

CVT - UNIT - V

LAPLACE TRANSFORM

PART - A (2 Marks)

1. State the conditions under which the Laplace transform of $f(t)$ exist.

i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a, b]$, where $a > 0$.

ii) $f(t)$ should be of exponential order

2. State and prove first shifting theorem.

Statement:

$$\text{If } \mathcal{L}[f(t)] = F(s)$$

Then

$$\mathcal{L}[e^{-at} f(t)] = F(s+a)$$

Proof:

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$$

$$\begin{aligned} L[e^{-at}f(t)] &= \int_0^\infty e^{-at} f(t) e^{-st} dt \\ &= \int_0^\infty e^{-t(s+a)} f(t) dt \end{aligned}$$

$$L[e^{-at}f(t)] = F(s+a).$$

③ State and prove change of scale property.

Statement: $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a}F\left[\frac{s}{a}\right]$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$L[f(at)] = \int_0^\infty e^{-st} f(at) dt$$

$$\text{Put } at = x \quad | \quad t \rightarrow 0 \Rightarrow x \rightarrow 0$$

$$adt = dx \quad | \quad t \rightarrow \infty \Rightarrow x \rightarrow \infty$$

$$\Rightarrow dt = \frac{dx}{a}$$

$$= \int_0^\infty e^{-sx/a} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t) dt \quad [\because x \text{ is dummy variable}]$$

$$L[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]$$

(3)

④ Is the linearity property applicable
to $L\left(\frac{1-\cos t}{t}\right)$

$$L\left[\frac{1-\cos t}{t}\right] = L\left[\frac{1}{t}\right] - L\left[\frac{\cos t}{t}\right]$$

$L\left[\frac{1}{t}\right]$ does not exist.

$$\lim_{t \rightarrow 0} \frac{1}{t} = \frac{1}{0} = \infty$$

$L\left[\frac{\cos t}{t}\right]$ does not exist.

$$\lim_{t \rightarrow 0} \frac{\cos t}{t} = \frac{1}{0} = \infty$$

\therefore Linearity property is not applicable to
 $L\left(\frac{1-\cos t}{t}\right)$

⑤ Does $L\left(\frac{\cos at}{t}\right)$ exist?

$$L\left[\frac{\cos at}{t}\right] = \lim_{t \rightarrow 0} \frac{\cos at}{t}$$

$$= \frac{1}{0}$$

$$L\left[\frac{\cos at}{t}\right] = \infty$$

Therefore, $L\left[\frac{\cos at}{t}\right]$ does not exist.

⑥ Find the $L^{-1} \left[\frac{1}{(s+1)^2 + 1} \right]$ (F)

$$L^{-1} \left[\frac{1}{(s+1)^2 + 1} \right] = e^t \sin t$$

⑦ Find the inverse Laplace transform of

$$\log \left(\frac{s+1}{s-1} \right)$$

Soln.

$$F(s) = \log \left(\frac{s+1}{s-1} \right)$$

$$L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$$

$$L^{-1} \left[\log \left(\frac{s+1}{s-1} \right) \right] = -\frac{1}{t} L^{-1} \left[\frac{d}{ds} \log \left(\frac{s+1}{s-1} \right) \right]$$

$$= -\frac{1}{t} L^{-1} \left[\frac{d}{ds} (\log(s+1) - \log(s-1)) \right]$$

$$= -\frac{1}{t} L^{-1} \left[\frac{1}{s+1} - \frac{1}{s-1} \right]$$

$$= -\frac{1}{t} \left[L^{-1} \left(\frac{1}{s+1} \right) - L^{-1} \left(\frac{1}{s-1} \right) \right]$$

$$\sinht = \frac{e^t - e^{-t}}{2}$$

$$= -\frac{1}{t} [e^{-t} - e^t]$$

$$= \frac{1}{t} [e^t - e^{-t}]$$

$$L^{-1} \left[\log \left(\frac{s+1}{s-1} \right) \right] = \frac{2}{t} \sinht$$

⑧ Find the Laplace transform of unit step function. ⑤

Soln: $\frac{USF}{u(t-a)} = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$\begin{aligned} L[u(t-a)] &= \int_0^a e^{-st} f(t) dt + \int_a^{\infty} e^{-st} f(t) dt \\ &= 0 + \int_a^{\infty} e^{-st} (1) dt \\ &= \left[\frac{e^{-st}}{-s} \right]_a^{\infty} \\ &= -\frac{1}{s} [e^{-\infty} - e^{-as}] \\ &= -\frac{1}{s} [0 - e^{-as}] \end{aligned}$$

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

⑨ State initial and final value theorem

Initial value theorem:

Statement: If $L[f(t)] = F(s)$

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Then

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow 0} s F(s)$$

Final value theorem:Statement : If $\mathcal{L}[f(t)] = F(s)$

Then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)$$

(10) State convolution theorem.

If $f(t)$ & $g(t)$ are function definition
for $t \geq 0$, then

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$$

(7)

PART-B

- ① Find the Laplace transform of the triangular wave function $f(t) = \begin{cases} t & 0 \leq t \leq a \\ 2a-t, & a \leq t \leq 2a \end{cases}$
 with $f(t) = f(t+2a)$.

Soln:

$$\begin{aligned}
 \mathcal{L}[f(t)] &= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t) dt + \int_0^{2a} e^{-st} f(t) dt \right] \\
 &\stackrel{?}{=} \frac{1}{1-e^{-2as}} \left[\int_0^{2a} e^{-st} f(t) dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t) dt + \int_0^{2a} e^{-st} f(t) dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st} dt + \int_0^{2a} (2a-t) e^{-st} dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left\{ \left[(t) \left(\frac{e^{-st}}{-s} \right) - (1) \left(\frac{e^{-st}}{s^2} \right) \right]_0^a + \right. \\
 &\quad \left. \left[(2a-t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{s^2} \right) \right]_0^{2a} \right\} \\
 &= \frac{1}{1-e^{-2as}} \left[-\frac{a^2 e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{1}{s^2} - 0 + \frac{e^{-2sa}}{s^2} \right. \\
 &\quad \left. + a^2 \frac{e^{-sa}}{s} - \frac{e^{-sa}}{s^2} \right], \\
 &\stackrel{?}{=} \frac{1}{s^2(1-e^{-2as})} \left[1 + e^{-2sa} - 2e^{-sa} \right] \boxed{(a-b)^2 = a^2 + b^2 - 2ab} \\
 &= \frac{1}{s^2 [1-e^{-2as}]} \cdot (1-e^{-as})^2
 \end{aligned}$$

GivenPeriod = $2a$

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$$\begin{aligned}
 L[\sinh(s)] &= \frac{(1 - e^{-as})^2}{s^2[1 - (e^{-as})^2]} \\
 &= \frac{(1 - e^{-as})^2}{s^2(1 - e^{-as})(1 + e^{-as})} \\
 &= \frac{(1 - e^{-as})}{s^2(1 + e^{-as})}
 \end{aligned}$$

$$\tanh\left(\frac{\theta}{2}\right) = \frac{1 - e^\theta}{1 + e^\theta}$$

$$L[\sinh(s)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

(i)

$$(ii) \text{ find } L[t^2 e^t \cos t]$$

Soln:

$$L[t^2 \sinh(s)] = \frac{d^2}{ds^2} L[\sinh(s)]$$

$$L[t^2 \cos t] = \frac{d^2}{ds^2} L[\cos t]$$

$$= \frac{d^2}{ds^2} \left[\frac{s}{s^2 + 1} \right]$$

$$L[\cos t] = \frac{s}{s^2 + 1}$$

$$= \frac{d}{ds} \left[\frac{(s^2 + 1) \cdot 1 - (s)(2s)}{(s^2 + 1)^2} \right]$$

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$$= \frac{d}{ds} \left[\frac{s^2 + 1 - 2s^2}{(s^2 + 1)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{1 - s^2}{(s^2 + 1)^2} \right]$$

$$= \frac{(s^2 + 1)^2(-2s) - (1 - s^2)2(s^2 + 1) \cdot 2s}{(s^2 + 1)^4}$$

$$= \frac{2s(s^2 + 1)[-(s^2 + 1) - 2(1 - s^2)]}{(s^2 + 1)^3}$$

$$= \frac{2s}{(s^2 + 1)^3} [-s^2 - 1 - 2 + 2s^2]$$

$$\boxed{L[t^2 \cos t] = \frac{2s}{(s^2 + 1)^3} (s^2 - 3)}$$

Rep.
 $s \rightarrow s+1$

$$L[e^t t^2 \cos t] = \frac{2(s+1)[(s+1)^2 - 3]}{[(s+1)^2 + 1]^3}$$

$$\boxed{L[e^t t^2 \cos t] = \frac{2(s+1)[(s+1)^2 - 3]}{[(s+1)^2 + 1]^3}}$$

Q2 (i) Find the Laplace transform of the

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Soln:

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-st}} \left[\int_0^{\pi/\omega} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} f(t) dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t dt + 0 \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} (-\sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= -\frac{1}{(1-e^{-\frac{2\pi s}{\omega}})(s^2+\omega^2)} \left[\frac{e^{-\frac{s\pi}{\omega}}}{\omega} (\delta \sin \omega \frac{\pi}{\omega} + \omega \cos \omega \frac{\pi}{\omega}) - e^0 (\delta \sin 0 + \omega \cos 0) \right]$$

$$= -\frac{1}{(1-e^{-\frac{2\pi s}{\omega}})(s^2+\omega^2)} \left[\frac{e^{-\frac{s\pi}{\omega}}}{\omega} (0 + \omega(-1)) - (0 + \omega) \right]$$

$\sin 0 = 0$
$\sin \pi = 0$
$\cos 0 = 1$
$\cos \pi = -1$

formula

$$\int e^{at} \sin bt da = \frac{e^{at}}{a^2+b^2} (a \sin b - b \cos b)$$

$$L[f(t)] = \frac{-1(\omega)}{(1-e^{-\omega t})(s^2+\omega^2)} [e^{-s\omega} + 1]$$

$$= \frac{\omega}{(s^2+\omega^2)} \cdot \frac{(1+e^{-s\omega})}{1-(e^{-s\omega/\omega})^2}$$

$$= \frac{\omega}{(s^2+\omega^2)} \cdot \frac{(1+e^{-s\pi/\omega})}{(1+e^{-s\pi/\omega})(1-e^{-s\pi/\omega})}$$

$$\therefore L[f(t)] = \frac{\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})}$$

② (ii) find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution I.F.O.

Soln:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a} \int_0^t \cos au \sin a(t-u) du$$

$$= \frac{1}{a} \int_0^t \sin(a(t-u)) \cos au du$$

Formula:

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

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$$A = at - au$$

$$B = au$$

$$A+B = at$$

$$A-B = at - 2au$$

$$= \frac{1}{2a} \int_0^t [\sin at + \sin(at - 2au)] du$$

$$= \frac{1}{2a} \left[\sin at u - \frac{\cos(at - 2au)}{-2a} \right]_0^t$$

$$= \frac{1}{2a} \left[t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a} \right]$$

$$\mathcal{L}^{-1} \left[\frac{s}{(s^2 + a^2)^2} \right] = \frac{\sin at}{2a}$$

(3) (i) Find the Laplace transform of the square wave function

$$f(t) = \begin{cases} E & , 0 \leq t \leq Q/2 \\ -E & , Q/2 \leq t \leq Q \end{cases}$$

$$f(t) = f(t-Q)$$

Solve:

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-qs}} \left[\int_0^Q e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-qs}} \left[\int_0^{Q/2} e^{-st} f(t) dt + \int_{Q/2}^Q e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-qs}} \left[E \int_0^{Q/2} e^{-st} dt - E \int_{Q/2}^Q e^{-st} dt \right]$$

$$= \frac{E}{1-e^{-qs}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{Q/2} - \left(\frac{e^{-st}}{-s} \right)_{Q/2}^Q \right]$$

$$= \frac{E}{1-e^{-qs}} \left[-\frac{e^{-qs/2}}{s} + \frac{1}{s} + \frac{e^{-qs}}{s} - \frac{e^{-qs/2}}{2} \right]$$

$$\begin{aligned}
 L[f(t)] &= \frac{E}{\delta(1-e^{-\alpha\delta})} \left[1 + e^{-\alpha\delta} - 2e^{-\alpha\delta/2} \right] \quad (13) \\
 &= \frac{E}{\delta(1-e^{-\alpha\delta})} \cdot (1-e^{-\alpha\delta/2})^2 \\
 &= \frac{E}{\delta} \cdot \frac{(1-e^{-\alpha\delta/2})^2}{1-(e^{-\alpha\delta/2})^2} \\
 &= \frac{E}{\delta} \cdot \frac{(1-e^{-\alpha\delta/2})^2}{(1+e^{-\alpha\delta/2})(1-e^{-\alpha\delta/2})} \\
 &= \frac{E}{\delta} \cdot \frac{(1-e^{-\alpha\delta/2})}{(1+e^{-\alpha\delta/2})}
 \end{aligned}$$

$$L[f(t)] = \frac{E}{\delta} \cdot \tanh\left(\frac{\alpha\delta}{4}\right)$$

(3) (ii) Find $L^{-1}\left[\frac{\delta}{(\delta^2+\alpha^2)(\delta^2+b^2)}\right]$ using convolution theo.

Soln:

$$\begin{aligned}
 L^{-1}\left[\frac{\delta}{(\delta^2+\alpha^2)(\delta^2+b^2)}\right] &= L^{-1}\left[\frac{\delta}{(\delta^2+\alpha^2)} \cdot \frac{1}{(\delta^2+b^2)}\right] \\
 &= L^{-1}\left[\frac{\delta}{(\delta^2+\alpha^2)}\right] * L^{-1}\left[\frac{1}{(\delta^2+b^2)}\right]
 \end{aligned}$$

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$$= \cos at + \frac{\sin bt}{b}$$

$$= \frac{1}{b} \int_0^t \cos au \sin b(t-u) du$$

$$= \frac{1}{b} \int_0^t \sin(bt-bu) \cos au du$$

$$A = bt - bu$$

$$B = au$$

$$A+B = (a-b)u + bt$$

$$A-B = (a+b)u + bt$$

$$= \frac{1}{2b} \int_0^t (\sin[(a-b)u+bt] + \sin[(a+b)u+bt]) du$$

$$= \frac{1}{2b} \left[-\frac{\cos[(a-b)u+bt]}{(a-b)} - \frac{\cos[(a+b)u+bt]}{(a+b)} \right]_0^t$$

$$= \frac{1}{2b} \left[\frac{\cos[(a-b)u+bt]}{a-b} + \frac{\cos[(a+b)u+bt]}{a+b} \right]_0^t$$

$$= \frac{1}{2b} \left[-\frac{\cos(at-bt+\pi)}{a-b} + \frac{\cos(at+bt+\pi)}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos(-bt)}{a+b} \right]$$

$$= \frac{1}{2b} \left[-\frac{\cos at}{a-b} + \frac{\cos at}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos bt}{a+b} \right]$$

$$= \frac{1}{2b} \left[\cos at \left(-\frac{1}{a-b} + \frac{1}{a+b} \right) + \right.$$

$$\left. \cos bt \left(\frac{1}{a-b} + \frac{1}{a+b} \right) \right]$$

$$\begin{aligned}
 &= \frac{1}{2b} \left[\cos \alpha t \left(\frac{a+b+\alpha-b}{a^2-b^2} \right) - \right. \\
 &\quad \left. \cos \beta t \left(\frac{\alpha+b-\alpha+b}{a^2-b^2} \right) \right] \\
 &= \frac{1}{2b(a^2-b^2)} [\cos \alpha t (-2b) - \\
 &\quad \cos \beta t (2b)] \\
 &= \frac{(-2b)}{2b(a^2-b^2)} (\cos \alpha t - \cos \beta t)
 \end{aligned}
 \tag{15}$$

$$L^{-1} \left[\frac{s}{(s^2+\alpha^2)(s^2+\beta^2)} \right] = \frac{\cos \alpha t - \cos \beta t}{b^2-a^2}$$

(4) (i) Find the Laplace Transform of
the function $f(t) = \begin{cases} K, & 0 \leq t \leq Q \\ -K, & Q \leq t \leq 2Q \end{cases}$
 $\Rightarrow f(t+2Q) = f(t)$.

Soln:

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1-e^{-as}} \left[\int_0^a e^{-st} f(t) dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\int_0^{2a} e^{-st} f(t) dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t) dt + \int_a^{2a} e^{-st} f(t) dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[K \int_0^a e^{-st} dt - K \int_a^{2a} e^{-st} dt \right]
 \end{aligned}$$

Given
Period = $2Q$

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$$= \frac{K}{(1-e^{-2as})} \left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a} \right]$$

$$= \frac{K}{s(1-e^{-2as})} \left[-e^{-as} + 1 + \frac{e^{-2as}}{s} - e^{-2as} \right]$$

$$= \frac{K}{s(1-e^{-2as})} [1 + e^{-2as} - 2e^{-as}]$$

$$= \frac{K}{s(1-e^{-2as})} (1-e^{-as})^2$$

$$= \frac{K(1-e^{-as})^2}{s[1-(e^{-as})^2]} = \frac{K \cdot (1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{K}{s} \cdot \frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\boxed{L[\vec{f}(t)] = \frac{K}{s} \tanh\left(\frac{as}{2}\right)}$$

(4) (ii) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

Soln:

$$\boxed{L\left[\frac{\vec{f}(t)}{t}\right] = \int_0^s F(s) ds}$$

w.k.t $F(s) = L[\vec{f}(t)] = L[\cos at - \cos bt]$

$$\begin{aligned}
 & \int [\frac{\cos at - \cos bt}{t}] = \int_s^{\infty} [\cos at - \cos bt] ds \\
 &= \int_s^{\infty} [t(\cos at) - t(\cos bt)] ds \\
 &= \int_s^{\infty} \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds \\
 &= \frac{1}{2} \int_s^{\infty} \left(\frac{2s}{s^2 + a^2} - \frac{2s}{s^2 + b^2} \right) ds \\
 &= \frac{1}{2} \left[\log(s^2 + a^2) - \log(s^2 + b^2) \right]_s^{\infty} \\
 &= \frac{1}{2} \left[\log \frac{s^2 + a^2}{s^2 + b^2} \right]_s^{\infty} \\
 &= \frac{1}{2} \left[\log \frac{s^2(1 + a^2/s^2)}{s^2(1 + b^2/s^2)} \right]_s^{\infty} \\
 &= \frac{1}{2} \left[\log(1) - \log \left(\frac{1 + a^2/s^2}{1 + b^2/s^2} \right) \right] \\
 &= \frac{1}{2} \left[0 - \log \left(\frac{(s^2 + a^2)/s^2}{(s^2 + b^2)/s^2} \right) \right] \\
 &= -\frac{1}{2} \log \left[\frac{s^2 + a^2}{s^2 + b^2} \right]
 \end{aligned}$$

$$\therefore \frac{1}{2} [\cos(a-t) - \cos(b+t)] = \frac{1}{2} \log \left[\frac{s^2 + b^2}{s^2 + a^2} \right]$$

(5) Find $\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theo.

(18)

Soln:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] &= \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] * \mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] \\ &= \cos at * \cos bt\end{aligned}$$

$A = au$
$B = bt - bu$
$A+B = (a-b)u + bt$
$A-B = (a+b)u - bt$

$$\begin{aligned}\cos A \cos B &= \\ \frac{1}{2} [\cos(A+B) &+ \\ \cos(A-B)]\end{aligned}$$

$$\begin{aligned}\sin(-\theta) &= -\sin \theta \\ \cos(-\theta) &= \cos \theta\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2} \int_0^t \cos au \cos b(t-u) du \\ &= \frac{1}{2} \int_0^t \cos au \cos(b(t-bu)) du \\ &= \frac{1}{2} \int_0^t (\cos[(a-b)u + bt] + \\ &\quad \cos[(a+b)u - bt]) du \\ &= \frac{1}{2} \left[\frac{\sin[(a-b)u + bt]}{a-b} + \right. \\ &\quad \left. \frac{\sin[(a+b)u - bt]}{a+b} \right]_0^t \\ &= \frac{1}{2} \left[\frac{\sin(a-tb + bt)}{a-b} + \frac{\sin(a+tbt - bt)}{a+b} \right. \\ &\quad \left. - \frac{\sin(bt)}{a-b} - \frac{\sin(-bt)}{a+b} \right] \\ &= \frac{1}{2} \left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right]\end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{2} \left[\sin at \left(\frac{1}{a-b} + \frac{1}{a+b} \right) + \right. \\
 &\quad \left. \sin bt \left(-\frac{1}{a-b} + \frac{1}{a+b} \right) \right] \\
 &= \frac{1}{2} \left[\sin at \left(\frac{a+b+a-b}{a^2-b^2} \right) + \sin bt \left(\frac{-a-b+a-b}{a^2-b^2} \right) \right] \\
 &= \frac{1}{2(a^2-b^2)} [2a \sin at - 2b \sin bt] \\
 &= \frac{a \sin at - b \sin bt}{a^2-b^2}
 \end{aligned}$$

$$\therefore L^{-1} \left[\frac{s^2}{(s+a^2)(s+b^2)} \right] = \frac{a \sin at - b \sin bt}{a^2-b^2}$$

(6) (i) Verify Initial value & Final value theorem
for $f(t) = 1 + e^t (\sin t + \cos t)$

Soln: (i) Initial value theorem

$$\lim_{s \rightarrow \infty} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$\begin{aligned}
 F(s) &= L[f(t)] = L[1 + e^t (\sin t + \cos t)] \\
 &= L(1) + L(e^t \sin t) + L(e^t \cos t)
 \end{aligned}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

(20)

$$\text{LHS (INT)} \quad \lim_{t \rightarrow 0} f(t)$$

$$\begin{aligned}\Rightarrow \lim_{t \rightarrow 0} f(t) &= \lim_{t \rightarrow 0} [1 + e^t (\sin t + \cos t)] \\ &= 1 + e^0 (\sin 0 + \cos 0) \\ &= 1 + (0+1) = 1+1\end{aligned}$$

$$\boxed{\text{LHS} = \lim_{t \rightarrow 0} f(t) = 2}$$

$$\text{RHS (INT)} \quad \lim_{s \rightarrow \infty} SF(s)$$

$$\begin{aligned}\Rightarrow \lim_{s \rightarrow \infty} SF(s) &= \lim_{s \rightarrow \infty} s \left[\frac{1}{s} + \frac{s+2}{(s+1)^2 + 1} \right] \\ &= \lim_{s \rightarrow \infty} \left[1 + \frac{s(s+2)}{(s+1)^2 + 1} \right] \\ &= \lim_{s \rightarrow \infty} \left[1 + \frac{s \cdot s (1+2/s)}{s^2 [(1+1/s)^2 + 1/s^2]} \right] \\ &= \lim_{s \rightarrow \infty} \left[1 + \frac{(1+2/s)}{(1+1/s)^2 + 1/s^2} \right] \\ &= 1 + \frac{(1+0)}{1+0} = 1+1\end{aligned}$$

$$\boxed{\text{RHS} = \lim_{s \rightarrow \infty} SF(s) = 2}$$

$$\therefore \text{LHS} = \text{RHS} = 2$$

$$\boxed{\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} SF(s)}$$

INT
verified

(iii) Final Value Theo

$$\boxed{\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)}$$

LHS (FVT) $\lim_{t \rightarrow \infty} f(t)$

$$\Rightarrow \lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} [1 + e^{(smt + cos t)}]$$

$$= 1 + e^0 = 1 + 0 = 1$$

$$\boxed{\text{LHS} = \lim_{t \rightarrow \infty} f(t) = 1}$$

RHS (FVT) $\lim_{s \rightarrow 0} sF(s)$

$$\Rightarrow \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \left[1 + \frac{s(s+2)}{(s+1)^2 + 1} \right]$$

$$= 1 + 0 = 1$$

$$\therefore \boxed{\text{RHS} = \lim_{s \rightarrow 0} sF(s) = 1}$$

$$\boxed{\text{LHS} = \text{RHS} = 1}$$

$$\boxed{\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)}$$

FVT verified

\therefore Initial value & Final value Theo verified.

⑥ (iii) find $\int \left[\frac{e^{at} - e^{bt}}{t} \right]$

(22)

Soln:

$$\int \left[\frac{f(t)}{t} \right] = \int_s^\infty F(s) ds$$

$$F(s) = \int \left[f(t) \right] = \int \left[e^{at} - e^{bt} \right]$$

$$\therefore F(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\begin{aligned} \Rightarrow \int \left[\frac{e^{at} - e^{bt}}{t} \right] &= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b} \right) ds \\ &= \left[\log(s+a) - \log(s+b) \right]_s^\infty \\ &= \left[\log \frac{(s+a)}{(s+b)} \right]_s^\infty \\ &= \left[\log \frac{s(1+a/s)}{s(1+b/s)} \right]_s^\infty \\ &= \log(1) - \log \left(\frac{1+a/s}{1+b/s} \right) \\ &= 0 - \left[\log \frac{(s+a)/s}{(s+b)/s} \right] \\ &= -\log \left[\frac{(s+a)}{s+b} \right] \end{aligned}$$

$$\int \left[\frac{e^{at} - e^{bt}}{t} \right] = \log \left[\frac{s+b}{s+a} \right]$$

(4)

Solve $y'' - 3y' + 2y = 1$, given that
 $y(0) = 1$, $y'(0) = 1$, by using Lap. Transform

(23)

Soln:

Given: $y''(t) - 3y'(t) + 2y(t) = 1$

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(1) \quad (1)$$

$$\begin{aligned} L[y(t)]^2 - s^2 y(0) - s y'(0) - 3[L[y(t)] - y(0)] \\ + 2L[y(t)] = \frac{1}{s} \end{aligned}$$

$$L[y(t)](s^2 - 3s + 2) - s - 1 + 3 = \frac{1}{s}$$

$$L[y(t)](s^2 - 3s + 2) - s + 2 = \frac{1}{s}$$

$$L[y(t)](s-1)(s-2) = \frac{1}{s} + s - 2$$

$$\Rightarrow L[y(t)] = \frac{1}{s(s-1)(s-2)} + \frac{s-2}{(s-1)(s-2)}$$

$$\Rightarrow L[y(t)] = \frac{1}{s(s-1)(s-2)} + \frac{1}{s-1} \rightarrow (1)$$

consider

$$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\Rightarrow 1 = A(s-1)(s-2) + B(s(s-2)) + C(s(s-1))$$

put $\boxed{s=1} \Rightarrow 1 = 0 + B(-1) + 0 \quad \therefore \boxed{B = -1}$

$$\boxed{s=2} \Rightarrow 1 = 0 + 0 + C(2)(1) \quad \therefore \boxed{C = y_2}$$

$$\boxed{s=0} \Rightarrow 1 = A(-1)(-2) + 0 + 0$$

$$\Rightarrow \boxed{A = y_1}$$

Sub (A), (B), (C) in Equ. ①

(24)

$$\mathcal{L}[y_1(t)] = \frac{y_2}{s} - \frac{1}{s-1} + \frac{y_2}{s-2} + \frac{1}{s-1}$$

$$\therefore y_1(t) = y_2 \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2} \mathcal{L}^{-1}\left(\frac{1}{s-2}\right)$$

$$+ \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= \frac{1}{2}(1) - \cancel{\mathcal{E}^{st}} + \frac{1}{2}e^{2t} + \cancel{\mathcal{E}^{st}}$$

$$y_1(t) = \frac{1}{2} + \frac{e^{2t}}{2}$$

- ⑧ Solve $y'' - 3y' + 2y = e^{3t}$, given that
 $y(0) = 1$, $y'(0) = 0$, using Laplace Transform.

Soln: Given: $y''(t) - 3y'(t) + 2y(t) = e^{3t}$

$$\mathcal{L}[y''(t)] - 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = \mathcal{L}(e^{3t})$$

$$s^2 \mathcal{L}[y(t)] - sy(0) - y'(0) - 3[s\mathcal{L}[y(t)] - y(0)] + 2\mathcal{L}[y(t)] = \frac{1}{s-3}$$

$$\mathcal{L}[y(t)] (s^2 - 3s + 2) - s - 0 + 3 = \frac{1}{s-3}$$

$$\mathcal{L}[y(t)] (s^2 - 3s + 2) = \frac{1}{s-3} + s - 3$$

$$= \frac{1 + (s-3)^2}{s-3}$$

$$= \frac{s^2 + 9 - 6s}{s-3}$$

$$= \frac{s^2 - 6s + 10}{s-3}$$

(25)

$$L[y(t)] (s-1)(s-2) = \frac{s^2 - 6s + 10}{s-3}$$

$$L[y(t)] = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

$$L[y(t)] = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \xrightarrow{\text{①}}$$

$$\Rightarrow s^2 - 6s + 10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

put $\boxed{s=1} \Rightarrow 1 - 6 + 10 = A(-1)(-2)$

$$\Rightarrow 5 = 2A \quad \therefore \boxed{A = 5/2}$$

put $\boxed{s=2} \Rightarrow 4 - 12 + 10 = B(-1)(1)$

$$\Rightarrow 2 = -B \quad \therefore \boxed{B = -2}$$

put $\boxed{s=3} \Rightarrow 9 - 8 + 10 = C(2)(1)$

$$\Rightarrow 1 = 2C \quad \therefore \boxed{C = 1/2}$$

Sub ④, ⑤, ⑥ values in equ ①

$$L[y(t)] = \frac{5/2}{s-1} + \frac{(-2)}{s-2} + \frac{1/2}{s-3}$$

$$\therefore y(t) = \frac{5}{2} t^{-1} \left(\frac{1}{s-1}\right) - 2t^{-1} \left(\frac{1}{s-2}\right) + \frac{1}{2} t^{-1} \left(\frac{1}{s-3}\right)$$

$$\therefore y(t) = \frac{5}{2} e^t - 2e^{2t} + \frac{1}{2} e^{3t}$$

(26)

⑨ Solve $(D^2 + 9)y = \cos 2t$, given
that $y(0) = 1$, $y(\frac{\pi}{2}) = -1$

Soln:

$$\text{Given: } (D^2 + 9)y = \cos 2t$$

$$y''(1-t) + 9y(1-t) = \cos 2t$$

$$\begin{aligned} D^2y &= y'' \\ Dy &= y' \end{aligned}$$

$$+ [y''(1-t)] + 9 + [y'(1-t)] = + (\cos 2t)$$

$$s^2 + [y'(1-t)] - s y(0) - y'(0) + 9 + [y(1-t)] = \frac{s}{s^2 + 4}$$

$$+ [y(1-t)] (s^2 + 9) - s(1) - c = \frac{s}{s^2 + 4}$$

$$+ [y(1-t)] (s^2 + 9) = \frac{s}{s^2 + 4} + s + c$$

$$L[y(1-t)] = \frac{s}{(s^2 + 4)(s^2 + 9)} + \frac{s}{s^2 + 9} + \frac{c}{s^2 + 9} \rightarrow ①$$

consider

$$\frac{1}{(s^2 + 4)(s^2 + 9)} = \frac{A}{(s^2 + 4)} + \frac{B}{(s^2 + 9)} \rightarrow ②$$

$$\text{put } \boxed{s^2 = -9} \Rightarrow 1 = A(s^2 + 9) + B(s^2 + 4)$$

$$1 = A(-9 + 9) + B(-9 + 4)$$

$$1 = 0 + B(-5)$$

$$\therefore \boxed{B = -\frac{1}{5}}$$

$$\text{put } \boxed{s^2 = 4} \Rightarrow 1 = 0 + A(-4 + 9)$$

$$1 = 5A \quad \therefore \boxed{A = \frac{1}{5}}$$

Sub ④ is ③ values in ②

(27)

$$\therefore \frac{1}{(s^2+4)(s^2+9)} = \frac{y_5}{s^2+4} - \frac{y_5}{s^2+9}$$

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{1}{5} \left(\frac{1}{s^2+4} \right) - \frac{1}{5} \left(\frac{1}{s^2+9} \right)$$

III by

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{1}{5} \left(\frac{s}{s^2+4} \right) - \frac{1}{5} \left(\frac{s}{s^2+9} \right)$$

→ ③

Sub ③ in Eqn ①

$$L[y(t)] = \frac{1}{5} \left(\frac{s}{s^2+4} \right) - \frac{1}{5} \left(\frac{s}{s^2+9} \right) + \frac{5}{s^2+9} + \frac{c}{s^2+9}$$

$$= \frac{1}{5} \left(\frac{s}{s^2+4} \right) + \frac{4}{5} \left(\frac{s}{s^2+9} \right) + \frac{c}{s^2+9}$$

$$y(t) = \frac{1}{5} t^{-1} \left(\frac{s}{s^2+4} \right) + \frac{4}{5} t^{-1} \left(\frac{s}{s^2+9} \right) + c t^{-1} \left(\frac{1}{s^2+9} \right)$$

$$y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{c}{3} \sin 3t$$

Given: $y(\pi/2) = -1$

→ ④

$$y(\frac{\pi}{2}) = \frac{1}{5} \cos 2\frac{\pi}{2} + \frac{4}{5} \cos 3\frac{\pi}{2} + \frac{c}{3} \sin 3\frac{\pi}{2} \\ = -1$$

(28)

$$\Rightarrow \frac{1}{5}(-1) + 0 + \frac{c}{3}(-1) = -1$$

$$-\frac{1}{5} - \frac{c}{3} = -1$$

$$\Rightarrow -\frac{c}{3} = -1 + \frac{1}{5} = -\frac{4}{5}$$

$$\therefore \boxed{c = \frac{12}{5}}$$

Sub \textcircled{c} value
in Eqn.

$$y(t) = \frac{\cos 2t}{5} + \frac{4}{5} \cos 3t + \frac{12}{5} \sin 3t$$

(10) Solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that
 $y=4$, $y'=-2$, when $t=0$, using
Replace Transform.

Soln:

$$\text{Given: } y''(t) + y'(t) = t^2 + 2t$$

$$+ [y''(s)] + [y'(s)] = L(t^2) + 2L(t)$$

$$s^2 L[y(t)] - sy(0) - y'(0) + sL[y(t)] -$$

$$y(0) = \frac{2}{s^3} + 2 \cdot \frac{1}{s^2}$$

$$\text{Given: } \overline{y(0)} = 4, \quad \overline{y'(0)} = -2$$

$\cos \pi = -1$
$\cos \frac{3\pi}{2} = 0$
$\sin \frac{3\pi}{2} = -1$

$$L[y(t)](s^2 + s) - 4s + 2 - 4 = \frac{2}{s^3} + \frac{2}{s^2} \quad (29)$$

$$L[y(t)]s(s+1) = \frac{2+2s}{s^3} + 4s + 2$$

$$L[y(t)] = \frac{2(s+1)}{s^3 s(s+1)} + \frac{4s}{s(s+1)} + \frac{2}{s(s+1)}$$

$$\therefore L[y(t)] = \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s(s+1)} \rightarrow (1)$$

consider

$$\frac{2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 2 = A(s+1) + B(s)$$

$$\text{put } [s=0] \Rightarrow 2 = A(1) \quad \therefore [A=2]$$

$$[s=-1] \Rightarrow 2 = B(-1) \quad \therefore [B=-2]$$

Sub (A) & (B) values in Eqn: (1)

$$L[y(t)] = \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s} - \frac{2}{s+1}$$

$$= \frac{2}{s^4} + \frac{2}{s+1} + \frac{2}{s}$$

$$y(t) = 2t^{-1}\left(\frac{1}{s^4}\right) + 2t^{-1}\left(\frac{1}{s+1}\right) + 2t^{-1}\left(\frac{1}{s}\right)$$

$$y(t) = 2 \cdot \frac{t^3}{s^3} + 2e^{-t} + 2$$