Assignment 2

September 14, 2020

1 Problem 3.1

1.1 Part 1

It is given that $X_t = \epsilon_t + 0.5\epsilon_{t-1}$

We need to evaluate $cov(X_t, X_{t+r}) = cov(\epsilon_t, \epsilon_{t+r}) + 0.5cov(\epsilon_{t-1}, \epsilon_{t+r}) + 0.5cov(\epsilon_t, \epsilon_{t+r-1}) + 0.25cov(\epsilon_{t-1}, \epsilon_{t+r-1})$

When r = 0 $cov(X_t, X_{t+r}) = cov(\epsilon_t, \epsilon_t) + 0.5cov(\epsilon_{t-1}, \epsilon_t) + 0.5cov(\epsilon_t, \epsilon_{t-1}) + 0.25cov(\epsilon_{t-1}, \epsilon_{t-1})$

Since the random variables are uncorrelated, the middle two terms cancel out and the resulting expression is

$cov(X_t, X_{t+r}) = cov(\epsilon_t, \epsilon_t) + 0.25cov(\epsilon_{t-1}, \epsilon_{t-1})$

$cov(X_t, X_{t+r}) = var(\epsilon_t) + 0.25var(\epsilon_{t-1})$

$cov(X_t, X_{t+r}) = 1.25\sigma^2$

When r = 1 $cov(X_t, X_{t+r}) = cov(\epsilon_t, \epsilon_{t+1}) + 0.5cov(\epsilon_{t-1}, \epsilon_{t+1}) + 0.5cov(\epsilon_t, \epsilon_t) + 0.25cov(\epsilon_{t-1}, \epsilon_t)$

Since the terms 1, 2, 4 cancel out,

$cov(X_t, X_{t+r}) = 0.5\sigma^2$

When r = -1 $cov(X_t, X_{t+r}) = cov(\epsilon_t, \epsilon_{t-1}) + 0.5cov(\epsilon_{t-1}, \epsilon_{t-1}) + 0.5cov(\epsilon_t, \epsilon_{t-2}) + 0.25cov(\epsilon_{t-1}, \epsilon_{t-2})$

Since the terms 1, 3, 4 cancel out,

$cov(X_t, X_{t+r}) = 0.5\sigma^2$

When |r| > 1 $cov(X_t, X_{t+r}) = 0$

1.2 Part 2

It is given that $X_t = \sum_{j=0}^{\infty} \rho^j \epsilon_{t-j}, |\rho| < 1$

$X_t = \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + ...$

$X_{t+r} = \epsilon_{t+r} + \rho\epsilon_{t+r-1} + \rho^2\epsilon_{t+r-2} + \rho^3\epsilon_{t+r-3} + ...$

$var(X_t) = \sigma^2 + \rho^2\sigma^2 + \rho^4\sigma^2 + \rho^6\sigma^2 + ...$

$var(X_t) = \frac{\sigma^2}{(1-\rho^2)}$

When r >= 0 $X_{t+r} = \rho^r X_t + \sum_{j=1}^{r} \rho^{r-j}\epsilon_{t+j}$

The random variables in the summation part are $\epsilon_{t+1}, \epsilon_{t+2}, ...$ and they are not correlated with any term in X_t

Therefore,

$cov(X_t, X_{t+r}) = cov(X_t, \rho^r X_t)$,

$cov(X_t, X_{t+r}) = \rho^r var(X_t)$,

$cov(X_t, X_{t+r}) = \rho^r \frac{\sigma^2}{(1-\rho^2)}$

When r < 0 $X_t = \rho^{-r} X_{t+r} + \sum_{j=r}^{-1} \rho^{-r+j}\epsilon_{t+r-j}$

The random variables in the summation part are $\epsilon_{t+r+1}, \epsilon_{t+r+2}, ..., \epsilon_t$ and they are not correlated with any term in X_{t+r}

Therefore,

$cov(X_t, X_{t+r}) = cov(\rho^{-r} X_{t+r}, X_{t+r})$

$cov(X_t, X_{t+r}) = \rho^{-r} var(X_{t+r})$

$cov(X_t, X_{t+r}) = \rho^{-r} \frac{\sigma^2}{(1-\rho^2)}$

Hence we get a generalized result as

$cov(X_t, X_{t+r}) = \rho^{|r|} \frac{\sigma^2}{(1-\rho^2)}$

[62]:
```
# residual <- rnorm(100)
```

[63]:
```
# X1 <- tail(residual, -1) + 0.5*head(residual, -1)
```

[64]:
```
# print(acf(X1, lag.max = 5, type = "covariance"))
```

[65]:
```
# rho <- -0.99999+1.99998*runif(1)
```

[66]:
```
# X2 <- c()
# for (i in c(1:length(residual))){
#     x2 <- 0
#     for (j in c(1:i)){
#         x2 = x2 + (rho**(i-j+1))*residual[j]
#     }
#     X2 = c(X2, x2)
# }
# print(acf(X2, type = "covariance"))
```

2 Problem 3.2

2.1 Part 1

A time series of iid random variables with mean 0 and variance 1 is both strictly stationary and weak stationary Since each term of the sequence is independent and identically distributed, the distribution is the same throughout. So the distribution of $\epsilon_t, \epsilon_{t+1}, \epsilon_{t+2}, ...$ is the same as $\epsilon_{t+k}, \epsilon_{t+k+1}, \epsilon_{t+k+2}, ...$ This gives the strict stationarity of the series.

$E(\epsilon_t^2) = 1$ which is finite. So the time series is also weakly stationary

2.2 Part 2

A time series of iid random variables from a cauchu distribution is strictly stationary but not weak stationary. Since each term of the sequence is independent and identically distributed, the distribution is the same throughout. So the distribution of $\epsilon_t, \epsilon_{t+1}, \epsilon_{t+2}, ...$ is the same as $\epsilon_{t+k}, \epsilon_{t+k+1}, \epsilon_{t+k+2}, ...$ This gives the strict stationarity of the series.

But $E(\epsilon_t^2)$ is infinite. So the time series is not weakly stationary

2.3 Part 3

A random walk time series is not stationary $X_t = X_{t-1} + \epsilon t - 1$ $X_t = X_0 + \sum_{i=0}^{t-1} \epsilon_i$ $X_{t+k} = X_0 + \sum_{i=0}^{t+k-1} \epsilon_i \implies cov(X_t, X_{t+k}) = \sum_{i=0}^{t-1} var(\epsilon_i) \implies cov(X_t, X_{t+k}) = t\sigma^2$

Since the autocovariance term is linearly dependent on time, the series is not stationary

2.4 Part 4

The series $X_t = Y\ where\ \mu = 0, \sigma = 1$ is a weakly stationary process

Since the distribution is not iid, the series is not identical. So it is not strictly stationary

$cov(X_t, X_{t+k}) = \sigma^2$ when k = 0 and 0 otherwise

Since the autocovariance is not a function of time, the series is weakly stationary

2.5 Part 5

The strict and weak stationarity of a vector of time series implies that the individual components are both strictly and weakly stationary

Also, the linear combination of time series that are both strictly and weakly stationary is both strictly and weakly stationary

So, X_t is both a strictly and weakly stationary time series

$X_t = U_t + U_{t-1} + V_t$

$cov(X_t, X_{t+k}) = 2c(k) + 2c(k-1) + c'(k) + cov(U_t, V_{t+k}) + cov(U_{t-1}, V_{t+k}) + cov(U_{t+k}, V_t) + cov(U_{t+k-1}, V_t)$

Due to the joint stationarity, cov(U, V) terms become a function of the lag coefficient

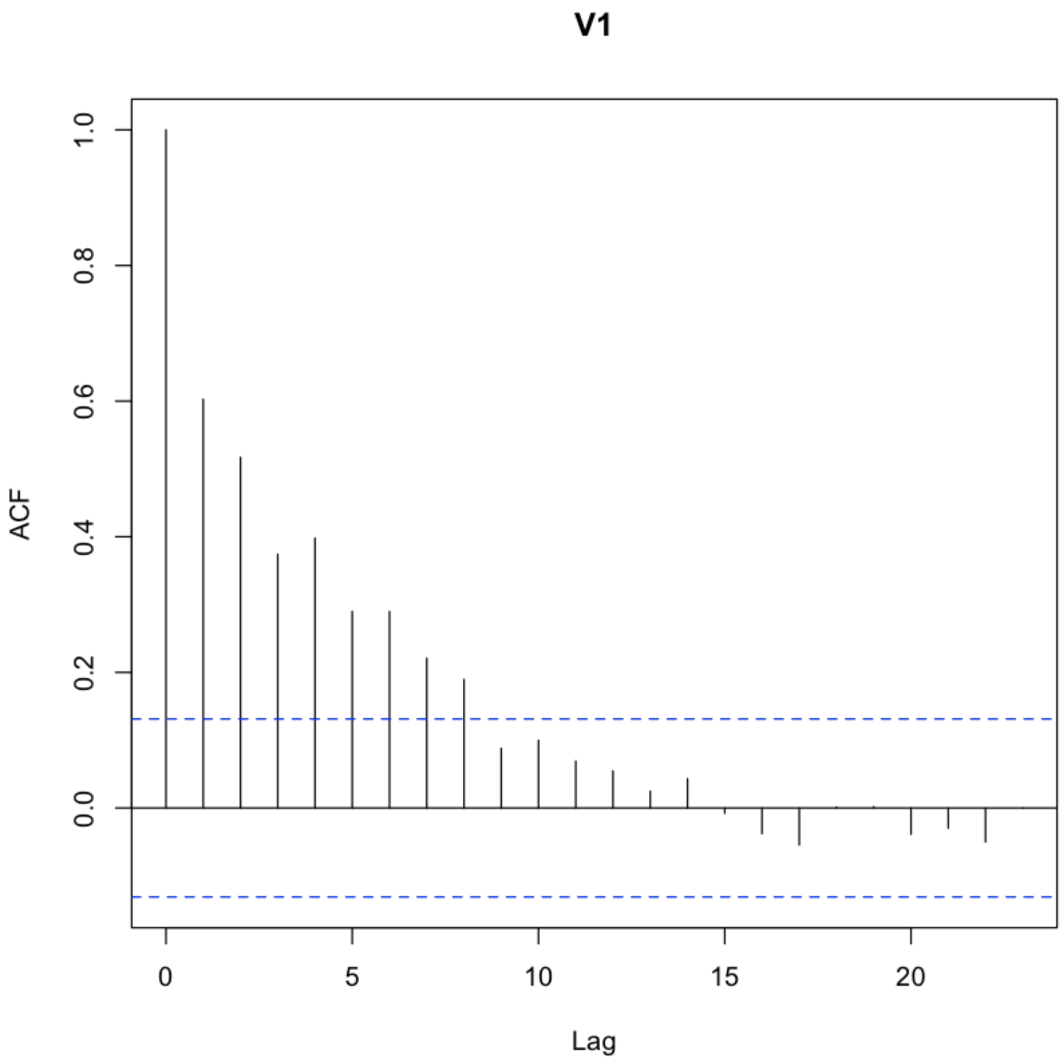
So, $cov(X_t, X_{t+k})$ is a function of k

3 Problem 3.3

3.1 ACF of Monthly Global Temperature

[67]:
```
data2 <- read.csv("https://www.stat.tamu.edu/~suhasini/teaching673/Data/
↪month_temp.txt", header = FALSE)
```

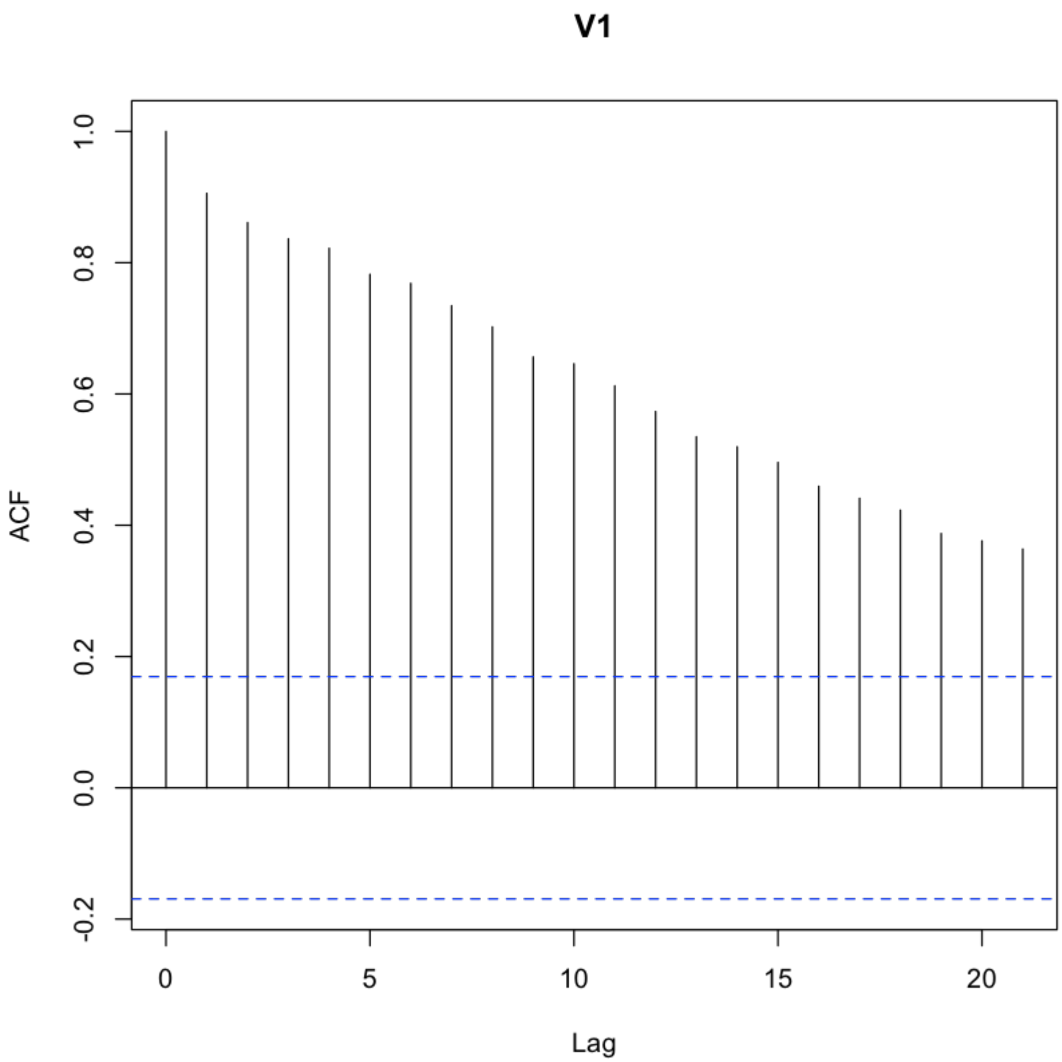
[68]:
```
acf(data2)
```



3.2 ACF of Yearly Global Temperature

[69]:
```
data1 <- read.csv('https://www.stat.tamu.edu/~suhasini/teaching673/Data/
↪global_mean_temp.txt', header = FALSE)
```

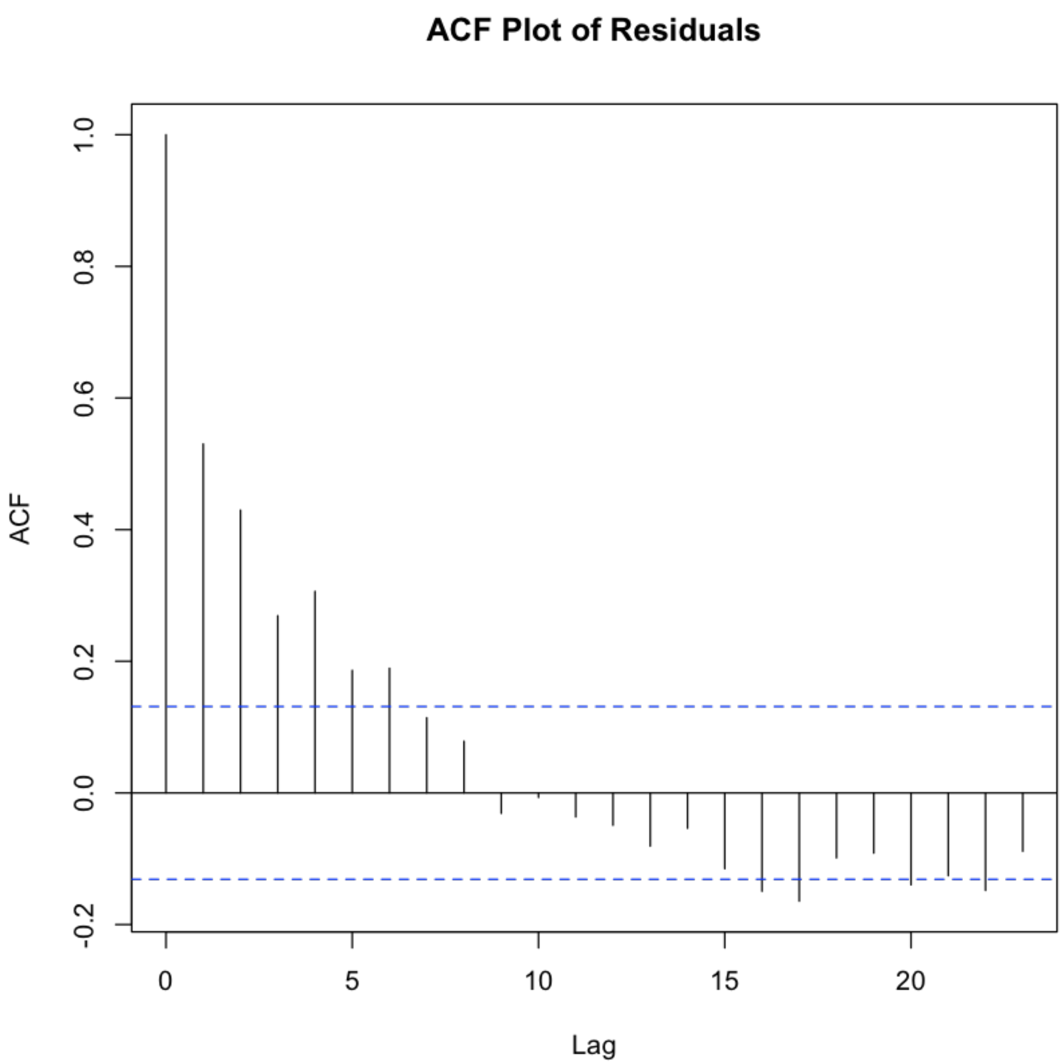
[70]:
```
acf(data1)
```



3.3 ACF plot of the residuals (after fitting a line through the data (using the command lsfit(..)$res)) of the yearly temperature data from 1880-2013.

[71]:
```r
data2 <- ts(data2, frequency = 1, start = c(1, 1))
Time <- c(1:length(data2))
model <- lm(data2 ~ Time)
out = summary(model)
data2_residual <- out$residuals
```

[72]:
```r
acf(data2_residual, main = "ACF Plot of Residuals")
```



4 Problem 3.4

4.1 Part 1

Problem 3.5

(i) $\operatorname{cov}(Y_t, Y_{t+k}) = \operatorname{cov}\left(\frac{1}{1+X_t^2}, \frac{1}{1+X_{t+k}^2}\right)$

$$= E\left[\frac{1}{(1+X_t^2)} \cdot \frac{1}{(1+X_{t+k}^2)}\right] - E\left[\frac{1}{1+X_t^2}\right] E\left[\frac{1}{1+X_{t+k}^2}\right]$$

$$= \sum_{x,y} \frac{P_{X_t, X_{t+k}}(x,y)}{(1+x^2)(1+y^2)} - \sum_x \frac{P_{X_t}(x)}{1+x^2} \sum_y \frac{P_{X_{t+k}}(y)}{1+y^2}$$

Since $P_{X_t, X_{t+k}}(x,y) = P_{X_0, X_k}(x,y)$

and $P_{X_t}(x) = P_{X_0}(x) = P_{X_{t+k}}(x)$

$$\operatorname{cov}(Y_t, Y_{t+k}) = \sum_{x,y} \frac{P_{X_0,X_k}(x,y)}{(1+x^2)(1+y^2)} - \left[\sum_x \frac{P_{X_0}(x)}{1+x^2}\right]^2$$

$= \operatorname{cov}(Y_0, Y_k)$

$=$ function of k

$\Rightarrow$ Y_t is a weakly stationary series.

4.2 Part 2

(ii) $$\mathrm{var}(\bar{Y}) \cancel{B} = \frac{1}{n^2}\sum_{t=1}^{n} \mathrm{var}(Y_t) + \frac{2}{n^2}\sum_{k=1}^{n-1}\sum_{t=1}^{n-k} \mathrm{cov}(Y_t, Y_{t+k})$$

Since Y_t is second order stationary,

$$\mathrm{cov}(Y_t, Y_{t+k}) = c(k)$$

$$\Rightarrow \mathrm{var}(\bar{Y}) = \frac{c(0)}{n} + \frac{2}{n}\sum_{k=1}^{n}\left(\frac{n-k}{n}\right)c(k)$$

$$\approx \frac{c(0)}{n} + \frac{2}{n}\sum_{k=1}^{\infty} c(k)$$

$$\mathrm{var}(\bar{Y}) = LRV(Y_t)$$

4.3 Part 3

(iii) Let $L_n = \sum_{t=1}^{n} \left[Y_t - g(\theta,t)\right]^2$

$$\frac{\partial L_n}{\partial \theta} = 2\sum_{t=1}^{n} \left[Y_t - g(\theta,t)\right]\left[\frac{\partial g(\theta,t)}{\partial \theta}\right]$$

Let $g'_\theta(\theta,t) = \frac{\partial g(\theta,t)}{\partial \theta}$

$$\frac{\partial L_n}{\partial \theta} = 2\sum_{t=1}^{n} \left[Y_t - g(\theta,t)\right] g'_\theta(\theta,t)$$

$\frac{\partial L_n}{\partial \theta} = 0$ at $\theta = \hat{\theta}_n$

$$\Rightarrow \sum_{t=1}^{n} \left[Y_t - g(\hat{\theta}_n,t)\right] g'_\theta(\hat{\theta}_n,t) = 0$$

Let us substitute an approximation

$$\frac{g(\hat{\theta}_n,t) - g(\theta_0,t)}{\hat{\theta}_n - \theta_0} = g'_\theta(\hat{\theta}_n,t)$$

$$\Rightarrow g(\hat{\theta}_n,t) = g'_\theta(\hat{\theta}_n,t)\left[\hat{\theta}_n - \theta_0\right] + g(\theta_0,t)$$

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Now we get,

$$\sum_{t=1}^{n}\left[g'_\theta(\hat{\theta}_n,t)[\hat{\theta}_n-\theta_0]-\varepsilon_t\right]g'_\theta(\hat{\theta}_n,t)=0$$

$$\Rightarrow \sum_{t=1}^{n}[g'_\theta(\hat{\theta}_n,t)]^2[\hat{\theta}_n-\theta_0] = \sum_{t=1}^{n}\varepsilon_t\, g'(\hat{\theta}_n,t)$$

$$\Rightarrow \hat{\theta}_n-\theta_0 = \frac{\sum_{t=1}^{n}\varepsilon_t\, g'(\hat{\theta}_n,t)}{\sum_{t=1}^{n}[g'(\hat{\theta}_n,t)]^2}$$

5 Problem 3.5

A function, c(x) is an autocovariance function of a stationary process if

i) c(0) >= 0
ii) |c(k)| <= c(0) for all k
iii) c(x) is an even positive semi-definite function

A function, c(x) is positive semi-definite if and only if $\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} c(k)e^{ik\omega} >= 0\ \forall\omega\ \epsilon\ [0, 2\pi]$

5.1 Part 1

c(0) = 1, c(1) = c(-1) = 0.5,

c(k) = 0 $\forall$ |k| > 1.

The function is non-negative at 0 .

The function has its global maximum at 0 .

The function is even as c(1) = c(-1) and every other value is 0.

$f(\omega) = 1 + 2 * 0.5 * cos(\omega)$.

$f(\omega) = 1 + cos(\omega)$.

The function is positive semi-definite as f(ω) >= 0 for all ω ϵ $(0, 2\pi)$.

Therefore, the sequence can be used as an autocovariance function of a stationary process.

5.2 Part 2

c(0) = 1, c(1) = 0.5, c(-1) = -0.5,

c(k) = 0 $\forall$ |k| > 1 .

The function is non-negative at 0 .

The function has its global maximum at 0 .

But the function is not even as c(1) $\neq$ c(-1).

Therefore, the sequence can not be used as an autocovariance function of a stationary process.

5.3 Part 3

c(0) = 1, c(1) = c(-1) = 0.5, c(2) = c(-2) = -0.8,

c(k) = 0 $\forall$ |k| > 2 .

The function is non-negative at 0 .

The function has its global maximum at 0 .

The function is even as c(1) = c(-1), c(2) = c(-2) and every other value is 0.

$f(\omega) = 1 + 2 * 0.5 * cos(\omega) + 2 * (-0.8) * cos(2\omega)$.

$f(\omega) = 1 + cos(\omega) - 1.6cos(2\omega)$.

But the function is not positive semi-definite as $f(\omega) < 0$ when $\omega = \pi$.

Therefore, the sequence can not be used as an autocovariance function of a stationary process.