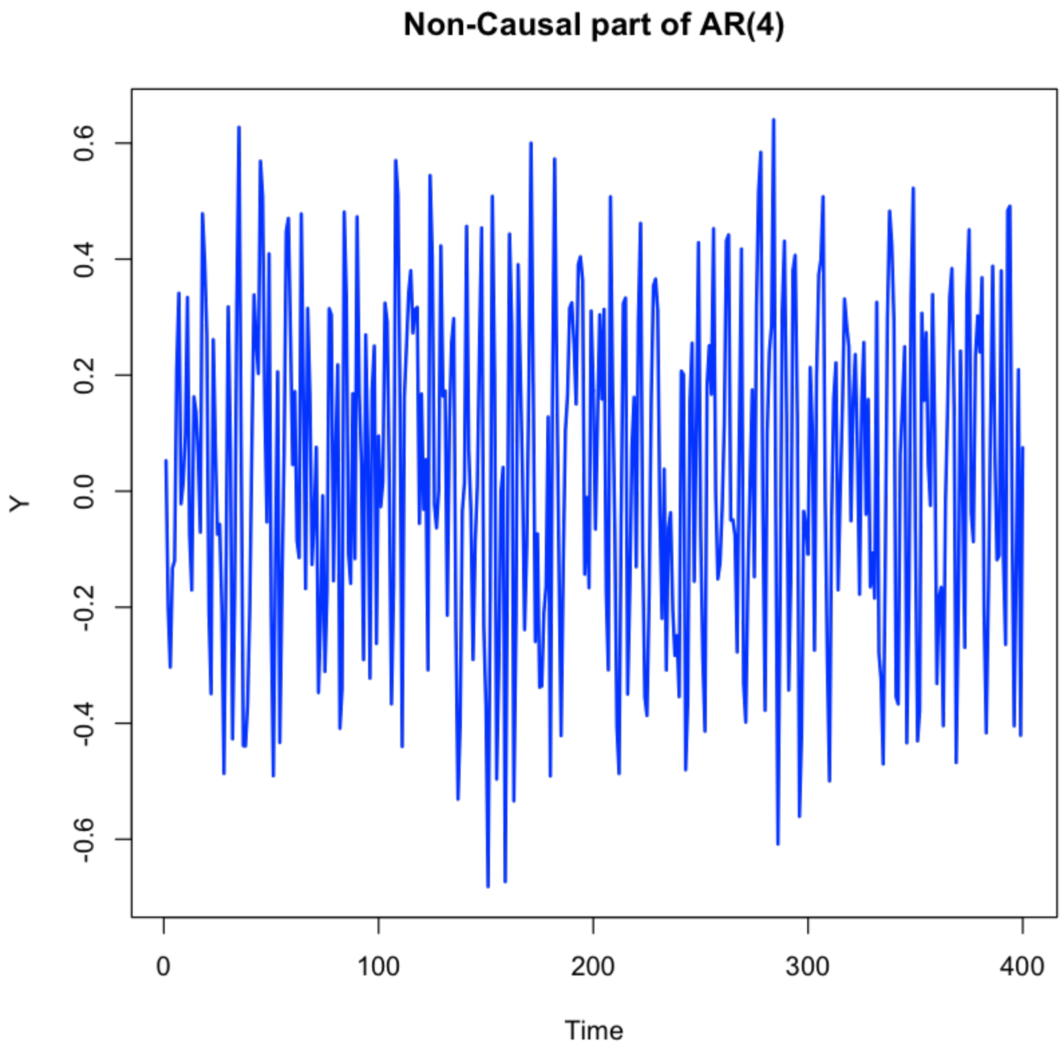
Then, the following causal AR(2) process is used to simulate the AR(4) process

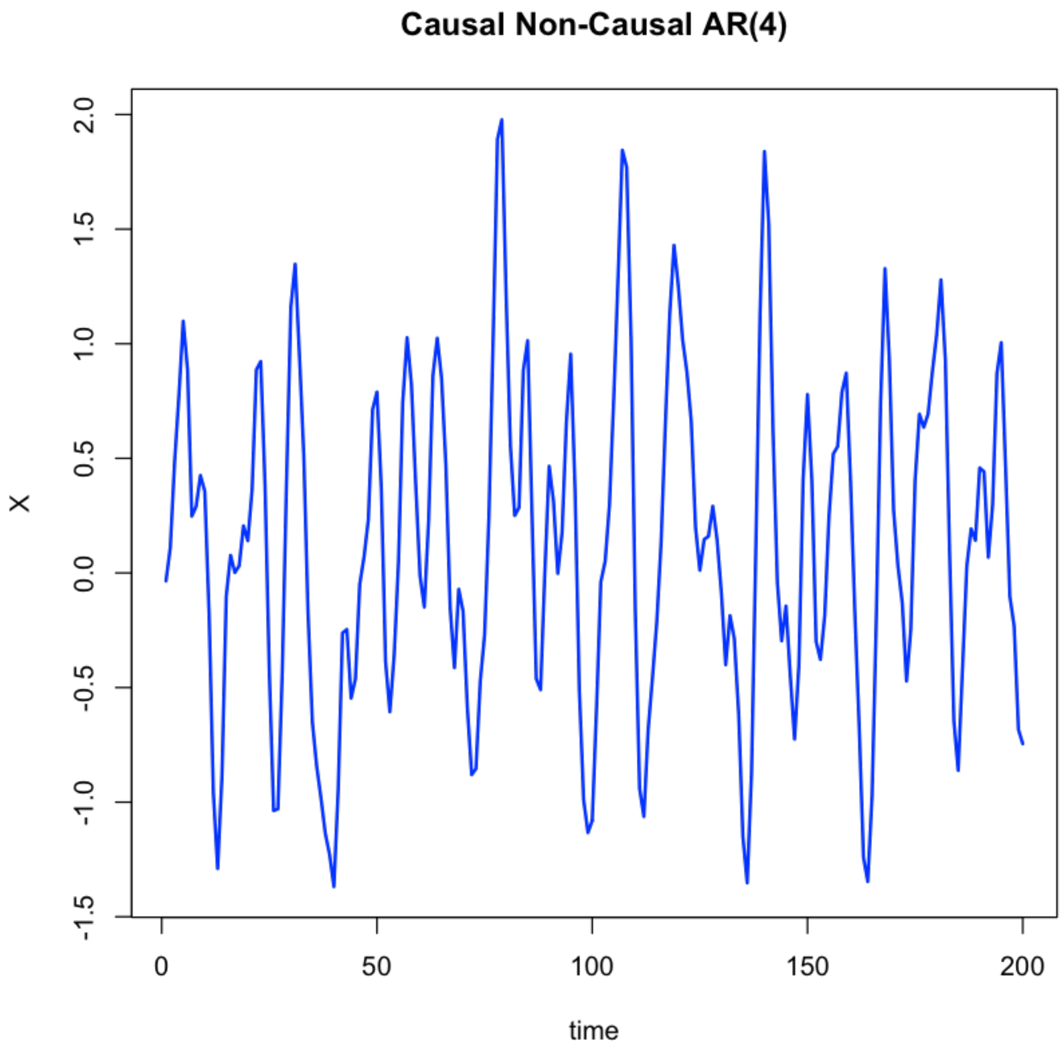
$X_t = 2rcos(\theta)X_{t-1} - r^2X_{t-2} + Y_t$

[233]:
```
ar2_non_causal <- function(r, omega, n, next_n, innovations){
    x = rep(0, n)
    for(j in c((n-2):1)){
        x[j] = (1/(r*r))*(-x[j+2]+2*r*cos(omega)*x[j+1]+innovations[j+2])
    }
    x = x[c(1:next_n)]
    return(x)
}

ar2_causal <- function(r, omega, n, next_n, y){
    x = rep(0, n)
    for(j in c(3:n)){
        x[j] = (2*r*cos(omega)*x[j-1] - (r**2)*x[j-2] + y[j])
     }
    x = x[c((n-next_n+1): n)]
    return(x)
}

ar4_causal_non_causal <- function(r1, omega1, r2, omega2, n){
    #innov <- rnorm(n+400, sd = 10)
    innov <- -1+2*runif(n+400)
    y <- ar2_non_causal(r1, omega1, n+400, n+200, innov)
    x <- ar2_causal(r2, omega2, n+200, n, y)
    time = c(1:n)
    plot(c(1:(n+200)), y, lwd=2,type="l",col="blue",ylim=c(min(y),max(y)),xlab␣
 ↪= "Time", ylab = "Y", main="Non-Causal part of AR(4)")
```

```
    plot(time,x,lwd=2,type="l",col="blue",ylim=c(min(x),max(x)), ylab =␣
 ↪"X",main="Causal Non-Causal AR(4)")
}
ar4_causal <- function(r1, omega1, r2, omega2, n){
    #innov <- rnorm(n+400, sd = 10)
    innov <- -1+2*runif(n+400)
    y <- ar2_causal(r1, omega1, n+400, n+200, innov)
    x <- ar2_causal(r2, omega2, n+200, n, y)
    time = c(1:n)
    plot(c(1:(n+200)), y, lwd=2,type="l",col="blue",ylim=c(min(y),max(y)),xlab␣
 ↪= "Time", ylab = "Y", main="First Causal part of AR(4)")
    plot(time,x,lwd=2,type="l",col="blue",ylim=c(min(x),max(x)), ylab = "X",␣
 ↪main="Causal AR(4)")
}
```

[234]:
```
ar4_causal_non_causal(1.5, 2*pi/5, 0.8, 2*pi/13, 200)
```


Non-Causal part of AR(4)
Y
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
100
200
300
400
Time



7.2 Part 2

This is a combination of 2 causal AR(2) processes

Let $r_1 e^{i\theta_1}$, $r_1 e^{-i\theta_1}$, $r_2 e^{i\theta_2}$, $r_2 e^{-i\theta_2}$ be the reciprocals of the roots of the equation

$r_1 = 0.8$

$\theta_1 = 2\pi/13$

$r_2 = 2/3$

$\theta_1 = 2\pi/5$

So, the AR(2) process with roots $2/3e^{2\pi/5}$ and $2/3e^{-2\pi/5}$ is simulated first with the characteristic equation

$1 - 4/3cos(2\pi/5)z + 4/9z^2 = 0$

It is represented by the following recursion equation

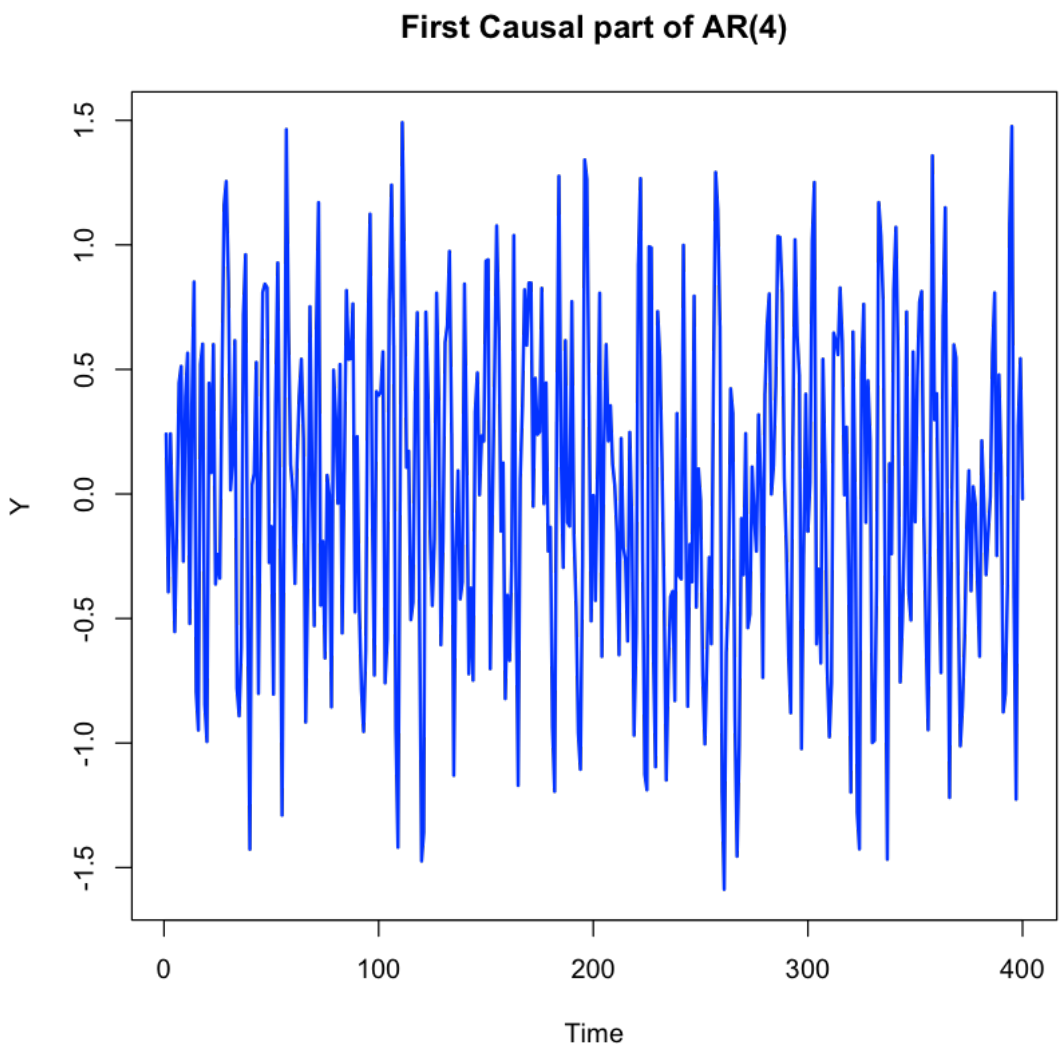
$Y_t = 2rcos(\theta)Y_{t-1} - r^2cos(\theta)Y_{t-2} + \epsilon_t$

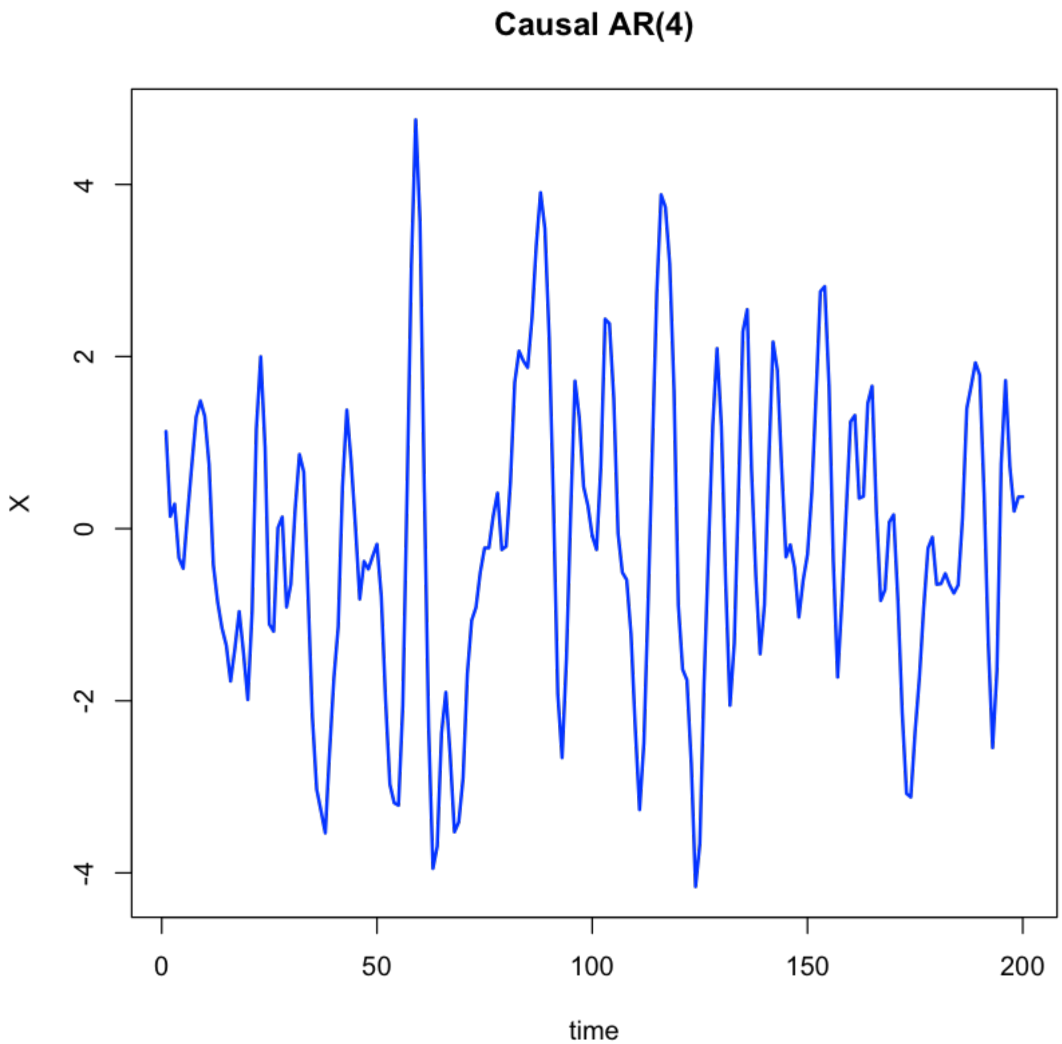
where ϵ_t ~ Uniform (-1, 1)

Then, the following causal AR(2) process is used to simulate the AR(4) process

$X_t = 2rcos(\theta)X_{t-1} - r^2X_{t-2} + Y_t$

[235]:
```
ar4_causal(2/3, 2*pi/5, 0.8, 2*pi/13, 200)
```





7.3 Inference:

The amplitude of the AR(4) process that has a non-causal component appears to have an amplitude less than a purely causal AR(4) process.