Solutions Manual for

Investment Science

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Chapter 2
The Basic Theory of Interest

1. (A nice inheritance) Use the "72 rule". Years = 1994–1776 = 218 years.

   (a) $i = 3.3\%$. Years required for inheritance to double = $\frac{72}{i} = \frac{72}{3.3} \approx 21.8$. Times doubled = $\frac{218}{21.8} = 10$ times. \$1 invested in 1776 is worth $2^{10} \approx \$1,000$ today.

   (b) $i = 6.6\%$. Years required to double = $\frac{72}{6.6} \approx 10.9$. Times doubled = $\frac{218}{10.9} = 20$ times. \$1 invested in 1776 is worth $2^{20} \approx \$1,000,000$ today.

2. (The 72 rule) Using $(1+r)^n = 2$ gives $n \ln(1+r) = \ln 2$. Using $\ln(1+r) \approx r$ and $\ln 2 = 0.69$. We have $nr \approx 0.69$ and thus $n \approx \frac{0.69}{r} \approx \frac{69}{i}$.

   Using instead $\ln(1+r) \approx r - \frac{1}{2}r^2 = r(1 - \frac{1}{2}r)$ we have $n \ln(1+r) = \ln 2$ or equivalently $nr \approx \frac{0.69}{1-r/2}$. For $r \approx 0.08$, we have $(1 - r/2)^{-1} \approx 1.042$. Therefore,

$$n \approx \frac{1}{r}(0.069)(1.042) = \frac{0.72}{r} = \frac{72}{i}$$

3. (Effective rates)

   (a) 3.04%

   (b) 19.56%

   (c) 19.25%.

4. (Newton's method) We have

$$f(\lambda) = -1 + \lambda + \lambda^2\,,\; f'(\lambda) = 1 + 2\lambda\,,\; \lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{f'(\lambda_k)}$$

i	λ_k	$f(\lambda_k)$	$f'(\lambda_k)$	λ_{k+1}
0	1	1	3	2/3
1	2/3	1/9	7/3	13/21
2	13/21	0.00227	2.23810	0.618033
3	0.618033	-2.2×10^{-6}	2.23607	0.618034
4	0.618034	0	2.23608	<u>0.618034</u>

5. (A prize)

$$\text{PV} = \$4,682,460.$$

6. (Sunk cost) The payment stream for apartment A is 1,000, 1,000, 1,000, 1,000 1,000, 1,000 while for B it is 1,900, 900, 900, 900, 900, 900. At any interest rate $PV_A < PV_B$ because the initial difference is less than the sum of the subsequent cash flow differences. Hence, they should not switch. For one year, the sign of the difference does depend on the interest rate. At 1% per month, the values are $PV_A = \$11{,}367.62$ and $PV_B = \$11{,}230.87$, so they should switch.

7. (Short cut) The cash flow with waiting is $(-1, 0, 0, x)$. We know that $PV = -1 + x/1.1^3 < -1 + 3/1.1^2$. Hence $x/1.1 < 3$ which means $x < 3.3$.

8. (Copy machines) Assume that the maintenance payments occur at the beginning of each year.

   Incremental IRR from A to B: (Cash flows in \$1000)

$$f(c) = 0 = -24 + 6c + 6c^2 + 6c^3 + 6c^4 + 10c^5,$$

   where

$$c = \frac{1}{1+r}.$$

   Using Newton's method, we get $c = 0.894112$, $r = 0.118$ Thus,

$$IRR_{A\to B} = 11.8\% > 10\%.$$

   Incremental IRR from B to C: (Cash flows in \$1000)

$$f(c) = 0 = -5 + 0.4c + 0.4c^2 + 0.4c^3 + 0.4c^4 + 2c^5\ ,\ c = \frac{1}{1+r}$$

   Using Newton's method ($c_0 = 1.1$), we get $c = 1.0862106$, $r = -0.079$

   Thus,

$$IRR_{B\to C} = -7.9\%$$

   A move from A to B is justified on the basis of IRR.

9. (An appraisal) Consider the PV of the two following payment streams:

   (a) Change roof now, then every 20 years:

$$PV_1 = \$20,000 \times \sum_{i=0}^{\infty} \frac{1}{(1.05)^{20i}} = \$32,097$$

   (b) Change roof in 5 years and then every 20 years:

$$PV_2 = \frac{PV_1}{(1.05)^5} = \$25,149$$

Taking the difference of these two we find the value of roof to be $PV_1 - PV_2 =$ $\$6,948$.

10. (Oil depletion allowance)

Yr	Barrels Produced	Gross Revenue	Net Income	Option 1	Option 2	Depletion Allowance	Taxable Income	After Tax Income
1	80	$1,600	$1,200	$352	$400	$400	$800	$840
2	70	1,400	1,000	$308	$350	$350	$650	$708
3	50	1,000	500	$220	$250	$250	$250	$388
4	30	600	200	$100	$150	$150	$50	$178
5	10	200	50	$25	$50	$50	$0	$50
					Total=	$1,200	PV=	$521.26

All numbers except years in thousands.

(a) Depletion $= \$1,200,000 > \$1,000,000$.

(b) PV $= \$521,260$. IRR $= 52.8\%$.

11. (Conflicting recommendations)

$$
\begin{aligned}
\mathrm{NPV}_1 &= 29.88 \\
\mathrm{NPV}_2 &= 31.84 > \mathrm{NPV}_1 \quad \text{recommend 2.} \\
\mathrm{IRR}_1 &= 15.2\% \\
\mathrm{IRR}_2 &= 12.4\% < \mathrm{IRR}_1 \quad \text{recommend 1.}
\end{aligned}
$$

See the solution for (Crossing) exercise for explanation.

12. (Domination) Equations for IRR are

$$A_i = B_i \sum_{j=1}^{n} c_i^j \quad i = 1,2$$

where $c_i = \frac{1}{1+r_i}$ which gives

$$\frac{1}{\sum_{j=1}^{n} c_i^j} = \frac{B_i}{A_i}.$$

Hence,

$$B_1/A_1 > B_2/A_2 \quad \text{implies} \quad \sum_{j=1}^{n} c_1^j < \sum_{j=1}^{n} c_2^j$$

which in turn implies

$$c_1 < c_2 \quad \text{or equivalently} \quad r_1 > r_2.$$

13. (Crossing)

(a) Let $P(c) = P_x(c) - P_y(c)$, a continuous function of c. Then

$$P(1) = P_x(1) - P_y(1) = \sum_{i=0}^{n} x_i - \sum_{i=0}^{n} y_i > 0.$$

Likewise

$$P(0) = P_x(0) - P_y(0) = x_0 - y_0 < 0.$$

By the intermediate value theorem there is a c such that $P_x(c) = P_y(c)$.

(b) We solve

$$-100 + 30(c + c^2 + \cdots + c^5) = -150 + 42(c + c^2 + \cdots + c^5)$$

This gives $c = 0.946$ and $r = 5.7\%$.

14. (Depreciation choice) Individual maximizes the PV of depreciation (in percentage terms)

(a) 25%, 38% and 37%.

$$PV_1 = 25 + \frac{38}{1+r} + \frac{37}{(1+r)^2}.$$

(b) 33⅓%.

$$PV_2 = \frac{100}{3}\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}\right)$$

Then $\mathrm{PV}_1 = \mathrm{PV}_2$ yields $\frac{1}{1+r} = 1$ which gives $r = 0$.

when $r > 0$, $PV_2 > PV_1$. Hence always use straight line method.

15. (An erroneous analysis)

ALL DOLLAR AMOUNTS IN THOUSANDS

Yr	Cost	Revenue	Before Tax Income	Deprec.	Taxable Income	Tax	After Tax Income	PV
0	$10,000							($10,000)
1	$310	$3,300	$2,990	$2,000	$990	$337	$2,653	$2,369
2	$310	$3,300	$2,990	$2,000	$990	$337	$2,653	$2,115
3	$310	$3,300	$2,990	$2,000	$990	$337	$2,653	$1,889
4	$310	$3,300	$2,990	$2,000	$990	$337	$2,653	$1,686
5	$310	$3,300	$2,990	$2,000	$990	$337	$2,653	$1,506
		NO INFLATION					Total PV=	($435)

	$10,000							($10,000)
	$310	$3,300	$2,990	$2,000	$990	$337	$2,653	$2,369
	$322	$3,432	$3,110	$2,000	$1,110	$377	$2,732	$2,178
	$335	$3,569	$3,234	$2,000	$1,234	$420	$2,814	$2,003
	$349	$3,712	$3,363	$2,000	$1,363	$464	$2,900	$1,843
	$363	$3,861	$3,498	$2,000	$1,498	$509	$2,989	$1,696
		4% INFLATION					Total PV=	$89

Chapter 3
Fixed-Income Securities

1. (Amortization) Use

$$A = \frac{rP}{1 - \frac{1}{(1+r)^n}} = \frac{.07 \times 25,000}{1 - \frac{1}{1.07^7}} = \$4,638.83.$$

2. (Cycles and annual worth) Let $d = 1/(1+r)$. Then

$$P_\infty = P\left\{1 + d^{n+1} + d^{2(n+2)} + \cdots\right\} = \frac{P}{1 - d^{n+1}}.$$

Also

$$A = \frac{rP}{1 - d^n}.$$

Hence

$$A = \frac{r(1 - d^{n+1})}{1 - d^n} P_\infty.$$

3. (Uncertain annuity)

(a) To find the life expectancy, we multiply each age of death by its probability. Thus the life expectancy is

$$\overline{L} = 90 \times .07 + 91 \times .08 + \cdots 101 \times .04 = 95.13 \text{ years.}$$

(b) To find the present value of an annuity that ends at age 95.13 we calculate the values for ages 95 and 96. From the standard formula

$$P = \frac{A}{r}\left\{1 - \frac{1}{(1+r)^n}\right\}$$

with $n = 5$ and $n = 6$ we find $P_{95} = \$39,927$ and $P_{96} = \$46,228$. Then we find $P = .87 \times P_{95} + .13 \times P_{96} = \$40,746$.

(c) To find the expected present value of the annuity we calculate the probabilities q_i of survival to various ages i. For example, $q_{90} = 1.0$, $q_{91} = q_{90} - .07 = .93$, $q_{92} = q_{91} - .08 = .85$, and so forth. For each year greater than 90 we evaluate $\$10,000 \times q_i/1.08^{i-90}$. Hence the expected present value is

$$\overline{\text{PV}} = \$10,000\left(\frac{.93}{1.08} + \frac{.85}{1.08^2} + \cdots + \frac{.04}{1.08^{11}}\right) = \$38,387.$$

Note that the expected present value of the annuity is less than the present value evaluated at the expected lifetime. This will always be the case.

4. (APR) First find the monthly payment M at the APR of 8.083% using the annuity formula, as

$$M = \frac{\frac{.08083}{12}(1+\frac{.08083}{12})^{360}}{(1+\frac{08083}{12})^{360}-1}\$203,150 = \$1,502.41.$$

Next find the initial balance for this monthly payment at the interest rate of 7.875%

$$B = \frac{12}{.07875}\left\{1 - \frac{1}{(1+\frac{.07875}{12})^{360}}\right\}\$1,502 = \$207,209.13.$$

The total fees are the difference between the initial balance and the amount of the loan

$$\text{Fees} = \$207,209.13 - \$203,150 = \$4,059.13.$$

5. (Callable Bond) After five years, the payment the company needs to make if exercising the call provision is

$$P_5^C = (1+0.05) \times \text{Face Value} = 105.$$

Exercising the call provision is advantageous, so

$$105 < P_5 = \frac{100}{(1+\lambda)^{15}} + \frac{10}{\lambda}\left(1 - \frac{1}{(1+\lambda)^{15}}\right).$$

Therefore, the YTM then is lower than 9.366%.

6. (The bi-weekly mortgage)

   (a) Monthly payment:

$$m = \frac{\frac{0.1}{12}(1+\frac{0.1}{12})^{360}}{(1+\frac{0.1}{12})^{360}-1} \times \$100,000 = \$877.57$$

   Total interest $= 360m - \$100,000 = \$215,925.20.$

   (b) Bi-weekly payment $= \frac{m}{2} = \$438.79.$
   Let n = number of periods. Then

$$\$100,000 = \$438.79 \times \frac{(1+\frac{0.1}{26})^n - 1}{\frac{0.1}{26}(1+\frac{0.1}{26})^n}$$

   Hence $n = 545$ or 20.95 years.
   Total interest $= 545 \times \$438.79 - \$100,000 = \$139,140.55.$
   Savings in total interest over monthly program = $76,784.65 or 35.6%

7. (Annual worth)

   A. Amortize the present value of \$22,847 over four years which gives $A_A = \$6,449$ per year.

   B. Amortize the present value of \$37,582 over six years which gives $A_B = \$7,845$.

$$A_A < A_B$$

   Car A should be selected.

8. (Variable rate mortgage)

   (a) $A = \$100,000 \times \frac{0.08(1.08)^{30}}{1.08^{30}-1} = \$8,882.74$

   (b) $P_5 = \$8882.74 \times \frac{(1.08)^{25}-1}{0.08(1.080)^{25}} = \$94,821.26$

   (c) $A' = \$94,821.26 \times \frac{0.09(1.09)^{25}}{1.09^{25}-1} = \$9,653.40$

   (d) $94,821.26 = 8882.74 \times \frac{(1.09)^n-1}{0.09(1.09)^n}$. This gives $n \approx 38$ years, which means that the total life of the mortgage is 43 years.

9. (Bond price) Straightforward use of the formula for a bond price (assuming coupons every six months) gives 91.17.

10. (Duration) Use the formula to obtain 6.84 years.

11. (Annuity duration) Using $\text{PV} = \frac{A}{r}$ we have

$$\begin{aligned} D &= -\frac{(1+r)}{\text{PV}}\frac{d\text{PV}}{dr} \\ &= -\frac{r(1+r)}{A}\cdot\left(-\frac{A}{r^2}\right) = \frac{1+r}{r} \end{aligned}$$

    Hence

$$D_M = \frac{D}{1+r} = \frac{1}{r}.$$

12. (Bond selection)

    (a)

$$\begin{aligned} P_A &= 885.84 \\ P_B &= 771.68 \\ P_C &= 657.52 \\ P_D &= 869.57 \end{aligned}$$

(b)

$$\begin{aligned} D_A &= 2.72 \\ D_B &= 2.84 \\ D_C &= 3.00 \\ D_D &= 1.00 \end{aligned}$$

(c) C is most sensitive to a change in yield.

(d)

$$V_A + V_B + V_C + V_D = \text{PV}$$

$$D_A V_A + D_B V_B + D_C V_C + D_D V_D = 2\text{PV},$$

where PV is the present value of the obligation.

(e) Use bond D.

$$V_C + V_D = \text{PV}$$

$$D_C V_C + D_D V_D = 2\text{PV}$$

where

$$\text{PV} = \frac{2,000}{1.15^2} = \$1,512.29.$$

Solving $V_C = \$756.15$ and $V_D = \$756.15$.

(f) None

13. (Continuous compounding)

$$\frac{dP}{d\lambda} = -\sum_{k=0}^{n} e^{-\lambda t_k} t_k C_k = -DP$$

14. (Duration limit) This follows directly from the Macaulay duration formula by setting $\lambda = my$ and noting that the second term in the formula goes to zero as n goes to infinity.

15. (Convexity value)

$$C = \frac{1}{P[1 + (\lambda/m)]^2} \frac{n(n+1)P}{m^2}.$$

We take $T = n/m$. Hence

$$C = \frac{1}{[1 + (\lambda/m)]^2} T(T + (1/m)).$$

As $m \to \infty$ we find $C = T^2$.

16. (Convexity theorem)

(a)

$$\begin{aligned} P(\lambda) &= \sum_t c_t d_t(\lambda) - d_{\bar{t}}(\lambda) \\ P''(\lambda) &= \sum_t c_t t(t+1) d_t(\lambda) d_2(\lambda) - \bar{t}(\bar{t}+1) d_{\bar{t}}(\lambda) d_2(\lambda) \end{aligned}$$

Therefore,

$$\begin{aligned} P''(0)(1+r)^2 &= \sum_t c_t t^2 d_t - \bar{t}^2 d_{\bar{t}} + \sum_t t d_t c_t - \bar{t} d_{\bar{t}} \\ &= \sum_t c_t t^2 d_t - \bar{t}^2 d_{\bar{t}} \quad \text{(by the second condition)} \end{aligned}$$

Since

$$\sum_t c_t d_t - d_{\bar{t}} = \sum_t \beta c_t d_t - \beta d_{\bar{t}} = 0 \quad \text{for all } \beta$$

and

$$\sum_t t d_t c_t - \bar{t} d_{\bar{t}} = \sum_t (\alpha t) c_t d_t - (\alpha \bar{t}) d_{\bar{t}} = 0 \quad \text{for all } \alpha$$

It follows that

$$P''(0)(1+r)^2 = \sum_t c_t (t^2 + \alpha t + \beta) d_t - (\bar{t}^2 + \alpha \bar{t} + \beta) d_{\bar{t}}.$$

(b) Let $\alpha = -2\bar{t}$, $\beta = \bar{t}^2 + 1$. Then $t^2 + \alpha t + \beta = (t - \bar{t})^2 + 1$ which has a minimum at $\bar{t}$ and has a value of 1 there.

$$\begin{aligned} P''(0)(1+r)^2 &\geq \sum_t c_t (\bar{t}^2 + \alpha \bar{t} + \beta) d_t - (\bar{t}^2 + \alpha \bar{t} + \beta) d_{\bar{t}} \\ &= \sum_t c_t d_t - d_{\bar{t}} = 0. \end{aligned}$$

Therefore, $P''(0) \geq 0$.

Chapter 4
The Term Structure of Interest Rates

1. (One forward rate)

$$f_{1,2} = \frac{(1+s_2)^2}{(1+s_1} - 1 = \frac{1.069^2}{1.063} - 1 = 7.5\%.$$

2. (Spot update) Use

$$f_{1,k} = \left\{\frac{(1+s_k)^k}{1+s_1}\right\}^{1/(k-1)} - 1.$$

Hence, for example,

$$f_{1,6} = \left\{\frac{(1.061)^6}{1.05}\right\}^{1/5} - 1 = .0632.$$

All values are

$f_{1,2}$	$f_{1,3}$	$f_{1,4}$	$f_{1,5}$	$f_{1,6}$
5.60	5.90	6.07	6.25	6.32

3. (Construction of a zero) Use a combination of the two bonds: let x be the number of 9% bonds, and y the number of 7% bonds. Select x and y to satisfy

$$\begin{aligned} 9x + 7y &= 0 \\ x + y &= 1. \end{aligned}$$

The first equation makes the net coupon zero. The second makes the face value equal to 100. These equations give $x = -3.5$, and $y = 4.5$. The price is $P = -3.5 \times 101.00 + 4.5 \times 93.20 = 65.9$.

4. (Spot rate project) All can be done on a spreadsheet with an optimizer, as shown below:

Maturity	Coupon	Buying Price w/ Acc. Int.	Model Price
15-Feb-12	6.625	$101.48	$101.49
15-Feb-12	9.125	$102.72	$102.71
15-Aug-12	7.875	$102.50	$102.50
15-Aug-12	8.25	$102.87	$102.87
15-Feb-13	8.25	$103.06	$103.06
15-Feb-13	8.375	$103.24	$103.24
15-Aug-13	8	$102.60	$102.59
15-Aug-13	8.75	$103.98	$103.99
15-Feb-14	6.875	$99.69	$99.69
15-Feb-14	8.875	$104.26	$104.26
15-Aug-14	6.875	$98.94	$98.94
15-Aug-14	8.625	$103.64	$103.63
15-Feb-15	7.75	$100.88	$100.90
15-Feb-15	11.25	$111.63	$111.64
15-Aug-15	8.5	$103.30	$103.30
15-Aug-15	10.5	$110.18	$110.18
15-Feb-16	7.875	$101.16	$101.17
15-Feb-16	8.875	$104.98	$104.97

Estimation	Coefficients
a_0	0.062009143
a_1	0.00627032
a_2	0.001099467
a_3	-0.000593607
a_4	4.95E-05

Maturity of Zero Bonds	Time from Valuation (Years)	Estimated Spot Interes Rate
15-Feb-12	0.2795	0.06383460
15-Aug-12	0.7781	0.06729211
15-Feb-13	1.2822	0.07073894
15-Aug-13	1.7781	0.07379215
15-Feb-14	2.2822	0.07633260
15-Aug-14	2.7781	0.07813531
15-Feb-15	3.2822	0.07918981
15-Aug-15	3.7781	0.07946645
15-Feb-16	4.2822	0.07905432

Valuation Date
5-Nov-11

5. (Instantaneous rates)

(a) $e^{s(t_2)t_2} = e^{s(t_1)t_1} e^{f_{t_1,t_2}(t_2-t_1)} \quad \Rightarrow \quad f_{t_1,t_2} = \dfrac{s(t_2)t_2 - s(t_1)t_1}{t_2 - t_1}$

(b) $r(t) = \lim\limits_{t \to t_1} \dfrac{s(t)t - s(t_1)t_1}{t - t_1} = \dfrac{\mathrm{d}(s(t)t)}{\mathrm{d}t} = s(t) + s'(t)t$

(c) We have

$$\mathrm{d}\ln x(t) = r(t)\mathrm{d}t = s(t)\mathrm{d}t + s'(t)t\,\mathrm{d}t = \mathrm{d}(s(t)t).$$

Hence,

$$\ln x(t) = \ln x(0) + s(t)t.$$

Finally,

$$x(t) = x(0)e^{s(t)t}.$$

This is in agreement with the invariance property of expectation dynamics. Investing continuously gives the same result as investing in a bond that matures at time t.

6. (Discount conversion) The discount factors are found by successive multiplication. For example $d_{0,2} = d_{01}d_{1,2} = .950 \times .940 = .893$. The complete set is .950, .893, .832, .770, .707, .646.

7. (Bond taxes) Let

   t be the tax rate
   x_i be the number of bond i bought
   c_i be the coupon of bond i
   p_i be the price of bond i

   To create a zero coupon bond, we require, first, that the after tax coupons match. Hence

   $$x_1(1-t)c_1 + x_2(1-t)c_2 = 0.$$

   which reduces to

   $$x_1c_1 + x_2c_2 = 0.$$

   Next, we require that the after tax final cash flow matches. Hence

   $$x_1[100-(100-p_1)t] + x_2[100-(100-p_2)t] = [100-(100-p_0)t].$$

   The price of the zero will be

   $$p_0 = x_1p_1 + x_2p_2.$$

   Using this last relation in the equation for final cash flow, we find

   $$x_1 + x_2 = 1.$$

   Combining $x_1 + x_2 = 1$, $c_1x_1 + c_2x_2 = 0$, and $p_0 = x_1p_1 + x_2p_2$, we find

   $$p_0 = \frac{c_2p_1 - c_1p_2}{c_2 - c_1}.$$

   Plugging in the given values we find $p_0 = 37.64$.

8. (Real zeros) We assume that with coupon bonds there is a capital gain tax at maturity. We replicate the zero-coupon bond after tax flows using bonds 1 and 2. Let x_i = amount of bond i required (for $i = 1,2$). We require

   (a) $100c_1(1-t)x_1 + 100c_2(1-t)x_2 = -(\frac{100-p_0}{n})t$

   (b) $(100-(100-p_1)t+100c_1(1-t))x_1 + (100-(100-p_2)t+100c_2(1-t))x_2 = 100-(\frac{100-p_0}{n})t$

   (c) $p_1x_1 + p_2x_2 = p_0$

Setting

$$\begin{aligned} p_1 &= 92.21 \\ p_2 &= 75.84 \\ t &= 30\% \\ c_1 &= .10 \\ c_2 &= .07 \\ n &= 10 \end{aligned}$$

We find $x_2 = 5.25428$ and $p_0 = 32.767$

9. (Flat forwards). For $i < j$

$$(1+r)^i(1+f_{i,j})^{j-i} = (1+r)^j.$$

Hence

$$(1+f_{i,j})^{j-i} = (1+r)^{j-i}$$

which implies $f_{ij} = r$.

10. (Orange County blues)

Additional clarifications::

(a) Assume portfolio is restructured annually to maintain a duration of 10 years.

(b) Assume value of money borrowed is maintained at \$12.5 billion every year.

(c) Assume Orange County makes interest on deposit at the rate which prevailed at the beginning of the given year.

Year

$$\begin{aligned}
\text{Year 1} \quad P &= 20(1.085) + \frac{-10(20)(-.005)}{1.085} - 12.5(.07) = 21.75 \\
r &= \frac{21.75 - 20}{20 - 12.5} = 23.33\% \\
\text{Year 2} \quad P &= (21.75)(1.08) + \frac{-10(21.75)(-.005)}{1.08} - 12.5(.065) = 23.68 \\
r &= \frac{23.68 - 21.75}{21.75 - 12.5} = 20.86\% \\
\text{Year 3} \quad P &= (23.68)(1.075) + \frac{-10(23.68)(-.005)}{1.075} - 12.5(.06) = 25.81 \\
r &= \frac{25.81 - 23.68}{23.68 - 12.5} = 19.02\% \\
\text{Year 4} \quad P &= (25.81)(1.07) + \frac{-10(25.81)(-.005)}{1.07} - 12.5(.055) = 28.14
\end{aligned}$$

$$r = \frac{28.14 - 25.81}{25.81 - 12.5} = 17.51\%$$

$$\text{Year 5} \quad P = (28.14)(1.065) - \frac{10(28.14)(.02)}{1.065} - 12.5(.05) = 24.06$$

$$r = \left(\frac{24.06 - 28.14}{28.14 - 12.5}\right) = -26.09\%$$

$$\text{Year 6} \quad P = (24.06)(1.085) - \frac{10(24.06)(.02)}{1.085} - 12.5(.07) = 20.80$$

$$r = \left(\frac{20.80 - 24.06}{24.06 - 12.5}\right) = -28.20\%$$

Money left after 6 years = 20.79 − 12.5 − 7.5 = .80

If invested in bank = 7.5(1.06)(1.055)(1.045)(1.4)(1.06) − 7.5 = 2.64

11. (Running PV example)

(a)

$d_{0,1}$	$d_{0,2}$	$d_{0,3}$	$d_{0,4}$	$d_{0,5}$	$d_{0,6}$	⇒ NPV = 9.497
.9524	.9018	.8492	.7981	.7472	.7010	

(b)

Year	0	1	2	3	4	5	6
Cash Flow	-40	10	10	10	10	10	10
Discount	.9524	.9469	.9416	.9399	.9362	.9381	
PV(n)	9.497	51.970	44.324	36.453	28.144	19.381	10.000

12. (Pure duration)

$$P(\lambda) = \sum_{k=0}^{n} x_k (1 + s_k/m)^{-k} = \sum_{k=0}^{n} x_k \left((1 + s_k^0/m)e^{\lambda/m}\right)^{-k}.$$

$$\frac{dP(\lambda)}{d\lambda} = \sum_{k=0}^{n} x_k \left(\frac{-k}{m}\right) \left((1 + s_k^0/m)e^{\lambda/m}\right)^{-k-1} (1 + s_k^0/m)e^{\lambda/m}$$

$$= \sum_{k=0}^{n} x_k \left(\frac{-k}{m}\right) (1 + s_k/m)^{-k}.$$

$$-\frac{1}{P}\frac{dP(\lambda)}{d\lambda} = \frac{\sum_{k=0}^{n} x_k \left(\frac{-k}{m}\right) (1 + s_k/m)^{-k}}{\sum_{k=0}^{n} x_k (1 + s_k/m)^{-k}} = D.$$

This D exactly corresponds to the original definition of duration as a cash flow weighted average of the times of cash payments. No modification factor is needed even though we are working in discrete time.

13. (Stream immunization)

Year	1	2	3	4	5	6	7	8
Spot	7.67	8.27	8.81	9.31	9.75	10.16	10.52	10.85
d	.929	.853	.776	.700	.628	.560	.496	.439
Obligations	500	900	600	500	100	100	100	50
PV_{obl}	464.5	767.7	465.6	350	62.8	56	99.6	21.9 = 2238.1
$\frac{x_k t_k}{(1+s_k)^{k+1}}$	431.4	1418.2	1284.1	1281.5	286.1	304.8	314.5	158.3 = 5478.9

Hence $D_{obl} = \frac{5478.9}{2238.1} = 2.45$ years.

We have $PV_1 = 65.98$, $D_1 = 7.11$ $PV_2 = 101.66$, $D_2 = 3.80$. We must solve

$$\begin{aligned} x_1 PV_1 &+ x_2 PV_2 &= PV_{obl} \\ x_1 PV_1 D_1 &+ x_2 PV_2 D_2 &= PV_{obl} D_{obl}. \end{aligned}$$

These become

$$\begin{aligned} 65.98x_1 &+ 101.66x_2 &= 2238.1 \\ 469.12x_1 &+ 386.31x_2 &= 5483.34. \end{aligned}$$

Giving $x_1 \cong -13.835 \quad x_2 \cong 30.995$.

14. (Mortgage division)

(a)

$$\begin{aligned} P(k) &= B - rM(k-1) = B - r\left[(1+r)^{k-1}M(0) - \left(\frac{(1+r)^{k-1}-1}{r}\right)B\right] \\ &= (1+r)^{k-1}(B - rM). \end{aligned}$$

We are looking for

$$V = \sum_{k=1}^{n} \frac{P(k)}{(1+r)^k} = \sum_{k=1}^{n} \frac{(1+r)^{k-1}(B-rM)}{(1+r)^k} = \frac{n(B-rM)}{(1+r)}.$$

(b) Plug the expression for B into the result found in part (a).

$$V = \frac{n}{1+r}\left[\frac{r(1+r)^n M}{(1+r)^n - 1} - rM\right] = \frac{n}{1+r}\left[\frac{rM}{(1+r)^n - 1}\right].$$

(c)

$$\begin{aligned} W &= \sum_{k=1}^{n} \frac{I(k)}{(1+r)^k} = \sum_{k=1}^{n} \frac{B - P(k)}{(1+r)^k} = B\left(\sum_{k=1}^{n} \frac{1}{(1+r)^k}\right) - V \\ &= \left(\frac{r(1+r)^n M}{(1+r)^n - 1}\right)\left(\frac{(1+r)^n - 1}{r(1+r)^n}\right) - V = M - V. \end{aligned}$$

(d) It should be clear that $V \to 0$. (Use L'Hopital's rule if it is not obvious.)

(e) We know $P(k) = (1+r)^{k-1}(B - rM)$. Clearly $B - rM > 0$. (This follows from part (a).) So $P(k)$ is increasing in k and $I(k) = B - P(k)$ must be decreasing in k. Remember that duration is a weighted sum of the times k, with the weights being proportional to to the cash flows at those times. Hence the duration of the stream determined by $P(k)$, which increases in k, should be the larger, because more relative weight is given to higher k's.

15. (Short-rate sensitivity) In general

$$P_{k-1}(\lambda) = c_{k-1} + \frac{P_k(\lambda)}{1 + r_{k-1} + \lambda}.$$

Differentiation at $\lambda = 0$ leads to

$$S_{k-1} = -\frac{P_k}{(1 + r_{k-1})^2} + \frac{S_k}{1 + r_{k-1}}.$$

Hence, $a_k = \frac{1}{(1+r_{k-1})^2}$, $b_k = \frac{1}{1+r_{k-1}}$. This process together with

$$P_{k-1} = c_{k-1} + \frac{P_k}{1 + r_{k-1}}$$

is initiated with $P_n = c_n$ and $S_n = 0$; and the two processes are worked backward to $k = 0$. S_0 is the final result.

Chapter 5
Applied Interest Rate Analysis

1. (Capital budgeting)

Project	Benefit-Cost Ratio
1	2
2	5/3
3	3/2
4	4/3
5	5/3

So, the approximate method based on cost-benefit ratios implies projects 1, 2, and 5 would be recommended.

The optimal set of projects is the same. Note: projects 1,2, and 3 provide the same total net present value and use the entire budget.

2. (The road) The zero-one problem is the same as in Example 5.2 with the following additional constraint:

$$(x_2 + x_4)(1 - (x_6 + x_7)) = 0$$

Excel's Solver yields an optimal solution with a total benefit of \$7,8000,000 for a cost of \$4,700,000 by funding projects 4,6, and 10.

3. (Two-period budget) The problem is to

$$\begin{aligned}
\text{maximize} \quad & 150x_1 + 200x_2 + 100x_3 + 100x_4 + 120x_5 + 150x_6 + 240x_7 \\
\text{subject to} \quad & 90x_1 + 80x_2 + 50x_3 + 20x_4 + 40x_5 + 80x_6 + 80x_7 + y \leq 250 \\
& 58x_1 + 80x_2 + 100x_3 + 64x_4 + 50x_5 + 20x_6 + 100x_7 \leq \\
& \quad 250 + (1.1)y \\
& x_i = 0 \text{ or } 1 \text{ for each } i \\
& y \geq 0.
\end{aligned}$$

Excel's Solver yields a maximal NPV of 610, achieved by funding projects 4,5,6, and 7, at a cost of 220 in the first year and 234 in the second year. Another plan with NPV of 610 is to fund projects 1,4,5, and 7, at a cost of 230 in the first year and 272 in the second year. Both plans are under the budget, but the first costs less.

4. (Bond matrix)

$$\mathbf{C} = \begin{bmatrix} 10 & 7 & 8 & 6 & 7 & 5 & 10 & 8 & 7 & 100 \\ 10 & 7 & 8 & 6 & 7 & 5 & 10 & 8 & 107 & \\ 10 & 7 & 8 & 6 & 7 & 5 & 110 & 108 & & \\ 10 & 7 & 8 & 6 & 7 & 105 & & & & \\ 10 & 7 & 8 & 106 & 107 & & & & & \\ 110 & 107 & 108 & & & & & & & \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 100 \\ 200 \\ 800 \\ 100 \\ 800 \\ 1200 \end{bmatrix}$$

(a) $\mathbf{p}^T$ and $\mathbf{x}^T$ are identified in Table 5.3

(b) We know the price of bond j is $p_j = \sum_{i=1}^{n} c_{ij}(1+s_n)^i$ so choosing $\mathbf{v} = [(1+s_n)^1, (1+s_n)^2, \ldots, (1+s_n)^n]^T$ solves the equation $\mathbf{C}^T\mathbf{v} = \mathbf{p}$

(c) To meet the obligation of period i exactly, we require

$$\sum_{j=1}^{m} C_{ij}x_j = b_i$$

or in matrix form $\mathbf{Cx} = \mathbf{b}$

(d) The price of the portfolio is $\mathbf{p}^T\mathbf{x} = \mathbf{v}^T\mathbf{Cx} = \mathbf{v}^T\mathbf{b}$ which shows that the present value of the portfolio must equal the present value of the liabilities.

5. (Trinomial lattice) The trinomial lattice spanning three periods (with four time points) contains $4^2 = 16$ nodes. In general, a trinomial lattice with n time points contains n^2 nodes.

   In a full trinomial tree spanning three periods there are 40 nodes. In general, a full trinomial tree with n time points contains $\sum_{i=0}^{n-1} 3^i = \frac{1}{2}(3^n - 1)$ nodes.

6. (A bond project)

		Cash Match	Cash Match Reinvest	Guard for D a(0)	Guard for D a(1)	Guard for D a(2)	Guard for D a(3)	Guard for D a(4)
	Cost:	$70,723.31	$70,558.12	$70,557.61	$70,558.01	$70,559.31	$70,559.47	$70,560.21
Maturity	Coupon	# Bought	# Bought	# Bought	# Bought	# Bought	# Bought	# Bought
15-Feb-12	6.625	0	376.91	186.8	132.24	23.04	32.29	10.98
15-Feb-12	9.125	0	0	0	0	0	0	0
15-Aug-12	7.875	165.25	0	0	0	0	0	73.46
15-Aug-12	8.25	0	0	0	0	0	0	0
15-Feb-13	8.25	0	0	0	83.82	309.81	249.41	198.52
15-Feb-13	8.375	0	0	0	0	0	0	0
15-Aug-13	8	208.26	0	0	0	0	0	0
15-Aug-13	8.75	0	0	0	0	0	65.79	0
15-Feb-14	6.875	0	0	0	0	0	0	92.79
15-Feb-14	8.875	0	0	0	0	0	0	0
15-Aug-14	6.875	0	0	0	0	0	0	0
15-Aug-14	8.625	0	0	0	0	0	0	0
15-Feb-15	7.75	172.38	320.27	511.5	480.76	224.6	201.97	171.4
15-Feb-15	11.25	0	0	0	0	0	0	0
15-Aug-15	8.5	4.14	0	0	0	0	0	0
15-Aug-15	10.5	0	0	0	0	0	0	0
15-Feb-16	7.875	144.32	0	0	0	134.78	141.98	147.44
15-Feb-16	8.875	0	0	0	0	0	0	0

7. (The fishing problem) The decisions at times after the initial time do not depend on d. At time 1 the upper and lower node values are

$$\begin{aligned} x_2 &= 14 + 14d \\ x_1 &= 7 + 7d \end{aligned}$$

respectively. Then the initial value is

$$x_0 = \max[14d(1+d), 7(1+d+d^2)]$$

The choice depends on d. The critical value of d is

$$d^* = \frac{\sqrt{5}-1}{2} \approx .618.$$

For $d < d^*$ we choose x_2.

For $r = 33\%$ we have $d = .75$ and for $r = 25\%$ we have $d = .8$, so solution is the same for both.

8. (Complexico mine)

   (a) Since we mine forever, we have $K_K = K_{K+1} =$ constant K.
   So $K = \frac{(g-dK)^2}{2000} + dK$ implies $K = 220$ every period.
   Thus, the initial value of the mine, $V_0 = 220x_0 = \$11$ million.

   (b) The amount of gold remaining in the mine in period n, x_n, equals $x_{n-1} - z_{n-1}$ where z_n equals the amount mined in period n. Using Excel and the equations from Example 5.5, we find $x_{10} = 2393$.

Thus, by part (a), the value of the mine in period 10 is found to be $220x_{10} =$ $\$526,460$ (at that time).

(c) The optimal extraction rate in each period $= \frac{g-dK}{1000} = 20\%$ so, after 10 years, 5369 ounces of gold remains with a value of \$1,181,116 (at that time).

9. (Little Bear Oil)

   (a) Set up a trinomial lattice with arcs:
   "up" = no pumping
   "middle" = normal pumping
   "down" = enhanced pumping
   The reserve values can be entered on each node. (At the final time the maximum reserve is 100,000 and the minimum is 26,214 barrels.)

   (b) Work backward to find PV = \$366,740. The optimal strategy is: enhanced pumping for the first two years, followed by normal pumping in the last year.

10. (Multiperiod harmony theorem) We can write

$$\begin{aligned} V_0 &= \max\left[x_0 + \frac{1}{1+s_1}\left\{x_1 + \frac{x_2}{1+s_1'} + \frac{x_3}{(1+s_2')^2} + \cdots + \frac{x_n}{(1+s_{n-1}')^{n-1}}\right\}\right] \\ &= \max_{x_0}\left[x_0 + \frac{1}{1+s_1}V_1(x_0)\right] \end{aligned}$$

achieved by x_0^*. Clearly,

$$\frac{V_1(x_0^*)}{V_0(x_0^*) - x_0^*} = 1 + s_1.$$

Suppose $\overline{x}_0$ gives

$$\frac{V_1(\overline{x}_0)}{V_0(\overline{x}_0) - \overline{x}_0} > 1 + s_1.$$

Then $V_0 - \overline{x}_0 > 0$ and thus

$$V_0 < \overline{x}_0 + \frac{V_1(\overline{x}_0)}{1+s_1}$$

which contradicts the definition of V_0.

11. (Growing annuity) Using the hint we have

$$S = \frac{1}{1+r} + \frac{S(1+g)}{(1+r)}$$

implying

$$S\left[1 - \left(\frac{1+g}{1+r}\right)\right] = \frac{1}{1+r}$$

or

$$S = \frac{1}{r-g}.$$

12. (Two-stage growth)

(a) This part follows easily from (b)

(b) Let $R = 1 + r$. then

$$\begin{aligned}
\text{NPV} &= D_1\left[\left(\frac{G}{R}\right)^0 + \left(\frac{G}{R}\right)^1 + \cdots + \left(\frac{G}{R}\right)^k\right] \\
&\quad + D_1\left[\left(\frac{g}{R}\right)^{k+1} + \left(\frac{g}{R}\right)^{k+2} + \cdots\right] \\
&= D_1\left[\frac{1-(\frac{G}{R})^k}{1-(\frac{G}{R})} + \frac{(\frac{g}{R})^{k+1}}{1-(\frac{g}{R})}\right].
\end{aligned}$$

Chapter 6
Mean-Variance Portfolio Theory

1. (Shorting with margin) The money invested is X_0. The money received at the end of a year is $X_0 - X_1 + X_0$. Hence,

$$R = \frac{2X_0 - X_1}{X_0}.$$

2. (Dice product) Let a and b be the outcomes of two die rolls. Then $Z = ab$. By independence, we know

$$\begin{aligned} E[ab] &= E[a]E[b] \\ \text{and } \mathrm{var}[Z] &= E[a^2]E[b^2] - (E[a]E[b])^2 \\ &\cong 79.97 \end{aligned}$$

3. (Two correlated assets) For solution method, see solution to problem called Two stocks (below).

   (a) α equals 19/23.

   (b) The minimum standard deviation is approximately 13.7%.

   (c) The expected return of this portfolio is approximately 11.4%.

4. (Two stocks) Let α, β equal the percent of investment in stock 1 and stock 2, respectively. The problem is

$$\min_{\alpha,\beta} \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + 2\alpha\beta\sigma_{12}$$

$$\text{subject to } \alpha + \beta = 1.$$

Setting up the Lagrangian, L, we have:

$$L = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + 2\alpha\beta\sigma_{12} - \lambda(\alpha + \beta - 1)$$

The first order necessary conditions are:

$$0 = \frac{\partial L}{\partial \alpha} = 2\alpha\sigma_1^2 + 2\beta\sigma_{12} - \lambda$$

$$0 = \frac{\partial L}{\partial \beta} = 2\beta\sigma_2^2 + 2\alpha\sigma_{12} - \lambda$$

$$1 = \alpha + \beta$$

which imply

$$\alpha = \left[\frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}\right]$$

The mean rate of return is just $\alpha m_1 + \beta m_2$.

5. (Rain insurance)

   (a) The expected rate of return equals $\frac{(.5)\cdot 3\cdot 10^6 + .5\cdot u}{10^6 + .5u}$

   (b) By inspection, it can be seen that buying 3 million units of insurance eliminates all uncertainty regarding the return. So, 3 million units of insurance results in a variance of 0 and a corresponding expected rate of return equal to $\frac{3}{2.5} - 1 = 20\%$.

6. (Wild cats)

   (a) The three assets are on a single horizontal line. The efficient set is a single point on the same line, but to the left of the left-most of the three original points.

   (b) Let w_i be the percentage of the total investment invested in asset i. Then, since the assets are uncorrelated, we have

   $$\text{var (total investment)} = \sum_{i=1}^{n} w_i^2\sigma_i^2$$

   where $\sum_{i=1}^{n} w_i = 1$. Setting up the Lagrangian,

   $$L = \sum_{i=1}^{n} w_i^2\sigma_i^2 - \lambda\left(\sum_{i=1}^{n} w_i - 1\right)$$

   the first-order necessary conditions imply

   $$w_i\sigma_i^2 = \frac{\lambda}{2} \quad i = 1, \ldots, n$$

   or $w_j = \frac{\lambda}{2\sigma_j^2}$.

   Since $\sum_{i=1}^{n} w_i = 1$, we have $\frac{1}{2}\lambda\left(\sum_{j=1}^{n}\frac{1}{\sigma_j^2}\right) = 1$

which implies

$$w_j = \frac{\overline{\sigma}^2}{\sigma_j^2} \quad j = 1, 2, \ldots, n$$

where

$$\overline{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1} .$$

The minimum variance is

$$\text{var}_{\text{min}} = \sum_{i=1}^{n} w_i^2 \sigma_i^2 = \sum_{i=1}^{n} \left(\frac{\overline{\sigma}^2}{\sigma_i^2} \right)^2 \sigma_i^2 = \overline{\sigma}^2 .$$

7. (Markowitz fun)

(a) First solve for the v_i's from

$$\begin{array}{ccccccc} 2v_1 & + & v_2 & & & = & 1 \\ v_1 & + & 2v_2 & + & v_3 & = & 1 \\ & & v_2 & + & v_3 & = & 1 \end{array}$$

This yields $v_1 = .5$, $v_2 = 0$, and $v_3 = .5$. This solution happens to be normalized, so also $w_1 = .5$, $w_2 = 0$, and $w_3 = .5$.

(b) In this case we solve

$$\begin{array}{ccccccc} 2v_1 & + & v_2 & & & = & .4 \\ v_1 & + & 2v_2 & + & v_3 & = & .8 \\ & & v_2 & + & v_3 & = & .8 \end{array}$$

This leads to $v_1 = .1$, $v_2 = .2$, and $v_3 = .3$. This solution must be normalized to get the final result $w_1 = 1/3$, $w_2 = 1/6$, and $w_3 = 1/2$.

(c) We find the v_i's by the formula $v_i = v_i^b - r_f v_i^a$, where v_i^a is the solution from part (a) and v_i^b is the solution from part (b). Thus

$$\begin{array}{ccccc} v_1 & = & .1 - .2 \times .5 & = & 0 \\ v_2 & = & .2 - 0 & = & .2 \\ v_3 & = & .3 - .2 \times .5 & = & .2 \end{array}$$

When normalized the solution is $w_1 = 0$, $w_2 = .5$, and $w_3 = .5$.

8. (Tracking)

(a)

$$\begin{aligned} \text{var}(r - r_M) &= \text{var}(r) - 2\,\text{cov}(r, r_M) + \text{var}(r_M) \\ &= \sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 \end{aligned}$$

So, to minimize $\mathrm{var}(r - r_M)$ subject to $\sum_{i=1}^{n} \alpha_i = 1$ set up the Lagrangian

$$L = \sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 + \lambda\left(\sum_{i=1}^{n} \alpha_i - 1\right)$$

The first order necessary conditions imply

$$\begin{aligned} 2 \sum_{j=1}^{n} \alpha_j \sigma_{ij} - 2\sigma_{im} + \lambda &= 0 \quad \text{for all } i \\ \sum_{i=1}^{n} \alpha_i &= 1 \end{aligned}$$

(b) Similar to (a) with the added constraint $\sum_{i=1}^{n} \alpha_i r_i = m$.

So the first order necessary conditions imply

$$2 \sum_{j=1}^{n} \alpha_j \sigma_{ij} - 2\sigma_{im} + \lambda + \mu r_i = 0 \quad \text{for all } i$$

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad \sum_{i=1}^{n} \alpha_i r_i = m$$

9. (Betting wheel) For every segment the payoff for a bet $B_i = 1/A_i$ will equal \$1. Thus, the payoff is \$1 independent of the wheel.

   Since, with this betting strategy, the reward is completely determined, the risk-free rate of return just equals

$$r = \frac{1}{\sum_{i=1}^{n} 1/A_i} - 1.$$

   Thus, the risk free rate of return in example 6.7 is 0.

10. (Efficient portfolio)

$$0 = \frac{\partial(\tan\theta)}{\partial w_k} = \frac{(\bar{r}_k - r_f)\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)^{1/2} - \left(\sum_{i=1}^{n} w_i \bar{r}_i - r_f\right) \frac{\partial}{\partial w_k}\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)^{1/2}}{\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)}$$

Using the hint, this implies

$$(\bar{r}_k - r_f) = \left(\frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j} \right) \sum_{j=1}^{n} \sigma_{kj} w_k.$$

Setting $\left(\frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j} \right) = \lambda$, we achieve the desired result.

Chapter 7
The Capital Asset Pricing Model

1. (Capital market line)

   (a) $\bar{r} = .07 + \frac{.23-.07}{.32}\sigma = .07 + .5\sigma$

   (b) i. $\sigma = .64$

   ii. Solve $w \times .07 + (1-w) \times .23 = .39$ giving $w = -1$. Hence, borrow \$1000 at the risk-free rate; invest \$2000 in the market

   (c) \$1182

2. (A small world)

   (a)
$$\begin{aligned}
\sigma_M^2 &= \frac{1}{4}(\sigma_A^2 + 2\sigma_{A,B} + \sigma_B^2) \\
\sigma_{AM}^2 &= \frac{1}{2}(\sigma_A^2 + \sigma_{A,B}) \quad \text{hence} \quad \beta_A = \frac{\sigma_A^2 + \sigma_{A,B}}{2\sigma_M^2} \\
\sigma_{AB}^2 &= \frac{1}{2}(\sigma_B^2 + \sigma_{A,B}) \quad \text{hence} \quad \beta_B = \frac{\sigma_B^2 + \sigma_{A,B}}{2\sigma_M^2}
\end{aligned}$$

   (b)
$$\begin{aligned}
\bar{r}_A &= .10 + \frac{5}{4}(.18 - .10) = 20\% \\
\bar{r}_B &= .10 + \frac{3}{4}(.18 - .10) = 16\%
\end{aligned}$$

3. (Bounds on returns)

   (a) Using the two-fund theorem and noting that the market portfolio cannot contain assets in negative amounts, we have

$$\tfrac{1}{2}\mathbf{w} + \tfrac{1}{2}\mathbf{v} = \begin{bmatrix} .7 \\ 0 \\ .3 \end{bmatrix} \quad \text{with a rate of return of } .1$$

$$2\mathbf{w} - \mathbf{v} = \begin{bmatrix} .4 \\ .6 \\ 0 \end{bmatrix} \quad \text{with a rate of return of } .16$$

so the expected rate of return of the market portfolio $\overline{r}_M$ is bounded by: $.1 \leq \overline{r}_M \leq .16$.

(b) Since $\overline{r}_M \geq \overline{r}_{\text{min var portfolio}}$, we have $.12 < \overline{r}_M \leq .16$.

4. (Quick CAPM derivation) From (6.9) we have

$$\sum_{i=1}^{n} \sigma_{Mi}\lambda w_i = \lambda\sigma_M^2 = \overline{r}_M - r_f.$$

Hence $\lambda = \frac{\overline{r}_m - r_f}{\sigma_m^2}$ Also,

$$\sum_{i=1}^{n} \sigma_{Mi}\lambda w_i = \lambda \operatorname{cov}(r_k, r_M) = \overline{r}_M - r_f.$$

Combining, we have

$$\overline{r}_k - r_f = \frac{\sigma_{kM}}{\sigma_M^2}(\overline{r}_M - r_f)$$

5. (Uncorrelated assets)

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{x_i\sigma_i^2}{\sum_{j=1}^{n} x_j\sigma_j^2}$$

since the assets are uncorrelated.

6. (Simpleland) The market consists of \$150 in shares of A and \$300 in shares of B. Hence, the market return is

$$r_M = (\frac{150}{450})r_A + (\frac{300}{450})r_B = \frac{1}{3}r_A + \frac{2}{3}r_B.$$

(a) $\overline{r}_M = \frac{1}{3} \times .15 + \frac{2}{3} \times .12 = .13$

(b) $\sigma_M = \left[\frac{1}{9}(.15)^2 + \frac{4}{9 \times 3}(.15)(.09) + \frac{4}{9}(.09)^2\right]^{\frac{1}{2}} = .09$

(c) $\sigma_{AM} = \frac{1}{3}\sigma_A^2 + \frac{2}{3}\rho_{AB}\sigma_A\sigma_B = \frac{1}{3}(.15)^2 + \frac{2}{9}(.15)(.09) = .0105.$

$\beta_A = \frac{\sigma_{AM}}{\sigma_M^2} = 1.2963$

(d) Since Simpleland satisfies the CAPM exactly, stocks A and B plot on the security market line. Specifically,

$$\overline{r}_A - r_f = \beta_A(\overline{r}_M - r_f).$$

Hence,

$$r_f = \frac{\overline{r}_A - \beta_A\overline{r}_M}{1 - \beta_A} = .0625.$$

7. (Zero-beta assets)

   (a) Let $\mathbf{p}$ be a portfolio such that $\mathbf{p} = (1-\alpha)\mathbf{w}_0 + \alpha\mathbf{w}_1$. Then, $\sigma_{\mathbf{p}}^2 = (1-\alpha)^2\sigma_0^2 + 2(1-\alpha)\alpha\sigma_{01} + \alpha^2\sigma_1^2$. So, since $0 = \left.\frac{d\sigma_{\mathbf{p}}^2}{d\alpha}\right|_{\alpha=0}$, we have $0 = -2\sigma_0^2 + 2\sigma_{01}$ which implies $A = 1$.

   (b) $0 = \sigma_{1,z} = (1-\alpha)\sigma_{01} + \alpha\sigma_1^2$ implies (using (a)), $\alpha = \left(\frac{\sigma_0^2}{\sigma_0^2 - \sigma_1^2}\right) < 0$.

   (c) The zero-beta portfolio is on the minimum variance set but below the minimum variance point.

   (d) $\rho = \frac{\sigma_{iM}}{\sigma_i\sigma_M} \quad \Rightarrow \quad \overline{r}_i = \overline{r}_z + \frac{\rho\sigma_i}{\sigma_M}(\overline{r}_M - \overline{r}_z) = .09 + .5\frac{.5}{.15}(.15 - .09)$
   That is, $\overline{r}_i = 10\%$.

8. (Wizards)

   (a)

$$\begin{aligned}
\mathrm{E}(r) &= \mathrm{E}\left(\frac{p}{c} - 1\right) = \mathrm{E}\left(\frac{1}{c}\right)\mathrm{E}(P) - 1 \\
&= \left(\frac{.5}{20} + \frac{.5}{16}\right)24 - 1 = \frac{9}{160}24 - 1 \\
&= \frac{7}{20} = 35\%
\end{aligned}$$

   (b)

$$\begin{aligned}
\sigma_M = \mathrm{E}[(r_p - \overline{r}_p)(r_M - \overline{r}_M)] &= \mathrm{E}\left[\left(\frac{p - \overline{p}}{c}\right)(r_M - \overline{r}_M)\right] \\
&= \mathrm{E}(1/c)\mathrm{E}(p - \overline{p})(r_M - \overline{r}_M)] \\
&= \left(\frac{9}{160}\right)20\sigma_M^2 = \frac{9}{8}20\sigma_M^2
\end{aligned}$$

   Hence

$$\beta = \frac{\sigma_{pM}}{\sigma_M^2} = \frac{\frac{9}{8}\sigma_M^2}{\sigma_M^2} = \frac{9}{8}$$

   (c) $\overline{r}_p = r_f + \beta(\overline{r}_M - r_f) = .09 + \frac{9}{8}(.24) = .36$ Thus the rate of return predicted by the CAPM method exceeds the project rate of return by 1%, so the project is <u>not</u> acceptable—but it is close.

9. (Gavin's problem)

   Note: $\sigma_{\alpha M} = \mathrm{cov}(\alpha r_f + (1-\alpha)r_M, r_M) = (1-\alpha)\sigma_M^2$

   $\overline{Q} = p(\alpha r_f + (1-\alpha)\overline{r}_M + 1)$

   $\mathrm{cov}(Q, r_M) = \mathrm{cov}(P(\alpha r_f + (1-\alpha)r_M + 1), r_M) = P(1-\alpha)\sigma_M^2.$

So, by method 1:

$$
\begin{aligned}
P &= \frac{\overline{Q}}{1+r_f+\beta(r_M-r_f)} = \frac{P(\alpha r_f+(1-\alpha)\overline{r}_M+1)}{1+r_f+\frac{\sigma_{\alpha M}}{\sigma_M^2}(r_M-r_f)} \\
&= \frac{P(\alpha r_f+(1-\alpha)\overline{r}_M+1)}{1+r_f+(1-\alpha)(r_M-r_f)} = P
\end{aligned}
$$

By method 2:

$$
\begin{aligned}
P &= \left(\frac{1}{1+r_f}\right)\left(\frac{\overline{Q}-\operatorname{cov}(Q,r_M)(\overline{r}_M-r_f)}{\sigma_M^2}\right) \\
&= \left(\frac{1}{1+r_f}\right)[P(\alpha r_f+(1-\alpha)\overline{r}_M+1)-P(1-\alpha)(\overline{r}_M-r_f)] \\
&= \left(\frac{P}{1+r_f}\right)(1+r_f) = P
\end{aligned}
$$

Chapter 8
Models and Data

1. (A simple portfolio)

   (a) The beta of the portfolio is a weighted combination of the individual betas:

   $$\beta = 0.2 \times 1.1 + 0.5 \times 0.8 + 0.3 \times 1 = .92.$$

   Hence, applying the CAPM to the portfolio we find

   $$\bar{r}_p = .05 + .92(.12 - .05) = 11.44\%.$$

   (b) Using the single-factor model, we have

   $$\begin{aligned} \sigma_e^2 &= \sum_{i=A}^{C} w_i^2 \sigma_{e_i}^2 = 0.2^2 \times 0.007^2 + 0.5^2 \times 0.023^2 + 0.3^2 \times .01^2 \\ &= 0.00033725 \\ \sigma^2 &= b^2 \sigma_M^2 + \sigma_e^2 = 0.42^2 \times 0.18^2 + 0.00033725 = 0.2776 \\ \sigma &= 16.7\%. \end{aligned}$$

2. (APT Factors) By the APT we have $\lambda_0 = r_f = 10\%$ and

   $$\begin{aligned} .15 &= .10 + 2\lambda_1 + \lambda_2 \\ .20 &= .10 + 3\lambda_1 + 4\lambda_2 \end{aligned}$$

   This yields $\lambda_1 = .02$ and $\lambda_2 = .01$.

3. (Principal components) The estimated covariance matrix of the four stocks is

   $$\mathbf{V} = \begin{bmatrix} 90.28 & 50.88 & 79.00 & 40.18 \\ 50.88 & 107.2 & 105.4 & 30.98 \\ 79.00 & 105.4 & 162.2 & 56.54 \\ 40.18 & 30.98 & 56.54 & 68.27 \end{bmatrix}$$

   The largest eigenvalue is 311.16 with corresponding eigenvector

   $$\mathbf{v} = [0.217,\ 0.263,\ 0.360,\ 0.153]$$

which has been normalized so that the components sum to one. Therefore, the first principal component is $0.217r_1 + 0.263r_2 + 0.360r_3 + 0.153r_r$. This can be considered as the weighted average of the returns of the four stocks, and resembles the return of the market portfolio.

The top part of Table 8.2 is reproduced here, showing the time values of the principle component. Note that its behavior is similar to that of the market.

Year	stock 1	stock 2	stock 3	stock 4	market	riskless	Principal Component
1	11.91	29.59	23.27	27.24	23.00	6.20	23.07
2	18.37	15.25	19.47	17.05	17.54	6.70	17.74
3	3.64	3.53	-6.58	10.20	2.70	6.40	.92
4	24.37	17.67	15.08	20.26	19.34	5.70	18.60
5	30.42	12.74	16.24	19.84	19.81	5.90	18.97
6	-1.45	-2.56	-15.05	1.51	-4.39	5.20	-6.21
7	20.11	25.46	17.80	12.24	18.90	4.90	19.48
8	9.28	6.92	18.82	16.12	12.78	5.50	13.16
9	17.63	9.73	3.05	22.93	13.34	6.10	11.08
10	15.71	25.09	16.94	3.49	15.31	5.80	16.76
aver	15.00	14.34	10.90	15.09	13.83	5.84	13.36
var	90.28	107.24	162.19	68.27	72.12		84.93

4. (Variance estimate)

$$\begin{aligned}
E(s^2) &= E\left[\frac{1}{n-1}\sum_{i=1}^{n}(r_i-\hat{\bar{r}})^2\right] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(r_i-\frac{1}{n}\sum_{j=1}^{n}r_j\right)^2\right] \\
&= E\left[\frac{1}{n-1}\sum_{i=1}^{n}\left((r_i-\bar{r})-\frac{1}{n}\sum_{j=1}^{n}(r_j-\bar{r})\right)^2\right] \\
&= \frac{1}{n-1}E\left[\sum_{i=1}^{n}\left((1-\frac{1}{n})(r_i-\bar{r})-\frac{1}{n}\sum_{j\neq i}(r_i-\bar{r})\right)^2\right] \\
&= \frac{\sigma^2 n}{n-1}\left\{(1-\frac{1}{n})^2+\frac{n-1}{n^2}\right\} \\
&= \sigma^2.
\end{aligned}$$

5. (Are more data helpful?)

(a)

$$\sigma(\hat{\bar{r}}) = \sigma(n\hat{\bar{r}}_n) = n\frac{\sigma_n}{\sqrt{n}} = n\frac{\sigma}{\sqrt{n}\sqrt{n}} = \sigma.$$

Hence $\sigma(\hat{\bar{r}})$ is independent of n.

(b) Assuming normality,

$$\sigma(\hat{\sigma}^2) = \sigma(n\hat{\sigma}_n^2) = n\frac{\sqrt{2}\sigma_n^2}{\sqrt{n-1}} = n\frac{\sqrt{2}}{\sqrt{n-1}}\frac{\sigma^2}{n} = \frac{\sqrt{2}\sigma^2}{\sqrt{n-1}}.$$

Part (a) shows that by using smaller periods to get more samples does not improve the estimate of $\bar{r}$. Part (b) shows that using smaller periods to get more samples *does* improve the estimate of σ^2.

6. (A record) Assuming a normal population,

(a)

$$\begin{aligned} \hat{\bar{r}}_m &= \frac{1}{n}\sum_{i=1}^{n} r_i = 1\% \\ \hat{\bar{r}}_{\text{yr}} &= 12\hat{\bar{r}}_m = 12\%. \end{aligned}$$

(b)

$$\begin{aligned} \hat{\sigma}_m^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\bar{r}}_m)^2 = 0.00072 \\ \hat{\sigma}_{\text{yr}} &= \sqrt{12}\hat{\sigma}_m = 9.29\%. \end{aligned}$$

(c)

$$\begin{aligned} \sigma(\hat{\bar{r}}_m) &= \frac{\hat{\sigma}_m}{\sqrt{n}} = .55\% \\ \sigma(\hat{\bar{r}}_{\text{yr}}) &= \sigma(12\hat{\bar{r}}_m) = 12\hat{\sigma}_m = 6.6\% \\ \sigma(\hat{\sigma}_m^2) &= \frac{\sqrt{2}\hat{\sigma}_m^2}{\sqrt{n-1}} = \frac{\sqrt{2}\times 0.00072}{\sqrt{23}} = 0.00021 \\ \sigma(\hat{\sigma}^2\text{yr}) &= \sigma(12\hat{\sigma}_m^2) = 12\sigma(\hat{\sigma}_m^2) = 0.0025. \end{aligned}$$

(d) From the previous exercise we know that the estimate of $\bar{r}$ will not be improved by having weekly, rather than monthly samples. All that matters is the total length of the period that is observed. However, the estimate in σ^2 can be improved. In fact, letting $\sigma^{\text{week}}(\hat{\sigma}_{\text{yr}}^2)$ denote the standard deviation in $\hat{\sigma}_{\text{yr}}^2$ based on weekly data, we expect that $\sigma^{\text{week}}(\hat{\sigma}_{\text{yr}}^2) = \sqrt{\frac{23}{104}}\sigma(\hat{\sigma}_{\text{yr}}^2) = .47\hat{\sigma}_{\text{yr}}^2 = .0012$.

7. (Clever, but no cigar) First divide the year into half-month intervals and index these time points by i. Let r_i be the return over the i-th full month (but some will start midway through the month). We let $\bar{r}$ and σ^2 denote the monthly expected return and variance of that return.

Now let ρ_i be the return over the i-th half-month period. Assume that these returns are uncorrelated. Then $\overline{\rho}_i = \overline{r}_m/2$ and $\sigma^2(\rho_i) = \sigma^2/2$. The return over any monthly period is a sum of two half-month returns; that is, the monthly return r_i is $r_i = \rho_i + \rho_{i+1}$. It is easy to see that $\text{cov}(r_i, r_{i+1}) = \frac{1}{2}\sigma^2$ and $\text{cov}(r_i, r_j) = 0$ for $|i - j| > 1$.

Now for Gavin's scheme we form the estimate

$$\hat{\overline{r}} = \frac{1}{24}\sum_{i=1}^{24} r_i.$$

We need to evaluate

$$\begin{aligned}
\sigma^2(\hat{\overline{r}}) &= \frac{1}{24^2}\left[\sum_{i=1}^{24}(r_i - \overline{r})\right]^2 \\
&= \frac{1}{24^2}\sum_{i,j=1}^{24} \text{cov}(r_i, r_j) \\
&= \frac{1}{24^2}\sum_{i=1}^{24} [\text{cov}(r_{i-1}, r_i) + \text{cov}(r_i, r_i) + \text{cov}(r_i, r_{i+1})].
\end{aligned}$$

Except at the two end periods, each i will give three terms as shown. We will ignore the slight discrepancy at the ends and assume that every i gives the three terms as shown in the summation. The terms are $\frac{1}{2}\sigma^2$, σ^2, and $\frac{1}{2}\sigma^2$, respectively. Hence we have

$$\begin{aligned}
\sigma^2(\hat{\overline{r}}) &= \frac{1}{24^2} \times 24(\tfrac{1}{2} + 1 + \tfrac{1}{2})\sigma^2 \\
&= \frac{1}{12}\sigma^2
\end{aligned}$$

which is identical to the result for twelve nonoverlapping months of data.

8. (General tilting)

   (a) Let $P = P^T = 1$, $Q = \sigma^2$, and $Q^{-1} = 1/\sigma^2$. Then

   $$\hat{\overline{r}} = p.$$

   (b) Let

   $$\mathbf{P} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad \mathbf{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}.$$

   Then $\mathbf{P}^T\mathbf{Q}^{-1}\mathbf{P} = \sigma_1^2 + \sigma_2^2$, and $\mathbf{P}^T\mathbf{Q}^{-1}\mathbf{p} = \frac{p_1}{\sigma_1^2} + \frac{p_2}{\sigma_2^2}$. Hence

   $$\hat{\overline{r}} = \left(\frac{p_1}{\sigma_2^2} + \frac{p_2}{\sigma_2^2}\right)\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}.$$

Chapter 9
General Theory

1. (Certainty equivalent) The possible incomes and their utility levels (found by taking the 1/4-th power) are

Income	80	90	100	110	120	130	140
Utility	2.99	3.08	3.16	3.23	3.31	3.38	3.44

The total utility is the average of these, which is 3.23. We must find C such that $C^{1/4} = 3.23$. Using an iterative process we find $C = \$108,61$.

2. (Wealth independence) The investment will be made if:

$$\mathrm{E}\left[U(W - w + x)\right] > \mathrm{E}\left[U(W)\right],$$

for our case we have:

$$\mathrm{E}\left[-e^{-aW}e^{-a(x-w)}\right] > \mathrm{E}\left[-e^{-aW}\right]$$

or equivalently,

$$-e^{-a(W-w)}\mathrm{E}\left[e^{-ax}\right] > -e^{-aW}.$$

Dividing the expression by $-e^{-aW}$, the investment will be made if:

$$e^{aw}\mathrm{E}\left[e^{-ax}\right] < 1$$

which is independent of W.

3. (Risk aversion invariance) The risk aversion coefficient for a utility function, $U(x)$ is:

$$a(x) = -\frac{U''(x)}{U'(x)}$$

For $V(x) = c + bU(x)$

$$\begin{aligned} V'(x) &= bU'(x) \\ V''(x) &= bU''(x) \end{aligned}$$

so the risk aversion coefficient for $V(x)$ is given by:

$$-\frac{V''(x)}{V'(x)} = -\frac{U''(x)}{U'(x)} = a(x).$$

4. (Relative risk aversion) Given $U(x)$, the relative risk aversion coefficient μ is defined as:

$$\mu(x) = -\frac{xU''(x)}{U'(x)}$$

   (a) $U(x) = \log(x)$

$$\begin{aligned} U'(x) &= \tfrac{1}{x} \\ U''(x) &= -\tfrac{1}{x^2} \\ \Rightarrow \mu(x) &= 1 \end{aligned}$$

   (b) $U(x) = \gamma x^{\gamma-1}$

$$\begin{aligned} U'(x) &= \gamma^2 x^{\gamma-1} \\ U''(x) &= \gamma^2 (\gamma - 1) x^{\gamma-2} \\ \Rightarrow \mu(x) &= 1 - \gamma \end{aligned}$$

   Relative risk aversion coefficients, μ, are constant for both utility functions.

5. (Equivalency) If results are consistent, we have that $V(x) = aU(x) + b$, and since $V(A') = A'$ and $V(B') = B'$ we must have

$$\begin{aligned} A' &= aU(A') + b \\ B' &= aU(B') + b \end{aligned}$$

   So solving both equations simultaneously we find parameters a and b:

$$\begin{aligned} a &= \frac{A' - B'}{U(A') - U(B')} \\ b &= \frac{B'U(A') - A'U(B')}{U(A') - U(B')}. \end{aligned}$$

6. (HARA) The hyperbolic absolute risk aversion function is given by:

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}, \quad b > 0.$$

   (a) Linear: We can write the HARA as:

$$U(x) = \frac{1}{\gamma}\left(ax(1-\gamma)^{\frac{1-\gamma}{\gamma}} + b(1-\gamma)^{\frac{1}{\gamma}}\right)^{\gamma}$$

   Choosing $\gamma = 1$ and $a = 1$ and using L'Hopital's rule we can write:

$$\begin{aligned} U(x) &= \lim_{\gamma \to 1} a\,x\,e^{\frac{1}{\gamma}\frac{\ln(1-\gamma)}{1/(1-\gamma)}} \\ &= x \end{aligned}$$

(b) Quadratic: By choosing $\gamma = 2$, HARA takes the form

$$U(x) = -\frac{1}{2}a^2x^2 + abx - \frac{1}{2}b^2$$

Choosing $a > 0$ and $b = 1/a > 0$ we have an equivalent form of the required quadratic form. Furthermore, by adding $b^2/2$, which is a legitimate transformation, we get the precise desired form:

$$U(x) = x - \frac{1}{2}cx^2,$$

where $c = a^2$

(c) Exponential: By choosing $b = 1$ and $\gamma = -\infty$ and using L'Hopital's rule we can write

$$\begin{aligned} U(x) &= \lim_{\gamma \to -\infty} \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+1\right)^{\gamma} \\ &= -1 \lim_{\gamma \to -\infty} e^{\gamma \ln\left(\frac{ax}{1-\gamma}+1\right)} \\ &= -\lim_{\gamma \to -\infty} e^{\frac{\ln\left(\frac{ax}{1-\gamma}+1\right)}{1/\gamma}} \\ &= -e^{-ax} \end{aligned}$$

(d) Power: If we let $b = 0$, HARA takes the form:

$$U(x) = \frac{(1-\gamma)^{1-\gamma}}{\gamma}a^{\gamma}x^{\gamma},$$

which for $\gamma < 1$ is of the required form

$$U(x) = cx^{\gamma}$$

(e) Logarithmic: Let $b = 0$ and $a = 1$, then HARA is given by:

$$U(x) = (1-\gamma)^{1-\gamma}\frac{x^{\gamma}}{\gamma}.$$

We can subtract the constant $c = (1-\gamma)^{(1-\gamma)}/\gamma$, and obtain an equivalent utility function

$$U(x) = (1-\gamma)^{1-\gamma}\left(\frac{x^{\gamma}-1}{\gamma}\right).$$

Now letting $\gamma = 0$ and using L'Hopital's rule we get

$$U(x) = \lim_{\gamma \to 0} \frac{x^{\gamma}-1}{\gamma}$$

$$
\begin{aligned}
&= \lim_{\gamma \to 0} \frac{e^{\gamma \ln x} - 1}{\gamma} \\
&= \lim_{\gamma \to 0} \frac{\ln x \, e^{\gamma \ln x}}{1} \\
&= \ln x.
\end{aligned}
$$

The Arrow-Pratt $a(x)$ risk aversion coefficient for HARA is

$$
\begin{aligned}
a(x) &= -\frac{U''(x)}{U'(x)} = -\frac{-a^2\left(\frac{ax}{1-\gamma} + b\right)^{\gamma-2}}{a\left(\frac{ax}{1-\gamma} + b\right)^{\gamma-1}} \\
&= a\left(\frac{ax}{1-\gamma} + b\right)^{-1} = \frac{a}{\frac{ax}{1-\gamma} + b} \\
&= \frac{1}{\frac{1}{1-\gamma}x + \frac{b}{a}},
\end{aligned}
$$

which is of the form $1/(cx + d)$, as required.

7. (The venture capitalist) The expected value e is given by

$$e = p + (1-p)9 = 9 - 8p$$

so,

$$p = \frac{9-e}{8}$$

On the other hand since $U(x) = \sqrt{x}$:

$$
\begin{aligned}
U(C) = \sqrt{C} &= pU(1) + (1-p)U(9) \\
&= p + 3(1-p) \\
&= 3 - 2p \\
&= 3 - 2\left(\frac{9-e}{8}\right)
\end{aligned}
$$

So

$$C = \left(3 - 2\left(\frac{9-e}{8}\right)\right)^2 = \frac{1}{16}(3+e)^2$$

Solving for e we get:

$$e = 4\sqrt{C} - 3$$

which agrees with the values of the table on example 9.3

8. (Certainty approximation) By definition, the certainty equivalent c is such that $\mathrm{E}[U(x)] = U(c)$. Substituting in the given approximations, we have

$$U(\bar{x}) + \frac{1}{2}U''(\bar{x})\mathrm{var}(x) \approx U(\bar{x}) + U'(\bar{x})(c - \bar{x}).$$

Solving for c yields

$$c \approx \bar{x} + \frac{U''(x)}{U'(x)}\mathrm{var}(x)$$

as required.

9. (Quadratic mean-variance) In general

$$\begin{aligned} \mathrm{E}[U(y)] &= \mathrm{E}[ay - \tfrac{1}{2}by^2] \\ &= a\mathrm{E}[y] - \tfrac{1}{2}b\mathrm{E}[y^2] \\ &= a\mathrm{E}[y] - \tfrac{1}{2}b(\mathrm{var}[y] + \mathrm{E}[y]^2) \end{aligned}$$

If the random payoff of the portfolio of the investor with unit wealth is R, it would maximize

$$\mathrm{E}[U(R)] = a\mathrm{E}[R] - \tfrac{1}{2}b(\mathrm{var}[R] + \mathrm{E}[R]^2)$$

Now, if the investor with wealth W purchases the same portfolio, its payoff must be WR and R should maximize:

$$\begin{aligned} \mathrm{E}[U(y_2)] &= a\mathrm{E}[RW] - \tfrac{1}{2}b_2(\mathrm{var}[RW] + \mathrm{E}[RW]^2) \\ &= aW\mathrm{E}[R] - \tfrac{1}{2}b_2(W^2\,\mathrm{var}[R] + W^2\mathrm{E}[R]^2) \\ &= W\left[a\mathrm{E}[R] - \tfrac{1}{2}b_2W(\mathrm{var}[R] + \mathrm{E}[R]^2)\right]. \end{aligned}$$

If the second investor has $b' = \frac{b}{W}$, the same R will solve this as the R using unit wealth.

10. (Portfolio optimization) The first-order conditions for portfolio optimization are:

$$\mathrm{E}[U'(x^*)d_i] = \lambda P_i \qquad \text{for all } i.$$

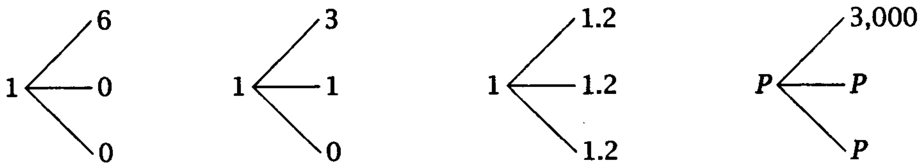
Dividing by P_i we get

$$\begin{aligned} \mathrm{E}[U'(x^*)(1+r_i)] &= \lambda & &\text{for all } i \text{ or,} \\ \mathrm{E}[U'(x^*)] + \mathrm{E}[U'(x^*)r_i] &= \lambda & &\text{for all } i. \end{aligned}$$

Taking the difference between asset i and the risk-free asset yields

$$\begin{aligned} \mathrm{E}[U'(x^*)] + \mathrm{E}[U'(x^*)r_i] - \mathrm{E}[U'(x^*)] + \mathrm{E}[U'(x^*)r_f] &= \lambda - \lambda \\ \mathrm{E}[U'(x^*)r_i] - \mathrm{E}[U'(x^*)r_f] &= 0 \\ \mathrm{E}[U'(x^*)(r_i - r_f)] &= 0. \end{aligned}$$

11. (Money-back guarantee) We have the three investment options from example 9.6 with price 1 and the new money-back guarantee alternative:



Residual Rights Film Venture Risk Free Money-Back Guarantee

Since we had three states and three 'securities', the new alternative can be replicated by combining the existing ones in certain amounts A, B, and C. It follows that the price of the Money-back guarantee security is $A+B+C$, so that there are no arbitrage opportunities:

$$\begin{aligned} 6A + 3B + 1.2C &= 3,000 \\ B + 1.2C &= P \\ 1.2C &= P \\ A + B + C &= P \end{aligned}$$

Solving the system gives us the price of the Money-back guarantee deal: $P =$ \$1,500

Alternatively we could have used the state prices from example 9.9 where we had $\psi_1 = 1/6$, $\psi_2 = 1/2$ and $\psi_3 = 1/6$ so that the price of the Money back guarantee is given by the single equation:

$$P = \frac{1}{6}3,000 + \frac{1}{2}P + \frac{1}{6}P,$$

with solution $P = \$1,500$.

12. (General positive state prices result) Let the $N \times S$ matrix $\mathbf{D}$ be the payoff matrix of N securities in S states. Security prices are given by some $\mathbf{q} \in E^N$ and a portfolio is characterized by $\mathbf{x} \in E^N$. The market value or cost c of a portfolio $\mathbf{x}$ is $\mathbf{q} \cdot \mathbf{x}$ and its payoff is given by $\mathbf{D}^T\mathbf{x} \in E^S$. Now consider the $(S+1) \times N$ matrix $\mathbf{A}$ which results from appending the negative of the price vector $\mathbf{q}$ as a row to $\mathbf{D}^T$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{D^T} \\ \hline -\mathbf{q} \end{bmatrix}$$

The first N elements of the vector $\mathbf{p}$ given by $\mathbf{Ax} = \mathbf{p}$ correspond to the payoff vector while the $(n+1)$ – th element is the negative of the cost of the portfolio $\mathbf{x}$.

If there is no arbitrage, then we must have no $\mathbf{p} \geq \mathbf{0}$ except $\mathbf{p} = \mathbf{0}$, since otherwise we could have non-positive cost ($c \leq 0$) with a positive probability of yielding positive payoff or negative cost ($c < 0$) with zero payoff which is instant money.

Using the stated matrix theory result it follows that there exists a vector $\mathbf{y} > \mathbf{0}$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{0}$. Note that this is the same as:

$$\begin{array}{ccccccccccccc}
D_{11}y_1 & + & D_{12}y_2 & + & D_{13}y_3 & + & \cdots & + & D_{1S}y_S & - & q_1y_{S+1} & = & 0 \\
D_{21}y_1 & + & D_{22}y_2 & + & D_{23}y_3 & + & \cdots & + & D_{2S}y_S & - & q_2y_{S+1} & = & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
D_{N1}y_1 & + & D_{N2}y_2 & + & D_{N3}y_3 & + & \cdots & + & D_{NS}y_s & - & q_Ny_{S+1} & = & 0
\end{array}$$

Note that by dividing each element of $\mathbf{y}$ by y_{S+1} we get positive state prices $\psi_i = y_i/y_{s+1}$ such that:

$$\begin{array}{ccccccccccccc}
D_{11}\psi_1 & + & D_{12}\psi_2 & + & D_{13}\psi_3 & + & \cdots & + & D_{1s}\psi_S & - & q_1 & = & 0 \\
D_{21}\psi_1 & + & D_{22}\psi_2 & + & D_{23}\psi_3 & + & \cdots & + & D_{2s}\psi_S & - & q_2 & = & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
D_{N1}\psi_1 & + & D_{N2}\psi_2 & + & D_{N3}\psi_3 & + & \cdots & + & D_{Ns}\psi_S & - & q_N & = & 0
\end{array}$$

Therefore, if there is no arbitrage, there are positive state prices.

13. (Quadratic pricing) From the earlier exercise we have $\mathrm{E}[U'(x^*)(r_i - r_f)] = 0$. If $U(x) = x - c/2x^2$ then, $U'(x) = 1 - cx$, so in this case we have

$$\mathrm{E}[(1 - cWR_M)(r_i - r_f)] = 0$$

or equivalently

$$\mathrm{E}[(1 - cWR_M)(R_i - R)] = 0.$$

Written out we have

$$\begin{aligned}
\overline{R}_i - R &= cW[\mathrm{E}(R_M R_i) - \overline{R}_M R] \\
&= cW[\mathrm{cov}(R_M, R_i) + \overline{R}_M(\overline{R}_i - R)].
\end{aligned}$$

This implies, $\overline{R}_i - R = \gamma\mathrm{cov}(R_M, R_i)$ for

$$\gamma = 1 - cW\overline{R}_M.$$

Appling this to R_M yields

$$\overline{R}_M - R = \gamma\mathrm{var}(R_M)$$

which shows that

$$\gamma = \frac{\overline{R}_M - R}{\mathrm{var}(R_M)}$$

and hence finally

$$R_i - R = \beta_i(\overline{R}_M - R).$$

14. (At the track) Gavin Jones will choose the fraction α of his money m to bet on the horse so as to maximize his expected utility:

$$\max \quad \mathrm{E}[U] = \frac{1}{4}\sqrt{m+4\alpha m} + \frac{3}{4}\sqrt{(1-\alpha)m},$$

(a) The first order necessary condition is:

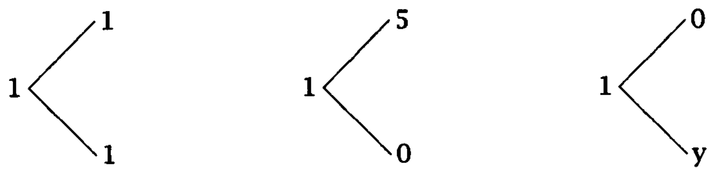
$$\frac{1}{2}\frac{m}{\sqrt{m+4\alpha m}} - \frac{3}{4}\frac{m}{\sqrt{(1-\alpha)m}} = 0$$

which yields:

$$\alpha = \frac{7}{52} = .1346$$

Gavin's maximizing choice is to bet 13.46% of his money and keep the rest in his pocket.

(b) We can summarize Gavin's world by the following three alternatives:



Keep money in pocket | Bet for No Arbitrage | Bet against No Arbitrage

Linear pricing holds if there aren't any arbitrage opportunities. Thus, we could replicate a four dollar bet against No Arbitrage by 'shorting' the bet in favor and keeping five dollars in the pocket. Dividing by four we get the implied payoff for a one dollar bet against No Arbitrage:

$$y = \frac{5}{4} = 1.25$$

15. (General risk neutral pricing) From log-optimal pricing we have

$$P = \mathrm{E}\left(\frac{d}{R^*}\right)$$

Now, using the expectation operation $\hat{\mathrm{E}}$ defined by

$$\hat{\mathrm{E}}(x) = \mathrm{E}\left(\frac{Rx}{R^*}\right)$$

where R is the risk free rate, we can multiply the log-optimal pricing formula by $\frac{R}{R} = 1$ without affecting it. This yields

$$P = \frac{R \cdot \mathrm{E}\left(\frac{d}{R^*}\right)}{R} = \frac{\mathrm{E}\left(\frac{R \cdot d}{R^*}\right)}{R} = \frac{\hat{\mathrm{E}}(d)}{R}$$

which is risk neutral pricing.

Chapter 10
Forwards, Futures, and Swaps

1. (Gold futures) Applying equation 10.2 we have

$$\begin{aligned} F &= \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)} \\ &= \frac{412}{d(0,9m)} + \frac{2/4}{d(0,9m)} + \frac{2/4}{d(3m,9m)} + \frac{2/4}{d(6m,9m)} \end{aligned}$$

where (with m = "months")

$$\begin{aligned} d(0,9m) &= \frac{1}{(1+\frac{.09}{4})^3} = .9354 \\ d(3m,9m) &= \frac{1}{(1+\frac{.09}{4})^2} = .9565 \\ d(0,9m) &= \frac{1}{(1+\frac{.09}{4})} = .9780. \end{aligned}$$

Combining these, we find

$$F = 440.45 + .5345 + .5227 + .5112 = \$442.02.$$

2. (Proportional carrying charges) Suppose at time zero you take out a loan for $S(0)$, purchase 1 unit of the commodity, and short $(1-q)^M$ units of the commodity at a forward price of F per unit. The total intial cash outlay of these transactions is zero. Each period you pay the carrying cost of the commodity by selling a fraction q of your commodity holdings. Hence the amount of commodity held at the end of M periods is $(1-q)^M$. At the final time you deliver your commodity holdings to make good on your short, receiving $F(1-q)^M$. You also repay your loan by paying $S(0)/d(0,M)$. The total profit from these transactions (which clear all accounts) is $F(1-q)^M - S(0)/d(0,M)$. To avoid arbitrage, this profit must be zero. Hence $F = S(0)(1-q)^{-M}/d(0,M)$.

3. (Silver contract) In general the spot and forward prices are related by:

$$S = Fd(0,M) - \sum_{k=0}^{M-1} c(k)d(0,k)$$

If the yield curve is flat, and with forward contracts being settled at the end of the month, this formula can be written for any maturity date M as:

$$S = \frac{F}{(1+\frac{r}{12})^M} - \sum_{k=0}^{M-1} \frac{c(k)}{(1+\frac{r}{12})^k}$$

So taking any two maturities, we can solve for the spot price and the interest rate. In particular, by taking the forward prices for April and July we have the equations:

$$\begin{aligned} S &= \frac{406.50}{(1+\frac{r}{12})} - \frac{20}{12} \\ S &= \frac{409.3}{(1+\frac{r}{12})^4} - \frac{20}{12} - \frac{20}{12}\frac{1}{(1+\frac{r}{12})} - \frac{20}{12}\frac{1}{(1+\frac{r}{12})^2} - \frac{20}{12}\frac{1}{(1+\frac{r}{12})^3} \end{aligned}$$

Solving the equations simultaneously we get: $S = 403.15$ and $r = 5\%$. Other contract dates give the same results.

4. (Continuous-time carrying charges) One way to solve this is to discretize time with intervals Δt , use the result of Exercise 2, and then take the limit as $\Delta t \to 0$. Instead, it is possible to solve directly with a similar method. At time 0 you borrow $S(0)$, buy one unit, and short e^{-qT} units—for a net zero cash outlay. You pay the carrying costs at each instant by selling a fraction q of your holdings. Hence the amount you hold satisfies $\frac{dx}{dt} = -qx$. Therefore at time T you have e^{-qT}. The profit from clearing all accounts at time T is $Fe^{-qT} - e^{rT}S(0)$. Therefore, ruling out arbitrage profits we find $F = S(0)e^{(r+q)T}$.

5. (Carrying cost proof) Consider the cashflows of the suggested transactions:

	Time 0 cashflow	Time k cashflow	Time M cashflow
Long 1 unit forward	0	0	$-F$
Short 1 unit spot	$+S$	0	0
Storage payments from asset lender	$c(0)$	$c(k)$	0
Investment of cashflows until time M	$-S-c(0)$	$-c(k)$	$\frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}$
Total	0	0	$\frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)} - F$

The transactions yield no net cashflow until the final period where we recieve $S/d(0,M)+\sum_{k=0}^{M-1} c(k)/d(k,M)$, from our investments and pay F for the asset, as required by our forward contract, which in turn we will deliver to Mr. X who lent us the asset in time 0. Thus if $F < S/d(0,M) + \sum_{k=0}^{M-1} c(k)/d(k,M)$, we have an arbitrage profit so the inequality must be false under the no arbitrage condition, which completes the proof of the *forward price formula with carrying costs* in section 10.3.

6. (Foreign currency alternative) In Example 10.12 the company hedges by shorting forward contracts for 500,000 Deutsche marks, assuring dollar receipts in 90 days of $\$500,000F_T$, where F_T is the forward dollar price for Deutsche marks in 90 days.

   Based on the no arbitrage condition we have that:

$$F_T = \frac{1 + r_{US}}{1 + r_G} S_0,$$

   where r_{US} is the 90 day U.S. risk-free rate, r_G is the 90 day interst rate in Germany, and S_0 is the spot price for Deutsche marks at time 0.

   To verify this consider taking a long position on a 90 day Deutsche mark forward, which has zero cost at time 0 and a value of $S_T - F_T$ at the maturity time T.

   On the other hand consider borrowing $F_T/(1+r_{US})$ dollars at time zero for 90 days and at the same time purchasing $1/(1+r_G)$ worth of risk free german bonds with a dollar cost of $S_0/(1+r_G)$. The total cost of this strategy is $S_0/(1+r_G)-F_T/(1+r_{US})$ with a total net payoff of $S_T - F_T$ at time T which is equal to the value at time T of the 90 day forward.

   It follows that if there is no arbitrage, the cost of both alternatives must be equal, therefore we must have $S_0/(1 + r_G) - F_T/(1 + r_{US}) = 0$, which gives us the the Deutsche mark forward price: $F_T = S_0(1 + r_{US})/(1 + r_G)$

   So the dollar value of the receipts at time T for the firm by hedging with forward contracts is:

$$500,000\frac{1 + r_{US}}{1 + r_G} S_0 \text{ dollars}$$

   If instead it borrows $500,000/(1 + r_{DM})$ Deutsche marks (to be repaid with the receivables at time T) and sells them into dollars, the receipts at time 0 are $500,000\,S_0/(1 + r_G)$ dollars, which invested for 90 days in U.S. T-bills will pay

$$500,000\frac{1 + r_{US}}{1 + r_{DM}} S_0 \text{ dollars.}$$

   Hence, the two procedures are equivalent.

7. (A bond forward) We solve first for F_t the current forward price of the bond:

$$\begin{aligned} F_t &= \frac{S}{d(0,2)} + \sum_{k=1}^{2} \frac{d(0,k)c(k)}{d(0,2)} \\ &= 920(1.04)^2 - \frac{80(1.04)^2}{1.035} - \frac{80(1.04)^2}{(1.04)^2} = \$831.47. \end{aligned}$$

Now we solve for the value of the forward contract:

$$f_t = (F_t - F_0)d(0,2) = \frac{831.47 - 940}{(1.04)^2} = -\$100.34.$$

8. (Simple formula) X/C units of the bond will pay X in coupons plus a final principal payment of $100X/C$. Hence a stream $(X, X, X \ldots, X)$ is worth $B(M,C)X/C - d(0,M)100X/C$. The result follows immediately.

9. (Equity swap)

   (a) The market price at time $i-1$ for the cash flow $S_i + d_i$ is S_{i-1}. Hence, $V_{i-1}(S_i + D_i) = S_{i-1}$. We have

$$\begin{aligned} V_{i-1}(r_i) &= V_{i-1}([S_i + d_i - S_{i-1}]/S_{i-1}) \\ &= 1 - V_{i-1}[1] \\ &= 1 - d(i-1, i). \end{aligned}$$

   (b) We just discount $V_i(r_i)$ back to time 0. Hence $V_0(r_i) = d(0,i-1)[1 - d(i-1,i)] = d(0,i-1) - d(0,i)$ because $d(0,i-1)d(i-1,i) = d(0,i)$.

   (c)

$$\begin{aligned} \sum_{i=1}^{M} V_0(r_i) &= [d(0,0) - d(0,1)] + [d(0,1) - d(0,2] + \cdots + [d(0,M-1) - d(0,M)] \\ &= 1 - d(0,M). \end{aligned}$$

   (d) Value $= \left\{\sum_{i=1}^{M} d(0,i)r - [1 - d(0,M)]\right\} N$. The first term can be reduced using the formula of the previous exercise.

10. (Forward vanilla) We have

$$d(0,i+1) = \frac{1}{(1+r_0)(1+r_1)\cdots(1+r_i)}.$$

Therefore

$$\begin{aligned} d(0,i+1)r_i &= \frac{r_i}{(1+r_0)(1+r_1)\cdots(1+r_i)} \\ &= \frac{1+r_i}{(1+r_0)(1+r_1)\cdots(1+r_i)} - d(0,i+1) \\ &= d(0,i) - d(0,i+1) \end{aligned}$$

Then we find

$$\sum_{i=0}^{M-1}[d(0,i)-d(0,i+1)] = 1-d(0,M),$$

which agrees with the text.

11. (Specific vanilla)

(a) The floating side of the swap is worth

$$\begin{aligned} V_{\text{Float}} &= [1-d(0,6)]\times \$10 \text{ million} \\ &= [1-\frac{1}{1.088^6}]\times \$10 \text{ million} \\ &= [1-.6029]\times \$10 \text{ million} = \$3.971 \text{ million} \end{aligned}$$

(b) The fixed side of the swap is worth

$$\begin{aligned} V_{\text{Fixed}} &= \sum_{1=0}^{M} d(0,i)r\times \$10 \text{ million} \\ &= 4.606r\times \$10 \text{ million} \end{aligned}$$

where the 4.606 was obtained by summing the discount rates implied by the term structure.

Setting $V_{\text{Fixed}} = V_{\text{Float}}$ we find $r = .0864 = 8.64\%$.

12. (Derivation) We obtain the mean-variance hedge formula by solving:

$$\max_h \mathrm{E}[x+h(F_T-F_0)] - r\,\mathrm{var}[x+hF_T]$$

which can be written as:

$$\max_h \mathrm{E}[x]+h(\overline{F}_T-F_0)-r(\mathrm{var}[x]+2h\,\mathrm{cov}[x,F_T]+h^2\,\mathrm{var}[F_T])$$

Taking the derivative with respect to h we get the first order condition:

$$\overline{F}_T-F_0-2r\,\mathrm{cov}[x,F_T]-2hr\,\mathrm{var}[F_T]=0$$

Solving for h we find the mean-variance hedge formula:

$$h=\frac{\overline{F}_T-F_0}{2r\,\mathrm{var}[F_T]}-\frac{\mathrm{cov}[x,F_T]}{\mathrm{var}[F_T]}$$

13. (Grapefruit hedge) The minimum-variance hedge is

$$\begin{aligned} h &= -\beta W = -\rho\frac{\sigma_G}{\sigma_O}\frac{S_G}{S_O}150{,}000 \\ &= -(.7)\left(\frac{.2}{.2}\right)\left(\frac{\$1.50/\text{lb orange}}{\$1.20/\text{lb grapefruit}}\right)150{,}000 \\ &= -131{,}250 \quad \text{lbs orange juice.} \end{aligned}$$

Hence the farmer should short 131250 lbs of orange juice. To check how effective this hedge is, we note that

$$\begin{aligned}\text{stdev}_{\text{new}} &= \sqrt{1-\rho^2}\sigma_{\text{old}} \\ &= \sqrt{1-.7^2}\sigma_{\text{old}} = .714\sigma_{\text{old}}.\end{aligned}$$

14. (Opposite hedge variance) We have the the future cashflow $y = S_T W + (F_T - F_0)h$.

    (a) The equal and opposite hedge h is given by an opposite equivalent dollar value of the hedging instrument. Therefore $h = -kW$ where k is the price ratio between asset and the hedging instrument. By taking this hedge position, the cashflow received in the future date can be written as

    $$\begin{aligned}y &= W\,S_T + (F_0 - F_T)\,W\,k \\ &= W(S_T + k(F_0 - F_T)),\end{aligned}$$

    where its variance σ_y^2 is given by

    $$\begin{aligned}\sigma_y^2 &= W^2\text{var}[S_T - k\,F_T + k\,F_0] \\ &= W^2(\sigma_S^2 - 2k\sigma_{ST} + k^2\sigma_F^2) \\ &= W^2\sigma_S^2\left(1 - 2k\frac{\sigma_{ST}}{\sigma_S^2} + k^2\frac{\sigma_F^2}{\sigma_S^2}\right).\end{aligned}$$

    Hence

    $$\sigma_y = W\sigma_S \times \left(1 - 2k\frac{\sigma_{ST}}{\sigma_S^2} + k^2\frac{\sigma_F^2}{\sigma_S^2}\right)^{\frac{1}{2}}.$$

    (b) In example 10.12 we have $k = K/M = .262$, thus the equal and opposite hedge for the receivable $W = 1,000,000$ Danish Krone is $h = -262,000$ Deutsche Marks and the variance of the future cashflow y is given by

    $$\begin{aligned}\sigma_y^2 &= W^2\sigma_K^2\left(1 - 2k\frac{\sigma_{KM}}{\sigma_K^2} + k^2\frac{\sigma_M^2}{\sigma_K^2}\right) \\ &= \text{var}[x]\left(1 - 2\frac{K}{M}\rho\frac{\sigma_M}{\sigma_K} + \frac{K^2}{M^2}\frac{\sigma_M^2}{\sigma_K^2}\right) \\ &= \text{var}[x]\left(1 - 2\frac{K}{M}.8 \times \frac{.03M}{.025K} + \frac{K^2}{M^2}\frac{(.03M)^2}{(.025K)^2}\right) \\ &= \text{var}[x]\left(1 - 2 \times .8 \times \frac{.03}{.025} + \frac{.03^2}{.025^2}\right) \\ &= .52\text{var}[x]\end{aligned}$$

where as before $x = 1,000,000 \times K$. Hence,

$$\sigma_y = .7211\,\mathrm{stdev}[x].$$

As expected the standard deviation with the equal and opposite hedge is greater than with the minimum variance hedge were we had $\sigma_y = .6\,\mathrm{stdev}[x]$

15. (Immunization as hedging) First we calculate the present value of the portfolio. (We shall use the continuous-time formula for discount factors in this exercise; however, the discrete-time version would give similar results.)

$$\begin{aligned}
\mathrm{PV}_{\mathrm{portfolio}} &= \sum_i c_i e^{-r(t_i)t_i} \\
&= -(1)d^{-(1.05)1} - (2)e^{-(1.053)2} - (1)e^{-(1.056)3} + 4.253e^{-(1.053)2} \\
&= 0.
\end{aligned}$$

Hence the portfolio is balanced in the sense of having assets equal to liabilities. The current spot price of the bond considered for the futures contract is

$$S = e^{-(.061)6}.$$

The corresponding forward price is then

$$F = Se^{(.05)1} = e^{-(.061)6}e^{(.05)1}.$$

A futures contract will not change the PV of the portfolio. One way to express the present value is to assume that we actually take delivery of the contract. This means that there will be cash flows at the end of one year (to buy at the contract price) and at the end of six years (when the bond pays its principal). Hence a contract to purchase $\$x$ par value zero-coupon bonds has present value

$$\mathrm{PV}_{\mathrm{Futures}} = xe^{-(.061)6} - xe^{-(.061)6}e^{(.05)1}e^{-(.05)1} = 0.$$

We now assume that the term structure is of the form $f(t) + \alpha$, where α represents a parallel shift. In this case

$$\mathrm{PV}_{\mathrm{portfolio}} = \sum_i c_i e^{-(f(t_i)+\alpha)t_i}.$$

We will set the derivative equal to zero. We have

$$\begin{aligned}
\frac{\mathrm{d\,PV}}{\mathrm{d}\alpha}\Big|_{\alpha=0} &= (1)(1)e^{-(.05)1} + (2)(2)e^{-(.053)2} + (3)(1)e^{-(.056)e} \\
&\quad -(2)(4.253)e^{-(.053)2} - 6xe{-}(.061)6 + (1)(x)e^{-(.061)6}e^{.05}e^{-.05} \\
&= -.566 - 3.468x.
\end{aligned}$$

Solving, we find $x = -.1632$. Hence the fund should short \$163,200 worth of the Treasury futures.

16. (Symmetric probability) Given the future wealth $W = a + hx + cx^2$, the investor's problem can be written as

$$\max_h \mathrm{E}[U(a + hx + cx^2)] \quad \text{where } U \text{ is strictly concave.}$$

(a) The first order necessary condition for the problem is

$$\mathrm{E}[U'(a + hx + cx^2)x] = 0$$

Let $f(x)$ be the probability density function of x, such that $f(x) = f(-x)$. Then the first order condition takes the form

$$\int_{-\infty}^{\infty} U'(a + hx + cx^2)x f(x)\,\mathrm{d}x = 0$$

or equivalently,

$$\begin{aligned}
\int_{-\infty}^{0} U'(a + hx + cx^2)x f(x)\,\mathrm{d}x + \int_{0}^{\infty} U'(a + hx + cx^2)x f(x)\,\mathrm{d}x &= 0 \\
\int_{0}^{\infty} -U'(a - hx + cx^2)x f(-x)\,\mathrm{d}x + \int_{0}^{\infty} U'(a + hx + cx^2)x f(x)\,\mathrm{d}x &= 0
\end{aligned}$$

However, since $f(x)$ is symmetric, $f(x) = f(-x)$, so we can writte the above condition as

$$-\int_{0}^{\infty} U'(a - hx + cx^2)x f(x)\,\mathrm{d}x + \int_{0}^{\infty} U'(a + hx + cx^2)x f(x)\,\mathrm{d}x = 0.$$

Rearranging by moving the first term to the right hand side of the equation we get

$$\int_{0}^{\infty} U'(a + hx + cx^2)x f(x)\,\mathrm{d}x = \int_{0}^{\infty} U'(a - hx + cx^2)x f(x)\,\mathrm{d}x$$

Clearly, this condition holds if $h = 0$. Furthermore, since U is strictly concave $h = 0$ is the only solution.

(b) The farmer's revenue is given by

$$R = 10C - \frac{C^2}{1000} + \frac{\overline{C} - C}{1000}h$$

where $\overline{C} = 3,000$.

If C has a symmetric distribution, then the distribution of $x = C - \overline{C}$ is such that $f(x) = f(-x)$, and we can write the farmer's revenue as

$$\begin{aligned}
R &= 10(x + 3,000) - \frac{(x + 3000)^2}{1000} - \frac{x}{1000}h \\
&= 21,000 + \left(4 - \frac{h}{1,000}\right)x - \frac{x^2}{1,000}
\end{aligned}$$

From part (a) we know that for an investor with strictly concave utility the optimal solution is:

$$4 - \frac{h}{1,000} = 0.$$

Hence, $h^* = 4,000$.

17. (Double symmetric probability) We have

$$\begin{aligned} \mathrm{var}(R) &= \mathrm{var}(Axy + Bx - hy) \\ &= A^2\mathrm{var}(xy) + B^2\mathrm{var}(x) + h^2\mathrm{var}(y) \\ &\quad +2AB\mathrm{cov}(xy, x) - 2Ah\mathrm{cov}(xy, y) - 2Bh\mathrm{cov}(x, y). \end{aligned}$$

Setting the derivative of the above (with respect to h) to zero we have

$$2h\ \mathrm{var}(y) - 2A\ \mathrm{cov}(xy, y) - 2B\ \mathrm{cov}(x, y) = 0.$$

Hence

$$h^* = \frac{B\ \mathrm{cov}(x, y) + A\ \mathrm{cov}(xy, y)}{\mathrm{var}(y)}.$$

Now we show that $\mathrm{cov}(xy, y) = 0$.

$$\mathrm{cov}(xy, y) = \mathrm{E}(xy^2) - \mathrm{E}(xy)\mathrm{E}(y) = \mathrm{E}(xy^2),$$

because $\mathrm{E}(y) = 0$ by symmetry.

$$\mathrm{E}(xy^2) = \int\int xy^2 f(x, y)\mathrm{d}x\,\mathrm{d}y.$$

Let $s = -x$, $\mathrm{d}s = -\mathrm{d}x$, $t = -y$, $\mathrm{d}t = -\mathrm{d}y$. Then

$$\begin{aligned} \mathrm{E}(xy^2) &= \int\int (-s)t^2 f(-s, -t)\mathrm{d}s\,\mathrm{d}t \\ &= -\int\int st^2 f(s, t)\mathrm{d}s\,\mathrm{d}t \\ &= -\mathrm{E}(xy^2). \end{aligned}$$

Since $\mathrm{E}(xy^2) = -\mathrm{E}(xy^2)$ it follows that $\mathrm{E}(xy^2) = 0$. Thus $\mathrm{cov}(xy, y) = 0$ and $h^* = \frac{R\sigma_{xy}}{\sigma_y^2}$.

18. (A general farm problem) In general the farmer's revenue will be

$$R = PC + h(P - P_0)$$

substituting for P in terms of D we now find

$$R = \left(10 - \frac{D}{100,000}\right) C + \frac{\bar{D} - D}{100,000} h$$

after some algebraic manipulation we can write the equation as

$$R = -\frac{(C-\bar{C})(D-\bar{D})}{100,000} + \left(10C - \frac{\bar{D}}{100,000}\right)\left(C - \bar{C}\right) - \left(\frac{h}{100,000} + \frac{\bar{C}}{100,000}\right)(D-\bar{D})$$

then, substituiting values for $\bar{C}$ and $\bar{D}$ we get

$$R = -\frac{(C-\bar{C})(D-\bar{D})}{100,000} + 7(C-\bar{C}) - \left(\frac{h}{100,000} + \frac{3}{100}\right)(D-\bar{D}),$$

which is of the form $R = Axy + Bx - hy$ and where the distributions of $x = C - \bar{C}$ and $y = D - \bar{D}$ are symmetric about (0,0) as required in Exercise 13. Thus, we can find the minimum variance position by applying the result $h = B\sigma_{xy}/\sigma_y^2$ which gives us

$$\frac{h}{100,000} + \frac{3}{100} = \frac{7\sigma_{CD}}{\sigma_D^2}$$

Solving for h we find

$$h = 100,000\left(-\frac{3}{100} + \frac{7\sigma_{CD}}{\sigma_D^2}\right)$$

as required.

(a) For the case where $D = 100C$ we know that $\rho_{CD} = \sigma_{CD}/\sigma_C\sigma_D = 1$ so then

$$\frac{\sigma_{CD}}{\sigma_D^2} = \frac{\sigma_C}{\sigma_D} = \frac{\sigma_C}{100\sigma_C} = \frac{1}{100}$$

Thus, the minimum variance hedging position is

$$\begin{aligned} h &= 100,000\left(-\frac{3}{100} + \frac{7}{100}\right) \\ &= 4,000 \end{aligned}$$

(b) If the crop size is not correlated with the total demand, namely $\sigma_{CD} = 0$, the minimum variance hedge is given by

$$\begin{aligned} h &= 100,000\left(-\frac{3}{100}\right) \\ &= -3,000 \end{aligned}$$

Variance is minimized by taking an opposite position to the expected crop size.

Chapter 11
Models of Asset Dynamics

1. (Stock lattice) If we consider that Δt is small, then by the formulas (11.1), we set

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}} = 1.105 \\
d &= \frac{1}{u} = 0.905 \\
p &= \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} = 0.65
\end{aligned}
$$

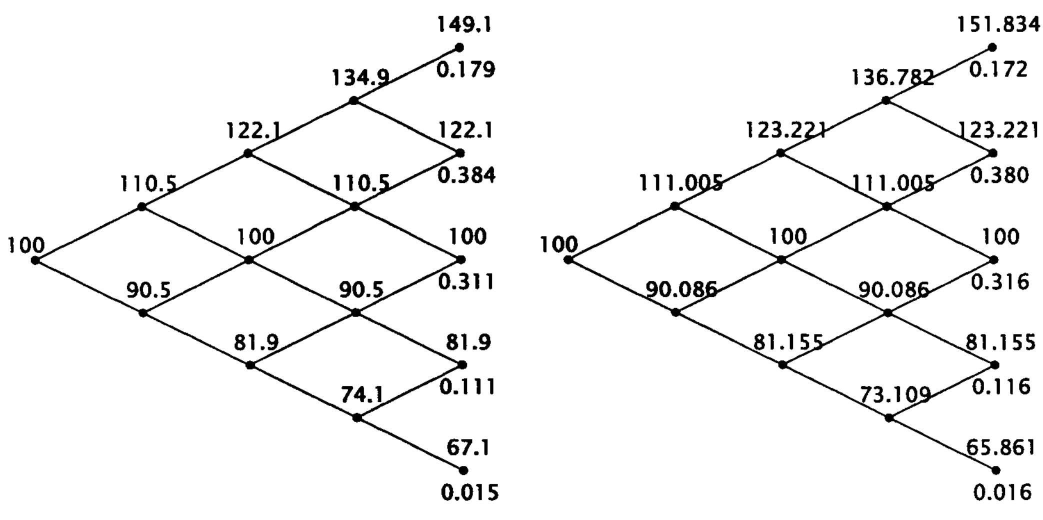


Figure 11.1 The binomial lattice. The numbers above the nodes are the stock prices. The numbers below the final nodes are the probabilities of achieving those nodes.

The binomial lattice for the stock is shown in the left in Figure 11.1. If we do not consider that Δt is small, then by the formulas (11.25), we get $p = 0.64367$, $u = 1.11005$, and $d = 0.90086$. For comparison, this binomial lattice is shown to the right of the first one.

The probabilities of the various final nodes are shown in the above figure under the nodes. For example, the probability of the top node is $p^4 = .64^4 = .179$.

2. (Time scaling) Each movement in k corresponds to a month, and each movement in K corresponds to a year. Let k_K denote the first month of year K. Then

$$W(K) = \sum_{i=0}^{11} w(k_{K-1} + i)$$

So,

$$\mathrm{E}[W(K)] = \mathrm{E}\Big[\sum_{i=0}^{11} w(k_{K-1} + i)\Big] = 12\nu$$

$$\mathrm{Var}[W(K)] = \mathrm{E}\Big[\sum_{i=0}^{11} w(k_{K-1} + i)\Big]^2 = 12\sigma^2$$

3. (Arithmetic and geometric mean)

(a) Proof: For $n = 2$,

$$(v_1 - v_2)^2 \geq 0 \Longrightarrow$$

$$v_1^2 + 2v_1v_2 + v_2^2 \geq 4v_1v_2.$$

That is:

$$\frac{1}{4}(v_1 + v_2)^2 \geq v_1v_2$$

So,

$$v_A \geq v_G$$

(b)

$$r_1 = 50\%,\ r_2 = -20\%$$

Arithmetic mean is

$$\frac{1}{2}(r_1 + r_2) = 15\%.$$

Geometric mean is

$$[(1 + r_1)(1 + r_2)]^{\frac{1}{2}} - 1 = 9.54\%$$

(c) The arithmetic mean rate of return essentially assigns a return based on simple interest, while the geometric mean rate of return is a measure of compound interest. Usually, the geometric mean rate of return is the most appropriate for measurement of investment performance.

4. (Complete the square)

$$
\begin{aligned}
w - \frac{(w-\overline{w})^2}{2\sigma^2} &= -\frac{1}{2\sigma^2}[-2\sigma^2 w + w^2 - 2w\overline{w} + \overline{w}^2 + 2\sigma^2\overline{w} - 2\sigma^2\overline{w} + \sigma^4 - \sigma^4] \\
&= -\frac{1}{2\sigma^2}[w - (\overline{w} + \sigma^2)]^2 + \overline{w} + \frac{\sigma^2}{2}
\end{aligned}
$$

So,

$$
\begin{aligned}
\overline{u} &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^w e^{-(w-\overline{w})^2/2\sigma^2} dw \\
&= e^{\overline{w}+\frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-[w-(\overline{w}+\sigma^2)]^2/2\sigma^2} dw \\
&= e^{\overline{w}+\frac{\sigma^2}{2}}
\end{aligned}
$$

5. (Log variance) Suppose that $u = e^w$, where w is distributed as $N(\overline{w}, \sigma^2)$

$$
\begin{aligned}
\mathrm{E}(u^2) &= \mathrm{E}(e^{2w}) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{2w} e^{-(w-\overline{w})^2/2\sigma^2} dw.
\end{aligned}
$$

Following the method in exercise (4), we have

$$
2w - \frac{(w-\overline{w})^2}{2\sigma^2} = -\frac{1}{2\sigma^2}[w - (\overline{w} + 2\sigma^2)]^2 + 2\sigma^2 + 2\overline{w}.
$$

Then

$$
\mathrm{E}(u^2) = e^{2\sigma^2 + 2\overline{w}}
$$

So,

$$
\mathrm{var}(u) = \mathrm{E}(u^2) - \overline{u}^2 = e^{2\overline{w}+\sigma^2}(e^{\sigma^2} - 1)
$$

6. (Expectations) We have

$$
\nu = \mu - \frac{1}{2}\sigma^2 = 0.2 - \frac{1}{2} \times 0.16 = 0.12
$$

So,

$$
\mathrm{E}[\ln S(1)] = 0.12
$$

$$
\mathrm{Stdev}[\ln S(1)] = 0.40
$$

$$
\mathrm{E}[S(1)] = e^{0.2} = 1.22
$$

$$
\mathrm{Stdev}[S(1)] = e^{0.2}(e^{0.16} - 1)^{\frac{1}{2}} = 0.51
$$

7. (Application of Ito's lemma) We have $G(t) = F(s,t) = S^{\frac{1}{2}}(t)$, then $\partial F/\partial S = \frac{1}{2}S^{-\frac{1}{2}}$, and $\partial^2 F/\partial S^2 = -\frac{1}{4}S^{-\frac{3}{2}}$. Therefore according to Ito's lemma

$$\begin{aligned} dG(t) &= (\tfrac{1}{2}S^{-\frac{1}{2}}aS - \tfrac{1}{8}S^{-\frac{3}{2}}b^2S^2)dt + \tfrac{1}{2}S^{-\frac{1}{2}}bS\,dz \\ &= (\tfrac{1}{2}a - \tfrac{1}{8}b^2)G\,dt + \tfrac{1}{2}bG\,dz \end{aligned}$$

8. (Reverse check) We have $a = \mu - \frac{1}{2}\sigma^2$, $b = \sigma$, $S = F(Q) = e^Q$. Thus

$$\begin{aligned} dS &= \left(\frac{\partial F}{\partial Q}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial Q^2}b^2\right)dt + \frac{\partial F}{\partial Q}b\,dz \\ &= \left(S(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}S\sigma^2\right)dt + S\sigma\,dz \\ &= \mu S\,dt + \sigma S\,dz. \end{aligned}$$

9. (Two simulations) Using $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, the equation in (11.20) can be expressed as

$$S(t_{k+1}) = [1 + (\nu + \frac{1}{2}\sigma^2\epsilon^2(t_k))\Delta t + \sigma\epsilon(t_k)\sqrt{\Delta t}]S(t_k)$$

Obviously, it is different from the expression in (11.19). But the expected values of the two expressions are identical to the first order:

$$E[S(t_{k+1})] = [1 + (\nu + \tfrac{1}{2}\sigma^2)\Delta t]S(t_k) = [1 + \mu\Delta t]S(t_k)$$

So, over the long run the two methods should produce similar results.

10. (A simulation experiment)

(a) A simulation shows that convergence is achieved only after a few thousand years. (See Fig. 11.2 for an example.)
We know that $\ln S(t)$ has a normal distribution with mean $\nu t = .10 - .30^2/2 = 0.055t$ and variance $\sigma^2 t = .09$. Hence $\frac{1}{t}\ln S(t)$ has mean ν and standard deviation $\frac{\sigma}{\sqrt{t}}$. As t goes to infinity, it is clear that the standard deviation goes to zero and

$$\frac{1}{t}\ln S(t) \to \nu.$$

(b) We have $\sigma = .30$. We would like the standard deviation of the simulation to be .005. Hence we must have $.30/\sqrt{t} = .005$. In other words, $t \approx 3,600$ years; or equivalently, about 43,000 months. This is consistent with the simulation experiment.

(c) This does not converge. The expected value is $\sigma^2 = .30^2$ but the simulation moves around that value. [For those who have studied statistics: The distribution of the quantity approaches a chi-squared distribution.]

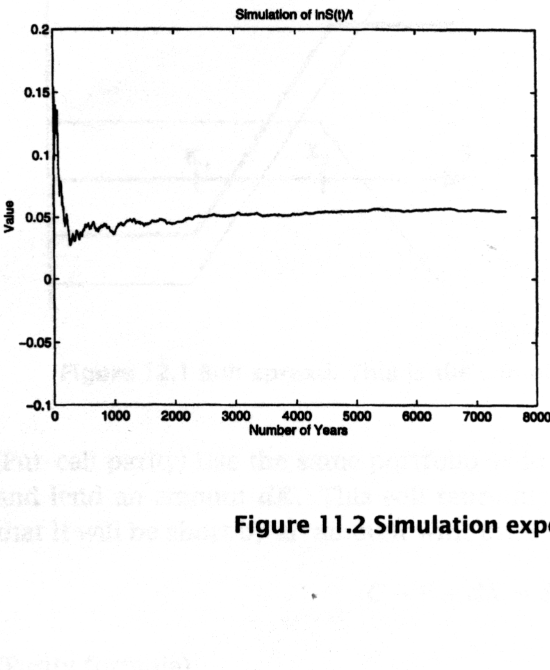


Figure 11.2 Simulation experiment.

Chapter 12
Basic Options Theory

1. (Bull spread) The initial cost of the spread is nonnegative since $C(K_1) \geq C(K_2)$ for $K_1 < K_2$ (see Exercise 4).

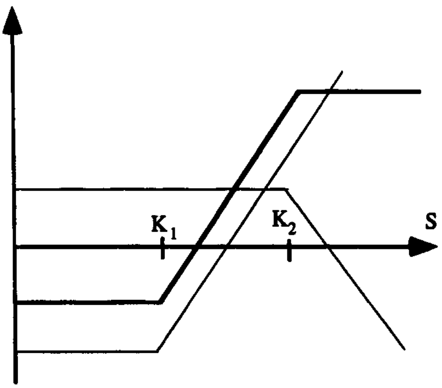


Figure 12.1 Bull spread. This is the combination of a call and a put.

2. (Put-call parity) Use the same portfolio as in the text: buy one call, sell one put, and lend an amount dK. This will reproduce the payment of the stock, except that it will be short by an amount with present value D. Hence

$$C - P + dK = S - D.$$

3. (Parity formula)

$$\begin{aligned} Q &= \max[0, S-K] - \max[0, K-S] + K \\ &= (S-K) - 0 + K = S \quad \text{if } S \geq K \\ &= 0 - (K-S) + K = S \quad \text{if } S \leq K. \end{aligned}$$

Therefore, $Q = S$.

4. (Call strikes)

(a) Assume $K_2 > K_1$, and suppose to the contrary that $C(K_2) > C(K_1)$. Buy option 1 and short option 2. Use option 1 to cover the obligations of option 2, since $\max[0, S - K_1] \geq \max[0, S - K_2]$ for all S. Keep profit of $C(K_2) - C(K_1)$.

(b) Assume $K_2 > K_1$ and suppose to the contrary that $K_2 - K_1 < C(K_1) - C(K_2)$. Buy option 2 and short option 1 to obtain $K_2 - K_1 + \varepsilon$ profit (where $\varepsilon > 0$). Use option 2 and profits $K_2 - K_1$ to cover option 1 since

$$\max[0, S - K_2] + (K_2 - K_1) = \max[K_2 - K_1, S - K_1] \geq \max[[0, S - K_1].$$

Hence you still keep ε profit. This arbitrage opportunity shows that the assumed inequality cannot hold.

(c) Assume $K_3 > K_2 > K_1$ and suppose to the contrary that

$$C(K_2) > \left(\frac{K_3 - K_2}{K_3 - K_1}\right) C(K_1) + \left(\frac{K_2 - K_1}{K_3 - K1}\right) C(K_3).$$

Buy $\left(\frac{K_3-K_2}{K_3-K_1}\right)$ of option 1 and $\left(\frac{K_2-K_1}{K_3-K_1}\right)$ of option 3 and short one unit of option 2. the profit is some $\varepsilon > 0$.

Notice that

$$\begin{aligned}
&\left(\tfrac{K_3-K_2}{K_3-K_1}\right)C(K_1) + \left(\tfrac{K_2-K_1}{K_3-K_1}\right)C(K_3) \\
\geq\ & \left(\tfrac{K_3-K_2}{K_3-K_1}\right)(S - K_1) + \left(\tfrac{K_2-K_1}{K_3-K_1}\right)(S - K_3) \\
=\ & \left(\tfrac{K_3-K_1}{K_3-K_1}\right)((S - K_2) + (K_2 - K_1)) + \left(\tfrac{K_2-K_1}{K_3-K_1}\right)((S - K_2) + (K_2 - K_3)) \\
=\ & S - K_2.
\end{aligned}$$

Also

$$\left(\tfrac{K_3-K_2}{K_3-K_1}\right)C(K_1) + \left(\tfrac{K_2-K_1}{K_3-K_1}\right)C(K_3) > 0.$$

Therefore it is possible to cover option 2 and make a profit of $\varepsilon > 0$. This is an arbitrage opportunity, so the original inequality cannot hold.

5. (Fixed dividend) The intitial present value of the dividend is $3e^{-(.10)(3.5)/12)} = 2.9138$. Hence we set $S^*(0) = 50 - 2.9138 = 47.086$. We assume (and this is an approximation) that S^* has the same volatility as S. Hence we assume that S^* follows a binomial process with $u = e^{\sigma\sqrt{\Delta t}} = e^{.2/\sqrt{12}} = 1.0594$ and $d = .9439$. The monthly return is $R = 1 + 0.1/12 = 1.0083$. We find the risk-neutral probability $q = \frac{R-d}{u-d} = 0.5577$. The process for S^* is shown in the upper part of Fig. 12.2. The stock price itself is expressed as $S(t) = S^*(t) + 3e^{-(.10)(3.5-t)/12)}$ for $t < 3.5$ and $S(t) = S^*(t)$ for $t \geq 3.5$. Hence we find the lattice for S by adding the appropriate amount to each node in the S^* lattice. The result is shown in the figure. The call option value is found by the normal backward process on the stock lattice. We find \$2.83 and \$2.51 for American and European options, respectively. See Fig. 12.2.

47.09	49.88	52.85	55.99	59.32	62.84	66.58
	44.44	47.09	49.88	52.85	55.99	59.32
		41.95	44.44	47.09	49.88	52.85
			39.60	41.95	44.44	47.09
				37.38	39.60	41.95
Random Component of Stock Price (S*)					35.28	37.38
						33.30

50.00	52.82	55.81	58.98	59.32	62.84	66.58
	47.38	50.05	52.87	52.85	55.99	59.32
		44.91	47.43	47.09	49.88	52.85
			42.59	41.95	44.44	47.09
				37.38	39.60	41.95
Stock Price (S)					35.28	37.38
						33.30

2.83	4.23	6.23	8.98	10.14	13.26	16.58
	1.11	1.80	2.87	4.23	6.40	9.32
		0.27	0.48	0.87	1.58	2.85
			0.00	0.00	0.00	0.00
				0.00	0.00	0.00
American Call Option					0.00	0.00
						0.00

2.51	3.70	5.32	7.47	10.14	13.26	16.58
	1.07	1.72	2.72	4.23	6.40	9.32
		0.27	0.48	0.87	1.58	2.85
			0.00	0.00	0.00	0.00
				0.00	0.00	0.00
European Call Option					0.00	0.00
						0.00

Figure 12.2 Call on a stock with dividend.

6. (Call inequality) The payoff of (A) purchase one call is always greater than or equal to the payoff off of (B) purchase one share of stock and sell K bonds. Hence the price of (A) must be greater than or equal to that of (B). Thus $C \geq S - KB(T)$. However the value of a call is always greater than or equal to zero. Thus $C \geq \max[0, S - KB(T)]$.

7. (Perpetual call) From Exercise 6, $C(S,T) \geq \max[0, S - KB(T)]$. Since $\lim_{T\to\infty} B(T) = 0$, we have $\lim_{T\to\infty} C(S,T) = C(S) \geq \lim_{T\to\infty} \max[0, S - KB(T)] = S$. Clearly the price of a call option must satisfy $C(S,T) \leq S$. Hence, in the limit $C = S$.

8. (A surprise)

   (a) PV(r) will increase if r is decreased.

   (b) The value of the Simplico mine with $r = 4\%$ is \$22.2 million, obtained by just changing r in the spreadsheet. Hence it moves in the opposite direction as that predicted by part (a). The reason is that when r is reduced, the costs are increased; whereas the value of the gold income is largely independent of r.

9. (My coin) This is like the example in the text. Draw a lattice with three stages. The payoffs of the four final nodes are 27, 27, 0, 0. Roll back one stage. The three nodes there have implied values 27, 9, 0. Roll back one more stage. The implied values there are 15, 3 which can be written as 12+3, 0+3. Hence the implied value of initial node is 4+3=7. The result can also be found by direct risk-neutral valuation (without rolling back) using $q = 1/3$ and $R = 1$. The risk-neutral probability of the two nodes with 27 are 1/27 and 6/27. Hence the total value is 7.

10. (The happy call) The payoff is

$$\max[.5S, S - K] = .5S + \max[0, .5S - K] = .5S + .5\max[0, S - 2K].$$

Hence, by linear pricing, we add the prices of the individual pieces to obtain $C_H = .5P + .5C_2$. Thus $\alpha = .5$, $\beta = 0$, $\gamma = .5$.

11. (You are a president) Notice that a \$1 increase in the S&P index index corresponds to a rate of return of $\frac{1}{414.74}$ in the index. Hence, for each dollar invested in the special offer, the payoff is \$1 plus $\frac{1}{4} \times \frac{1}{414.74}$ of a call option on the S&P 100 index with strike price \$414.74 and expiration in November.

The price of the \$1 portion of the payoff is its present value. Assuming 3.5 months until maturity, this value is

$$P_1 = \frac{1}{1 + .0311\frac{3.5}{12}} = .991010707.$$

The value of one call with strike price 414.74 is found by interpolation to be

$$C = 13 - \frac{(13 - 7.5)4.74}{10} = 10.393.$$

The value of the fractional call that is offered is therefore

$$P_2 = \frac{1}{4} \times \frac{1}{414.74} \times 10.393 = .0662647.$$

Hence the total value of the offer (per dollar invested) is

$$V = P_1 + P_2 = .991010707 + .0662647 = \$.9972754.$$

We conclude that, from the data we have, the offer is low by about 0.3%-which is not bad.

12. (Simplico invariance) Note that changing u or d amounts to changing the standard deviation of gold price and therefore would have an influence on the lease value of the Simplico gold mine. However, the initial gold price \$400 is relatively very

high compared to the extraction cost \$200 of gold, and so changes in gold price volatility will have minimal effect on the value of the lease. As noted in the chapter, the lease can be thought of as a series of call options. These options are very much "in the money" and fluctuations in the gold price will have little influence on the option prices.

13. (Change of period length) Fig. 12.3 shows the result. The two answers differ only by a cent or so. The value of the American put option is \$4.73.

One-month intervals

	0	1	2	3	4	5
	36.00	39.26	42.81	46.68	50.90	55.51
		33.01	36.00	39.26	42.81	46.68
			30.27	33.01	36.00	39.26
Stock Price				27.76	30.27	33.01
					25.46	27.76
						23.35

	4.74	2.69	1.10	0.17	0.00	0.00
		6.99	4.43	2.10	0.36	0.00
			9.73	6.99	4.00	0.74
American Put Option				12.24	9.73	6.99
					14.54	12.24
						16.65

	4.36	2.51	1.04	0.17	0.00	0.00
		6.40	4.11	1.98	0.36	0.00
			8.94	6.46	3.74	0.74
European Put Option				11.71	9.46	6.99
					14.28	12.24
						16.65

Half-month intervals

	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
	36.00	38.27	40.69	43.26	45.99	48.90	51.98	55.27	58.76	62.47	66.41
		33.86	36.00	38.27	40.69	43.26	45.99	48.90	51.98	55.27	58.76
			31.85	33.86	36.00	38.27	40.69	43.26	45.99	48.90	51.98
				29.96	31.85	33.86	36.00	38.27	40.69	43.26	45.99
					28.18	29.96	31.85	33.86	36.00	38.27	40.69
						26.50	28.18	29.96	31.85	33.86	36.00
Stock Price							24.93	26.50	28.18	29.96	31.85
								23.45	24.93	26.50	28.18
									22.06	23.45	24.93
										20.75	22.06
											19.51

	4.73	3.26	2.03	1.08	0.45	0.11	0.00	0.00	0.00	0.00	0.00
		6.30	4.57	3.04	1.76	0.80	0.22	0.00	0.00	0.00	0.00
			8.15	6.21	4.40	2.78	1.42	0.46	0.00	0.00	0.00
				10.04	8.15	6.14	4.23	2.43	0.95	0.00	0.00
					11.82	10.04	8.15	6.14	4.00	1.95	0.00
						13.50	11.82	10.04	8.15	6.14	4.00
							15.07	13.50	11.82	10.04	8.15
American Put Option								16.55	15.07	13.50	11.82
									17.94	16.55	15.07
										19.25	17.94
											20.49

	4.41	3.094	1.956	1.058	0.442	0.109	0	0	0	0	0.00
		5.82	4.309	2.912	1.711	0.794	0.224	0	0	0	0.00
			7.444	5.804	4.191	2.685	1.396	0.461	0	0	0.00
				9.216	7.534	5.799	4.056	2.387	0.947	0	0.00
					11.04	9.405	7.668	5.833	3.914	1.946	0.00
						12.83	11.29	9.644	7.884	6.006	4.00
							14.54	13.1	11.56	9.909	8.15
European Put Option								16.15	14.8	13.36	11.82
									17.68	16.42	15.07
										19.12	17.94
											20.49

Figure 12.3 Change of period length. Note that changing from monthly to bi-monthly periods has only a small effect on the computed value of the put option.

	0	1	2	3	4	5	6	7	8	9	10
	305.78	371.05	450.07	545.93	662.68	805.98	984.44	1212.45	1516.03	1940.64	0.00
		256.62	310.56	375.08	451.97	543.14	650.54	775.99	921.25	1091.61	0.00
			212.01	255.59	306.83	366.28	433.65	506.54	576.57	614.03	0.00
				171.11	204.76	243.00	284.76	326.57	358.26	345.39	0.00
					133.83	157.87	183.13	206.32	218.80	194.28	0.00
K-Value						100.36	115.23	127.64	131.29	109.28	0.00
							71.05	77.47	77.62	61.47	0.00
								46.26	45.34	34.58	0.00
									26.24	19.45	0.00
										10.94	0.00
											0.00

Figure 12.4 Average value Complexico. The overall value does not change much, but the computation is slightly more complicated than the original version.

14. (Average value Complexico) The Complexico gold mine of Example 12.8 is modified to reflect the average value of gold price in adjacent periods and the discounting of revenue received at the end of each period. As before $V_i(x_i^*) = K_i x_i^*$ for all i with $K_{10} = 0$. However, the following adjustments are made

$$K_i = \frac{\left(\frac{g_i + \hat{g}_{i+1}}{2R} - \frac{\hat{K}_{i+1}}{R}\right)^2}{2{,}000} + \frac{\hat{K}_{i+1}}{R} \qquad i = 1, 2 \ldots, 9.$$

Gold prices follow a process described by $u = 1.2$, $d = 0.9$, and $q = 0.6667$ the K-values, shown in Fig. 12.4, are calculated by working back from period 9.

The value of the lease is $50,000 \times 305.78 = \$15,288,786$. the value in the original version was \$16,220,000.

15. ("As you like it" option) We use the solutions for the call and the put in the examples in the text. At the end of the third year, we know that the value will be the maximum of the call or the put value. Hence, we just copy those values from the examples and then work backward with the standard risk-neutral discounted valuation process. The table below shows the details. The value is $A = \$6.73$, which is greater than either the put or the call value.

6.73	8.34	11.18	14.71
	4.83	4.92	6.96
		4.81	2.45
			7.86

16. (Tree harvesting)

						Growth					
1	1.6	1.5	1.4	1.3	1.2	1.15	1.1	1.05	1.02	1.01	
1	1.6	2.4	3.36	4.368	5.242	6.028	6.631	6.962	7.101	7.172	

5	6	7.2	8.64	10.37	12.44	14.93	17.92	21.5	25.8	30.96
	4.5	5.4	6.48	7.776	9.331	11.2	13.44	16.12	19.35	23.22
		4.05	4.86	5.832	6.998	8.398	10.08	12.09	14.51	17.41
			3.645	4.374	5.249	6.299	7.558	9.07	10.88	13.06
				3.281	3.937	4.724	5.669	6.802	8.163	9.796
	Lumber Price				2.952	3.543	4.252	5.102	6.122	7.347
						2.657	3.189	3.826	4.592	5.51
							2.391	2.87	3.444	4.132
								2.152	2.583	3.099
									1.937	2.325
										1.743

23.13	31.13	40.64	51.95	65.41	81.45	100.6	123.4	150.7	**183.2**	**222**
	20.68	28.06	36.81	47.2	59.55	74.23	91.7	112.5	**137.4**	**166.5**
		18.73	25.57	33.67	43.26	54.65	68.16	**84.19**	**103.1**	**124.9**
			17.27	23.64	31.17	40.07	50.62	**63.15**	**77.29**	**93.68**
				16.32	22.28	29.32	**37.59**	**47.36**	**57.97**	**70.26**
	Optimal Value				15.88	21.49	**28.19**	**35.52**	**43.48**	**52.69**
	Bold= cut					**16.02**	**21.14**	**26.64**	**32.61**	**39.52**
							15.86	**19.98**	**24.46**	**29.64**
								14.98	**18.34**	**22.23**
									13.76	**16.67**
										12.5

The first two rows show the growth rate and the cumulative growth by year. The first lattice shows the price of lumber using an up factor of $u = 1.2$ and a down factor of $d = .9$. The bottom lattice shows the optimal value as a function of position in the lattice. The risk-neutral probability of an up move is $q = \frac{R-d}{u-d} = (1.1 - 0.9)/(1.2 - 0.9) = .6667$. The value at a typical node is the maximum of either (1) the market value of the trees or (2) the risk-neutral discounted value of continuing to the next time. For example, the value at the top node of the second to last column (in millions) is

$$\max[25.8 \times 7.101, (q222 + (1-q)166.5)/R - 2] = 183.2.$$

In this case the maximum was obtained in the first portion of the maximization, meaning that the trees should be harvested. The value is entered in bold face to indicate that.

Chapter 13
Additional Option Topics

1. (Numerical evaluation of normal distribution) The equation is best implemented on a calculator, spreadsheet, or other computing device. The answer for the call stated in the exercise is $C = \$2.57$.

2. (Perpetual put)

   (a) For the expression $P(S) = a_1 S + a_2 S^{-\gamma}$ we have

   $$\begin{aligned} P'(S) &= a_1 - \gamma a_2 S^{-\gamma-1} \\ P''(S) &= \gamma(\gamma+1) a_2 S^{-\gamma-2}. \end{aligned}$$

   Substituting in the Black-Scholes equation

   $$\tfrac{1}{2}\sigma^2 \gamma(\gamma+1) a_2 S^{-\gamma} + r a_1 S - r\gamma a_2 S^{-\gamma} - a_1 r S - a_2 r S^{-\gamma} = 0.$$

   Canceling terms we find

   $$\tfrac{1}{2}\sigma^2 \gamma(\gamma+1) - r\gamma - r = 0.$$

   Hence, $\gamma = 2r/\sigma^2$ satisfies the equation. Since a_1 and a_2 are arbitrary, this represents two independent solutions to the second-order differential equation; and hence is the general solution.

   (b) $P(\infty) = 0$ implies $a_1 = 0$.
   $P(G) = K - G$ implies $a_2 G^{-\gamma} = K - G$ leading to $a_2 = (K - G)/G^{-\gamma}$. Hence $P(S) = (K - G)\,(S/G)^{-\gamma}$.

   (c) It makes sense to maximize $P(S)$ since the maximization is independent of S. We maximize $(K-G)G^{\gamma}$. Differentiation with respect to G gives the condition

   $$-G^{-\gamma} + \gamma(K - G)G^{\gamma-1} = 0.$$

   Thus $G = \gamma K/(\gamma + 1)$ and

   $$P(S) = K\left(1 - \frac{\gamma}{1+\gamma}\right)\left(\frac{1+\gamma S}{\gamma K}\right)^{-\gamma}.$$

3. (Sigma estimation) Using the spreadsheet implementation of the previous exercise, we adjust σ by trial and error to obtain the give call premium. The result is $\sigma = .251$.

4. (Black-Scholes approximation) For $S = Ke^{-rT}$ we find

$$\begin{aligned} d_1 &= \frac{\ln(e^{-rT}) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{\sigma\sqrt{T}}{2}. \\ d_2 &= -\frac{\sigma\sqrt{T}}{2}. \end{aligned}$$

Hence

$$\begin{aligned} C &\approx S\left[\tfrac{1}{2} + \tfrac{\sigma\sqrt{T}}{2\sqrt{2\pi}}\right] - Se^{rT}e^{-rT}\left[\tfrac{1}{2} - \tfrac{\sigma\sqrt{T}}{2\sqrt{2\pi}}\right] \\ &= \frac{S\sigma\sqrt{T}}{\sqrt{2\pi}} \approx .4S\sigma\sqrt{T}. \end{aligned}$$

For delta we have

$$\Delta = N(d_1) \approx \tfrac{1}{2} + \tfrac{\sigma\sqrt{T}}{2\sqrt{2\pi}} \approx \tfrac{1}{2} + .2\sigma\sqrt{T}.$$

For the call of the example

$$Ke^{-rT} = 60e^{-.5/12} = 57.55.$$

The value at that value of S is (by the above approximation) $.4\times.2\times57.55\times\sqrt{5/12} = 2.972$. The value of Δ at the base point is

$$\Delta \approx .5 + .2 \times .2\sqrt{5/12} = .5258.$$

The difference in S is $62 - 57.55 = 4.44$. Hence the final value is

$$C \approx 2.97 + 4.44 \times .5258 = \$5.30.$$

5. (Delta) Using the options calculator the price of the call at \$63 is \$6.557. Hence, $\Delta \approx 6.557 - 5.798 = .759$.

   Changing T to $T = 5/12 + .1$ we obtain a call price of \$6.490. Hence $\Theta \approx (6.490 - 5.798)/0.1 = 6.02$.

6. (A special identity) This is just the Black-Scholes equation.

7. (Gamma and theta)

$$\begin{aligned}
\Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} \\
&= \frac{\partial N(d_1)}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} \\
&= \frac{N'(d_1)}{S\sigma\sqrt{T}}.
\end{aligned}$$

For theta, use Exercise6 to write

$$\begin{aligned}
\Theta &= rC - rS\Delta - \tfrac{1}{2}\sigma^2 S^2 \Gamma \\
&= rSN(d_1) - rKe^{-rT}N(d_2) - rSN(d_1) - \tfrac{1}{2}\sigma S N'(d_1)/\sqrt{T} \\
&= -\frac{SN'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2).
\end{aligned}$$

8. (Great Western CD) It is possible to set up a lattice of possible S&P returns in the standard way; however the payoff of the CD depends on the path through this lattice. Hence it is necessary to use a full tree representation. We use two-month periods so that the tree will not be too large. In the tree a move across corresponds to an up move; a move down to the nearest nonzero entry corresponds to a down move.

   The parameters are $u = e^{.2\sqrt{1/6}} = 1.085$, $d = 1/u$. The tree of S&P returns is shown in Fig. 13.1.

   The tree can be expressed in a filled in form as shown in Fig. 13.2. In this version, each row represents a distinct path trough the tree. This version simplifies the computation of the average value along a path (formed from the cells beginning in the second column).

   The risk-neutral valuation is shown in Fig. 13.3. The final column is the total return of the CD, defined as the average of the S&P returns of the path, or 1, whichever is larger. The risk-free rate is set arbitrarily at first. It is adjusted (with a solving program or by trial and error) until the original value is 1.0. Note that the risk-neutral probabilities will vary as the risk-free rate is adjusted. The final equivalent interest rate is 7.63%.

sig=	0.2	q=	0.55741874
u=	1.0850756	qq=	0.44258126
d=	0.92159478		

RR=	1.01272205	(Two-month risk-free return)
Val=	0.99999999	(Risk-neutral result from final sheet.)
Error=	6.491E-09	(Difference of Val from 1. Drive to zero.)
Yearly Rate=	7.63%	(RR converted to yearly rate.)

1.00000000	1.08507560	1.17738905	1.27755612	1.38624497	1.50418059	1.63214965
						1.38624497
					1.27755612	1.38624497
						1.17738905
				1.17738905	1.27755612	1.38624497
						1.17738905
					1.08507560	1.17738905
						1.00000000
			1.08507560	1.17738905	1.27755612	1.38624497
						1.17738905
					1.08507560	1.17738905
						1.00000000
				1.00000000	1.08507560	1.17738905
						1.00000000
					0.92159478	1.00000000
						0.84933693
		1.00000000	1.08507560	1.17738905	1.27755612	1.38624497
						1.17738905
					1.08507560	1.17738905
						1.00000000
				1.00000000	1.08507560	1.17738905
						1.00000000
					0.92159478	1.00000000
						0.84933693
			0.92159478	1.00000000	1.08507560	1.17738905
						1.00000000
					0.92159478	1.00000000
						0.84933693
				0.84933693	0.92159478	1.00000000
						0.84933693
					0.78274448	0.84933693
						0.72137322
	0.92159478	1.00000000	1.08507560	1.17738905	1.27755612	1.38624497
						1.17738905
					1.08507560	1.17738905
						1.00000000
				1.00000000	1.08507560	1.17738905
						1.00000000
					0.92159478	1.00000000
						0.84933693
			0.92159478	1.00000000	1.08507560	1.17738905
						1.00000000
					0.92159478	1.00000000
						0.84933693
				0.84933693	0.92159478	1.00000000
						0.84933693
					0.78274448	0.84933693
						0.72137322
		0.84933693	0.92159478	1.00000000	1.08507560	1.17738905
						1.00000000
					0.92159478	1.00000000
						0.84933693
				0.84933693	0.92159478	1.00000000
						0.84933693
					0.78274448	0.84933693
						0.72137322
			0.78274448	0.84933693	0.92159478	1.00000000
						0.84933693
					0.78274448	0.84933693
						0.72137322
				0.72137322	0.78274448	0.84933693
						0.72137322
					0.66481379	0.72137322
						0.61268892

Figure 13.1 Great Western CD. This is the tree of possible returns on the S&P 500. The parameters and the solution are shown at the top.

Copy of previous sheet but with duplicated numbers.
Each row shows the values obtained on path to final node.

1	1.0850756	1.17738905	1.27755612	1.38624497	1.50418059	1.63214965
1	1.0850756	1.17738905	1.27755612	1.38624497	1.50418059	1.38624497
1	1.0850756	1.17738905	1.27755612	1.38624497	1.27755612	1.38624497
1	1.0850756	1.17738905	1.27755612	1.38624497	1.27755612	1.17738905
1	1.0850756	1.17738905	1.27755612	1.17738905	1.27755612	1.38624497
1	1.0850756	1.17738905	1.27755612	1.17738905	1.27755612	1.17738905
1	1.0850756	1.17738905	1.27755612	1.17738905	1.0850756	1.17738905
1	1.0850756	1.17738905	1.27755612	1.17738905	1.0850756	1
1	1.0850756	1.17738905	1.0850756	1.17738905	1.27755612	1.38624497
1	1.0850756	1.17738905	1.0850756	1.17738905	1.27755612	1.17738905
1	1.0850756	1.17738905	1.0850756	1.17738905	1.0850756	1.17738905
1	1.0850756	1.17738905	1.0850756	1.17738905	1.0850756	1
1	1.0850756	1.17738905	1.0850756	1	1.0850756	1.17738905
1	1.0850756	1.17738905	1.0850756	1	1.0850756	1
1	1.0850756	1.17738905	1.0850756	1	0.92159478	1
1	1.0850756	1.17738905	1.0850756	1	0.92159478	0.84933693
1	1.0850756	1	1.0850756	1.17738905	1.27755612	1.38624497
1	1.0850756	1	1.0850756	1.17738905	0.92159478	1.17738905
1	1.0850756	1	1.0850756	1.17738905	1.0850756	1.17738905
1	1.0850756	1	1.0850756	1.17738905	1.0850756	1
1	1.0850756	1	1.0850756	1	1.0850756	1.17738905
1	1.0850756	1	1.0850756	1	1.0850756	1
1	1.0850756	1	1.0850756	1	0.92159478	1
1	1.0850756	1	1.0850756	1	0.92159478	0.84933693
1	1.0850756	1	0.92159478	1	1.0850756	1.17738905
1	1.0850756	1	0.92159478	1	1.0850756	1
1	1.0850756	1	0.92159478	1	0.92159478	1
1	1.0850756	1	0.92159478	1	0.92159478	0.84933693
1	1.0850756	1	0.92159478	0.84933693	0.92159478	1
1	1.0850756	1	0.92159478	0.84933693	0.92159478	0.84933693
1	1.0850756	1	0.92159478	0.84933693	0.78274448	0.84933693
1	1.0850756	1	0.92159478	0.84933693	0.78274448	0.72137322
1	0.92159478	1	1.0850756	1.17738905	1.27755612	1.38624497
1	0.92159478	1	1.0850756	1.17738905	1.27755612	1.17738905
1	0.92159478	1	1.0850756	1.17738905	1.0850756	1.17738905
1	0.92159478	1	1.0850756	1.17738905	1.0850756	1
1	0.92159478	1	1.0850756	1	1.0850756	1.17738905
1	0.92159478	1	1.0850756	1	1.0850756	1
1	0.92159478	1	1.0850756	1	0.92159478	1
1	0.92159478	1	1.0850756	1	0.92159478	0.84933693
1	0.92159478	1	0.92159478	1	1.0850756	1.17738905
1	0.92159478	1	0.92159478	1	1.0850756	1
1	0.92159478	1	0.92159478	1	0.92159478	1
1	0.92159478	1	0.92159478	1	0.92159478	0.84933693
1	0.92159478	1	0.92159478	0.84933693	0.92159478	1
1	0.92159478	1	0.92159478	0.84933693	0.92159478	0.84933693
1	0.92159478	1	0.92159478	0.84933693	0.78274448	0.84933693
1	0.92159478	1	0.92159478	0.84933693	0.78274448	0.72137322
1	0.92159478	0.84933693	0.92159478	1	1.0850756	1.17738905
1	0.92159478	0.84933693	0.92159478	1	1.0850756	1
1	0.92159478	0.84933693	0.92159478	1	0.92159478	1
1	0.92159478	0.84933693	0.92159478	1	0.92159478	0.84933693
1	0.92159478	0.84933693	0.92159478	0.84933693	0.92159478	1
1	0.92159478	0.84933693	0.92159478	0.84933693	0.92159478	0.84933693
1	0.92159478	0.84933693	0.92159478	0.84933693	0.78274448	0.84933693
1	0.92159478	0.84933693	0.92159478	0.84933693	0.78274448	0.72137322
1	0.92159478	0.84933693	0.78274448	0.84933693	0.92159478	1
1	0.92159478	0.84933693	0.78274448	0.84933693	0.92159478	0.84933693
1	0.92159478	0.84933693	0.78274448	0.84933693	0.78274448	0.84933693
1	0.92159478	0.84933693	0.78274448	0.84933693	0.78274448	0.72137322
1	0.92159478	0.84933693	0.78274448	0.72137322	0.78274448	0.84933693
1	0.92159478	0.84933693	0.78274448	0.72137322	0.78274448	0.72137322
1	0.92159478	0.84933693	0.78274448	0.72137322	0.66481379	0.72137322
1	0.92159478	0.84933693	0.78274448	0.72137322	0.66481379	0.61268892

Figure 13.2 Great Western CD. This is the tree of possible returns on the S&P 500, but shown in duplicated form. Each row of the sheet represents a path through the tree.

Risk-Neutral Valuation. Cells in final column
are average of path from previous sheet, or 1, which ever is maximum.

0.99999999	1.05651643	1.12874264	1.19883352	1.25972483	1.30897436	1.343766
						1.30278188
					1.23390736	1.26501114
						1.23020182
				1.15660322	1.19953532	1.23020182
						1.1953925
					1.13577814	1.16331241
						1.13374757
			1.07291097	1.12532407	1.16785823	1.19812173
						1.16331241
					1.10410105	1.13123232
						1.10166748
				1.03773908	1.07490761	1.10166748
						1.07210264
					1.02075628	1.04485584
						1.01974532
		0.99591923	1.03741115	1.08222665	1.11273766	1.16855689
						1.07442068
					1.07490761	1.10166748
						1.07210264
				1.01078778	1.04571417	1.07210264
						1.0425378
					0.99585418	1.01529099
						1
			0.97228912	0.99230094	1.01880965	1.04485584
						1.01529099
					0.98743777	1
						1
				0.97503335	0.98743777	1
						1
					0.98743777	1
						1
	0.95756423	0.98487685	1.02256418	1.06993082	1.11176026	1.14131009
						1.10650077
					1.04800309	1.07442068
						1.04485584
				0.99230094	1.01880965	1.04485584
						1.01529099
					0.98743777	1
						1
			0.96572111	0.98036815	0.99713007	1.01760903
						1
					0.98743777	1
						1
				0.97503335	0.98743777	1
						1
					0.98743777	1
						1
		0.95069003	0.96278475	0.97503335	0.98743777	1
						1
					0.98743777	1
						1
				0.97503335	0.98743777	1
						1
					0.98743777	1
						1
			0.96278475	0.97503335	0.98743777	1
						1
					0.98743777	1
						1
				0.97503335	0.98743777	1
						1
					0.98743777	1
						1

Figure 13.3 Great Western CD. This is the risk-neutral valuation. The interest rate used is adjusted until the value at the originating node is 1.0

9. (The control variate method) We have

$$\hat{x} = x_{\text{avg}} + a(y_{avg} - \overline{y}).$$

Let $\hat{\sigma}^2$ denote the variance of $\hat{x}$. Then

$$\hat{\sigma}^2 = \text{var}(x_{\text{avg}}) + 2a\,\text{cov}(x_{\text{avg}}, y_{\text{avg}}) + a^2\text{var}(y_{\text{avg}}).$$

Minimization leads immediately to

$$a = -\frac{\text{cov}(x_{\text{avg}}, y_{\text{avg}})}{\text{var}(y_{\text{avg}})}.$$

The average values of x and y each contain factors of $1/n$ (where n is the number of samples). These factors cancel out, giving

$$a = -\frac{\text{cov}(x, y)}{\text{var}(y)}.$$

10. (Control variate application) The simulation can be carried out by moving through the lattice, selecting the subsequent nodes according to the risk-neutral probabilities. As the simulation is carried out, the sample averages of the unknown x and the control variate y are calculated. The covariance between x and y and the variance of y are estimated from the samples as well. For instance the estimate of var(Y) is $\frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{y})^2$, where n is the number of samples. We use these estimates to select the parameter a (which may change with n, but in practice tends to be quite stable).

    The results of such a simulation using the standard call option as control variate are shown in Fig. 13.4. The column headed **x** is the average of the call price itself, formed without using the control variate. The column headed by **y** is the corresponding estimate of the control variate. The column headed by **x estimate** is the estimate of x corrected for the control variate. The final two columns give the standard deviation (as computed from the formula using sample covariances and variances) of the two estimates. Notice that using the control variate reduces the standard deviation by about half.

    The same experiment was repeated using the normalized average price as a control variate. Specifically, we used the form

$$y = \{\frac{1}{6}(S_0 + S_1 + S_2 + S_3 + S_4 + S_5) - K\}/R^5.$$

    The expected value of the one period return is $qu + (1 - q)d = R$. Hence the expected value of y is $\hat{\text{E}}(y) = \{(1+R+R^2+R^3+R^5+R^6)62-60\}/R^5 = 3.171732$. The results are shown in Fig. 13.5.

Iteration	x average	y average	x estimate	s. d. x	s. d. x est.
10	3.158118	4.743947	3.794307	1.030501	0.316124
1000	3.653756	5.756972	3.698205	0.109900	0.049447
2000	3.604331	5.592042	3.733517	0.077883	0.035440
3000	3.656639	5.652068	3.755232	0.063773	0.029491
4000	3.640437	5.621538	3.754041	0.054939	0.025571
5000	3.638296	5.606736	3.759379	0.049411	0.022962
6000	3.658783	5.647381	3.758900	0.045209	0.021016
7000	3.674058	5.687055	3.753941	0.041919	0.019409
8000	3.676383	5.686797	3.756212	0.039091	0.018108
9000	3.691382	5.710738	3.759152	0.036871	0.017136
10000	3.704037	5.738365	3.757839	0.035089	0.016229
11000	3.718614	5.769309	3.756808	0.033499	0.015512
12000	3.711784	5.752076	3.758661	0.032023	0.014826
13000	3.706227	5.731418	3.763586	0.030758	0.014231
14000	3.699005	5.712500	3.765958	0.029615	0.013731
15000	3.700939	5.731989	3.758046	0.028654	0.013260
16000	3.717370	5.759209	3.760736	0.027762	0.012829
17000	3.715498	5.764219	3.756334	0.026949	0.012413
18000	3.717579	5.772618	3.754232	0.026197	0.012059
19000	3.710736	5.760131	3.753709	0.025454	0.011755
20000	3.704428	5.748116	3.753481	0.024790	0.011440
21000	3.709078	5.759194	3.752544	0.024218	0.011139
22000	3.712701	5.760125	3.755701	0.023681	0.010881
23000	3.711221	5.757569	3.755511	0.023149	0.010639
24000	3.715592	5.764618	3.756327	0.022676	0.010417
25000	3.709831	5.751397	3.757279	0.022227	0.010197
26000	3.709455	5.755037	3.755071	0.021803	0.010003
27000	3.711328	5.753543	3.757682	0.021378	0.009816
28000	3.709042	5.753736	3.755284	0.020993	0.009635
29000	3.715879	5.762562	3.757673	0.020644	0.009469
30000	3.720317	5.768952	3.758857	0.020312	0.009307

Figure 13.4 Use of option as control variate. Here x is the price of the Asian option and y is the price of the standard call option. The variable x_{extimate} is the best estimate based on the control variate. Note that the standard deviation of the estimated is about one-half that of the standard method.

Iteration	x average	y average	x estimate	s.d. x	s.d. x est.
10	4.876330	3.581876	4.613649	1.307191	0.406418
1000	3.944883	3.364028	3.791650	0.115925	0.030036
2000	3.870906	3.288819	3.778136	0.081095	0.021743
3000	3.890886	3.318208	3.774783	0.065760	0.017543
4000	3.877007	3.310910	3.766802	0.056428	0.015108
5000	3.883702	3.303992	3.779569	0.050450	0.013663
6000	3.868370	3.289963	3.775283	0.045982	0.012489
7000	3.861318	3.295386	3.763566	0.042472	0.011447
8000	3.833471	3.270019	3.755566	0.039765	0.010670
9000	3.825892	3.260656	3.755504	0.037429	0.010090
10000	3.816983	3.257190	3.749236	0.035434	0.009529
11000	3.796201	3.231682	3.748723	0.033782	0.009116
12000	3.791059	3.226795	3.747471	0.032274	0.008714
13000	3.794953	3.232800	3.746567	0.031016	0.008352
14000	3.792672	3.227149	3.748758	0.029971	0.008066
15000	3.789359	3.223121	3.748666	0.028898	0.007781
16000	3.793232	3.228202	3.748510	0.027967	0.007525
17000	3.786084	3.220943	3.747083	0.027143	0.007287
18000	3.795214	3.229812	3.749176	0.026419	0.007085
19000	3.792336	3.226089	3.749248	0.025734	0.006905
20000	3.787858	3.220191	3.749452	0.025084	0.006733
21000	3.780361	3.210898	3.749332	0.024481	0.006580
22000	3.781657	3.211316	3.750269	0.023964	0.006417
23000	3.777782	3.206126	3.750533	0.023409	0.006279
24000	3.776637	3.203082	3.751806	0.022926	0.006149
25000	3.770144	3.197675	3.749590	0.022436	0.006016
26000	3.762955	3.189925	3.748547	0.021979	0.005902
27000	3.760509	3.184809	3.750164	0.021554	0.005798
28000	3.760340	3.187178	3.748111	0.021153	0.005681
29000	3.758398	3.187070	3.746250	0.020768	0.005572
30000	3.762451	3.191677	3.746657	0.020405	0.005475
40000	3.761654	3.184537	3.751530	0.017695	0.004758
50000	3.771508	3.199892	3.749207	0.015838	0.004248
60000	3.766341	3.192636	3.749798	0.014449	0.003880
100000	3.759438	3.182984	3.750538	0.011194	0.003009
150000	3.754023	3.177719	3.749287	0.009136	0.002455
200000	3.750090	3.173041	3.749055	0.007910	0.002125
250000	3.752277	3.175374	3.749395	0.007076	0.001901
300000	3.750777	3.173457	3.749413	0.006460	0.001736
400000	3.766341	3.192636	3.749798	0.014449	0.003880
500000	3.766341	3.192636	3.749798	0.014449	0.003880
1000000	3.748908	3.171466	3.749118	0.003538	0.000950

Figure 13.5 Use of average stock price as control variate. Here x is the price of the Asian option and y is the normalized average price of the stock. Note that the standard deviation of the estimated is about one-fourth that of the standard method.

Notice that this control variate reduces the standard deviation of the estimate by about one fourth. This reduces the number of iterations to obtain a given accuracy by a factor of 16.

The best estimate of the value of the Asian call is $3.749 (but the last place is not fully reliable even after 1 million iterations.

11. (Pay-later options)

(a) We set up a standard price lattice using the parameters $u = e^{\sigma\sqrt{\Delta t}} = e^{.2\sqrt{1/12}} = 1.06$. This lattice is shown at the top of Fig. 13.6.

0	1	2	3	4	5	6	7	8	9	10
12.0	12.7	13.5	14.3	15.1	16.0	17.0	18.0	19.0	20.2	21.4
	11.3	12.0	12.7	13.5	14.3	15.1	16.0	17.0	18.0	19.0
		10.7	11.3	12.0	12.7	13.5	14.3	15.1	16.0	17.0
			10.1	10.7	11.3	12.0	12.7	13.5	14.3	15.1
				9.5	10.1	10.7	11.3	12.0	12.7	13.5
	Stock Price				9.0	9.5	10.1	10.7	11.3	12.0
						8.5	9.0	9.5	10.1	10.7
							8.0	8.5	9.0	9.5
								7.6	8.0	8.5
									7.1	7.6
										6.7

0.53	0.76	1.08	1.49	2.02	2.67	3.44	4.32	5.28	6.29	7.38
	0.25	0.38	0.57	0.85	1.23	1.75	2.40	3.20	4.09	5.04
		0.09	0.15	0.24	0.38	0.61	0.95	1.45	2.13	2.97
			0.02	0.03	0.06	0.10	0.19	0.34	0.62	1.12
				0	0	0	0	0	0	0
	Standard Option				0	0	0	0	0	0
						0	0	0	0	0
							0	0	0	0
								0	0	0
									0	0
										0

0.00	0.04	0.13	0.29	0.55	0.95	1.54	2.33	3.27	4.27	5.33
	-0.05	-0.06	-0.06	-0.04	0.04	0.23	0.59	1.19	2.07	3.00
		-0.04	-0.07	-0.10	-0.14	-0.19	-0.22	-0.17	0.11	0.93
			-0.01	-0.03	-0.05	-0.09	-0.16	-0.28	-0.51	-0.92
				0	0	0	0	0	0	0
	Pay Later Option				0	0	0	0	0	0
	Premium =$2.04					0	0	0	0	0
							0	0	0	0
								0	0	0
									0	0
										0

Figure 13.6 Pay-later option. The pay-later premium is set so that the original value is zero.

Next we calculate $R = 1 + 0.10/12$. This gives $q = 0.44$ as the risk-neutral probability for an up move. The standard lattice using discounted risk-neutral valuation is shown as the second lattice in the figure. The resulting price of the option is \$0.53.

(b) The paylater option lattice is set up exactly the same way except that the final values are $S-K-C$, for those cells where $S-K$ is positive, and zero otherwise. The value C is unknown. We use a solving routine in the spreadsheet (or trial and error) to adjust the value of C so that the initial price is zero. In this case that value is \$2.04. The corresponding lattice with that value is shown as the third lattice in the figure.

(c) Obviously the pay-later option premium is higher than that of a standard option: The premium is not paid until later (meaning there is interest rate advantage, and more importantly, no premium is paid if the option does not end up in the money.

u=	1.2		q=	0.73		Fac=	1.05		PUT=	0.72					
d=	0.84		qq=	0.27		P=	0.72		RATE=	.101					
RR=	1.1		Free=	0.1		Pay=	11.9								
			A=	0.13											
						Balance									
90	87.2	84.1	80.7	77	72.8	68.3	63.3	57.8	51.7	45.1	37.7	29.6	20.7	10.8	-0
100	120	143	172	205	246	294	353	422	505	605	724	867	1038	1243	1488
	83.5	100	120	143	172	205	246	294	353	422	505	605	724	867	1038
		69.8	83.5	100	120	143	172	205	246	294	353	422	505	605	724
			58.3	69.8	83.5	100	120	143	172	205	246	294	353	422	505
	House Price			48.7	58.3	69.8	83.5	100	120	143	172	205	246	294	353
					40.7	48.7	58.3	69.8	83.5	100	120	143	172	205	246
						34	40.7	48.7	58.3	69.8	83.5	100	120	143	172
							28.4	34	40.7	48.7	58.3	69.8	83.5	100	120
								23.7	28.4	34	40.7	48.7	58.3	69.8	83.5
									19.8	23.7	28.4	34	40.7	48.7	58.3
										16.5	19.8	23.7	28.4	34	40.7
											13.8	16.5	19.8	23.7	28.4
												11.5	13.8	16.5	19.8
													9.63	11.5	13.8
														8.05	9.63
															6.72
0.72	0.06	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	2.81	0.23	0.01	0	0	0	0	0	0	0	0	0	0	0	0
		10.9	0.93	0.03	0	0	0	0	0	0	0	0	0	0	0
			19.5	3.72	0.13	0	0	0	0	0	0	0	0	0	0
				25.9	11.7	0.52	0	0	0	0	0	0	0	0	0
					30.2	17.2	2.12	0	0	0	0	0	0	0	0
						32.7	20.6	6.69	0	0	0	0	0	0	0
	Put Value						33.5	22.1	9.04	0	0	0	0	0	0
								32.9	22	9.4	0	0	0	0	0
									31	20.2	7.91	0	0	0	0
										27.7	16.9	4.71	0	0	0
											23.2	12.2	0	0	0
												17.5	6.16	0	0
													10.5	0	0
														2.38	0
															0

Figure 13.7 California housing put. The put value is remarkably small.

12. (California housing put)

The calculation is shown on the spreadsheet in Fig. 13.7. The up and down factors for housing prices are $u = e^{\sigma}$, $d = e^{-\sigma}$, using the small Δt approximation (although it is not fully justified here). This produces the house price lattice.

The mortgage interest rate of 10% defines an amount A that is the yearly payment per dollar of loan. This is $A = .13$. This defines the yearly balance which is initially recorded in the first full row of the spreadsheet.

The value of the put is calculated in the usual backward style using the appropriate risk-neutral probabilities ($q = .73$) and the risk-free rate of 10%. As an example, the last entry in the third to last column is $\max[20.7 - 9.63(1.05), (q \times 0 + (1-q)2.38)/1.1] = 10.5$.

The put value is .72 for a loan of \$90; which is quite small.

We then assume that the bank would like to charge for the put. Hence the new loan is for \$90.72. At 10% the payments of this loan are \$11.9 per year. These payments spread over fifteen years is equivalent to an interest rate of 10.1%. (See the example on APR in Chapter 3.)

This new rate will change the balance structure of the first row. In fact, the values shown in the spreadsheet correspond to this 10.1% rate. In theory, these new balances will change the put value and the whole process must be iterated until convergence. However, only this single step is required in this example, because the put value is so small, and the value does not change except in far out decimal places. (We have also not accounted for the fact that the full \$.72 may not be recovered if the put option is exercised. The value of this "put on put" is likely to be extremely small, however, and can safely be ignored.)

13. (Forest value) This solution is identical to that of Exercise 16 in Chapter 12, except that the risk-neutral probability is $q = \frac{R-d+c}{u-d} = (1.1 - .9 + .05)/(1.2 - .9) = .833$. The value obtained from the modified spreadsheet is \$42.42 millions.

14. (Mr. Smith's put) This is straightforward. It turns out that it is never optimal to exercise the put early. The appropriate lattices are shown in Fig. 13.8.

0	1	2	3	4	5	6
0.625	0.644	0.664	0.684	0.705	0.726	0.748
	0.607	0.625	0.644	0.664	0.684	0.705
		0.589	0.607	0.625	0.644	0.664
			0.571	0.589	0.607	0.625
				0.554	0.571	0.589
	Stock Price				0.538	0.554
						0.522
0.017	0.007	0.002	0	0	0	0
	0.023	0.01	0.003	0	0	0
		0.031	0.015	0.004	0	0
			0.041	0.023	0.007	0
				0.053	0.032	0.011
	Put Option				0.065	0.046
						0.078

Figure 13.8 Mr. Smith's put. The lattice calculations are standard.

Chapter 14
Interest-Rate Derivatives

1. (A callable bond)

   The calculation of the bond value in part (a) is straightforward. The calculation is shown in the two upper lattices of the spreadsheet in Fig. 1. The value is 91.72.

									0.31	
								0.26	0.23	
							0.21	0.19	0.17	
						0.18	0.16	0.15	0.13	
	Short rate lattice				0.15	0.13	0.12	0.11	0.10	
				0.12	0.11	0.10	0.09	0.08	0.07	
			0.10	0.09	0.08	0.08	0.07	0.06	0.06	
		0.09	0.08	0.07	0.06	0.06	0.05	0.05	0.04	
	0.07	0.06	0.06	0.05	0.05	0.04	0.04	0.03	0.03	
0.06	0.05	0.05	0.04	0.04	0.04	0.03	0.03	0.03	0.02	
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	
Part (a)										106.00
									86.94	106.00
								77.13	92.03	106.00
							72.68	84.89	96.28	106.00
	Bond valuation					71.58	81.99	91.60	99.75	106.00
					72.77	81.91	90.22	97.22	102.54	106.00
				75.63	83.82	91.15	97.25	101.82	104.75	106.00
			79.77	87.21	93.76	99.12	103.06	105.51	106.46	106.00
		84.95	91.76	97.65	102.37	105.77	107.78	108.43	107.79	106.00
	90.98	97.24	102.54	106.69	109.58	111.18	111.52	110.70	108.81	106.00
91.72	103.47	108.23	111.84	114.26	115.46	115.50	114.46	112.45	109.59	106.00
Part (b)										106.00
	Assume callable in year 5								86.94	106.00
	Assume callable after coupon payment							77.13	92.03	106.00
							72.68	84.89	96.28	106.00
						71.58	81.99	91.60	99.75	106.00
					72.77	81.91	90.22	97.22	102.54	106.00
				75.63	83.82	91.15	97.25	101.82	104.75	106.00
			79.77	87.20	93.74	99.07	102.96	105.29	106.00	106.00
		84.90	91.66	97.44	101.93	104.88	106.00	106.00	106.00	106.00
	90.71	96.72	101.54	104.78	106.00	106.00	106.00	106.00	106.00	106.00
90.95	102.11	105.88	107.92	107.99	106.00	106.00	106.00	106.00	106.00	106.00

Figure 14.1 Callable bond.

To find the value with the call feature, we construct a third lattice in a similar way, except that at each step we compare the discounted risk-neutral value of the next period with the option of calling the bond, which gives 106. The value is 90.95.

2. (General adjustable formula) First find the payment require, form Chapter 3,

$$P_{ks} = \frac{(r_{ks} + p)(1 + r_{ks} + p)100}{(1 + r_{ks} + p)^{n-k} - 1}.$$

After one period the remaining principal will be loaned again. This amount is

$$L_{ks} = 100[1 + r_{ks} + p] - P_{ks}.$$

Then the recursion for the value is found by discounted risk-neutral valuation as

$$V_{ks} = \left[\frac{L_{ks}\left(1 + \frac{1}{100}[.5V_{(k+1),s} + .5V_{(k+1),(s+1)}]\right) + P_{ks}}{1 + r_{ks}}\right] - 100.$$

3. (Bond futures option) You could set up a futures price lattice as shown in Example 14.4. Then use this as the underlying price lattice and carry out a standard option backward evaluation.

4. (Adjustable-rate CAP) The same spreadsheet as in the example can be used. It is only necessary to change the formula for the interest rate used to determine the loan payments, so that it is capped at 11%. The formulas in the evaluation lattice do not change, since the bank should still discount by the actual rate. The resulting two lattices are shown in Fig. 14.2. The answer is $4.20.

		Year						Year			
0	1	2	3	4	5	0	1	2	3	4	5
					100						0
	Payment Rate			111	100		Value per 100			-7.494	0
			58.39	111	100				-6.073	-2.496	0
		40.92	58.39	111	100			-2.635	0.033	1.294	0
	32.23	40.35	57.13	108.6	100		1.76857	2.709	2.634	1.876	0
25.71	30.39	38.57	55.39	106.6	100	4.201	4.22306	3.681	2.835	1.912	0

Figure 14.2 Auto loan with a CAP of 11%.

5. (Forward construction) The elementary price lattice is constructed using the equations in the text. The value of a zero-coupon bond is found by summing the elementary prices corresponding to the maturity date. The spot rate is then determined from the value of the zero. The results are shown in Fig. 14.3

										0.310	
									0.258	0.232	
								0.215	0.193	0.174	
	Short Rate Lattice						0.179	0.161	0.145	0.131	
						0.149	0.134	0.121	0.109	0.098	
					0.124	0.112	0.101	0.091	0.082	0.073	
				0.104	0.093	0.084	0.076	0.068	0.061	0.055	
			0.086	0.078	0.070	0.063	0.057	0.051	0.046	0.041	
		0.072	0.065	0.058	0.052	0.047	0.043	0.038	0.034	0.031	
	0.060	0.054	0.049	0.044	0.039	0.035	0.032	0.029	0.026	0.023	
	0	1	2	3	4	5	6	7	8	9	10
											0.000
										0.001	0.003
									0.002	0.006	0.015
								0.004	0.014	0.030	0.047
	Elementary Prices						0.009	0.029	0.054	0.077	0.092
						0.020	0.057	0.094	0.118	0.127	0.122
					0.046	0.107	0.151	0.167	0.160	0.138	0.111
				0.101	0.190	0.223	0.212	0.177	0.136	0.098	0.068
			0.220	0.310	0.293	0.232	0.166	0.112	0.072	0.045	0.027
		0.472	0.444	0.315	0.200	0.120	0.069	0.039	0.021	0.012	0.006
	1.000	0.472	0.224	0.107	0.051	0.025	0.012	0.006	0.003	0.001	0.001
Bond Price		0.943	0.888	0.833	0.779	0.727	0.676	0.627	0.580	0.535	0.492
Spot Rates		1.060	1.061	1.063	1.064	1.066	1.067	1.069	1.070	1.072	1.073
Spot %		6.000	6.146	6.294	6.443	6.593	6.744	6.894	7.045	7.194	7.347

Figure 14.3 Construction of forward prices and determination of spot rates.

yr	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
spot	7.67	8.27	8.81	9.31	9.75	10.16	10.52	10.85	11.15	11.42	11.67	11.89	12.09	12.27	
a's	7.67	8.829	9.799	10.66	11.3	11.93	12.33	12.73	13.05	13.27	13.51	13.57	13.67	13.72	
State															
13														15.62	
12													15.42	15.47	
11												15.15	15.26	15.31	
10											14.93	15.00	15.11	15.16	
9		**Short Rate**								14.52	14.78	14.85	14.96	15.01	
8									14.13	14.38	14.63	14.70	14.81	14.86	
7								13.66	13.99	14.23	14.49	14.56	14.66	14.71	
6							13.09	13.52	13.85	14.09	14.34	14.41	14.52	14.57	
5						12.54	12.96	13.39	13.72	13.95	14.20	14.27	14.37	14.42	
4					11.76	12.42	12.83	13.25	13.58	13.81	14.06	14.13	14.23	14.28	
3				10.99	11.64	12.29	12.70	13.12	13.44	13.67	13.92	13.99	14.09	14.13	
2			10.00	10.88	11.53	12.17	12.58	12.99	13.31	13.54	13.78	13.85	13.95	13.99	
1		8.92	9.90	10.77	11.41	12.05	12.45	12.86	13.18	13.40	13.64	13.71	13.81	13.85	
0	7.67	8.83	9.80	10.66	11.30	11.93	12.33	12.73	13.05	13.27	13.51	13.57	13.67	13.72	
14															1E-05
13														3E-05	2E-04
12													6E-05	3E-04	0.001
11												1E-04	7E-04	0.002	0.004
10											3E-04	0.002	0.004	0.008	0.012
9		**Forward**								7E-04	0.003	0.008	0.014	0.019	0.024
8		**Equations**							0.002	0.007	0.015	0.024	0.031	0.035	0.036
7								0.004	0.014	0.027	0.039	0.047	0.05	0.047	0.041
6							0.009	0.027	0.048	0.063	0.069	0.067	0.059	0.048	0.037
5						0.02	0.052	0.081	0.096	0.095	0.083	0.067	0.051	0.036	0.025
4					0.044	0.098	0.131	0.136	0.12	0.095	0.07	0.048	0.032	0.02	0.012
3				0.097	0.175	0.196	0.175	0.136	0.096	0.064	0.04	0.024	0.014	0.008	0.005
2			0.213	0.291	0.263	0.196	0.131	0.082	0.048	0.027	0.015	0.008	0.004	0.002	0.001
1		0.464	0.427	0.291	0.175	0.098	0.053	0.027	0.014	0.007	0.003	0.002	8E-04	4E-04	2E-04
0	1	0.464	0.213	0.097	0.044	0.02	0.009	0.004	0.002	8E-04	3E-04	1E-04	7E-05	3E-05	1E-05
P0	1	0.929	0.853	0.776	0.7	0.628	0.56	0.496	0.439	0.386	0.339	0.297	0.26	0.227	0.198
Forward Rate		7.67	8.27	8.81	9.31	9.75	10.16	10.52	10.85	11.15	11.42	11.67	11.89	12.09	12.27

Figure 14.4 Term match using the Black-Derman-Toy model.

6. (Ho-Lee volatility) From node $(k-1, s)$ the short rate next period is either $a_k + b_k s$ or $a_k + b_k(s+1)$. The variance is

$$\begin{aligned} \text{var} &= \tfrac{1}{2}(a_k + b_k s)^2 + \tfrac{1}{2}(a_k + b_k(s+1))^2 - \tfrac{1}{4}[a_k + b_k(s+1) + a_k + b_k s]^2 \\ &= \tfrac{1}{2}b_k^2 - \tfrac{1}{4}b_k^2 \\ &= \tfrac{1}{4}b_k^2. \end{aligned}$$

   Hence the standard deviation is $b_k/2$.

7. (Term match) This can be solved by a simple modification of the spreadsheet in the Ho-Lee example. We merely change the formula for the short rate to $a_k e^{.01s}$. The new values of the a_k's are found by optimization as in the earlier example. The resulting spreadsheet is shown in Fig. 14.4

8. (Swaps) Fig. 14.5 shows three binomial lattices. The first is the short-rate lattice. The second is the lattice of values for the floating rate payment stream (in hundred thousands of dollars). Each node value is equal to the amount of interest to be paid at the end of the year plus the risk-neutral sum of the values in the next two

nodes—all of this discounted by the risk-free rate. For example, the value at the top of the last column is $100 \times .260/1.260 = 20.629$; and the value at the top of the second to last column is $[100 \times .20 + .5 \times 20.629 + .5 \times 15.250]/1.20 = 31.612$. The final value is \$3.979 million.

					0.260
Short rate lattice				0.200	0.180
			0.154	0.138	0.125
		0.118	0.106	0.096	0.086
	0.091	0.082	0.074	0.066	0.060
0.070	0.063	0.057	0.051	0.046	0.041
					20.629
				31.612	15.250
Floating rate value			37.308	23.721	11.077
		39.874	28.214	17.421	7.939
	40.474	30.240	20.838	12.586	5.634
39.790	30.676	22.378	15.115	8.982	3.969
					6.857
				13.108	7.322
Fixed rate value			19.313	14.179	7.683
		25.751	21.002	15.019	7.954
	32.558	28.012	22.330	15.654	8.153
39.790	35.313	29.785	23.338	16.124	8.297

Figure 14.5 Determination of swap interest rate.

The third lattice is constructed with a given fixed interest rate A. The values are the present values of the fixed-rate payments. For example the value at the top of the second to last column is $[100A + .5 \times 6.857 + .5 \times 7.322]/1.2$. The value of A is adjusted until the initial value of this lattice is equal to that of the one above. This value is $A = 8.64\%$.

9. (Swaption) To find the value of the swaption, we merely put the minimum of the two value lattices (from the previous exercise) in the third column. (The case where this minimum indicates exercise of the swaption is shown in bold in Fig. 14.6.) We then value the lattice backward in the usual way. The value of the

		25.751		1.628	
	32.558	28.012		Swaption	Value
38.162	31.829	**22.378**			

Figure 14.6 Evaluation of a swaption

swaption is the difference between the value obtained and the value without the swaption.

10. (Change of variable)

(a) Ler r^{+-} be the value for an up move followed by a down. We have

$$\begin{aligned} r^{+-} &= r + \sigma(r,t)\sqrt{\Delta t} - \sigma(r^{+}, t+\Delta t)\sqrt{\Delta t} \\ r^{-+} &= r - \sigma(r,t)\sqrt{\Delta t} + \sigma(r^{-}, t+\Delta t)\sqrt{\Delta t} \end{aligned}$$

Clearly these are not equal in general.

(b) For

$$w(r,t) = \int_0^r \frac{dy}{\sigma(y,t)}$$

we have

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{1}{\sigma(r,t)} \\ \frac{\partial^2 w}{\partial r^2} &= -\frac{1}{\sigma(r,t)^2}\frac{\partial \sigma(r,t)}{\partial r} \\ \frac{\partial w}{\partial t} &= \int_0^r \frac{-1}{\sigma(y,t)^2}\frac{\partial \sigma(y,t)}{\partial t} dy. \end{aligned}$$

Using these in Ito's lemma, we find

$$dw = \left[\frac{\mu(r,t)}{\sigma(r,t)} - \int_0^r \frac{1}{\sigma(r,t)^2}\frac{\partial \sigma(r,t)}{\partial t} dy - \frac{1}{2}\frac{\partial \sigma(r,t)}{\partial r}\right] dt + dz.$$

Clearly the volatility term is constant (equal to 1).

(c) The transformation is

$$w(r,t) = \int_1^r \frac{dy}{\sigma y} = \frac{\ln r}{\sigma}.$$

Note that we integrate from 1 rather than 0.

Also we find

$$dw = \frac{1}{\sigma}[\mu - \tfrac{1}{2}\sigma^2]dt + dz.$$

11. (Ho–Lee term structure) Clearly

$$\begin{aligned} F(t) &= -\frac{1}{t}\{\ln A(0,t) - rt\} \\ &= -\frac{1}{t}\left\{\tfrac{1}{2}at^2 - \tfrac{1}{6}\sigma^2 t^3 - rt\right\} \\ &= r - \tfrac{1}{2}at + \tfrac{1}{6}\sigma^2 t^2. \end{aligned}$$

12. (Continuous zero) Since the cash flow is all at time T we have

$$V(0) = \hat{\mathrm{E}}\left[\exp\left(\int_0^T -r(s)\,\mathrm{d}s\right)\right]$$

First find that $r(s) = r_0 + \sigma\hat{z}(s)$. Hence $r(s)$ is a normal random variable. Next we need $\int_0^T -r(s)\,\mathrm{d}s = -r_0T - \sigma\int_0^T \hat{z}(s)\,\mathrm{d}s = -r_0T - \sigma Z(T)$. The variable $Z(T)$ is basically a sum of normal random variables, so it is normal. It has zero mean, since each $\hat{z}(s)$ has zero mean. We need the variance of $Z(T)$.

Let $S(t) = \hat{\mathrm{var}}\, Z(t)$. Then

$$S(t) = \hat{\mathrm{E}}\left[\int_0^t z(s)\,\mathrm{d}s\right]^2.$$

Hence

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = 2\hat{\mathrm{E}}\left[z(t)\int_0^t z(s)\,\mathrm{d}s\right]$$

And

$$\frac{\mathrm{d}^2S(t)}{\mathrm{d}t^2} = 2\hat{\mathrm{E}}[z(t)^2] + 2\hat{\mathrm{E}}\left[\frac{\mathrm{d}z}{\mathrm{d}t}\int_0^t z(s)\,\mathrm{d}s\right].$$

Since changes in $z(t)$ are independent of $z(s)$ for $s < t$ the second term on the right of the above is zero. Hence

$$\frac{\mathrm{d}^2S(t)}{\mathrm{d}t^2} = 2\hat{\mathrm{var}}\, z(t) = 2t.$$

And we find $S(t) = t^3/3 + at + b$. Since $S(0) = 0$ and $S'(0) = 0$ it follows that $a = b = 0$. Hence $S(T) = T^2/3$.

Since $-Z(T)$ is normal, with mean zero and variance $T^3/3$ we have

$$\hat{\mathrm{E}}[\exp(-\sigma Z(T))] = \exp[\sigma^2T^3/6].$$

Hence

$$V(0) = \exp\left[-r_0T + \sigma^2T^3/6\right]$$

which agrees with the Ho-Lee example.

Chapter 15
Optimal Portfolio Growth

1. (Simple wheel strategy) At each turn, your money will either be multiplied by $3\gamma + (1-\gamma) = 1 + 2\gamma$ or by $1-\gamma$, each with probability one-half. Hence over the long run, the factor is $(1+2\gamma)^{n/2}(1-\gamma)^{n/2}$. We wish to maximize $(1+2\gamma)(1-\gamma) = 1 + \gamma - 2\gamma^2$. This is easily found to give $\gamma = 1/4$.

2. (How to play the state lottery)

   (a) Suppose Victor buys one ticket. His expected logarithm is then

$$\begin{aligned} \mathrm{E}\ln &= 10^{-6}\ln[10^7 + (10^5 - 1)] + (1 - 10^{-6})\ln[10^5 - 1] \\ &\approx 10^{-6}\ln 10^7 + (1 - 10^{-6})[\ln 10^5 - 10^{-5}] \\ &\approx 10^{-6}\ln 10^7 + \ln 10^5 - 10^{-6}\ln 10^5 - 10^{-5} \\ &= \ln 10^5 + [7\ln 10 - 5\ln 10 - 10]10^{-6} \\ &< \ln 10^5. \end{aligned}$$

   Hence, Victor should not buy a lottery ticket (even at the 10 to 1 odds in his favor!).

   (b) Let α be the fraction of tickets purchased. The optimal solution maximizes

$$10^{-6}\ln[10^7\alpha + 10^5 - \alpha] + (1 - 10^{-6})\ln[10^5 - \alpha]$$

   This implies

$$\frac{10^{-6}(10^7 - 1)}{10^7\alpha + 10^5 - \alpha} + \frac{1 - 10^{-6}}{10^5 - \alpha} = 0$$

   Approximately,

$$\frac{10}{10^7\alpha + 10^5} = \frac{1}{10^5}$$

   which has solution $\alpha = 11/100$. Victor should invest only 11 cents of his \$100,000 wealth, despite the tempting odds.

3. (Easy policy) The expected logarithm is

$$\tfrac{1}{2}\ln(2\alpha + (1-\alpha)) + \tfrac{1}{2}\ln(\tfrac{\alpha}{2} + (1-\alpha)).$$

Differentiation and setting to zero gives

$$\frac{1}{1+\alpha} - \frac{\frac{1}{2}}{1-\frac{\alpha}{2}} = 0.$$

This has solution $\alpha = \frac{1}{2}$.

4. (A general betting wheel)

   (a) If sector j occurs on the wheel, the total return is $r_j\alpha_j + 1 - \sum_{i=1}^n \alpha_i$. where $1 - \sum_{i=1}^n \alpha_i$ is the fraction of wealth not invested. So, maximizing the expected log of returns for all sectors gives the problem

   $$\max_{\alpha_1,\ldots\alpha_n} \sum_{j=1}^n p_j \ln\left(r_j\alpha_j + 1 - \sum_{i=1}^n \alpha_i\right).$$

   (b) These are the first-order conditions of differentiating the objective with respect to α.

   (c) Setting $\alpha_i = p_i$ in the equation from (b) implies

   $$\frac{p_k r_k}{r_k p_k + 1 - \sum_{i=1}^n p_i} = \sum_{j=1}^n \frac{p_j}{r_j p_j + 1 - \sum_{i=1}^n p_j}.$$

   But since $\sum_{i=1}^n p_i = 1$, this requirement reduces to

   $$\frac{p_k r_k}{p_k r_k} = \sum_{j=1}^n \frac{p_j}{r_j p_j}$$

   which holds if, as assumed, $\sum_{j=1}^n \frac{1}{r_j} = 1$.

   (d) Since $r_1 = 1, r_2 = 2, r_3 = 6$, we have $\sum_{j=1}^3 \frac{1}{r_j} = 1$. Thus by part (c) an optimal strategy is $\alpha_i = p_i$; that is, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{6}$. So the optimal growth rate is $e^m = 1.06991$; (see Example 15.5).

5. (More on the wheel) From the previous problem, the first-order conditions imply

$$\frac{p_k r_k}{r_k\alpha_k + \alpha_0} = \sum_{j=1}^n \frac{p_j}{r_j\alpha_j + \alpha_0} \quad \text{for } k = 1, 2, \ldots n-1$$

where $\alpha_0 = 1 - \sum_{i=1}^n \alpha_i$. Dividing both sides by r_k and summing over $k = 1, 2, \ldots, n-1$ we have

$$\sum_{k=1}^{n-1} \frac{p_k}{r_k\alpha_k + \alpha_0} = \left(\sum_{i=1}^{n-1} \frac{1}{r_k}\right)\left(\sum_{j=1}^n \frac{p_j}{r_j\alpha_j + \alpha_0}\right).$$

(a) Using the fact that $\sum_{k=1}^{n-1} \frac{1}{r_k} = 1 - \frac{1}{r_n}$ and $\alpha_n = 0$, we can simplify to get

$$\frac{p_n r_n}{\alpha_0} = \sum_{k=1}^{n} \frac{p_k}{r_k \alpha_k + \alpha_0}.$$

Thus, we have

$$\frac{p_k r_k}{r_k \alpha_k + \alpha_0} = \frac{p_n r_0}{\alpha_0}$$

which implies

$$\alpha_k = \left(\frac{p_k}{p_n r_n} - \frac{1}{r_k} \right) \alpha_0.$$

Finally, since $\sum_{k=1}^{n} \alpha_k = 1 - \alpha_0$ we find $\alpha_0 = p_n r_n$ and

$$\alpha_k = p_k - \frac{p_n r_n}{r_k} \quad \text{for } k = 1, 2, \ldots, n-1.$$

(b) For the specific wheel

$$\begin{aligned} \alpha_1 &= \frac{1}{2} - \frac{2}{3 \cdot 3} = \frac{5}{18} \\ \alpha_2 &= 0 \\ \alpha_3 &= \frac{1}{6} - \frac{2}{3 \cdot 6} = \frac{1}{18} \end{aligned}$$

6. (Volatility pumping) Since $\nu_i = \mu_i - \frac{1}{2}\sigma_i^2$ for each stock, we have $\mu_i = .23$. Thus,

$$\begin{aligned} \nu &= \sum_{i=1}^{n} w_i \mu_i - \tfrac{1}{2} \textstyle\sum_{i,j}^{n} w_i \sigma_{ij} w_j \\ &= .23 - \frac{1}{2n^2} \left[\sigma_{i \neq j}(n^2 - n) + n\sigma_{ii} \right]. \end{aligned}$$

Since $\sigma_{i \neq j} = .08$ and $\sigma_{ii} = \sigma_i^2 = .16$, we have $\nu = .19 - \frac{.04}{n}$.

7. (Dow Jones Average puzzle) Keeping the certificates in his drawer is equivalent to following a "buy and hold" strategy. The Dow Jones Average, on the other hand, "sells" some portion of stocks when they get high (and split) and the proceeds of the "sale" are used to buy other stocks in the Average. This is a weak form of pumping, and so we expect that the Dow Jones Average will out-perform the buy and hold strategy (and it does).

8. (Power utility)

(a) We know that returns combine according to weight in the portfolio. Hence, letting $B(t)$ be the value of a bond at time t, we have

$$\begin{aligned} \frac{dX(t)}{X(t)} &= w\frac{dS(t)}{S(t)} + (1-w)\frac{dB(t)}{B(t)} \\ &= (r + w(\mu - r))\, dt + w\sigma\, dz. \end{aligned}$$

It follows that $X(t)$ is lognormal with mean $rt + w(\mu - r)t - \frac{1}{2}w^2\sigma^2 t$ and standard deviation $w\sigma\sqrt{t}$. In other words,

$$X(t) = X(0)e^{rt+w(\mu-r)t-\frac{1}{2}w^2\sigma^2 t+wn\sigma\sqrt{t}}.$$

(b) We have immediately that

$$U(X(t)) = \frac{1}{\gamma}X(y)^\gamma = \frac{X(0)^\gamma}{\gamma}e^{\gamma[rt+w(\mu-r)t-\frac{1}{2}w^2\sigma^2 t+wn\sigma\sqrt{t}]}.$$

Hence the expected value is

$$\mathrm{E}[U(X(t)] = \frac{1}{\gamma}\mathrm{E}[X(y)^\gamma] = \frac{X(0)^\gamma}{\gamma}e^{\gamma[rt+w(\mu-r)t-\frac{1}{2}w^2\sigma^2 t]+\frac{1}{2}\gamma^2 w^2\sigma^2 t}.$$

(c) The first-order conditions are

$$\frac{d\mathrm{E}[U(X(t)]}{dw} = 0.$$

Or equivalently

$$0 = \mathrm{E}[\mathrm{U}(X(t))]\,[\gamma t(\mu - r) - w\sigma^2\gamma t + w\sigma^2\gamma^2 t].$$

Hence

$$\gamma t(\mu - r) = w\gamma t\sigma^2(1-\gamma)$$

which yields

$$w = \frac{\mu - r}{\sigma^2(1-\gamma)}$$

as stated.

9. (Discrete-time log-optimal pricing formula)

(a) The log-optimal portfolio is defined by the problem

$$\max \mathrm{E}[\ln(1 + \sum_{i=1}^{n} r_i\alpha_i - (1 - \sum_{j=1}^{n}\alpha_j)r_f)].$$

The first-order conditions are

$$\mathrm{E}\left[\frac{r_i - r_f}{1 + r_0}\right] = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

This is equivalent to

$$\mathrm{E}[r_i P_0] - r_f \mathrm{E}[P_0] = 0.$$

Using the relation $\mathrm{E}[r_i P_0] = \mathrm{cov}(r_i, P_0) + \overline{r}_i \mathrm{E}[P_0]$, the above can be written as

$$\overline{r}_i - r_f = -\frac{\mathrm{cov}(r_i, P_0)}{\mathrm{E}[P_0]}$$

which is the stated result.

(b) We have

$$(\mu_i - r_f)\Delta t = -\frac{\mathrm{cov}(n_i\sqrt{\Delta t}, 1/(1+\mu_0\Delta t + n_0\sqrt{\Delta t}))}{\mathrm{E}[1/(1+\mu_0\Delta t + n_0\sqrt{\Delta t})]}.$$

Using, in the numerator, the approximation $1/(1+x) \approx 1-x$, and noting that the denominator approaches 1, we find

$$(\mu_i - r_f)\Delta t = \sigma_{i,0}\Delta t.$$

Hence,

$$\mu_i - r_f = \sigma_{i,0}.$$

Chapter 16
General Investment Analysis

1. (A state tree)

		a	b	c
Security	1	1.2	1.0	0.8
	2	1.2	1.3	1.4

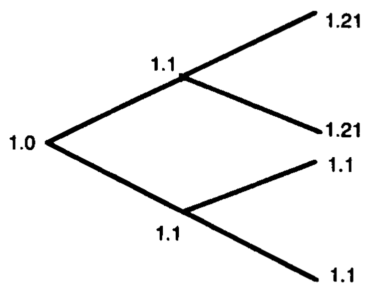
To find the short-term riskless asset, we note that for any portfolio with weights α_1, α_2 with $\alpha_1 + \alpha_2 = 1$, the payoff factor for state 1 is 1.2. Hence if there is a riskless return it must be 1.2. We therefore solve the equations

$$\begin{aligned} \alpha_1 + 1.3\alpha_2 &= 1.2 \\ .8\alpha_1 + 1.4\alpha_2 &= 1.2 \end{aligned}$$

This has solution $\alpha_1 = 1/3$, $\alpha_2 = 2/3$. Since $\alpha_1 + \alpha_2 = 1$ it is a portfolio, and hence defines a riskless asset.

Yes, there is an arbitrage: buy security 2 and subtract the same amount of security 1.

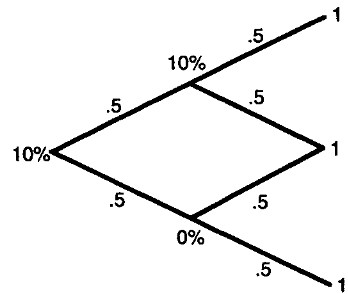
2. (Node separation)



A tree must be used to represent the growth of \$1 because the middle node separates.

3. (Bond valuation)

   (a) The appropriate lattice is



The value at the top node at time 1 is $v_{11} = (.5 + .5)/1.1 = .9091$. The value at the lower node is $v_{10} = (.5 + .5)/1 = 1.0$. Finally, the value at the initial node is

$$v_{00} = .5 \times .5(.9091 + 1.0)/1.1 = \$0.8678.$$

   (b)

$$v_{00} = \tfrac{1}{4}\frac{\$1}{1.21} + \tfrac{1}{4}\frac{\$1}{1.21} + \tfrac{1}{4}\frac{\$1}{1.1} + \tfrac{1}{4}\frac{\$1}{1.1} = \$0.8678.$$

4. (Optimal option valuation) In each period we have the the following maximization problem

$$\max_{\alpha}\left\{p_1\sqrt{\alpha u + (1-\alpha)R_0} + p_2\sqrt{\alpha + (1-\alpha)R_0} + p_3\sqrt{\alpha d + (1-\alpha)R_0}\right\}.$$

The solution is $\alpha = 1.01$. The corresponding risk-neutral probabilities are $q_1 = .218$, $q_2 = .635$, and $q_3 = .148$. The option lattice is almost identical to that of the example in the text. The value of the call is found to be \$5.8070.

5. (Gold correlation) We wish to find the q_{ij}'s. The gold fluctuation is modeled as a binomial lattice with $u = 1.2$, $d = .9$. The interest rate has $u = 1.1$ and $d = .9$ with risk-neutral probabilities of .5. The initial interest rate is 4%. The risk-neutral probability for gold is

$$q_{u_g} = \frac{1 + r - d}{u - d} = .46667.$$

To find the risk-neutral probabilities for each of the four successor nodes, we need a total of four equations, but we only have three equations so far (the two individual risk-neutral probabilities and the fact that the sum of all four probabilities is 1.) We must first find the real probabilities and then use the invariance theorem to get the final equation.

Let S' and r' denote the $S(k+1)/S(k)$ and $r(k+1)/r(k)$, respectively. Then we calculate

$$\begin{aligned}
\mathrm{E}[\ln S'] &= .6\ln(1.3) + .4\ln(0.9) = .067\\
\mathrm{E}[\ln r'] &= .7\ln(1.1) + .3\ln(.09) = .035\\
\mathrm{var}[\ln S'] &= .6(\ln 1.2)^2 + .4(\ln 0.9)^2 - \mathrm{E}[\ln S']^2 = .02\\
\mathrm{var}[\ln r'] &= .7(\ln 1.1)^2 + .3(\ln 0.9)^2 - \mathrm{E}[r']^2 = .008\\
\mathrm{cov}(\ln S', \ln r') &= \rho(\mathrm{var}(\ln S')\mathrm{var}(\ln r'))^{\frac{1}{2}} = -.005
\end{aligned}$$

We can now solve for the p_{ij}'s (letting i = gold, j = interest rate). We let "1" correspond to an "up" move.

$$\begin{aligned}
p_{11} + p_{12} &= .6\\
p_{11} + p_{21} &= .7\\
p_{21} + p_{22} &= .4\\
p_{11}(\ln 1.2)(\ln 1.1) &+ p_{12}(\ln 1.2)(\ln 0.9) + p_{21}(\ln 0.9)(\ln 1.1)\\
+p_{22}(\ln 0.9)^2 &= \mathrm{cov}(\ln S', \ln r') + \mathrm{E}(\ln S')(\mathrm{E}(\ln r')
\end{aligned}$$

This yields $p_{11} = .33$ $p_{12} = .27$ $p_{21} = .37$ $p_{22} = .03$

We now have the following set of equations for the q_{ij}'s

$$\begin{aligned}
q_{11} + q_{12} &= q_{u_g} = .46667\\
q_{11} + q_{21} &= q_{u_r} = .5\\
q_{11} + q_{12} + q_{21} + q_{22} &= 1\\
\frac{q_{11}q_{22}}{q_{12}q_{21}} &= \frac{p_{11}p_{22}}{p_{12}p_{21}}
\end{aligned}$$

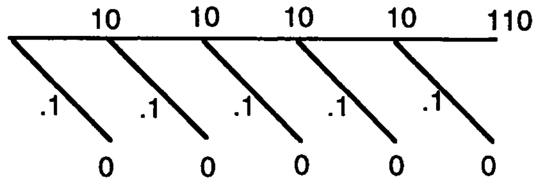
The solution is $q_{11} = .1$ $q_{12} = .36$ $q_{21} = .4$ $q_{22} = .14$.

6. (Complexico mine) This problem can be solved by combining the methods for solving example 16.4 and example 12.8. A set of spreadsheets is required. The answer is \$14.898 (in millions).

7. (Simultaneous solution) We use a numerical Black-Scholes calculator and find S and σ so that the prices of the two options match the given data. Using a time to maturity of $T = .25$ and interest rate of 7% we find $S = \$16.81$ and $\sigma = 20.6\%$. It would be more appropriate to use a formula for options on futures, rather than the standard option formula.

8. (Default risk) We may treat the default risk as risk-neutral risk because it is independent of the interest-rate process and because we seek the zero-level price.

(a)



In this case

$$P = 100 \times \left(\frac{.9}{1.1}\right)^5 + \sum_{k=1}^{5} \left(\frac{.9 \times 10}{1.1}\right)^k = \$65.17.$$

(b) Evaluation can be done recursively using

$$v_{t,s} = c_{t,s} + \frac{1}{1+r_{t,s}} \frac{1}{2}(.9v_{t+1,s} + .9v_{t+1,s+1}).$$

This produces $v_{0,0} = \$63.25$.

9. (Automobile choice) We set $a_0 = 1/1000$. Hence $a_k = a_0(1.05)^{-5} = 1/(1000 \cdot (1.05)^k)$. We know from the CE formula that for uncertain cash flows at time j, the CE at time $j-1$ is equal to the discounted CE at time j.

   For car B:

$$\mathrm{CE_B} = -35{,}000 - \frac{1}{a_4 R^4} \ln\left(.5e^{-a_4 12{,}000} + .5e^{-a_4 8{,}000}\right) = -\$27{,}761.78.$$

   For car A: We must use two stages. The certainty equivalent of the second car, as evaluated when that car is purchased is

$$\mathrm{CE_{A2}} = -20{,}000 - \frac{1}{a_4 R^2} \ln\left(.5e^{-a_4 10{,}000} + .5e^{-a_4 5{,}000}\right) = -\$14{,}718.54.$$

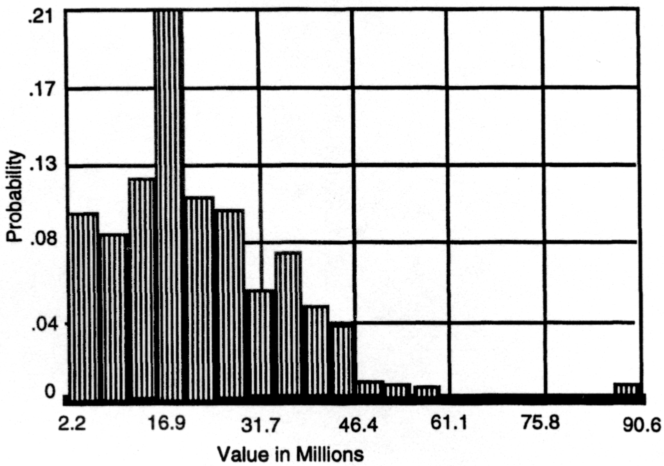
   The total certainty equivalent at year 2 is therefore either $-\$14{,}177.19 + \$10{,}000 = \$4{,}177.19$ or $-\$14{,}177.19 + \$5{,}000 = \$9{,}177.19$ each with probability .5.

   Hence the overall certainty equivalent of car A is

$$\mathrm{CE_A} = -20{,}000 - \frac{1}{a_2 R^2} \ln\left(e^{-a_2 10{,}000 + \mathrm{CE_{A2}}} + .5e^{-a_2 5{,}000 + \mathrm{CE_{A2}}}\right) = \$28{,}132.52.$$

   Hence car B is preferred and the difference in certainty equivalence is \$370.74.

10. (Continuco mine simulation) The results of one simulation are shown below. The corresponding (average) value is \$22.5 million.



11. (Gavin's final) At time 1 there is one period to go so the CAPM applies. this gives

$$V_1 = \frac{\mathrm{E}_1[x_{2|1}]}{1 + r + \beta_2(\overline{r}_2 - r)}$$

where

$$\beta_2 = \mathrm{cov}[x_{2|1}/V_1, r_2].$$

Now consider the period from time 0 to time 1. We have

$$V_0 = \frac{\mathrm{E}_0[V_1]}{1 + r + \beta_1(\overline{r}_1 - r)}$$

where

$$\beta_1 = \mathrm{cov}[V_1/V_0, r_1].$$

Substituting the expression for V_1 and accounting for $\mathrm{E}_0[\mathrm{E}_1[x_{2|1}] = \mathrm{E}_0[x_{2|0}]$ and that β_2 is not random, we obtain the result stated.