Diffusion through a permeable barrier

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1 Introduction

Diffusion processes, right from its inception in Einstein's celebrated works, had offered crucial clues to various scientific phenomenon, such as the confirmation of hypothesis of discreteness of matter, mechanisms of intra-membrane movements, conductivity in semiconductors, etc. Thanks to its striking correlation to 'random walk' model, the empirical equations to understand diffusion processes has its origin in the theory of probability and statistics, and thus have also aided in modelling several physico-chemical and socio-economic processes. The ubiquitous nature of diffusion processes and the enormous physical insight(or 'biological insight') they offer, motivated us take cue from them and extend an existing understanding of a specific natural phenomenon. Typically, a diffusion process is described by the stochastic evolution of a dynamic variable. In most cases, and in our case as well, the dynamic variable is the displacement of a diffusing particle. One of the most fascinating results in mathematical statistics – the central limit theorem, brings about a fundamental implication on the mean square displacement(MSD) of the particle – It grows linearly with time. This particular result, serves as a statistical signature of a normal diffusion process. Einstein in his seminal paper, had derived a simple expression relating the viscosity of the medium to the diffusion coefficient and put forward a convincing evidence for the discrete matter hypothesis. Ever since he derived this result, deviations from normality have been reported in the form of a departure from the usual linear variation of MSD in time to a power law or similar such variation. This might sound like a violation of the rigorously derived Central limit theorem, but the very assumptions of the central limit theorem might not be satisfied in the natural process studied. Thus, these 'anomalous' diffusion, offers interesting clues about the nature of the physical process under study. They are modelled by mathematical abstractions such as Continuous Time Random Walk [1] and Fractional Brownian Motion [2] and several natural processes have been well described by them.

The various avenues where diffusion processes appropriately model natural phenomenon includes membrane biophysics, movement of ions across biological channels, diffusion of ions/holes in semiconductors etc. The deceiving simplicity of the diffusion process and its applications in successful explanation of various natural phenomenon has inspired both theoretical endeavours in pure mathematics and physics, and experimental studies in several artificially constructed environments such as the one we chose to work on. Our interests in the physical insights offered by the diffusion processes, prompted us to work on actual physical situations that have had limited theoretical understanding. An interesting paper by Granick [3] reporting anomalous diffusion of nanospheres in artificially prepared actin filaments networks attracted us into further exploration of the statistical nature of the diffusive processes, when Einstein's assumptions are not valid.

2 Abstraction of a permeable barrier

The first question we asked ourselves was this: A particle is diffusing in a region and is hindered by a permeable barrier. What could be a mathematically profound description of such a process? A strong physical insight into this simple problem would aid in the firm understanding of various situations described above. Analytical solutions, if obtainable for such a problem would place this concept on a structured mathematical base. We proceeded to form equations and describe the process.

A typical diffusion process is mathematically described by the Langevin equation:

$$\frac{dx}{dt} = F(x,t) + \eta(t) \tag{1}$$

Where $\eta(t)$ is the noise term. Diffusion can also be characterized by the Fokker Planck equation, which describes the probability distribution of the stochastic variable:

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} - \frac{\partial\left(P(x,t)V^{'}(x,t)\right)}{\partial x} \tag{2}$$

The striking similarity of this equation with the Schroedinger equation in quantum mechanics has been analysed in the past [4]. Solutions of the diffusion equation can be obtained by analysing the Eigen value problem, as in quantum mechanics. In fact, attempts have been made in the past to provide a 'classical' derivation to the Schroedinger equation [5]. This provided us with a template to work with: various physical situations such as 1D box potential, harmonic potential etc. The usual operator formalism of quantum mechanics also holds and the partial differential equation can be well characterized under this operator formalism.

Diffusion through a permeable barrier has been analysed in the past, in various physical situations [6],[7]. Powles et al, in their rapid communication, had conceived the so called 'Leather barrier Boundary Conditions', at the location of permeable barrier. They had claimed that the rate of 'flow' of concentration is proportional to the difference in concentration, and it is physically reasonable to draw parallel from heat diffusion equation:

$$\frac{1}{\mathscr{P}}\frac{\partial P}{\partial x}\rfloor_{x=0} = P_{+} - P_{-} \tag{3}$$

They had referred to a paper by Tanner, who had analysed a similar problem in a different context. Tanner himself had described such a situation from results in heat diffusion problem. As this boundary condition was not derived from concrete mathematical arguments, we intended to derive this boundary condition from first principles.

We expected that a potential field would characterize this boundary condition quantitatively. A permeable barrier introduces a transition at the boundary, which we expected to be described by the step potential, which is not physically plausible. We expected a delta potential to appropriately model a permeable barrier. The idea of conceiving boundary conditions using potential is not new [8], but the analogy of Fokker Planck Equation to the Schroedinger equation offers interesting physical analogues to work with. We could quite easily arrive at the boundary condition using the dirac delta potential field as follows:

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} + \frac{\partial\left(P(x,t)\alpha\delta^{'}(x,t)\right)}{\partial x} \tag{4}$$

The continuity equation is:

$$\nabla.\vec{J} = -\frac{\partial P}{\partial t} \tag{5}$$

$$J = \alpha\delta'(x)P(x,t) + D\frac{\partial P}{\partial x} \tag{6}$$

Integrating J over a small region gives:

$$\alpha\frac{\partial P}{\partial x} = D(P_{\varepsilon} - P_{-\varepsilon}) \tag{7}$$

Which gives the boundary condition above stated with

$$\mathscr{P} = \frac{-\alpha}{D} \tag{8}$$

The problem of scattering and tunnelling through a delta potential is well understood in quantum mechanics. As mentioned above, the analogy of quantum mechanics prompted us to abstract a permeable barrier by a dirac delta, with an expectation that we would obtain 'bound states' that would describe physically realisable probability density functions. We proceeded by the usual Eigen function based analysis, hoping to obtain bound states for Fokker Planck equation in a delta potential.

Going by the usual procedure, we assumed an incident wave (eigen function), which is partially reflected and transmitted, as is done in quantum mechanics. Using scattering matrix, or any such method, we tried to find the 'poles' of the solutions. The stationary state solutions of Fokker Planck equation are obtained by solving the following equations:

$$P_k(x,t) = e^{ikx - Dk^2 t} \tag{9}$$

Here, it is to be noted that, the solutions represent probability density and they don't behave like wave functions in quantum mechanics. Hence, every eigen function must decay in time, i.e. their eigen values should be negative.

Analysing the delta barrier problem, the reflection and transmission coefficients thus obtained are:

$$R = \frac{-ik}{2 - \frac{ik}{\mathscr{P}}} \tag{10}$$

$$T = \frac{2}{2 - \frac{ik}{\mathcal{P}}} \tag{11}$$

To obtain the 'bounded state' solution, the incident wave coefficient is made zero. The solution thus obtained has an exponentially increasing time dependence, which is physically impossible. This irked us, as this was unexpected and we were forced to check back our assumptions. The delta potential assumption is completely fine, but we had our doubts in the transformation of the Fokker Planck Equation to a Schroedinger equation. Such a transformation yields terms of first and second derivatives of the potential, in the 'effective potential' term of Fokker Planck Equation [9]

In this case, since we have a delta potential, which means the effective potential term contains first and second derivatives of the dirac delta. Singularities of second order often are tough to handle mathematically, as their analytic behaviour is not easily described. So, we proceeded to understand the delta potential as a limit of a finite potential field. We tried to find the solutions for a 1D box potential and obtain the solutions for the delta potential as a limit of the 1D potential well.

Once again, we arrived at exponentially growing solutions as the bound state solutions, which made us to rethink our procedure yet again. The box potential had not done any favour for us, it had only made equations more messier. The box potential introduces two singularities in the effective potential, at the starting and ending point of the box.

Thus, the original intention of introducing a delta potential had failed, as we had not resolved the issue of singularities. So, we considered the following symmetric 'ramp' potential:

The boundary conditions are slightly modified in this case, as the potential has a finite gradient. But, in the delta function limit, we get back the usual boundary conditions obtained earlier.

For the case of a symmetric ramp potential, the transmission and reflection coefficients are quite complicated. The Eigen functions in this case are:

$$P_E(x,t) = e^{\frac{-m_1 \pm \sqrt{m_1^2+4DE}x}{2D}} \tag{12}$$

for region with positive slope. E is the eigen value of the partial differential equation.

$$P_E(x,t) = e^{\frac{m_1 \pm \sqrt{m_1^2+4DE}x}{2D}} \tag{13}$$

for region with negative slope

In the limit of a very large slope$m_1$and a small slopem_2, we see that the exponential tail of the eigen function converges to the delta solution. Thus, physical arguments leads us to the asymptotic behaviour of the solution, which is shown in figure.

Thus, it is erroneous to assume that the quantum mechanics picture of a bound state is valid here. The poles obtained obviously need not characterize physically realisable states. The only possible bound state in this problem is the dirac delta distribution.

This is understandable: When a classical brownian particle encounters a delta function well, it is confined there forever; It cannot tunnel through(neither did we expect that), unlike a quantum particle. If it has to cross the barrier, it has to be at the expense of some energy, i.e activated diffusion.

Powles had claimed that the boundary conditions are obtained by considering the barrier to be having a different diffusion coefficient, and considering the limit of zero thickness of the barrier. We had analysed such a situation as well, but they didn't yield any mathematically exciting result. For the case of a box type variation of diffusion coefficient as shown, we obtain the following result a spectrum of Eigen values for even solutions as follows: For the case of infinite such barriers, we need to work out the asymptotic behaviour of the transition matrix T^n, which turned out to be very complicated (Not mentioned here). Our initial idea of conceiving a mathematical representation of a permeable barrier using a delta function potential was unsuccessful, and we were forced to drop this idea. The conclusions here is that we cannot extend the quantum mechanics analogue beyond a limit to understand solutions of Fokker Planck Equation. Nonetheless, Gaussian solutions are still possible to obtain, which obey the boundary conditions, which is worked out.

3 Diffusion in an infinite array of barriers

The solution to the diffusion equation in the presence of a permeable barrier is worked out. The boundary conditions of Powles is assumed and the solution is obtained by using laplace transform.

For the case of a single barrier in each side of the origin, the solution in laplace domain is :

$$P(x,s) = \frac{e^{-\sqrt{\frac{s}{D}}x} + \frac{1}{\mathscr{P}}\sqrt{\frac{s}{D}}e^{-\sqrt{\frac{s}{D}}x}}{\sqrt{\frac{s}{D}}\frac{1}{\mathscr{P}}e^{-\sqrt{\frac{s}{D}}\frac{L}{2}}L + 2\sqrt{\frac{D}{s}}} \tag{14}$$

This is qualitatively interpreted in the high permeability and low permeability limit as follows:

For high permeability, we get back the usual Gaussian solution:

$$P(x,s) = \frac{1}{2\sqrt{sD}} e^{-\sqrt{\frac{s}{D}}x} \tag{15}$$

(Gives Gaussian upon inversion)

For low permeability,

$$P(x,s) = \frac{D\mathscr{P}}{ls\sqrt{s}e^{-\sqrt{\frac{s}{D}}\frac{L}{2}}} \tag{16}$$

whose inverse is Gaussian term with an error function, which is not much significant.

The solutions are discontinuous at the barriers, while the partial derivatives, describing the probability flux are continuous. This indicates that the solution is still Gaussian, with the probability density redistributed at the barrier, thanks to its permeability.

The provision to express the boundary conditions in terms of potential field, leads us to a few interesting results. The case of infinite number of barriers, periodically placed correspond to a periodic potential, which has interesting implications on the Eigen value spectrum of the diffusion equation. A quantum particle in a periodic potential can take up energy eigen values, which occur as distinct continuous bands. This is a direct consequence of the 'Bloch's theorem' , which is extensively used in solid state physics [10]

Bloch' theorem is a result of the operator formalism of quantum mechanics. In the case of quantum mechanics, the hamiltonian operator becomes periodic if the potential is periodic. Similarly the Fokker Planck Operator also becomes periodic if an array of delta barriers is considered. A formal proof of Bloch's theorem can be worked out for the Fokker Planck operator, but it is quite straight-forward.

An array of delta barriers is similar in principle, to the Dirac Kronig Penney Model for electron transport in metals in solid state physics. The eigen value spectrum for this potential is described by a transcendental equation. We tried to obtain the band structure for the Fokker Planck equation in the case of an infinite array of delta potentials and we obtained the following transcendental equation:

$$\frac{\frac{A}{a}\sum\limits_{K=-\infty}^{\infty}\frac{(k+K)^2}{C+(k+K)^2}}{1+\frac{A}{a}\sum\limits_{K=-\infty}^{\infty}\frac{kK+K^2}{C+(k+K)^2}}=\frac{1-\frac{A}{a}\sum\limits_{K=-\infty}^{\infty}\frac{kK+K^2}{C+(k+K)^2}}{\frac{A}{a}-\sum\limits_{K=-\infty}^{\infty}\frac{kK^2+K^3}{C+(k+K)^2}} \tag{17}$$

The transcendental equation, as shown above, contains diverging infinite series, for which no solution is possible. However, when two diverging series are divided, the quotient thus obtained might well be converging. If one of the diverging series is factorisable into a product of converging and diverging series, there could be a well-defined solution for the equation for this equation. Unfortunately, we couldn't factor out the series at all. Thus, the final transcendental equation obtained stays as such, and couldn't be further simplified.

We describe a specific diffusion problem of a colloidal nano-sphere in an actin suspension, which had been studied by Granick. As mentioned earlier, the original idea was to conceive a permeable barrier mathematically and analyse the problems of Granick and similar such diffusion set-up where the particle is hindered by permeable obstacles. Failure in mathematically constructing a permeable barrier, didn't prevent us from analysing the problem that Granick put forward.

4 Effect of Entropic Forces

Granick had claimed that, he obtained a probability distribution which transitions from a Gaussian to an exponential at a size, characteristic of the distance of the particle from the cage size in actin mesh. The exponential tail didn't revert back to a Gaussian over the time scales they considered, but over a very large time scale, the 'hop diffusion' might bring back the Gaussian distribution, which was not their concern. The interesting fact was that the MSD was linear in time, over the whole time scale they had considered. We began by trying to describe the actin mesh primitively, as a usual barrier, whose behaviour can be accounted for, by the tanner boundary condition. As was mentioned earlier, the boundary condition created only a discontinuity in the probability density, while retaining the continuity in the spatial partial derivative. It modifies the time evolution of the probability density, while causing no change in the nature of the function, i.e. the probability density remains Gaussian, and can never be exponential.

The Anomalous yet Brownian diffusion, as they had named it, initially posed a few doubts in front of us; The paper didn't clearly state the method they had followed to obtain the MSD. This question might be trivially dismissed, but several authors have cautioned against interpreting non-ergodicity as spatial heterogeneity, if the data is not analysed for ergodicity [11]. Among the various mathematical models to describe anomalous diffusion, CTRW processes, might well be non-ergodic, but FBM isn't. We had questions regarding the nature of method followed by Granick – Ensemble average or time average, to obtain the MSD. In fact, such an exponential tail is obtained if CTRW is considered to account for the anomaly. We tried to understand how ergodicity could well play a role in diffusion processes. CTRW and FBM have been analysed in the past for ergodicity, and a measure for ergodicity breaking has already been introduced

[12].

Amidst all these, Granick had himself denied CTRW to appropriately model his diffusion data. He had used the 'Exchange and Persistence Time' analysis [13], and did not observe a decoupling of the two, which clearly meant that the process is not a CTRW. So, although there could be questions regarding the nature of MSD measurements, the process was not a CTRW, and it called for a different physical concept to account for the exponential tail, which grows as square root of the time.

In their discussion at the end of their article, Granick had mentioned that the exponential tail could be a result of interaction of the particle with the actin filaments. The interacting forces could not be friction, as they coated the nano-sphere with some protein to avoid undesirable impacts of non-specific adsorption. The actin filament as such, does not create a potential field, that interacts with the nano-sphere. They attributed exponential tail to 'entropic forces' arising out of transverse fluctuations of the filaments. The concept of entropic force was quite new to us, and we had a tough time understanding the origins of these forces.

Meanwhile, we encountered a seminal paper by Zwanzig, in which he had described a model for the passage of ligand through the myoglobin protein [14]. Zwanzig had claimed that the problem can be understood as 'Diffusion through a fluctuating bottleneck'. The model is specific for a particular physical situation, but he had referred to one of his own paper, in which he had analysed rate processes with dynamical disorders [15] . He also mentioned that the bottleneck, being geometrical does not involve potential barriers, but entropic barrier. He had rigorously described entropic barriers, which is a result of geometric constraints on the diffusing particle. Earlier works by Ficks and Jacobi [16], put forward the 'Fick-Jacobi equation' for diffusion in a region with varying cross sections, and they gave heuristic derivations for the equation. Zwanzig had laid this problem on a firm physical foundation in one of his yet another seminal papers [17].

The Fick's Jacobi equation is:

$$\frac{\partial G(x,t)}{\partial t} = D\frac{\partial}{\partial x}\mathscr{A}(\mathrm{x})\frac{\partial}{\partial \mathrm{x}}\mathscr{A}(\mathrm{x})\mathrm{G}(\mathrm{x},\mathrm{t}) \tag{18}$$

The Fokker Planck Equation can well be written as:

$$\frac{\partial G(x,t)}{\partial t} = D\frac{\partial}{\partial x}\mathrm{e}^{-\beta \mathrm{V}(\mathrm{x})}\frac{\partial}{\partial}\mathrm{e}^{\beta \mathrm{V}(\mathrm{x})}\mathscr{G}(,) \tag{19}$$

The term exponential in the potential cues at the equilibrium Boltzmann distribution of the thermodynamic states. This roughly suggests that this can be extended to consider the effects of entropy, by generalising it into free energy A(x). In the absence of any potential barrier, the free energy is purely entropic. And, if the free energy is purely entropic, it has to be completely and only related to the cross sectional area. So, the area of cross section is directly related to the entropy as:

$$\mathscr{A}(\mathrm{x}) = \mathrm{e}^{\mathrm{S}(\mathrm{x})} \tag{20}$$

This was the physical arguments used by Zwanzig, to conceive the so-called Entropic barrier. So, a region with varying cross sections 'interacts' with the diffusing particle through entropy. Thus, an opening whose geometry is fluctuating in time, can also generate 'entropic force' that interacts with the diffusing particle. Effectively, in the problem of fluctuating bottleneck, Zwanzig has accounted for a time varying entropic force.

As described above, the picture of entropic forces and its effects became quite clear, and we proceeded to obtain analytical expression describing the exponential tail obtained in Granick's problem, by accounting for entropic force contribution.

Granick's paper had claimed that the probability distribution has an exponential tail, which does not revert back to the usual Gaussian, for a large time scale. The distribution is Gaussian until a specific displacement and then crosses over to exponential after that point. The value of displacement, for which the distribution transitions is characteristic of the distance between the particle and actin mesh. The usual Gaussian does not evoke a surprise, but the exponential tail does.

Granick had stated that a packet of diffusive processes 'superimposed' with an appropriate weighting of their standard deviation gives an effective exponential distribution. But this weighting did not follow from any physically plausible arguments. Our solution seems to provide a reason for this to occur, but it isn't precise yet, as we had considered a source 'generating' probability. Our next endeavour was to consider physical situations which give rise to the probability 'source' at a specific point. We expected a selectively permeable barrier to be the 'source' which takes in flux from one side and pumps it into the other.

Selectively permeable barriers occur in various contexts of physics, chemistry and biology [18] and typically they have been addressed, by using heuristic boundary conditions such as the one in [6]. However, we considered introducing a new mathematical abstraction for permeable barriers, based on physical intuition. Here, we propose a model for a permeable barrier, founded upon convincing physical arguments and analogy. This model is then applied to the environment described in Granick's experimental studies, where the actin meshes are abstracted as permeable barriers. Also, since the actin filaments are rearranging, the diffusion coefficient is stochastic as well, and hence the diffusing diffusivity [19] [20] [21] model is also relevant, but not complete to describe the anomaly. Our solution uses both these models and we nearly get back the probability distribution that Granick had obtained. However, the model is approximate and its application might be quite limited. In the end, we also discuss possible extensions of the problem.

5 Plausibility argument for the construction of a permeable barrier

Permeable barriers to diffusion typically occur in membrane biophysics, where the cell membrane has selective permeability [18]. In the past, there has been attempts to mathematically understand diffusion in the presence of permeable barriers, using appropriate boundary conditions [6]:

$$\frac{1}{\mathscr{P}} \frac{\partial P}{\partial x} \rfloor_{x=0} = P_+ - P_- \tag{21}$$

These boundary conditions, were directly drawn from analogies in heat diffusion problems [22] and numerical solutions were thus obtained. Such a description had also been used to approximately fit and analyse experimental data in the past [23], but the situations described there did not involve selective permeability. We intended to account for selective permeability, using analogies and plausibility arguments.

We drew an analogy from heat diffusion problems, with conduction and insulation processes concurring, which uses the 'Robin boundary conditions' to conserve heat flux [22]. In fact, a similar such 'partially reflected' boundary condition has been described in the context of diffusion in the past [8], but the idea of understanding a selectively permeable membrane using such a condition does not seem to exist in literature. The Robin boundary condition is the following:

$$D\frac{\partial n(0,t)}{\partial x} = Kn(0,t) \tag{22}$$

The boundary condition is self explanatory and it is, in a way, a conservation equation that equates the flux due to conduction at the boundary to the flux that passes out due to convection. In the context of diffusion, the incoming flux of the diffusing entity is partially absorbed and partially reflected. Totally reflection and total absorption can be obtained in the limits of this boundary condition.

Now, in our model, we claim that the permeable barrier acts as a partial absorber and partial reflector, and the absorbed flux appears as a 'probability source' on the other side, thus conserving the overall probability flux. On the other side, the diffusion is as though the particle has started just at the barrier and has no memory of its past, and is described by the usual Fokker Planck equation, with a source term

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} - S(x,t) \tag{23}$$

Since the barrier is selectively permeable, the particle escaping out of the confined region is almost completely debarred from crossing over to the other side of the membrane. Thus a reflecting boundary condition is applied on the other side of the barrier. Thus, we assume that the barrier segregates the particle into two independent regions, with the diffusion process in each of them decoupled.

One more physical argument favouring our approach is the idea that a selectively permeable barrier induces confinement of the particle within a domain. If a diffusing particle is bounded by barriers, its diffusion becomes confined. This, again is a physically reasonable claim, as the particle would 'feel' that it is confined, but it is occasionally absorbed at the boundary. In the mathematical framework, confinement quantizes the probability 'eigen states' available to the particle. The elaborate mathematical discussion is provided as well.

The set-up just described is similar in spirit to the reaction-diffusion equation in one dimension:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} - f(x,t,P) \tag{24}$$

In such a case, a typical reaction process is depicted thus: Concentration of a chemical is diffusing according to the diffusion equation and reaction occurs with some rate at a specific location, leading to the depletion of the concentration. In this regards, our approach is equivalent to the reaction-diffusion theory and can be understood as a rate process, by which the particle is 'reacting' at some location and hence its probability density is 'depleted' there and it appears on the other side. Interestingly, the correspondence of the Robin boundary conditions with the reaction-diffusion equation has been studied in the past [24].

We now describe the effect of a decaying point source on the nature of diffusion process, in a region with a diffusing diffusivity [19], and then proceed to show its relevance with our permeable barrier construction.

6 Diffusion in the presence of a decaying point source

We describe the anomaly introduced by a decaying point source in the diffusion of a particle in a region with stochastically varying diffusivity. The Fokker-Planck Equation in the presence of a exponentially decaying point source at the origin is:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} - P_0 \delta(x) e^{-\lambda t} \tag{25}$$

Considering a static diffusion coefficient, the solution can be obtained in Laplace domain:

$$sP(x,s) - p(x,0) = D\frac{d^2 P(x,s)}{dx^2} + P_0 \frac{\delta(x)}{s+\lambda} \tag{26}$$

To obtain the boundary conditions, we study the equation in a small limiting region around the origin:

$$s\int_{\varepsilon}^{\varepsilon} P(x,s)dx = D(\frac{dP(x,s)}{dx})\rfloor_{-\varepsilon}^{\varepsilon} + \frac{P_0}{s+\lambda} \tag{27}$$

The integral on the left is a continuous function. So, in the limit $\varepsilon \longrightarrow 0$, the integral on the left tends to zero, and the boundary condition is:

$$D\triangle\frac{dP(x,s)}{dx} = \frac{-P_0}{s+\lambda} \tag{28}$$

The differential equation in each side of the origin is thus:

$$sP(x,s) = D\frac{d^2 P(x,s)}{dx^2} \tag{29}$$

$$\Longrightarrow P(x,s) = A(s)e^{\pm\sqrt{\frac{s}{D}}} \tag{30}$$

Matching with the boundary conditions, we obtain:

$$A(s) = \frac{P_0}{2D(s+\lambda)\sqrt{\frac{s}{D}}} \tag{31}$$

So, the final solution in the laplace domain is:

$$P(x,s) = \frac{P_0}{2\sqrt{sD}(s+\lambda)} e^{-\sqrt{\frac{s}{D}} \bmod x} \tag{32}$$

Formally inverting the transform, we obtain:

$$P(x,t) = P_0 \int\limits_0^t \frac{e^{-\frac{x^2}{ADt_1}}}{\sqrt{4\Pi Dt_1}} e^{-\lambda(t-t_{1)}} dt_1 \tag{33}$$

Alternatively, the solution could have been directly obtained by the operator formalism, which yields the same result:

The solution, as we see, is obtained as a convolution. This integral is not solvable exactly.

7 Solutions of the diffusion equation with Robin Boundary conditions

The physical picture we are trying to describe is:

The Robin boundary condition is as follows:

$$D\frac{\partial P(\pm L,t)}{\partial x} = KP(\pm L,t) \tag{34}$$

With the boundary conditions applied at x = +-L.The factor K in the boundary conditions is a measure of 'permeability' of the barrier. As mentioned before the two limits of the factor K yield the absorbing and reflecting boundary conditions.

As K$\longrightarrow$ 0,

$$\frac{\partial P(\pm L,t)}{\partial x} = 0 \tag{35}$$

This yields the reflecting boundary conditions. In the other extreme, If K$\longrightarrow \infty$, P(x,t) $\longrightarrow 0$ to conserve the net flux, which yields the absorbing boundary conditions.

As is the usual procedure, the separable solutions are:

$$P_k(x,t) = e^{ikx - Dk^2 t} \tag{36}$$

The Eigen functions can be alternatively classified as odd and even solutions:

$$P_e(x,t) = cos(kx)e^{-Dk^2 t} \tag{37}$$

$$P_o(x,t) = sin(kx)e^{-Dk^2 t} \tag{38}$$

As in the case of the standard Storm Liouville problem, the eigen functions thus obtained must satisfy the boundary conditions. So, the eigen values are quantized by the following relations:

For even solutions,

$$tan(kL) = -\frac{K}{kD} \tag{39}$$

For odd solutions,

$$tan(pL) = \frac{Dp}{K} \tag{40}$$

Solutions of these transcendental equation can be obtained only graphically:

It can be clearly infered that Eigen value spectrum is discrete but countably infinite. Finally, all solutions to the boundary value problem can be written as:

$$f(x,t) = \sum_n c_n cos(k_n x)e^{-Dk_n^2 t} + \sum_n b_n sin(k_n x)e^{-Dp_n^2 t} \tag{41}$$

Now, at time t=0, we assume that the particle position distribution is uniform and hence is even. So, the evolution of this distribution with time is expressible only in terms of the even eigen functions.

The eigen functions are orthogonal and the corresponding coefficients c_n are obtained by the usual inner product notion:

$$f(x,t)=\sum_{n=1}^{\infty} c_n e^{-Dk_n^2 t} cos(k_n x) \tag{42}$$

$$c_n \int_{-L}^{L} cos^2(k_n x)dx = \int_{-L}^{L} f(x,0)cos(k_n x)dx \tag{43}$$

$$c_n = \frac{1}{L}\frac{\frac{sin(k_n L)}{k_n L}}{1+\frac{sin(2k_n L)}{2k_n L}} \tag{44}$$

If the particle starts at the origin with probability one at t=0,

$$f(x,0) = \delta(x) \tag{45}$$

$$c_n = \frac{1}{L}\frac{1}{1+\frac{sin(k_n L)}{k_n L}} \tag{46}$$

Thus the solutions are described by a series of time decaying exponentials. The Robin boundary conditions, as mentioned is equivalent to the reaction-diffusion equation and the correspondence can be formally obtained 18:

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} - \bar{K}P(x,t)\delta(x) \tag{47}$$

where x=0 is the position of the barrier. Now, integrating over a small region around origin:

$$\lim_{\varepsilon\to 0}\frac{\partial\int_{-\varepsilon}^{\varepsilon}P(x,t)}{\partial t}=D\int\limits_{-\varepsilon}^{\varepsilon}\frac{\partial^2P(x,t)}{\partial x^2}dx-\bar{K}\int_{-\epsilon}^{\epsilon}P(x,t)\delta(x)dx \tag{48}$$

$$D\left(\frac{\partial P}{\partial x}\right)_{-\varepsilon}^{\varepsilon}=\bar{K}P(0,t) \tag{49}$$

On the other side of the barrier, the flux should be zero (While considering a reaction boundary, this holds. In our case, the flux coming out serves as a source on the other side and when the particle is present in the region outside the barrier, it has no 'memory' of what happened inside the barrier). So:

$$D\left(\frac{\partial P}{\partial x}\right)_{-\varepsilon}=0 \tag{50}$$

Hence:

$$D\frac{\partial P(0,t)}{\partial t}=K\bar{P(0,t)} \tag{51}$$

Which is the Robin boundary condition with $\bar{K}=K$.

Also, the Robin boundary conditions can be obtained by a potential field with 'killing' [8].(See appendix)

So, the absorption rate at the boundary x=L is described by a multi-exponential term as follows:

$$g(L,t)=\sum_{n=1}^{\infty}Kc_ne^{-Dk_n^2t}cos(k_nL) \tag{52}$$

The motivation of the considering an exponentially decaying point source is clear right now, as the results just derived, show a multi-exponential decay rate, which appear at the other side. We now proceed to explain the anomalous probability distribution curve that Granick had obtained, using the just developed model for selectively permeable barriers.

8 Diffusion past of Confinement

We claim that the exponential tail of Granick's result, are a direct consequence of the selective permeability of the membrane. The nano-sphere is confined in a cage formed by the actin filaments. Outside the cage, the actin filaments are randomly fluctuating and hence a diffusing diffusivity model is quite relevant. Considering a 'hybrid' of both yields a result similar to the one obtained by Granick indicating the relevance of our model.

From the expressions derived above, we obtain the decay rate as a function of time, which is multi-exponential. This absorption rate is the source rate at any time on the other side of the barrier. The multi-exponential decay is a simple extension of the case of a single exponential term, which was described before. The formal solution is:

$$P(x,t) = \int_0^t \int_{-\infty}^{\infty} \left(< e^{-\int_0^{t_1} D(t^{'})p^2 dt_1 + ipx} > \sum_{n=1}^{\infty} K\, c_n e^{-Dk_n^2(t-t_1)} cos(k_n L) \right) dp\, dt_1 \tag{53}$$

$$P(x,t) = \sum_{n=1}^{\infty} \int \int e^{ipx} \left(4 \frac{\sqrt{1+\frac{4Fp^2}{\omega^2}}\, e^{-\omega t_1 \left(\sqrt{1+\frac{4Fp^2}{\omega^2}}-1\right)}}{\left(\sqrt{1+\frac{4Fp^2}{\omega^2}}+1\right)^2} K c_n cos(k_n L) e^{-Dk_n^2 t} \right) dtdp \tag{54}$$

$$\sqrt{1+\frac{4Fp^2}{\omega^2}} = \alpha \tag{55}$$

$$P(x,t) = \sum \int_{n=1}^{\infty} e^{ipx} \frac{4\alpha}{(\alpha+1)^2} \left(\frac{e^{-\omega t(\alpha-1))} - e^{-Dk_n^2 t}}{Dk_n^2 t - \omega(\alpha-1)} \right) K c_n cos(k_n L) dp \tag{56}$$

The graphs are plotted numerically: In the long time limit, only the first eigen value is significant which yields a single exponential time dependence at long times. If the dimensions of the confinement is quite small, then we obtain a limiting solution, which resembles a Gaussian. This is similar to Granick's curve, and can be, in principle' extended to the three-dimensional case.

9 Conclusion

We had thus described a model for selectively permeable membrane, which seems to be quite relevant when applied to Granick's experimental results, although it is quite a crude approximation. Our model can be formally extended under various physical contexts; The measure of permeability – K, can well be a function of time, or might be fluctuating. Fluctuating permeability can be analysed by the methods devised by Zwanzig to describe rate processes with dynamic disorders. In fact, such a case, where a particle diffuses and then encounters a fluctuating bottleneck at a specific location and reacts, has been analysed by Tachiya and Seki [25] and approximate solution has been arrived at. Our model could well serve as a fine mathematical representation of a selectively permeable barrier which occurs in various contexts of science.

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10 Figures

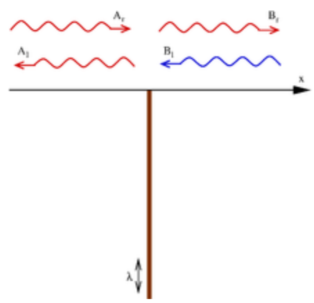

Figure 1: Scattering from a delta potential

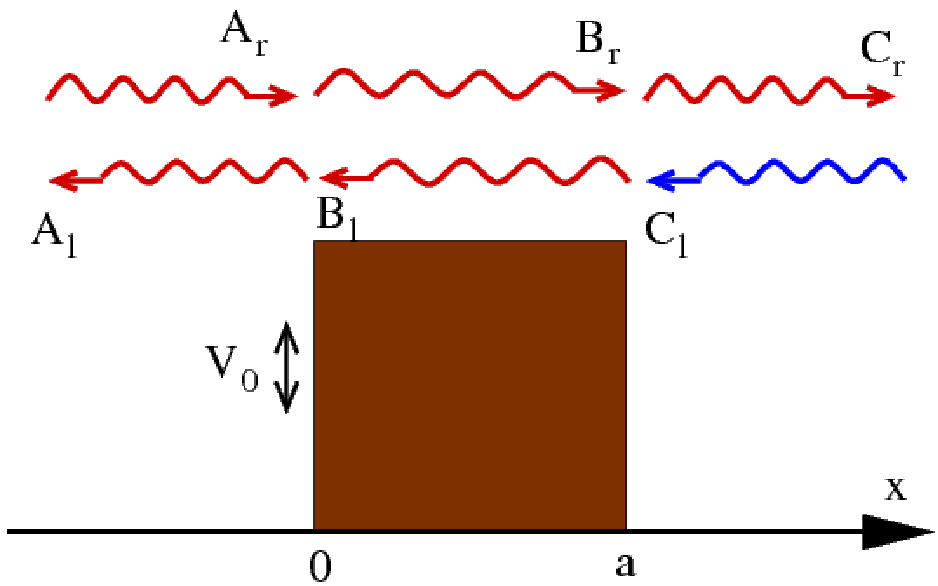

Figure 2: Transmission and reflection in 1D box Potential

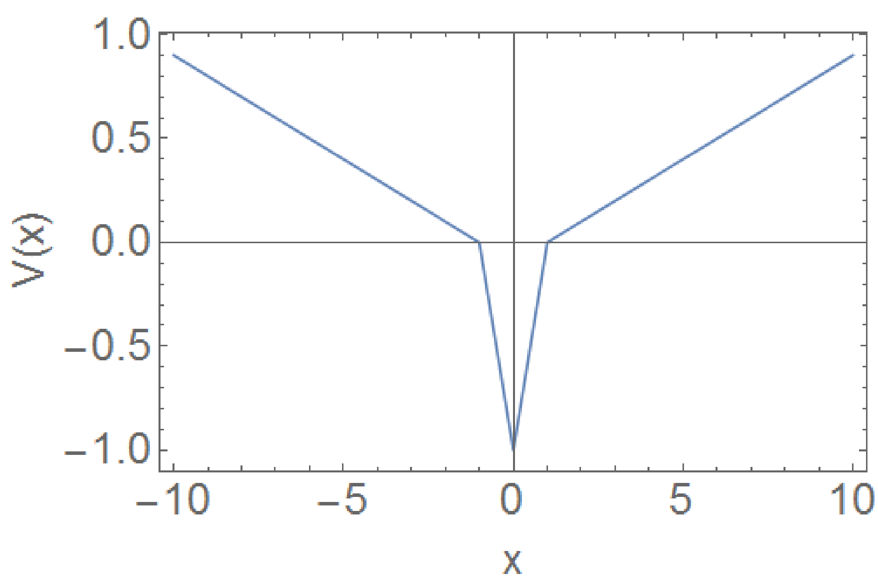

Figure 3: Modified Ramp Potential

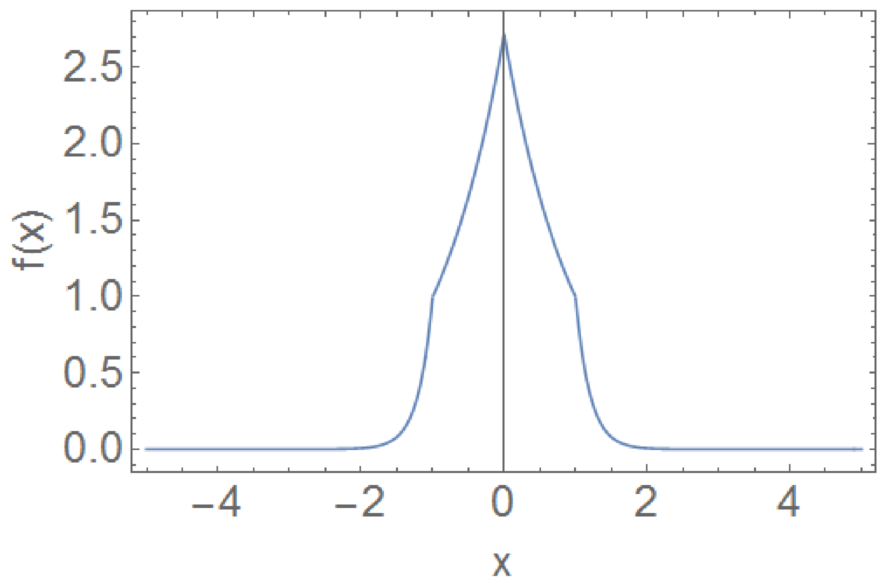

Figure 4: Asymptotic solution in a ramp potential

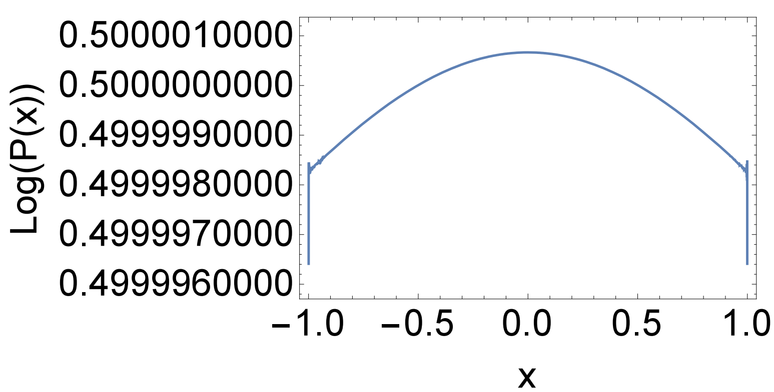

Figure 5: Solution with in the selectively permeable barrier