Assignment 4

October 11, 2020

1 Exercise 6.1

1.1 Part i

1.1.1 Part a

$X_t = \frac{7}{3}\ X_{t-1} - \frac{2}{3}\ X_{t-2} + \epsilon_t$

The characteristic equation of the above expression is

$1 - \frac{7}{3}\ z + \frac{2}{3}\ z^2 = 0$

The roots are 1/2 and 3

Comparing with the expression $(1 - az)(1 - bz) = 0$

a = 2, b = 1/3

So, the series can be written as the sum of a causal and a non-causal time-series

Using the result (4.10) in the textbook, the following expression is obtained

$X_t = \frac{-3}{5}\sum_{j=0}^{\infty}(3^{-j-1}\epsilon_{t-j} + 2^{-j}\epsilon_{t+1+j})$

$\implies X_t = \frac{-3}{5}(\sum_{j=0}^{\infty} 3^{-j-1}\epsilon_{t-j} + \sum_{j=-\infty}^{-1} 2^{j+1}\epsilon_{t-j})$

$X_{t+k} = \frac{-3}{5}\sum_{j=0}^{\infty}(3^{-j-1}\epsilon_{t+k-j} + 2^{-j}\epsilon_{t+k+1+j})$

$\implies X_{t+k} = \frac{-3}{5}(\sum_{j=-k}^{\infty} 3^{-j-k-1}\epsilon_{t-j} + \sum_{j=-\infty}^{-k-1} 2^{k+1+j}\epsilon_{t-j})$

if k > 0

$cov(X_t, X_{t+k}) = \frac{9}{25}\ var(\epsilon_t)\ (\sum_{j=0}^{\infty} 3^{-k-2j-2} + \sum_{j=-k}^{-1} 3^{-k-j-1}2^{j+1} + \sum_{j=-\infty}^{-k-1} 2^{2j+2+k})$

This can be rewritten as

$cov(X_t, X_{t+k}) = \frac{9}{25}\ \sigma^2\ (\sum_{j=0}^{\infty}(3^{-k-2j-2} + 2^{-k-2j}) + \sum_{j=0}^{k-1} 3^{-k+j}2^{-j})$

$cov(X_t, X_{t+k}) = \frac{9}{25}\sigma^2\ (3^{-|k|}/8 + 2^{-|k|} * 4/3 + 2 * 3^{-|k|}(1.5^{|k|} - 1))$

1.1.2 Part b

$X_t = \frac{4\sqrt{3}}{5}\ X_{t-1} - \frac{4^2}{5^2}\ X_{t-2} + \epsilon_t$

The characteristic equation of the above expression is

1 - $\frac{4\sqrt{3}}{5}\ z + \frac{4^2}{5^2}\ z^2 = 0$

The roots are not real since the discriminant is negative

Comparing with the equation,

$1 - 2rcos(\theta) + r^2 = 0$

$r = 4/5$

Solving for θ,

$cos(\theta) = \frac{\sqrt{3}}{2}$

$\implies \theta = \pi/6$

Substituting the above values in equation (4.12) in the textbook,

$X_t = \frac{5}{4}\sum_{j=0}^{\infty} 2 * 0.8^{j+1} sin((j+1)\pi/6)\epsilon_{t-j}$

$\implies X_t = 2\sum_{j=0}^{\infty} 0.8^j sin((j+1)\pi/6)\epsilon_{t-j}$

$\implies X_{t+k} = 2\sum_{j=0}^{\infty} 0.8^j sin((j+1)\pi/6)\epsilon_{t+k-j}$

$\implies X_{t+k} = 2\sum_{j=-k}^{\infty} 0.8^{j+k} sin((j+k+1)\pi/6)\epsilon_{t-j}$

$\implies cov(X_t, X_{t+k}) = 4\sigma^2 \sum_{j=0}^{\infty} 0.8^{2j+|k|} sin((j+|k|+1)\pi/6) sin((j+1)\pi/6)$

1.1.3 Part c

$X_t = X_{t-1} - 4X_{t-2} + \epsilon_t$

The characteristic equation of the above expression is

$1 - z + 4z^2 = 0$

The roots are not real since the discriminant is negative

Comparing with the equation,

$1 - 2rcos(\theta) + r^2 = 0$

$r = 2$

Solving for θ,

$cos(\theta) = \frac{1}{4}$

$sin(\theta) = \frac{\sqrt{15}}{4}$

Substituting $b = re^{i\theta}$ and $a = re^{-i\theta}$ in the following equation,

$\frac{1}{(1-za)(1-zb)} = \frac{1}{(b-a)}(\frac{b}{(1-bz)} - \frac{a}{(1-az)})$

The solution of an AR(2) when both the roots are not real and lie inside the circle becomes

$X_t = \frac{1}{2rsin(\theta)}\sum_{j=0}^{\infty} 2r^{-j} sin(j\theta)\epsilon_{t+j+1}$

Substituting the values gives

$X_t = \frac{2}{\sqrt{15}}\sum_{j=0}^{\infty} 2^{-j} sin(j\theta)\epsilon_{t+j+1}$

where $\theta = sin^{-1}(\sqrt{15}/4)$

$\implies X_{t+k} = \frac{2}{\sqrt{15}} \sum_{j=0}^{\infty} 2^{-j} sin(j\theta) \epsilon_{t+k+j+1}$

Rewriting X_t

$X_t = \frac{2}{\sqrt{15}} \sum_{j=-k}^{\infty} 2^{-j-k} sin((j+k)\theta) \epsilon_{t+k+j+1}$

$\implies cov(X_t, X_{t+k}) = \frac{4}{15}\ \sigma^2\ \sum_{j=0}^{\infty} 0.5^{2j+|k|} sin((j+|k|)\theta) sin(j\theta)$

Since at j = 0 of the above equation is redundant

$cov(X_t, X_{t+k}) = \frac{4}{15}\ \sigma^2\ \sum_{j=0}^{\infty} 0.5^{2j+|k|} sin((j+|k|+1)\theta) sin((j+1)\theta)$

The causal solution with the same autocovariance as above has the following characteristics for the reciprocal of its roots

$r = 1/2$

$\theta = cos^{-1}(1/4)$

So, the corresponding characteristic equation is

$1 - z/4 - z^2/4 = 0$

$\implies X_t = 0.25X_{t-1} + 0.25X_{t-2} + \epsilon_t$

1.2 Part ii

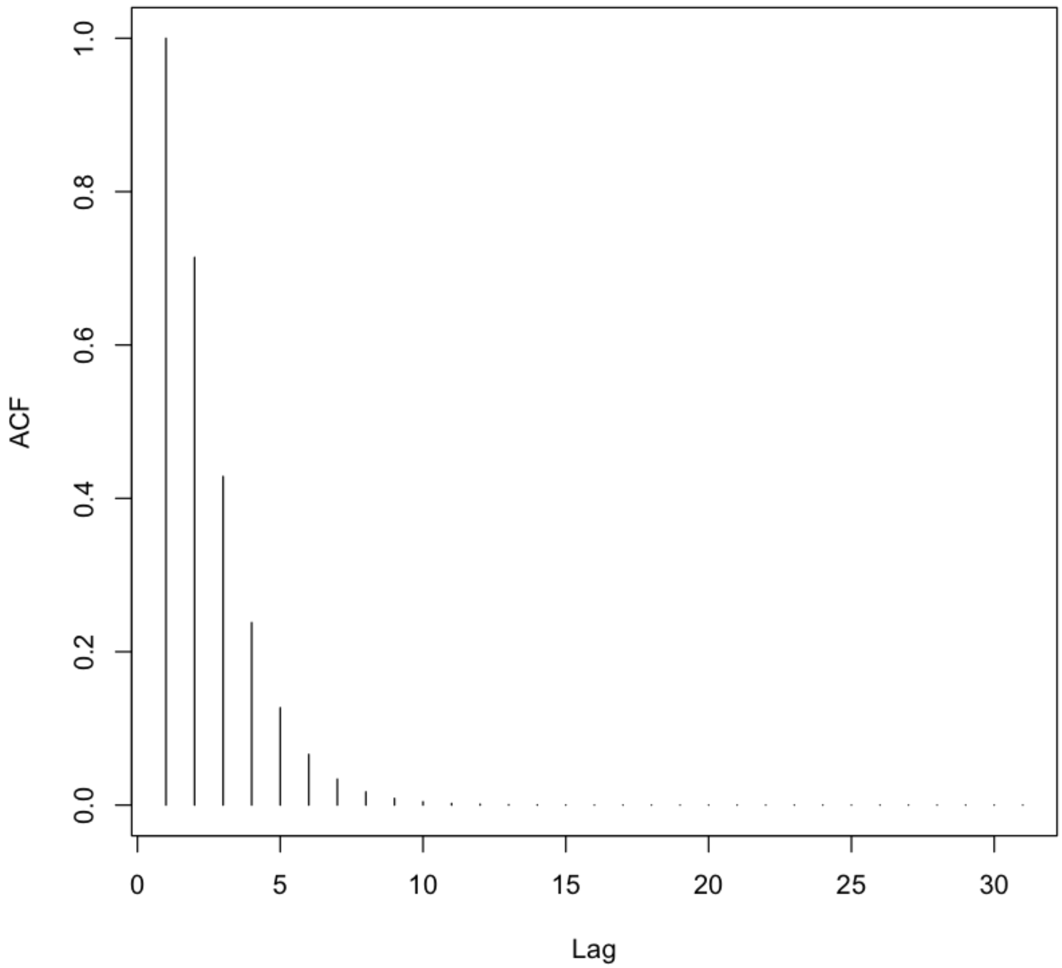
1.2.1 Part a

Since one of the two parts of the equation is non-causal, a causal component that gives an equivalent acf plot is used to obtain the following plot

So instead of the roots being 1/2 and 3, they need to be 2 and 3

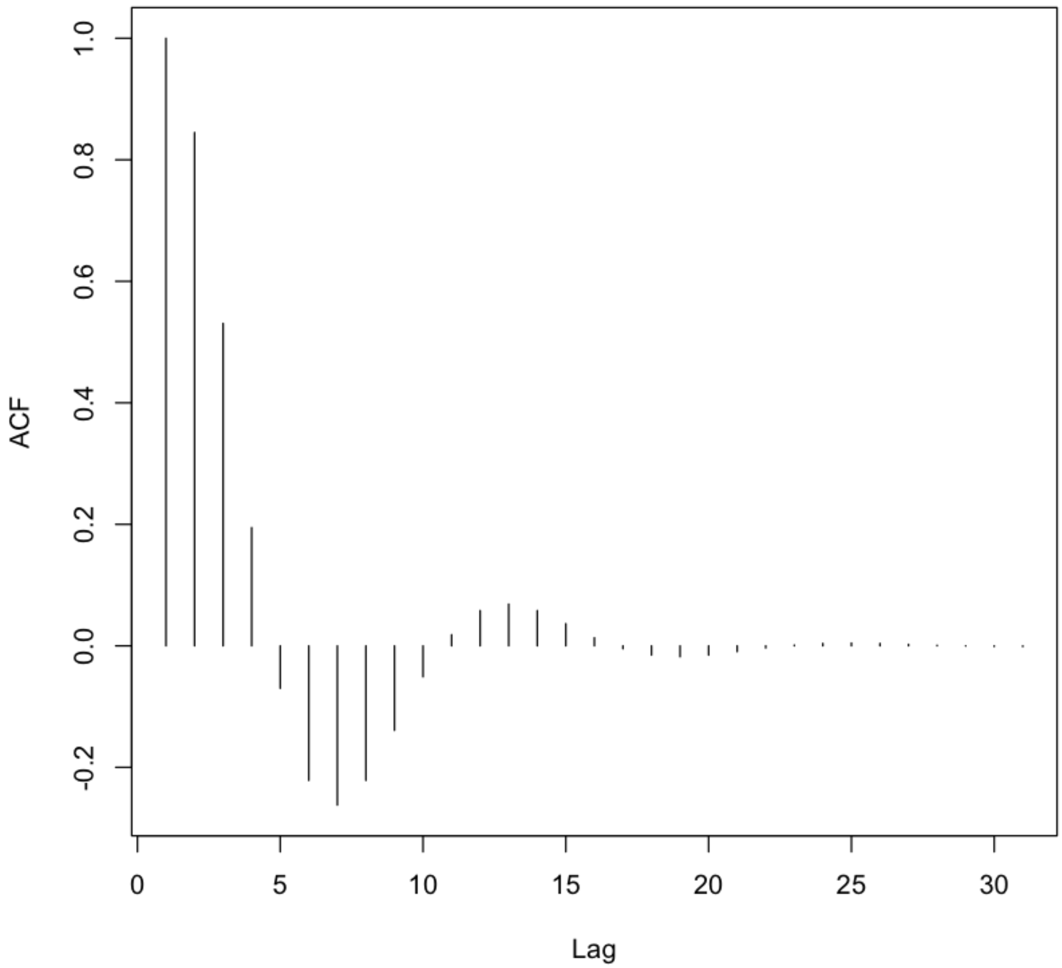
So, the equation becomes $X_t = 5/6X_{t-1} - 1/6X_{t-2} + \epsilon_t$

[56]:
```
plot_a <- ARMAacf(ar=c(5/6,-1/6), ma = c(), 30)
plot(plot_a, type = "h", xlab = "Lag", ylab = "ACF")
```



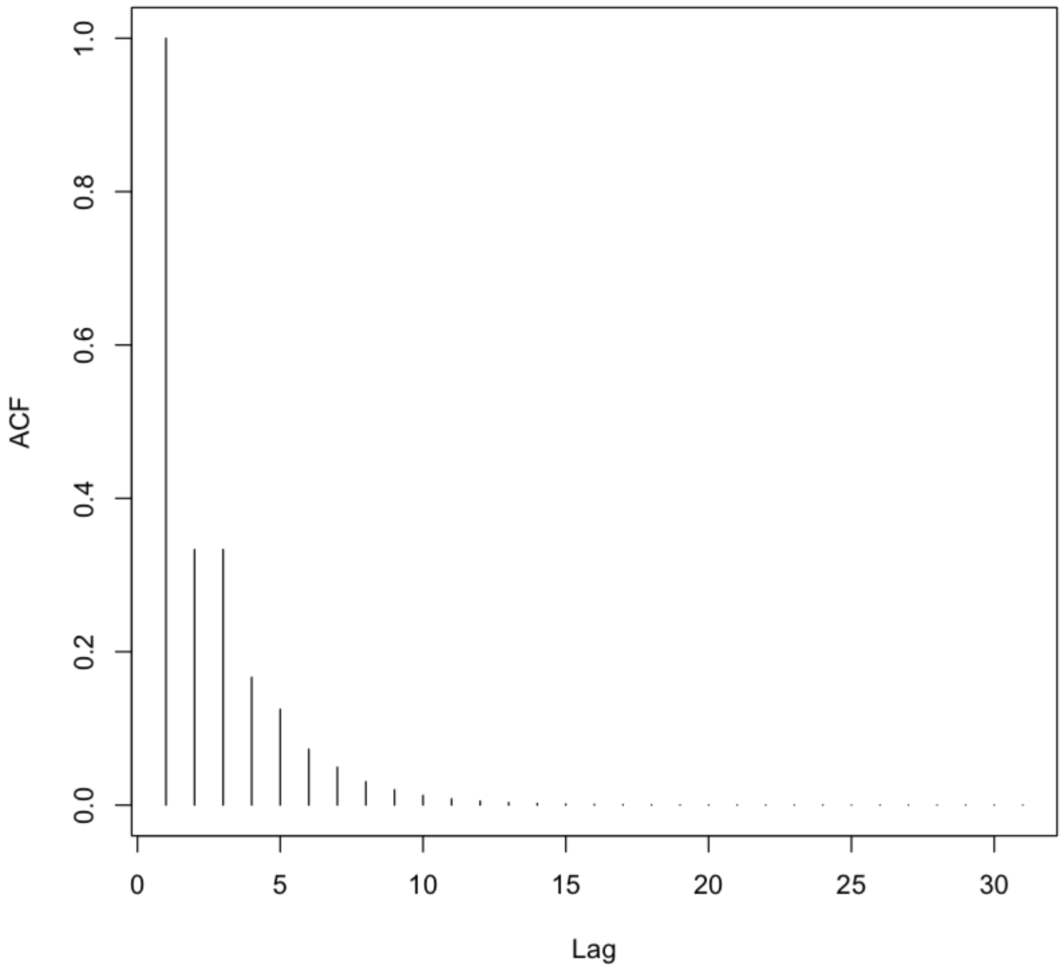
1.2.2 Part b

```
[55]: plot_b <- ARMAacf(ar=c(4*sqrt(3)/5,-16/25), ma = c(), 30)
      plot(plot_b, type = "h", xlab = "Lag", ylab = "ACF")
```



1.2.3 Part c

[52]:
```
plot_c <- ARMAacf(ar=c(0.25, 0.25), ma = c(), 30)
plot(plot_c, type = "h", xlab = "Lag", ylab = "ACF")
```



2 Exercise 6.2

A causal stationary AR(2) process with pseudo-period 17 has the form $X_t = 2rcos(2\pi/17)X_{t-1} - r^2X_{t-2} + \epsilon_t$ with 0 < r < 1

The process is plotted for r = 0.5 and r = 0.95. This gives,

$$X_t = cos(2\pi/17)X_{t-1} - 0.25X_{t-2} + \epsilon_t$$

and

$$X_t = 1.9cos(2\pi/17)X_{t-1} - 0.9025X_{t-2} + \epsilon_t$$

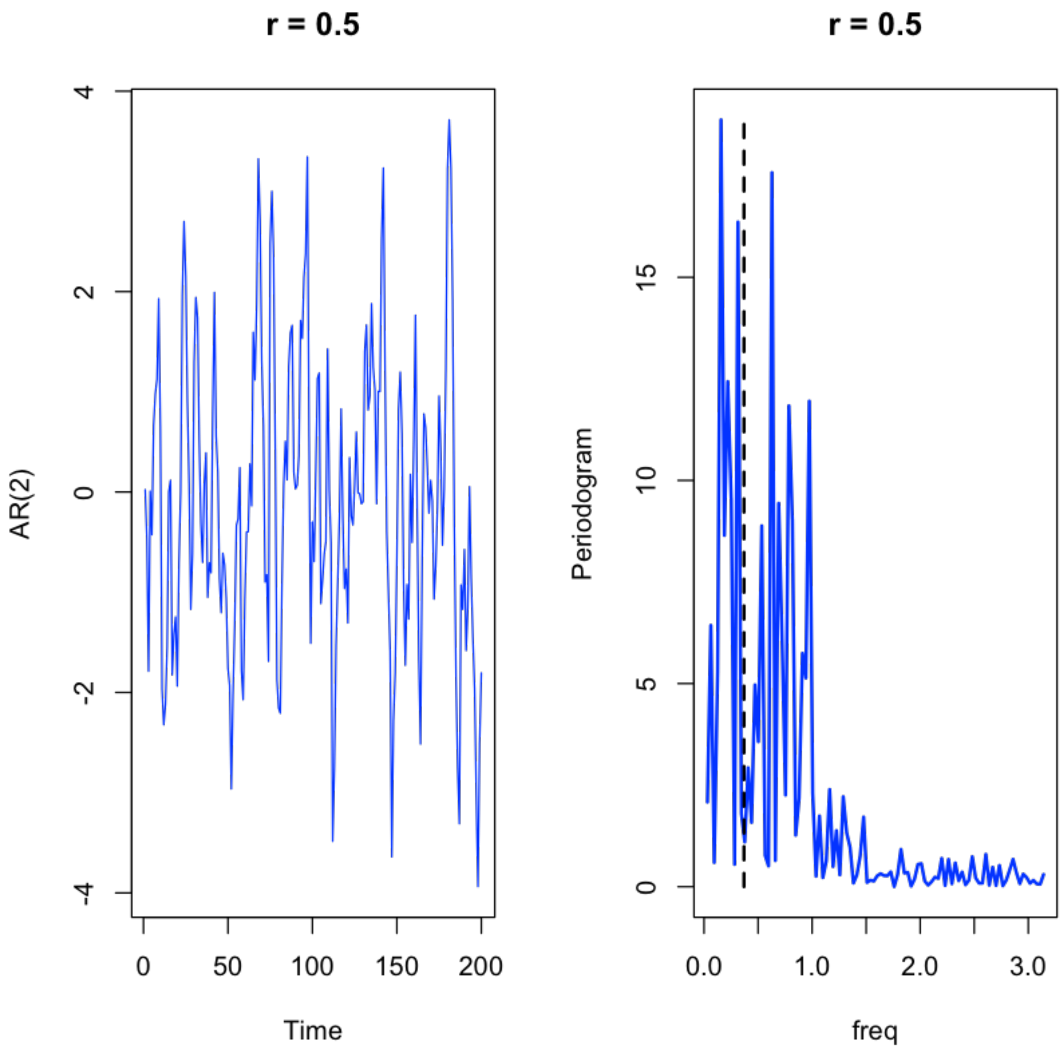
The above time series are plotted below and their periodograms are plotted to their right.

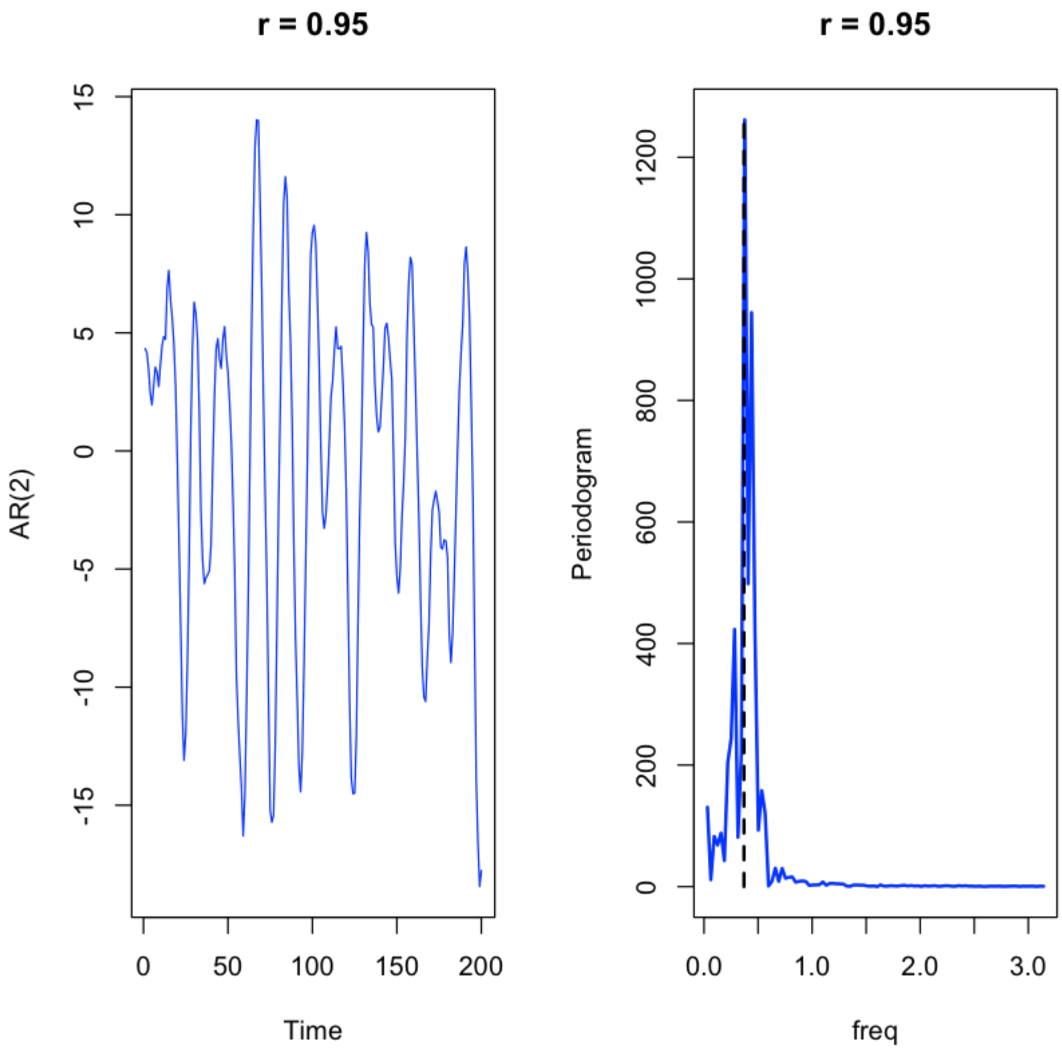
The periodogram's peak does not occur at 17 as 200/17 is not an integer.

But the peak is closer when r is increased closer to 1

[199]:
```
plot_function <- function (r, omega, n, label){
    ts.sim <- arima.sim(list(order = c(2,0,0), ar = c(2*r*cos(omega), -r*r)), n␣
↪= 200)
    F = abs(fft(ts.sim)/sqrt(n))**2
    F = F[c(1:(n/2))]
    freq = 2*pi*c(1:(n/2))/n
    par(mfrow=c(1,2))
    ts.plot(ts.sim, col = "blue", ylab = "AR(2)", main = label)
    plot(freq, F, lwd=2, col="blue", type="l", ylab="Periodogram", main = label)
    lines(c(omega, omega),c(0,max(F)),lwd=2,lty=2)
    return (n/which.max(F))
}

period_max1 <- plot_function(0.5, 2*pi/17, 200, "r = 0.5")
period_max2 <- plot_function(0.95, 2*pi/17, 200, "r = 0.95")
```


r = 0.5
AR(2)
Time
r = 0.5
Periodogram
freq



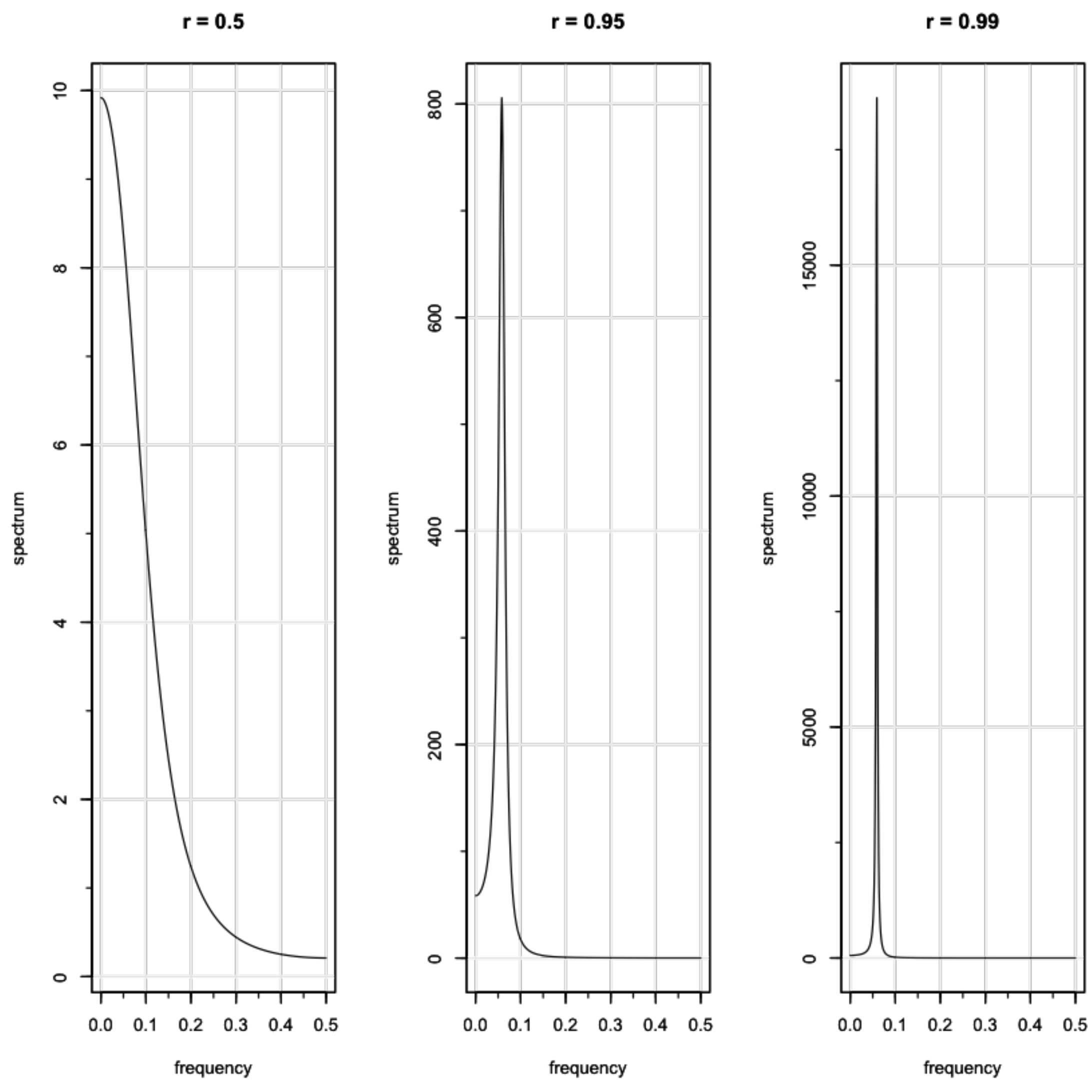
[208]:
```
print("Period obtained from r = 0.5")
print(period_max1)
print("Period obtained from r = 0.95")
print(period_max2)
```

```
[1] "Period obtained from r = 0.5"
[1] 40
[1] "Period obtained from r = 0.95"
[1] 16.66667
```

[64]:
```
install.packages("astsa")
library("astsa")
```

```
Updating HTML index of packages in '.Library'
Making 'packages.html' … done
```

[86]:
```r
par(mfrow=c(1,3))
arma.spec(ar = c(cos(2*pi/17), -0.25), main = "r = 0.5")
arma.spec(ar = c(1.9*cos(2*pi/17), -0.9025), main = "r = 0.95")
arma.spec(ar = c(1.98*cos(2*pi/17), -0.9801), main = "r = 0.99")
```



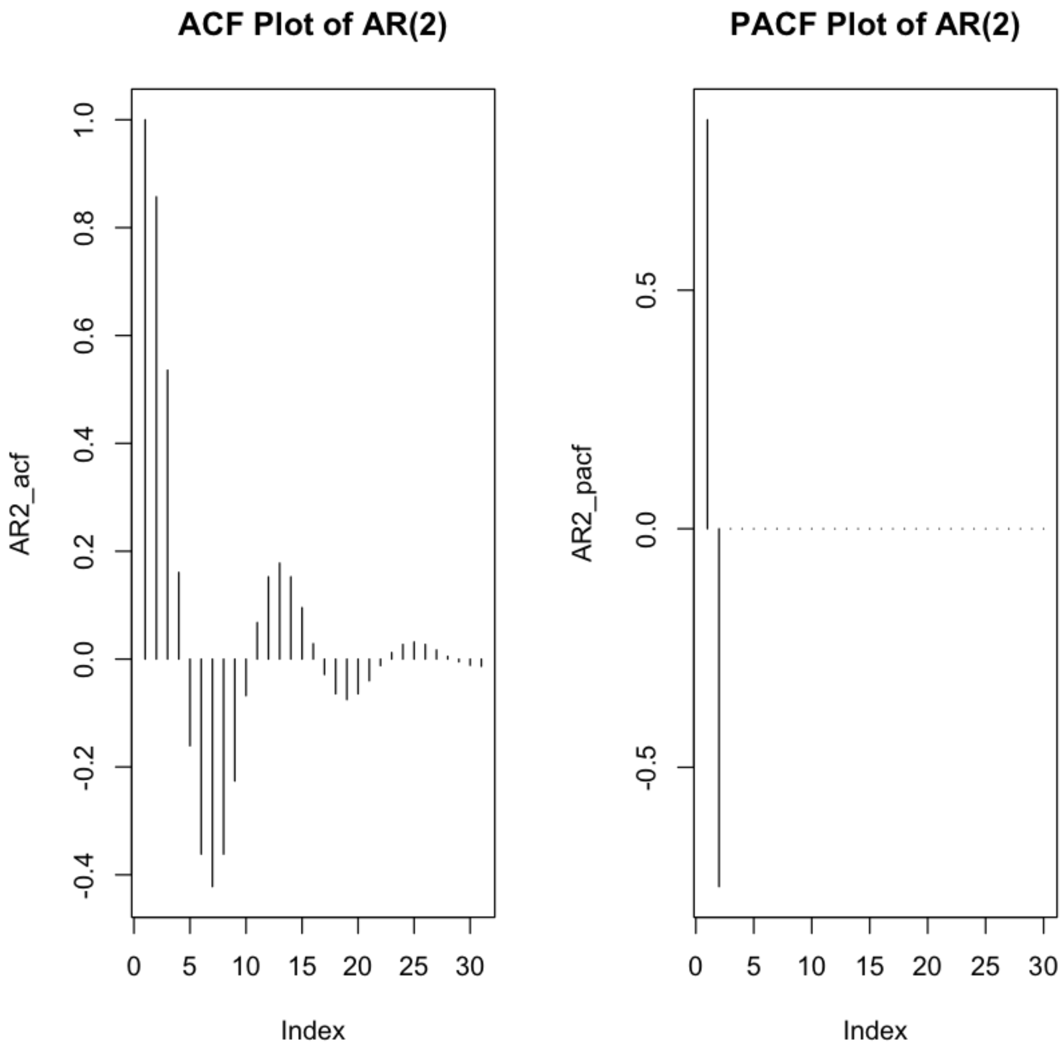
The periodogram given by the function has a peak at frequency greater than 0.5 whereas the periodogram drawn without the aid of the function gave a frequency less than 0.5. The actual frequency is $\frac{2\pi}{17}$ which is almost equal to 0.37.

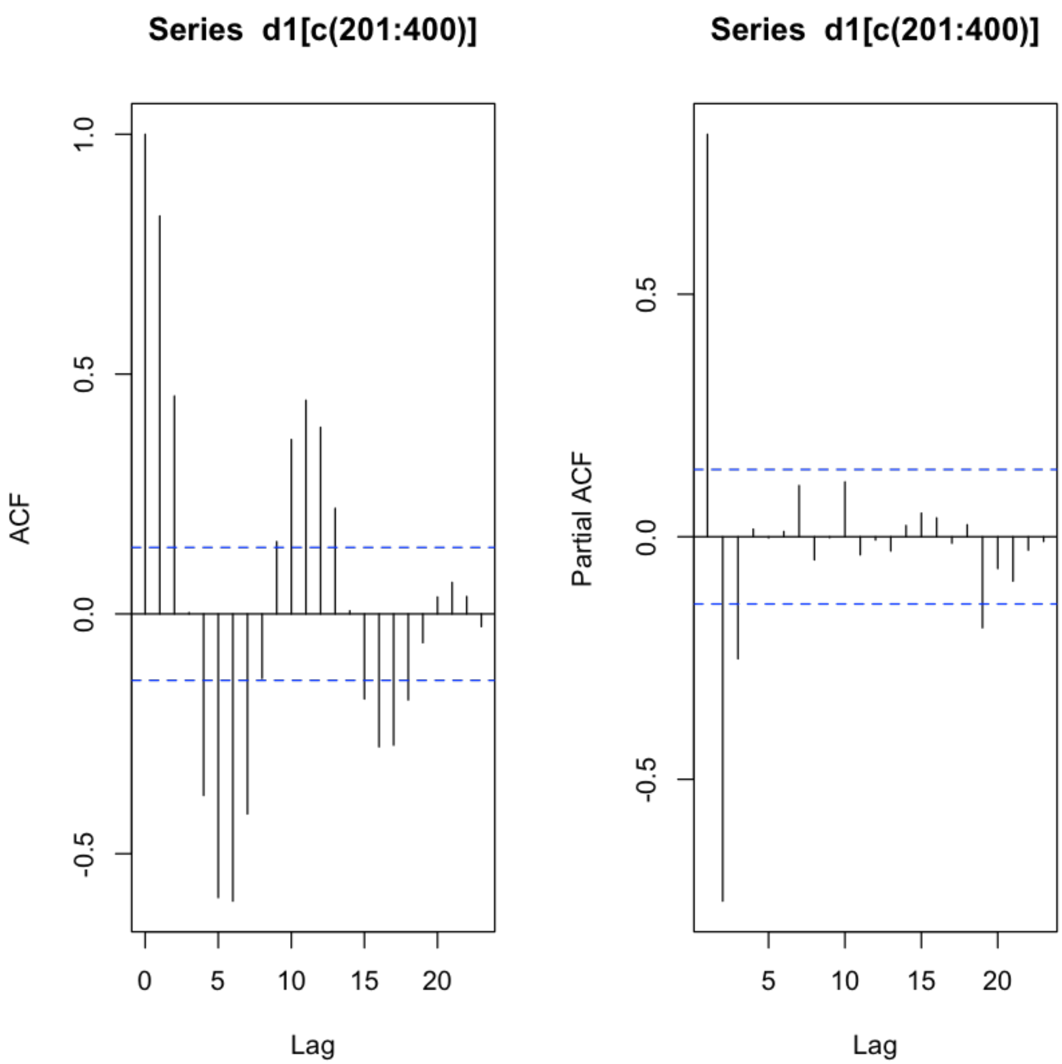
3 Exercise 6.4

3.1 Part i

[38]:
```
AR2_acf = ARMAacf(ar=c(1.5,-0.75),ma=0,30)
AR2_pacf = ARMAacf(ar=c(1.5,-0.75),ma=0, pacf = T,30)
par(mfrow = c(1, 2))
plot(AR2_acf, main = "ACF Plot of AR(2)", type = "h")
plot(AR2_pacf, main = "PACF Plot of AR(2)", type = "h")

d1 <- rep(0, 400)
epsilon1 <- rnorm(400)
for (i in 3:400){
    d1[i] = 1.5*d1[i-1] - 0.75*d1[i-2] + epsilon1[i]
}
acf(d1[c(201:400)])
pacf(d1[c(201:400)])
```


ACF Plot of AR(2)
PACF Plot of AR(2)
AR2_acf
AR2_pacf
Index
Index
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0.5
0.0
-0.5
0 5 10 15 20 25 30
0 5 10 15 20 25 30

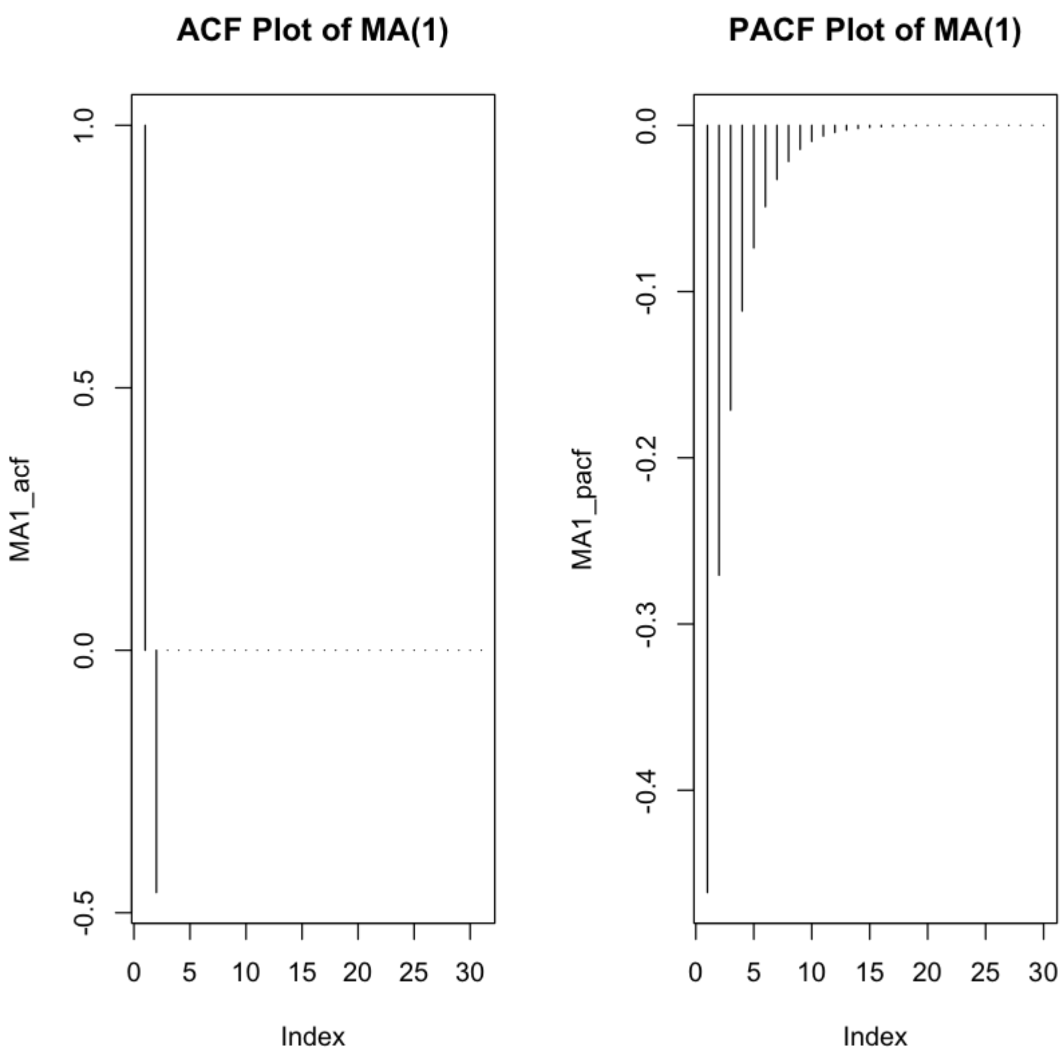


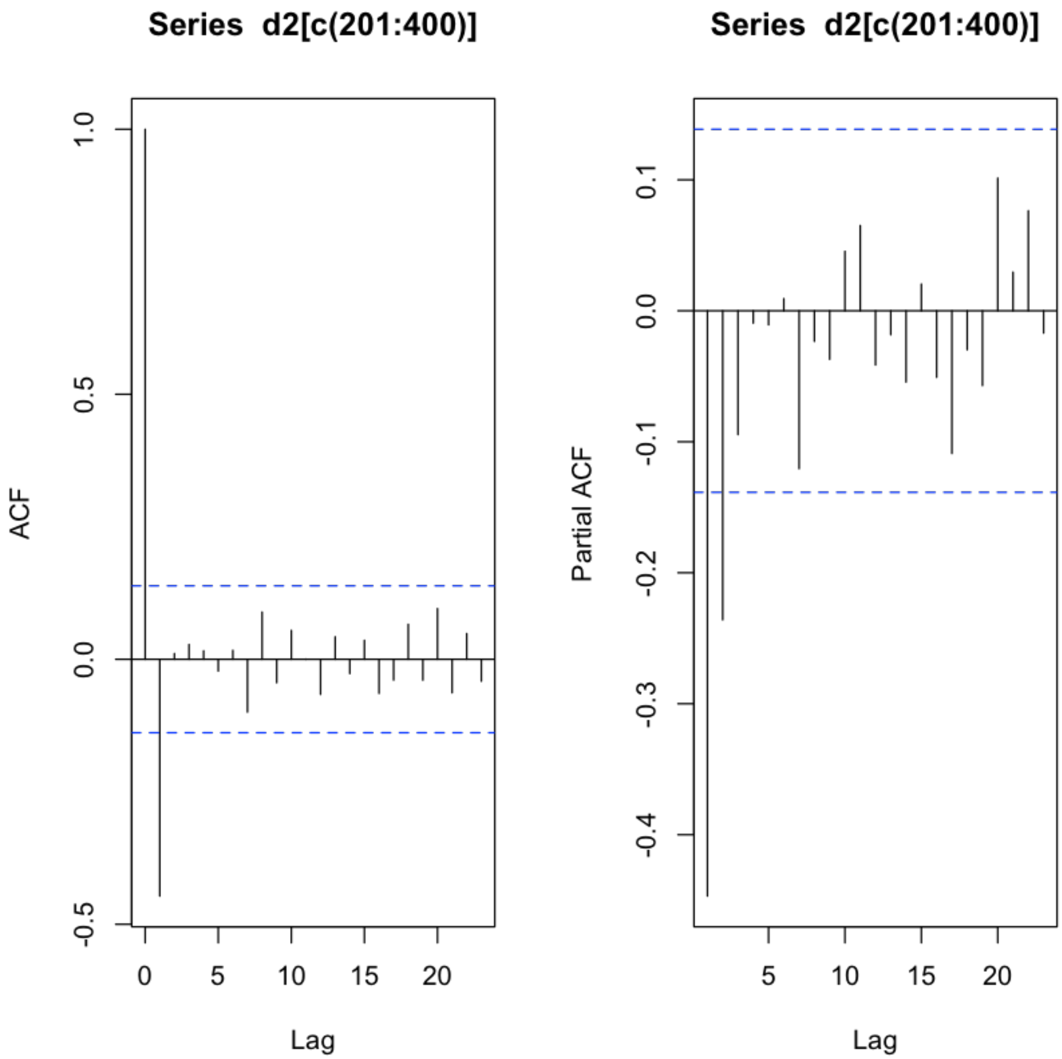
3.2 Part ii

[39]:
```
MA1_acf <- ARMAacf(ar=0,ma=c(-1.5),30)
MA1_pacf <- ARMAacf(ar=0,ma=c(-1.5),pacf=T,30)
par(mfrow = c(1, 2))
plot(MA1_acf, main = "ACF Plot of MA(1)", type = "h")
plot(MA1_pacf, main = "PACF Plot of MA(1)", type = "h")

d2 <- rep(0, 400)
epsilon2 <- rnorm(400)
for (i in 2:400){
    d2[i] = epsilon2[i] - 1.5*epsilon2[i-1]
}
```

```
acf(d2[c(201:400)])
pacf(d2[c(201:400)])
```





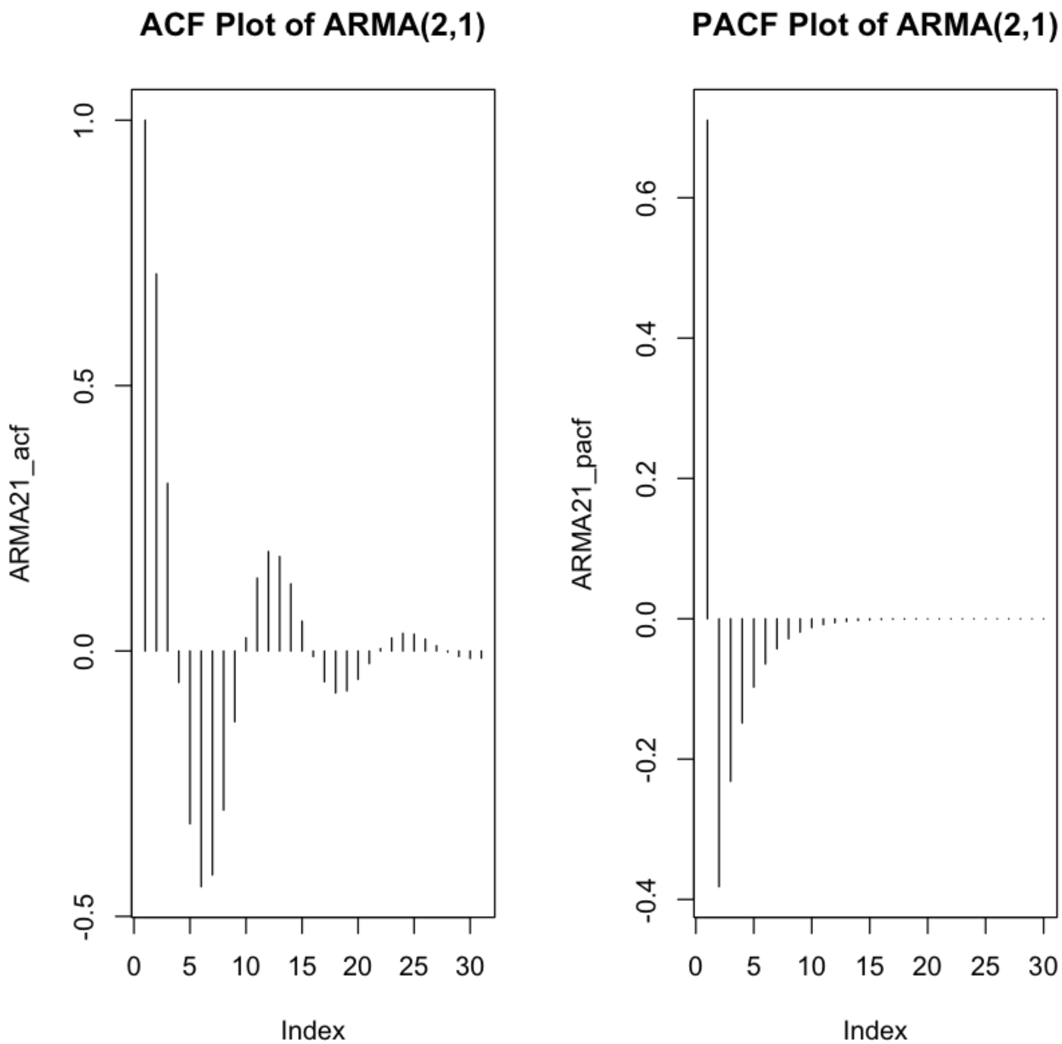
3.3 Part iii

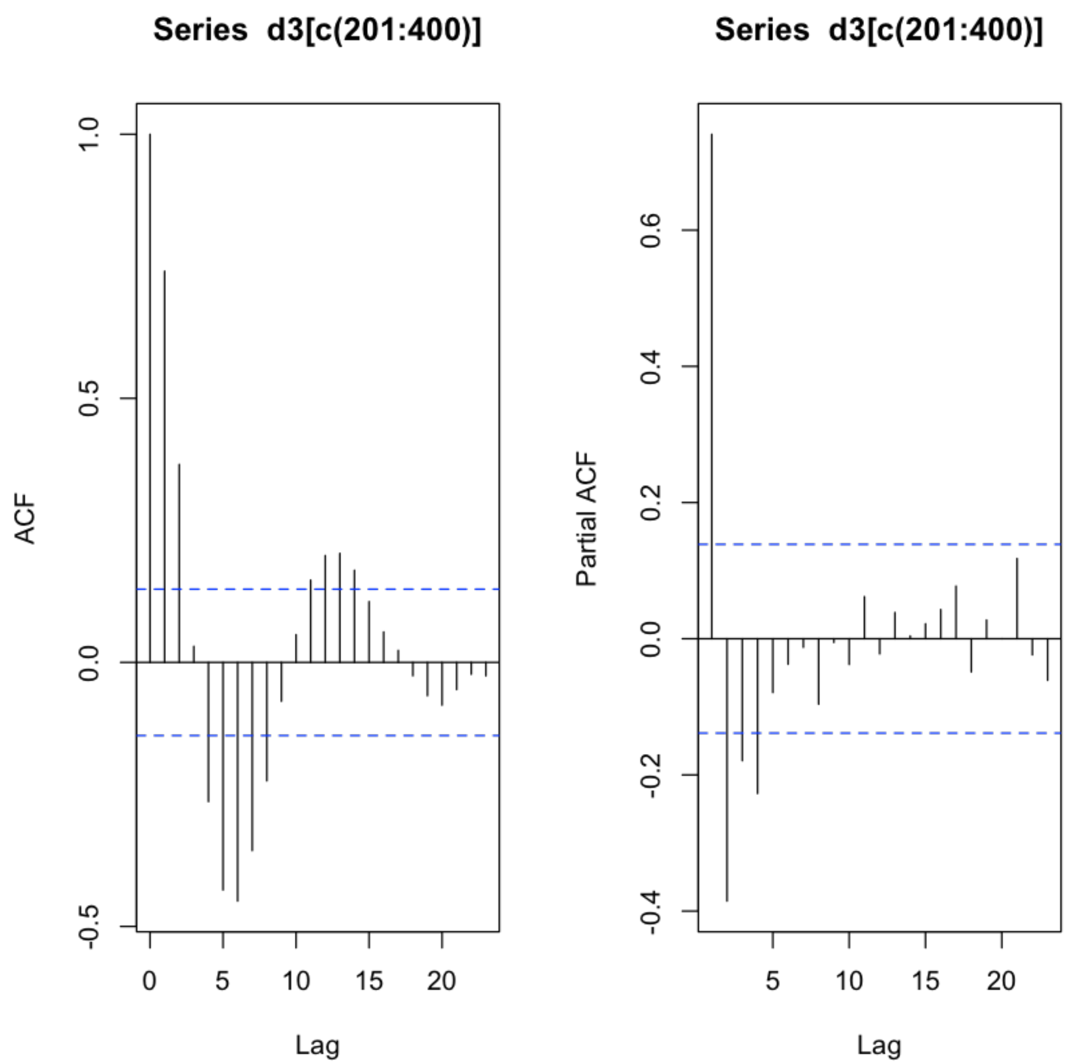
[40]:

```
ARMA21_acf <- ARMAacf(ar=c(1.5,-0.75), ma=c(-1.5), 30)
ARMA21_pacf <- ARMAacf(ar=c(1.5,-0.75), ma=c(-1.5), pacf = T, 30)
par(mfrow = c(1, 2))
plot(ARMA21_acf, main = "ACF Plot of ARMA(2,1)", type = "h")
plot(ARMA21_pacf, main = "PACF Plot of ARMA(2,1)", type = "h")

d3 <- rep(0, 400)
epsilon3 <- rnorm(400)
for (i in 3:400){
    d3[i] = 1.5*d3[i-1] - 0.75*d3[i-2] +epsilon3[i] - 1.5*epsilon3[i-1]
}
```

```r
acf(d3[c(201:400)])
pacf(d3[c(201:400)])
```

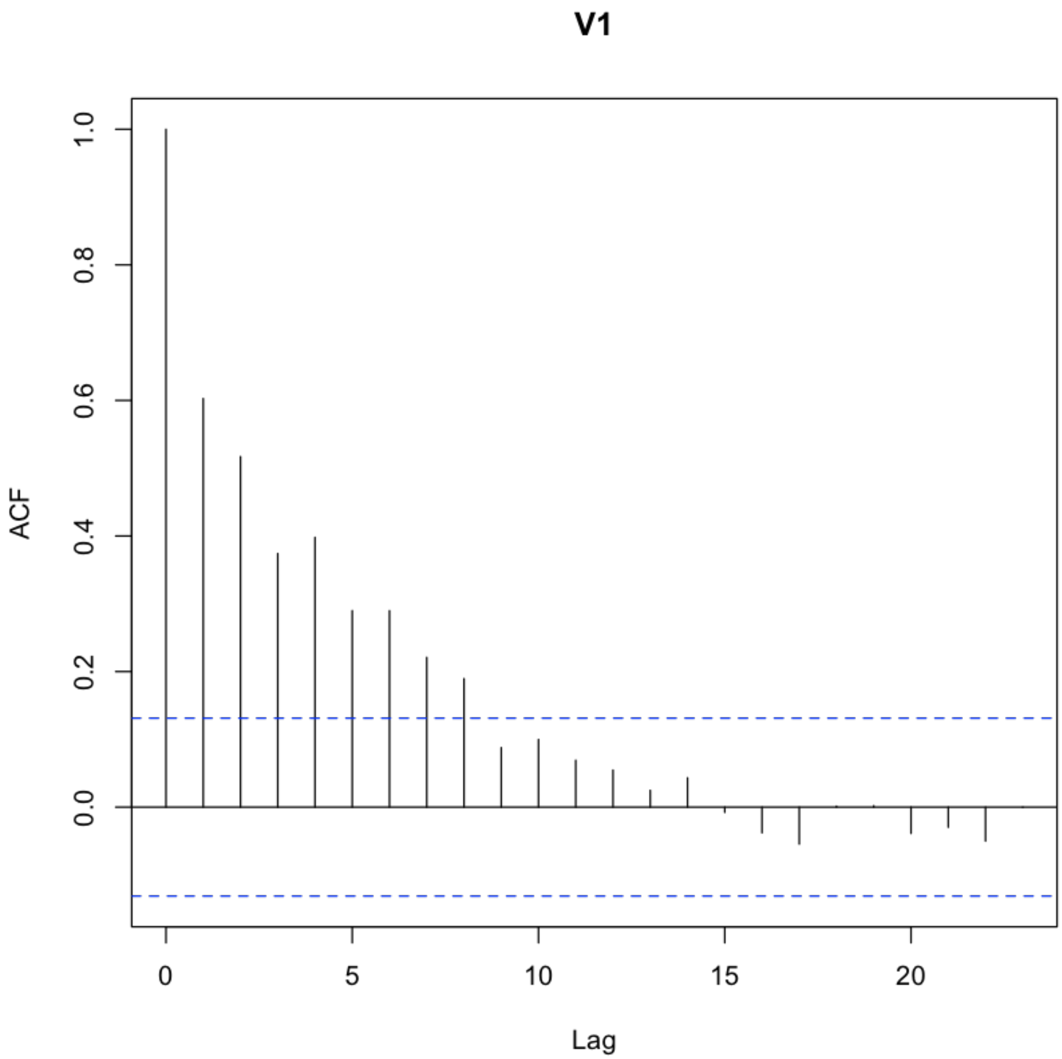




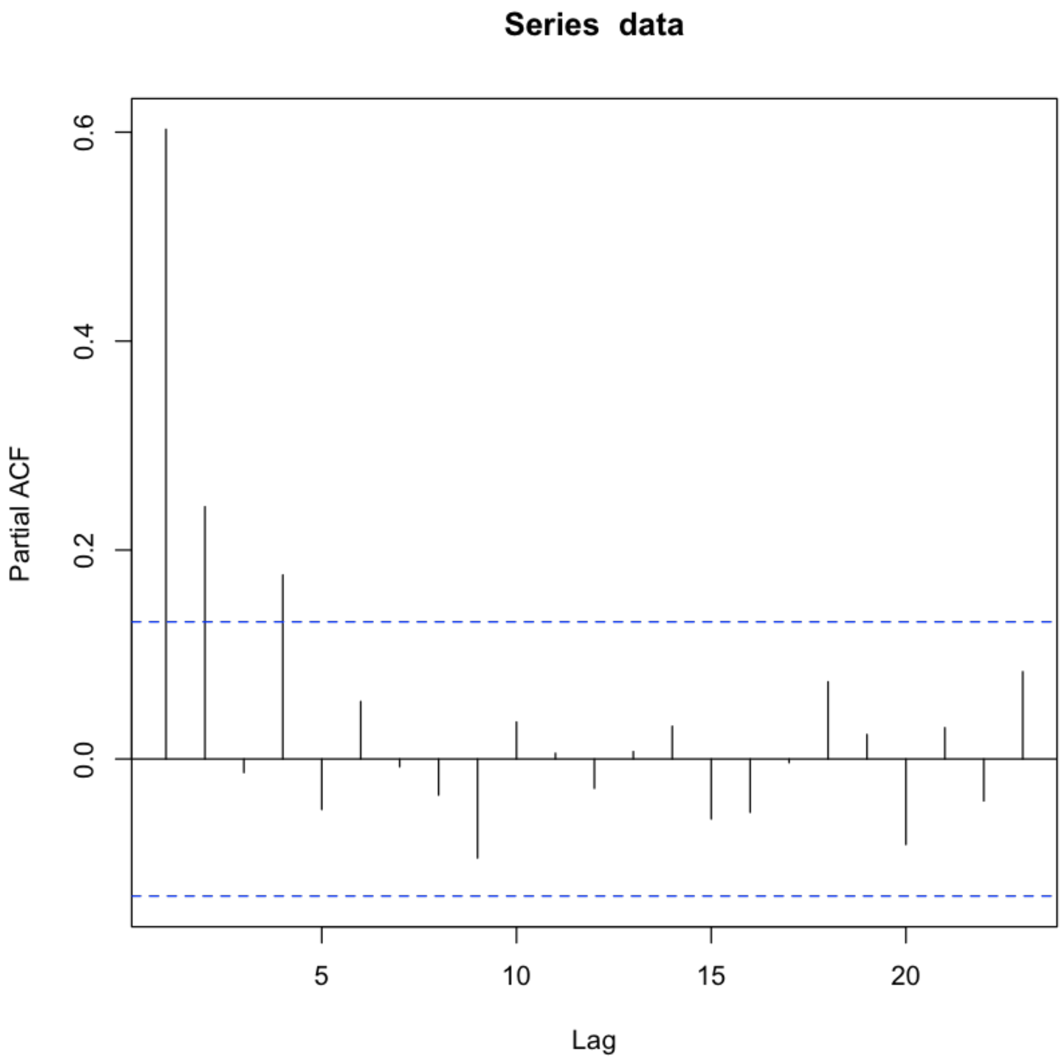
4 Exercise 6.5

[15]:
```
data <- read.csv("https://www.stat.tamu.edu/~suhasini/teaching673/Data/
↪month_temp.txt", header = FALSE)
```

[23]:
```
acf(data)
pacf(data)
```



V1
ACF
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
Lag



The model appears to be an AR model as the pacf plot dips abruptly after a lag of 4 whereas the acf plot steadily drops.

5 Exercise 6.6

(i) L_n is a lower triangular matrix and it ~~[illegible]~~ is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & & & \\ -\phi_1 & 1 & 0 & 0 & & & \\ -\phi_2 & -\phi_1 & 1 & & & & \\ 0 & -\phi_2 & -\phi_1 & 1 & & & \\ 0 & 0 & -\phi_2 & -\phi_1 & 1 & & \\ 0 & 0 & 0 & & & \ddots & \\ \vdots & & & & & & \\ \cdots & & & & -\phi_2 & \phi_1 & 1 \end{bmatrix}$$

It has a bandwidth of 2.

(ii) $P_{\{X_{t-j}\}_{j\neq 0}} X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2}$

So it can be written in the following form

$$\begin{bmatrix} P_{\{X_{t-j}\}_{j\neq 0}} X_t \\ P_{\{X_{t-j}\}_{j\neq 1}} X_{t-1} \\ P_{\{X_{t-j}\}_{j\neq 2}} X_{t-2} \\ \vdots \\ \vdots \end{bmatrix} = \left(I - L_n'\right) \begin{bmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ \vdots \\ \vdots \end{bmatrix}$$

where I is an identity matrix of size n

Or it can written as

$$\begin{bmatrix} P_{\{x_{t-j}\}_{j\neq t-1}}\, x_1 \\ P_{\{x_{t-j}\}_{j\neq t-2}}\, x_2 \\ \vdots \\ P_{\{x_{t-j}\}_{j\neq t-n}}\, x_n \end{bmatrix} = (\underline{I} - L_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

[ ]: