Asymptotic analysis of two dimensional potentials around a singularity

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1 Introduction

An accurate reconstruction of wave-packets from complex trajectories requires two important hassles to be overcome: Inclusion of all significant non-classical contributions to the wave-packet and removal of resultant stokes divergence. Non-classical contributions are usually obtained from complex trajectories whose time contours circumnavigate singularity times in topologically distinct ways. It is hence necessary to understand the topology of the Riemann surface, corresponding to trajectory manifolds, around a given singularity time. For the case of 1D potentials, the Riemann surface around singularity times is well understood [5] in terms of asymptotic analysis. Riemann surfaces corresponding to the trajectory manifolds of non-harmonic two dimensional potentials show rich and intricate multi-valuedness, due to the complexity of the underlying dynamics. In order to understand the topology of these Riemann surfaces, we studied asymptotic potentials that nearly approximated the given physical potential near one of its singularities. We present the findings of our study in this report.

2 Asymptotic potentials

We start with the specific case of the double slit potential, and consider a particular form as follows:

$$V(x,y) = \left(V_0 - \frac{mw^2y^2}{2} + \frac{m^2w^4y^4}{16V_0}\right)\left(\frac{1}{cosh^2(x)}\right) \tag{1}$$

The singularities of this potential arise from the singularity of the quartic potential (along the y direction) at ∞, and from the singularities of the eckart potential (along the x direction) at $x^* = (2n+1)\frac{i\pi}{2}$. Numerically we observed that all singularity times (that we could probe) arose from the singularities of the eckart potential along the x direction. In order to understand and analyse the local topology around a singularity time, we approximated the double slit potential in (1) by:

$$V(x,y) = -C_1 \frac{C_2 y - y_*}{(x-x^*)^2} \tag{2}$$

where C_1, C_2 are constants depending on the potential parameters. The above expression is obtained by truncating the Laurent series expansion of the eckart potential to leading order (see [5]) and by truncating the Taylor series expansion of the quartic potential at first order. It is to be noted here that y_*, the y coordinate at the singularity cannot be ascertained upfront; It could only be found numerically for each singularity time on a case to case basis. Unlike in the 1D case, an analytical expression for singularity times doesn't seem to exist for the double slit potential. Without loss of generality, we can translate the position of the singularity to the origin and consider the following potential:

$$V(x,y) = -\frac{y}{x^2} \tag{3}$$

To understand the Riemann surface geometry around a singularity time, one ought to follow the calculus outlined in [5] to obtain the asymptotic time dependence of position and momentum about a singularity time. In doing so, we arrived at the following asymptotic time dependence for x and y momentum around a singularity time:

$$\begin{aligned} x(t) &\sim (t-t_*)^{\frac{2}{3}} \quad p_x(t) \sim (t-t_*)^{-\frac{1}{3}} \\ y(t) &\sim (t-t_*)^{\frac{2}{3}} \quad p_y(t) \sim (t-t_*)^{-\frac{1}{3}} \end{aligned} \tag{4}$$

However, these expressions imply that the Riemann surface is finitely sheeted, while numerical studies suggest they are infinitely sheeted. Intrigued by the apparent failure of this calculus we studied several other model potentials of the form

$$V(x,y) = -\frac{y^n}{x^m}$$

The asymptotic time dependence obtained from the calculus in [5] and from numerical studies are tabulated in table 1.

Table 1: Asymptotic time dependence of momentum

Model Potential	**Analytically calculated**		**Numerically observed**	
$V(x,y)$	p_x	p_y	p_x	p_y
$-\frac{y}{x^2}$	$f_1(t)(t-t_*)^{-\frac{1}{3}}$	$f_2(t)(t-t_*)^{-\frac{1}{3}}$	$\widetilde{f_1}(t)(t-t_*)^{-\frac{1}{2}}$	$\widetilde{f_2}(t)(t-t_*)$
$-\frac{y}{x}$	$\sim$	$\sim$	$\widetilde{g_1}(t)(t-t_*)^{-\frac{1}{3}}$	$\widetilde{g_2}(t)$
$-\frac{y}{x^3}$	$h_1(t)(t-t_*)^{-\frac{1}{2}}$	$h_2(t)(t-t_*)^{-\frac{1}{2}}$	$\widetilde{h_1}(t)(t-t_*)^{-\frac{3}{5}}$	$\widetilde{h_2}(t)$

Here, the functions $f_1, f_2, h_1, h_2, \widetilde{g_1}$ and $\widetilde{g_2}$ are holomorphic, $\widetilde{f_1}, \widetilde{f_2}$ are infinitely sheeted. The functions $\widetilde{h_1}$ and $\widetilde{h_2}$ can have a maximum of 5 sheets in their Riemann surface. For the potential $V(x,y) = -\frac{y}{x}$, the calculus in [5] doesn't yield an asymptotic dependence.

As can be seen, the analytically calculated momentum dependence doesn't match with the numerically observed momentum dependence. In the next section, we attempt to explain this discrepancy.

3 General theory of asymptotic time dependence

In the context of describing a connection between nonlinear evolution equations and ordinary differential equations of Painleve type, Ablowitz et al [1], provided an algorithm to determine whether a given non-linear ODE admits solutions with movable branch points. This algorithm is similar to the calculus outlined in [], and the similarity shall be detailed in this section.

For the case of the 2D potential in (3), the Newton's equation of motion becomes a coupled second order non-linear differential equation in two variables:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{2y}{x^3} \\ \frac{d^2y}{dt^2} &= +\frac{1}{x^2} \end{aligned} \tag{5}$$

From the algorithm outlined in [], we found that the the asymptotic time dependence in (4), doesn't turn out to be coming from a *leading order* term of the non linear ODE (5). Instead,

$$\begin{aligned} x(t) &\sim (t-t_*)^{\frac{1}{2}} \\ y(t) &\sim (t-t_*) \end{aligned} \tag{6}$$

corresponds to the leading order term. The numerical result in the table above also reflects this dependence. This was the reason why the calculus had apparently failed for the 2D problem. In other words, the asymptotic time dependence obtained in (4) doesn't capture the dominant behaviour of the solution around a singularity time.

Now, the asymptotic dependence (6) doesn't explain the infinite sheetedness of the Riemann surface, i.e. this algorithm doesn't help identify the function $f_2(t)$ in table 1, which has an infinite sheeted Riemann surface. Infinite sheeted Riemann surfaces typically arise from logarithmic branch points. Ablowitz et al also suggest a way to identify if a non-linear ODE admits a solution with a simple logarithmic branch point. This can be done by transforming the given ODE into another by an appropriate substitution. If the resultant ODE has a non-exponential solution, then the original ODE has a simple logarithmic branch point.

From our analyses, it doesn't seem that there exists a solution to (5), which has a simple logarithmic branch point. Further, any transformation of the form

$$x(t) = u(t)\ (t-t_*)^a\ log^b(t-t_*)$$

resulted in a system of non algebraic ODEs, with logarithmic terms. For such ODEs, an asymptotic analysis of the form described in [] cannot be applied. Thus, it appears that the infinite sheetedness of the Riemann surface around a singularity time of the potential (3) (and hence (1)) doesn't originate from a simple logarithmic branch point.

Although this procedure couldn't describe the asymptotic time dependence for the asymptotic potential in (3), for other model potentials mentioned in table 1, the procedure proved successful. The numerically observed asymptotic behaviour was obtained from the leading order terms of the corresponding non-linear ODE. So, in conclusion, we believe that except for some pathological potentials, the algorithm outlined by Ablowitz et al can be extended as a general theory of asymptotic time dependence around singularity times.

While a simple algebraic/logarithmic branch point couldn't be attributed to the infinite sheetedness of the Riemann surface for the dynamics of (5), it seems that it is possible, in principle, to obtain an asymptotic solution for the potential (3) around a singularity time, that describes this intricate multi-valuedness in its full detail. In the next section we sketch a procedure to obtain such an asymptotic solution, without deriving it explicitly.

4 Analytical structure of the asymptotic potential

In a study on the relation between the integrability of the system and its analytic structure, Tabor and Weiss [6], had shown that the singularity structure of a dynamical system can understand the behaviour of a system in its non-integrable regimes. They analyse the specific case of the Lorenz system and follow a procedure similar to the algorithm outlined in [1], to obtain the leading order behaviour. After identifying the conditions on the parameters for the solutions to be of the Painleve type, they provide a general asymptotic expansion around a singularity time, which is not of Painleve type.

For the Lorenz system, the expansion has the specific form:

$$\begin{aligned} X(t) &= \frac{2i}{t}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj}\ t^j\ (t^2\ ln(t))^k \\ Y(t) &= -\frac{2i}{t^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_{kj}\ t^j\ (t^2\ ln(t))^k \\ Z(t) &= -\frac{2}{t^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{kj}\ t^j\ (t^2\ ln(t))^k \end{aligned} \tag{7}$$

Here, we denote $t - t_*$ as t for notational simplicity. After a series of transformation, they obtain the leading order logarithmic term for the x-coordinate:

$$X(t) = \frac{2i}{t}\ \frac{\alpha t(ln(t))^{\frac{1}{2}}}{sn(\alpha t(ln(t))^{\frac{1}{2}}, k)} + O(t) \tag{8}$$

The double series expansion in (7) is generic and also appears in the asymptotic expansion of other systems such as the Henon-Heiles Hamiltonian [2] and duffing oscillator [3].

For the case of the potential in (3), we have rightly identified the leading order behaviour from our numerical studies, and verified it with the algorithm in [1]. We now have to expand on this leading order term and obtain an appropriate double series as above. We start by writing:

$$\begin{aligned} x(t) &= t^{\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{kj}\ t^j\ (t^2\ ln(t))^k \\ y(t) &= t\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_{kj}\ t^j\ (t^2\ ln(t))^k \end{aligned} \tag{9}$$

From here, we need to apply a series of transformation as in [6] to reduce the problem to a set of coupled differential equations that are analytically solvable. This procedure would eventually result in obtaining the leading order logarithmic term, that aptly captures the infinite-sheetedness of the Riemann surface, described in sections before.

Thus, to obtain the right asymptotic behaviour around a singularity time for a generic 2D potential, the calculus in [5] has to be expanded as outlined in this section. The steps in this extended calculus can be summarized as follows:

1. Determine the asymptotic dependence of $x(t)$ and $y(t)$ that corresponds to the leading order term of the non-linear ODE associated with the given 2D potential.

2. Check for the existence of a simple logarithmic branch point by an appropriate transformation of the ODE. If all possible transformations result in non-algebraic ODEs, then the system doesn't have a simple logarithmic branch point.
3. Define a double series as in (9) with the appropriate leading order asymptotic dependence.
4. Apply a series of transformation to reduce the problem to a set of coupled differential equations that are almost analytically solvable.
5. Obtain the correct asymptotic behavior from the resultant differential equations.

5 Theory of complex analytic differential equations

The theory of analytic differential equations in the complex plane, provides another approach to obtaining asymptotic behaviour of the dynamics around a singularity time. This approach requires an extensive and rigorous understanding of the theory which is beyond the scope in this report. We only sketch a brief outline of the general procedure to be followed in analysing a complex differential equation that evolves in complex time. This procedure would also help in identifying the correct asymptotic time dependence, although it is non-trivial and considerably tedious to obtain the right analytical form of the time dependence. For a rigorous discussion on analytic differential equations, refer [4].

The procedure is as follows:

1. Formulate the Hamiltonian dynamics of (5) as a complex ODE system, as follows:
$$\frac{d\mathbf{X}}{dt} = F(\mathbf{X})$$
2. Convert the function F to a holomorphic function by an appropriate scaling of dt. This is done by multiplying the above equation with the highest order pole in the function F, and scaling dt, to obtain:
$$\frac{d\mathbf{X}}{d\tau} = F(\mathbf{X})$$
3. Remove the explicit time dependence of the ODE and reduce to its *Pfaffian* equation. By a Pfaffian equation, we mean an equation of the form,
$$\omega = P(x, y)dx + Q(x, y)dy = 0$$
4. Convert the ODE defined on $\mathbb{C}^2$ to one defined on $\mathbb{CP}^2$.
5. Analyse the singularities on the line at ∞.
6. Observe monodromy, if any. This would correspond to the case where singularity time becomes an algebraic branch point
7. If monodromy is absent, then the asymptotic time dependence about the singularity has to be deduced from the complicated Pfaffian equation.

For an elaborate description of steps 3-7, refer [4]. The procedure mentioned is described as a standard technique in analysing complex valued ODE evolving in complex time. This approach extensively describes the geometry of the complex Hamiltonian dynamics.

Usually, it seems that the procedure above is impractical to obtain the asymptotic time dependence about a singularity time. At best, the time dependence can be obtained as an exponential or *Dulac* series, which corresponds to the biseries described in the previous section.

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