Theory of Feynman path integrals:

A functional calculus approach

Vijay Ganesh

DECLARATION

I hereby declare that the work presented in the project report entitled "**Theory of Feynman path integrals: A functional calculus approach**" contains my own ideas in my own words. At places, where ideas and words are borrowed from other sources, proper references, as applicable, have been cited. To the best of our knowledge this work does not emanate or resemble to other work created by person(s) other than mentioned herein. The work was created on this 5^{th} **day of April, 2019**.

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Abstract

In this report, we shall provide an exposition of the mathematical theory of Feynman path integrals, which is renown in quantum physics. In this process we also provide an exact derivation for the Feynman integral for a special class of initial wave-functions and potentials.

Contents

1 Introduction

One of the holy grails of modern physics is to solve the time dependent Schrodinger equation,

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{1}$$

to obtain the wave-function either exactly or under appropriate approximation. Since the inception of old quantum theory, several attempts to reconcile classical mechanics with quantum mechanics have been made. One such attempt by Richard Feynman [7] lead to the notion of a path integral and a new formulation of quantum mechanics. Ever since the Feynman path integral was introduced, it had been used extensively to obtain analytical solutions to many problems in physics. While they prove to be intuitive, elegant and novel to work with, a mathematically sound description of the theory of path integrals is necessary and this has always been emphasized [3, 4] in literature. Since the space of continuous paths evolving with time is infinite dimensional, this theory is closely connected to functional calculus, or more specifically, to the theory of integration in function spaces [5]. Over the course of this report, we provide an exact derivation of the path integral expression for a specific class of potentials and initial wave-functions.

In section 2, the basic postulates of quantum mechanics are reviewed and the notion of a unitary propagator is introduced.

In section 3, an integral representation is provided for the propagator. An alternate representation, referred to as the Dyson series is also provided.

In section 4, a brief theory of fresnel integrable functions is provided.

In section 5, a derivation for the path integral is provided using fresnel integrals.

2 Unitary propagator

One of the most important aspects of classical mechanics that is absent in quantum mechanics is *determinism*: A particle with a given position and velocity moves to a specific location in space with a specific velocity after a given time, according to Newton's laws of motion. Quantum mechanics is *probabilistic* and the time dynamics of a system is governed by the evolution of a *quantum state* whose description is encapsulated into the Schrodinger equation.

At the realm in which classical mechanics operates, the measurement process doesn't interfere with the observation and quantification of physical quantities such as energy, momentum, position etc. Whereas, one of the fundamental claims of quantum mechanics is that measurement process disturbs the system and therefore it is impossible to obtain arbitrary precision while simultaneously measuring, say for instance, position and momentum. This limitation is called the *Heisenberg Uncertainty principle*.

Thus, our central concerns in quantum mechanics are to obtain statistical averages of *observables* such as position, momentum, energy etc. from the given quantum *state* of the system. To provide a mathematically rigorous description of quantum mechanics, the first step is to axiomatize the above fundamental philosophies in terms of mathematically concrete postulates:

Postulate 1: The state space of a quantum mechanical system has the structure of a complex separable Hilbert space $\mathcal{H}$. Associated with any observable A of the system is a self–adjoint operator $\hat{A}$ on the state space.

Postulate 2: The possible outcome of measuring the observable A are precisely the points of the spectrum $\sigma(\hat{A})$ of the operator $\hat{A}$. If the corresponding projection–valued measure and spectral decomposition of $\hat{A}$ is given as E_A and $\left\{E_A^{(\lambda)}\right\}$, then the probability of observing the measured value in a borel subset $\Delta \subset \mathbb{R}$ is given as:

$$w(\Delta, A, \psi) = \int d(\psi, E_A^{(\lambda)}) = ||E_A(\Delta)\psi||^2 \tag{2}$$

Postulate 3: If the measured value from an experiment is in the borel set Δ, then the system evolves to the state given by $E_A(\Delta)\psi/||E_A(\Delta)\psi||$ after measurement.

For postulate 4 and 5, we need the definition of a *unitary propagator*:

Definition: Unitary propagator

A unitary propagator is a family of operators $\{U(t,s) : t, s \in \mathbb{R}\}$ such that:

(a) The map $(t, s) \mapsto U(t, s)$ is strongly continuous as a map from $\mathbb{R}^2$

(b) $U(t,s)U(s,r) = U(t,r)$ holds for all $r,s,t \in \mathbb{R}$ and $U(t,t) = I$ for all $t \in \mathbb{R}$

Postulate 4: The time evolution of any state of the system is described by a *unitary propagator* U i.e.

$$\psi_t = U(t,s)\psi_s \tag{3}$$

The fifth postulate requires the notion of a conservative system:

Definition: Conservative system

A physical system is said to be *conservative* if its propagator satisfies $U(t+\tau, s+\tau) = U(t,s)$.

For a conservative system, the Unitary propagator becomes a one-parameter family $\{U(t) : t \in \mathbb{R}\}$, where $U(t) = U(t+\tau,\tau)$. Thus the Stone's theorem can be applied to this family:

Stone's theorem:

Let $(U_t)_{t\in R}$ be a family of strongly continuous unitary operators on a Hilbert space $\mathcal{H}$ that form a unitary group. Then there exists a unique operator $A : D_A \mapsto \mathcal{H}$ that is self-adjoint on D_A and such that $\forall t \in \mathbb{R}, U_t = e^{itA}$. Here, D_A is the domain of A.

The fifth postulate is the fundamental dynamical postulate of quantum theory and identifies the operator $-A$ from Stone's theorem, with the Hamiltonian $\hat{H}$ of the conservative system, as follows:

Postulate 5: The propagator of a *conservative* system with the Hamiltonian $\hat{H}$ is given by $U(t) = e^{-it\hat{H}}$ for any $t \in \mathbb{R}$.

Much of the endeavours in modern quantum dynamics has been to obtain suitable representation of the unitary propagator $U(t)$. In the next section, we provide a derivation for an integral representation, which was extended informally as an integral over the 'path space' (path integral) by Richard Feynman.

3 Representation of the propagator

The Hamiltonian $\hat{H}$ of any physical system can be written as:

$$\hat{H} = \hat{H}_0 + \hat{V},$$

where

$$\hat{H}_0 = \sum_{j=1}^{n} -\frac{1}{2m_j}\frac{\partial^2}{\partial x_j^2}$$

is the free particle Hamiltonian and $\hat{V}$ is a scalar/vector valued potential. Here, x_j's are the coordinates along each degree of freedom. Our derivation, for the most part, would be for a single degree of freedom, but in principle, the same derivation holds for any number of degrees of freedom.

We state the following theorem without proof, as the proof is beyond the scope of this report:

Trotter's formula:

Suppose A, B are self adjoint operators and $C := A + B$ is essentially self adjoint. Then the corresponding unitary groups of A, B and C are related by:

$$e^{it\bar{C}} = \underset{n\to\infty}{s\text{—}lim}\ (e^{itA/n}e^{itB/n})^n \tag{4}$$

where $s - lim$ denotes the limit in the strong operator topology.

We can apply the Trotter formula to the above Hamiltonian $\hat{H}$, for the class of potentials V for which $\hat{H}$ is essentially self adjoint, to get a limiting expression for the unitary propagator $U(t)$:

$$U(t) = \underset{n\to\infty}{s\text{—}lim}\ (e^{it\hat{H}_0/n}e^{it\hat{V}/n})^n \tag{5}$$

For any $\psi \in L^2(\mathbb{R}^n)$, the action of the propagator can be obtained as a limit of the following approximating sequence:

$$U(t)\psi = lim_{N\to\infty}\ \psi_t^{(N)} \tag{6}$$

where the approximating sequence is defined as follows:

$$\begin{aligned} \psi_t^{(0)} &:= \psi \\ \psi_t^{(N)} &:= e^{-iH_0t/N}e^{-iVt/N}\psi_{t(N-1)/N}^{(N-1)} \end{aligned} \tag{7}$$

Since the operator $e^{-iVt/N}$ is simple scalar multiplication, we need to obtain a suitable representation only for the operator $e^{-iH_0t/N}$; The following theorem provides us an integral representation for the propagator of the free Hamiltonian, e^{-iH_0t}.

Theorem 3.1:

For the free particle Hamiltonian, $\hat{H}_0 = \sum_{j=1}^n -\frac{1}{2m_j}\frac{\partial^2}{\partial x_j^2} = \sum_{j=1}^n \frac{1}{2m_j}P_j^2$ as described before, the following holds:

$$(U(t)\psi)(x) = \lim_{k\to\infty}\prod_{j=1}^n\left(\frac{m_j}{2\pi it}\right)^{\frac{1}{2}}\int_{\mathbb{R}^n} exp(\frac{i}{2t}\sum_{j=1}^n m_j|x_j - y_j|^2)\psi(y)f_k(y)dy \tag{8}$$

for all $\psi \in L^2(\mathbb{R}^n$, where $\{f_k\}$ is an arbitrary sequence of continuous functions such that $|f_k(x)|\leq 1$ and $\lim_{k\to\infty} f_k(x) = 1$ for almost all $x \in \mathbb{R}^n$.

Proof:

Without loss of generality, we can assume that the masses m_j to be $1/2$ and hence set $\hat{H}_0 := P^2 = \sum_{j=1}^n P_j^2$. (This transformation is unitary and hence P^2 is unitarily equivalent to H_0.)

Before we proceed to obtain the integral representation (8), we prove the following results that concerns with the momentum operator P.

(i) The position operator Q and the momentum operator P are unitarily equivalent.

We prove the result in $1D$, but it can in principle be proved in higher dimensions.

$$\hat{P}\psi = -i\hbar\frac{d\psi}{dx}$$
$$\hat{Q}\psi = x\psi$$

Let F be the fourier transform operator. Then we have:

$$F(\hat{P}\psi) = \int_{-\infty}^{\infty} -i\hbar e^{-ikx}\frac{d\psi}{dx}dx = \hbar k\int_{-\infty}^{\infty} e^{-ikx}\psi dx$$
$$= pF\psi(p) = \hat{Q}(F\psi) \qquad \forall\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

(ii) The identity $f(P)\psi(x) = (2\pi)^{1/2}\int_{\mathbb{R}}(Ff)(y-x)\psi(y)dy$ holds for any Borel function $f : \mathbb{R} \mapsto \mathbb{C}$ and $\psi \in L^2(\mathbb{R})$

For any Borel function f, $f(P) = F^{-1}f(Q)F$, where F is the Fourier transform operator. This identity can be proved from the existence of a Borel functional calculus for a self-adjoint operator [9]. The proof of this result is beyond the scope of this report.

From this result, we have:

$$(f(P)\psi)(x) = (F^{-1}f(Q)F\psi)(x) \quad \forall x \in \mathbb{R}$$

Now,

$$F^{-1}f(Q)F\psi(x) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{ixy}f(y)F\psi(y)dy$$

Setting $\eta = F\psi$, we can simplify this as:

$$(2\pi)^{1/2}(f(P)(x) = \langle\theta_x, F\psi\rangle \quad where \quad \theta_x(y) = e^{-ixy}\overline{f(y)}$$

Defining $\widetilde{U_x}$ to be the following unitary operator: $\widetilde{U_x}\psi(y) = e^{ixy}\psi(y)$ we get:

$$(2\pi)^{1/2}(f(P)(x) = \left\langle\widetilde{U_x}^*\theta_0, F\psi\right\rangle,$$

provided the inner product is well-defined, which can be shown as follows:

Firstly, it can be seen that $e^{iaP}\psi(x) = U_a\psi(x)$, where U_a is the unitary translation operator: $U_a\psi(x) = \psi(x+a)$. We also have that $e^{iaQ}F\psi = Fe^{iaP}\psi = FU_a\psi$. Let T_n denote the operator of multiplication by $\chi_{(-n,n)}$, so that $T_n\theta_x \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus the inner product above can be obtained as the following limit:

$$\left\langle\widetilde{U_x}\theta_0, F\psi\right\rangle = \lim_{n\to\infty}\langle T_n\theta_x, F\psi\rangle$$

$\langle T_n\theta_x, F\psi\rangle$ can be evaluated as follows:

$$\langle T_n\theta_x, F\psi\rangle = \left\langle T_n\widetilde{U_x}^*\theta_0, F\psi\right\rangle = \left\langle T_n\theta_0, \widetilde{U_x}F\psi\right\rangle$$
$$= \left\langle T_n\theta_0, e^{ixQ}F\psi\right\rangle = \langle T_n\theta_0, FU_x\psi\rangle = \left\langle F^{-1}T_n\theta_0, U_x\psi\right\rangle$$

Thus, finally we get,

$$(2\pi)^{1/2}(f(P)(x) = \lim_{n\to\infty}\langle T_n\theta_x, F\psi\rangle =$$
$$\int_{\mathbb{R}}\overline{F^{-1}\overline{f(y)}}\psi(x+y)dy = \int_{\mathbb{R}}\overline{F^{-1}\overline{f(y-x)}}\psi(y)dy = \int_{\mathbb{R}}Ff(y-x)\psi(y)dy$$

Now, consider the class of functions defined on $\mathbb{R}$ as:

$$u_\epsilon := u_\epsilon(k) = e^{-i|k^2|(t-i\epsilon)}, \quad \epsilon \geq 0, \quad u := u_0$$

These are clearly borel functions, as they are continuous; So, the identity in (ii) holds for the each of them. Now the unitary propagator $U(t)$ can be obtained as a limit of these functions:

$$\begin{aligned} U(t) =& u(P) = \underset{\epsilon\to 0^+}{s\text{–}lim}\ \ u_\epsilon(P) \\ (u_\epsilon(P)\psi)(x) =& \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} F u_\epsilon(y-x)\psi(y)dy \\ =& \frac{1}{(4\pi i(t-i\epsilon))^{1/2}} \int_{\mathbb{R}} e^{\frac{i(x-y)^2}{4(t-i\epsilon)}} \psi(y)dy, \qquad \forall \psi \in L^2(\mathbb{R}) \end{aligned}$$

It is to be noted here that while $U(t)$ is unitary, u_ϵ is not unitary for $\epsilon > 0$. Also, the function $u_\epsilon(P)\psi \in L^2(\mathbb{R})$ for $\epsilon > 0$, as it is square integrable. Since $L^2(\mathbb{R})$ is a banach space, by its completeness property, we get that the sequence defined by $f_k = u_{\epsilon_k}(P)\psi$ converges pointwise to $u(P)\psi$ as $\epsilon_k \to 0$ [10]. Choosing ψ such that $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have the following inequality:

$$\left| \frac{e^{i(x-y)^2/4(t-i\epsilon)}}{4\pi i(t-i\epsilon)^{1/2})} \right| |\psi(y)| \leq \frac{|\psi(y)|}{4\pi(t^2+\epsilon^2)}$$

Since ψ is integrable, by the dominated convergence theorem, we get the following relation:

$$(U(t)\psi(y))(x) = (4\pi i t)^{1/2} \int_{\mathbb{R}} e^{i\frac{(x-y)^2}{4t}} \psi(y)dy$$

Now, for a general $\psi \in L^2(\mathbb{R})$ and an arbitrary sequence $\{f_k\}$ as in the statement of the theorem, the integral

$$\int_{\mathbb{R}} e^{\frac{i}{4t}(x-y)^2} \psi(y) f_k(y)dy$$

exists for each k, and can be bounded by the Cauchy-Schwartz inequality.

Thus, after the unitary transformation back to the general masses m_j, we get:

$$(U(t)\psi)(x) = \lim_{k\to\infty} \prod_{j=1}^{n} \left(\frac{m_j}{2\pi i t}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{2t}\sum_{j=1}^n m_j|x_j-y_j|^2} \psi(y) f_k(y)dy.$$

The Feynman integral:

Consider the open balls $B_k \subseteq \mathbb{R}$, defined as $B_k = \{x \in \mathbb{R} : |x| \leq k\}$ and set f_k in Theorem 1.1 to be the characteristic functions on B_j. An integral expression for the approximating sequence (7) can be obtained from (8).

$$\begin{aligned} \psi_t^{(1)} &= e^{-iH_0 t} e^{-iVt} \psi \\ &= \left(\frac{m_j}{2\pi i t}\right)^{1/2} \lim_{k\to\infty} \int_{B_k} e^{\frac{i}{2t}\sum_{j=1}^n m_j|x_j-y_j|^2)} e^{-iV(y)t} \psi(y)dy \end{aligned}$$

Continuing thus for each x, we obtain:

$$\psi_t^{(N)} = \prod_{j=1}^{n} \left(\frac{m_j}{2\pi i \delta_N} \right)^{N/2} \lim_{j_1, j_2, \dots j_N \to \infty} \int_{B_{j_1} \times B_{j_2} \times \dots B_{j_N}} e^{iS_N(y^{(0)}, \dots y^{(N)}; t)} \psi(y^{(0)}) dy^{(0)} dy^{(1)} \dots dy^{(N-1)} \tag{9}$$

Here, $\delta_N = \frac{t}{N}$, $y^{(N)} = x$ and

$$S_N(y^{(0)}, \dots y^{(N)}; t) = \sum_{k=0}^{N-1} \sum_{j=1}^{n} \frac{m_j}{2\delta_N} |y_j^{(k+1)} - y_j^{(k)}|^2 - V(y^{(k)}) \delta_N \tag{10}$$

For a piecewise linear path γ with vertices at $y^{(k)}$, i.e. $y^{(k)} = \gamma(kt/N)$, we get:

$$\int_0^t |\dot{\gamma}(y^{(0)}, y^{(1)}, \dots y^N; \tau)|^2 d\tau = \sum_{k=0}^{N-1} \frac{1}{\delta_N} |y^{(k+1)} - y^{(k)}|^2$$

In classical mechanics, action is a functional defined over a continuous path γ as follows:

$$S[\gamma] = \int_0^t \left(\sum_{j=1}^{n} \frac{1}{2} m_j \dot{\gamma}_j(s)^2 - V(\gamma(s)) \right) ds \tag{11}$$

We can clearly observe that for a piecewise linear path, the first term in the summation in equation (10) and the first term in the action integral given by equation (11) coincide. If the difference in the second term is small, we can expect that in the limit of $N \to \infty$, the second term also coincides, in which case, equation transforms as:

$$(U(t)\psi)(x) = \lim_{N \to \infty} \prod_{j=1}^{n} \left(\frac{m_j}{2\pi i \delta_N} \right)^{N/2} \times \lim_{j_1, j_2, \dots j_N \to \infty} \int_{B_{j_1} \times B_{j_2} \times \dots B_{j_N}} e^{iS(y^{(0)}, \dots y^{(N)}; t)} \psi(y^{(0)}) dy^{(0)} dy^{(1)} \dots dy^{(N-1)} \tag{12}$$

Since any continuous path can be obtained as the limit of a sequence of piecewise linear paths, it is natural to ask if the integral in equation (12) can be replaced by an integral over the set of all continuous paths ending at x, as follows:

$$(U(t)\psi)(x) = \int_{\gamma(t)=x} e^{iS(\gamma)} \psi(\gamma(0)) D\gamma \tag{13}$$

This idea is due to Richard Feynman, and the integral in (13) is referred to as the Feynman path integral. It is to be noted here that the path space(the space of continuous paths ending at x) is infinite dimensional, and this integral cannot be understood using the usual tools of finite-dimensional analysis. The motivation of the further part of this report is to provide a class of initial wave-functions and potentials for which equation (13) can be derived formally.

Dyson Expansion:

The Dyson series expansion is another representation of the unitary propagator. In the derivation of the integral representation before, the Hamiltonian H didn't have any explicit time dependence. Dyson series expansion, in the context of quantum physics, is primarily aimed at obtaining an expression for the unitary propagator for interacting system with a time-dependent potential; The following theorem provides this expansion:

Theorem 3.2:

Let $H : \mathbb{R} \mapsto \mathcal{B}(\mathcal{H})$ be a strongly continuous Hermitian valued function onto the space of bounded linear operators on a Hilbert space $\mathcal{H}$.
Set $\psi_t := \phi + \sum_{n=1}^{\infty} U_n(t,s)\phi$, where

$$U_n(t,s)\phi = (-i)^n \int_s^t dt_1 \int_s^{t_1} dt_2 ...... \int_s^{t_{n-1}} H(t_1)H(t_2)....H(t_n)\phi \tag{14}$$

for any $\phi \in \mathcal{H}$. The series $\sum_n U_n(t,s)$ converges in the operator norm to $U(t,s)$ and so $U(t,s)\phi = \psi_t$. This $U(t,s)$ defines a unitary propagator and ψ_t solves the Schrodinger equation with $\psi_s = \phi$.

Before we proceed to the proof of this theorem, we make the following remark:

Remarks:

Suppose that $(A, \mathcal{A}, \mu)$ is a σ-finite measure space and E is a banach space. A function $f : A \mapsto E$ is said to be Bochner integrable if there exists a sequence of μ-simple functions $f_n : A \mapsto E$ such that the following holds:

(i) $\lim_{n\to\infty} f_n = f$, μ almost everywhere.

(ii) $\lim_{n\to\infty} \int_A ||f_n - f|| d\mu = 0$

If f is μ-bochner integrable, then the limit $\int_A f d\mu := \lim_{n\to\infty} \int_A f_n d\mu$ exists and is called the Bochner integral of f.
If f is μ-bochner integrable, then for all linear functional T on E, we have:

$$T(\int_A f d\mu) = \int_A T f d\mu$$

So, if E is a Hilbert space, then by the Riesz representation theorem, any linear functional acting on the Bochner integral can be interpreted as the usual Lebesgue integral over an inner product.

c*Proof:*

There are several parts in this proof. First, we show that the each $U_n(t,s)$ is well defined and strongly continuous as a function defined on $\mathbb{R}^2$. Then, we show that $U_n(t,s)$ converges uniformly to $U(t,s)$ on compact sets, so that $U(t,s)$ is also strongly continuous. Then, we show that $U(t,s)$ is a unitary propagator and then that ψ_t is a solution to the Schrodinger equation.

Strong continuity of $U_n(t,s)$: (By Induction)

We first show that $U_n(\cdot,s)\phi$ is continuous for any given $s\in\mathbb{R}$ and $\phi\in\mathcal{H}$.

$$U_1(t,s)\phi = -i\int_s^t dt_1 H(t_1)\phi$$

$$\|U_1(t_1,s)\phi - U_1(t_0,s)\phi\| = \left|\int_{t_0}^t dt_1 H(t_1)\phi\right|$$

Interpreting this as a Bochner integral, we have the following inequality:

$$\|U_1(t_1,s)\phi - U_1(t_0,s)\phi\| \leq \int_{t_0}^t dt_1 \, \|H(t_1)\phi\|$$

Now, since the space $\mathcal{B}(\mathcal{H})$ has the strong operator topology and H is strongly continuous, the evaluation map $(H(\cdot),\phi)\mapsto H(\cdot)\phi$ is continuous on $\mathbb{R}$. Thus, the integral on the right hand side becomes bounded and hence $U_1(\cdot,s)$ is continuous.

For $U_n(\cdot,s)$, the continuity can be proved using the following recursive relation:

$$U_{n+1}(t,s)\phi = -i\int_s^t dt_1 H(t_1)U_n(t_1,s)\phi \tag{15}$$

Consider the compact set $K_T=\{(s,t): s^2+t^2\leq T^2\}\subseteq\mathbb{R}^2$; The interval $[-T,T]$ lies inside this compact set and hence the family $\mathcal{S}=\{H(t): t\in[-T,T]\}\subseteq\mathcal{B}(\mathcal{H})$ is pointwise bounded for each $\phi\in\mathcal{H}$, which follows from the continuity of H and compactness of K_T. So, by the uniform boundedness principle on $\mathcal{S}$, we have that: $||H(t)||\leq C_T \ \forall t\in[-T,T]$.

Now,

$$||U_1(t,s)\phi||\leq \int_s^t dt_1||H(t_1)\phi||\leq C_T||\phi|||t-s|\leq 2TC_T||\phi||$$

$$||U_{n+1}(t,s)\phi||\leq \int_s^t dt_1||H(t_1)U_n(t_1,s)\phi||\leq C_T^n||\phi||\frac{|t-s|^n}{n!}\leq \frac{(2TC_T)^n}{n!}||\phi|| \tag{16}$$

as $|t-s|\leq 2T$ for $(s,t)\in K_T$.

We can now show the strong continuity of $U_n(t,s)$:

$$\begin{aligned}
||U_n(t,s)-U_n(t_0,s_0)|| &\leq ||U_n(t,s)-U_n(t_0,s)||+||U_n(t_0,s)-U_n(t_0,s_0)|| \\
||U_n(t,s)\phi-U_n(t_0,s)\phi|| &\leq |\int_{t_0}^t dt_1||H(t_1)U_{n-1}(t_1,s)\phi||,
\end{aligned}$$

So,

$$||U_n(t,s)-U_n(t_0,s)|| \leq \frac{C_T(2TC_T)^{n-1}}{(n-1)!}|t-t_0|$$

Similarly, we can show an estimate for the second time in the above inequality, so that,

$$||U_n(t,s)-U_n(t_0,s_0)|| \leq \frac{C_T(2TC_T)^{n-1}}{(n-1)!}\{|t-t_0|+|s-s_0|\}.$$

Thus $U_n(t,s)$ is strongly continuous as a function from $\mathbb{R}^2$ with the $||\cdot||_1$ norm. By the equivalence of norms on $\mathbb{R}^2$, we get that $U_n(t,s)$ is strongly continuous in the euclidean norm as well.

Uniform convergence of $U_n(t,s)$:

In the previous part of the proof, we showed that, for each n and T, the operator $U_n(t,s)$ is bounded, as shown above. So, the sequence defined by $U^{(N)}(t,s) := I + \sum_{n=1}^{N} U_n(t,s)$ is exponentially bounded on compact sets K_T:

$$||U^{(N)}(t,s)|| \leq ||I|| + \sum_{n=1}^{N} ||U_n(t,s)|| \leq \sum_{n=0}^{N-1} \frac{(2TC_T)^{n-1}}{(n-1)!}$$

and hence converges uniformly in the operator norm to $U(t,s)$. Since each $U_n(t,s)$ is strongly continuous, we get that $U^{(N)}(t,s)$ is strongly continuous for each N and thus by uniform convergence, $U(t,s)$ becomes strongly continuous.

Claim: $U(t,s)$ is a unitary propagator.

We already showed that $U(t,s)$ is strongly continuous. Now, we need to show that $U(t,s)$ is unitary and that property (ii) in the definition of unitary propagator holds.

First we show $U_n(t,s)^* = U_n(s,t)$ as follows:

For $\phi, \psi \in \mathcal{H}$,

$$\begin{aligned}
\langle \psi, U_1(t,s)^* \phi \rangle &= \overline{\langle \phi, U_1(t,s)\psi \rangle} \\
\langle \phi, U_1(t,s)\psi \rangle &= -\, i \int_s^t \langle \phi, H(t_1)\psi \rangle \, dt_1 = -i \int_s^t \langle H(t_1)\phi, \psi \rangle \, dt_1 \\
\overline{\langle \phi, U_1(t,s)\psi \rangle} &= i \int_s^t \langle \psi, H(t_1)\phi \rangle \, dt_1 = -i \int_t^s \langle \psi, H(t_1)\phi \rangle \, dt_1
\end{aligned}$$

But,

$$-i \int_t^s \langle \psi, H(t_1)\phi \rangle \, dt_1 = \langle \psi, U_1(s,t)\phi \rangle$$

Since ϕ, ψ were arbitrary, we have that $U_1(t,s)^* = U_1(s,t)$. We can retrace the same proof and use induction to show that:

$$U_n(t,s)^* = U_n(s,t) \quad \forall s,t \in \mathbb{R}$$

To show: $U(t,s)U(s,r) = U(t,r) \qquad \forall t,s,r \in \mathbb{R}$

$$\begin{aligned}
U^{(N)}(t,s)U^{(N)}(s,r) &= \sum_{i=0}^{N} U_i(t,s) \sum_{j=0}^{N} U_j(s,r) \\
&= \sum_{n=0}^{N} \sum_{j=0}^{n} U_j(t,s)U_{n-j}(s,r) + \sum_{n=N+1}^{2N} \sum_{j=n-N}^{N} U_j(t,s)U_{n-j}(s,r)
\end{aligned}$$

From the bound obtained before for $s, t, r \in K_T$, we have:

$$||U_j(t,s)U_{n-j}(s,r)|| \leq C_T^j \frac{(2T)^j}{j!} C_T^{n-j} \frac{(2T)^{n-j}}{(n-j)!}|| = \frac{(2TC_T)^n}{j!\,(n-j)!}$$

$$|| \sum_{n=N+1}^{2N} \sum_{j=n-N}^{N} U_j(t,s)U_{n-j}(s,r)|| \leq \sum_{n=N+1}^{2N} \sum_{j=n-N}^{N} ||U_j(t,s)U_{n-j}(s,r)||$$

$$\leq \sum_{n=N+1}^{2N} \frac{(2TC_T)^n}{n!} \sum_{j=n-N}^{N} \frac{n!}{j!\,(n-j)!} \leq \sum_{n=N+1}^{2N} \frac{(4TC_T)^n}{n!}$$

Thus, the norm of the second term is bounded by the tail of an exponential series, so vanishes in the limit $N \to \infty$.

Now, we show by induction that: $\sum_{j=0}^{n} U_j(t,s)U_{n-j}(s,r) = U_n(t,r)$
For $n = 0$, we get:

$$U_0(t,s)U_0(s,r) = I = U_0(t,r)$$

For $n > 0$, by induction hypothesis,

$$\begin{aligned}
U_{n+1}(t,r) &= -i \int_r^t dt_1 H(t_1) U_n(t_1,r) \\
&= -i \int_r^s dt_1 H(t_1) U_n(t_1,r) - i \int_s^t dt_1 H(t_1) U_n(t_1,r) \\
&= U_{n+1}(s,r) + \sum_{j=0}^{n} \left\{ -i \int_r^s dt_1 H(t_1) U_j(t_1,s) \right\} U_{n-j}(s,r) \\
&= U_{n+1}(s,r) + \sum_{j=0}^{n} \left\{ -i \int_s^t dt_1 H(t_1) U_j(t_1,s) \right\} U_{n-j}(s,r) \\
&= U_{n+1}(s,r) + \sum_{j=0}^{n} U_{j+1}(t,s) U_{n-j}(s,r)
\end{aligned}$$

from which we can conclude that $U(t,s)U(s,r) = U(t,r) \quad \forall t,s,r \in \mathbb{R}$.

Also, with this we can also show that $U(t,s)$ is unitary:
$U_n(t,s)^* = U_n(s,t)$ so we get $U(t,s)^* = U(s,t)$ due to uniform convergence. Thus,

$$U(t,s)^* U(t,s) = U(s,t)U(t,s) = U(s,s) = I$$

This completes the proof of the claim.

Claim: $\psi_t := U(t,s)\phi$ is a solution to the Schrodinger equation, with $\phi = \psi_s$

Interpreting the recursive relation (15) in the sense of the Bochner integral, we obtain the following equality:

$$\frac{d}{dt} U^{(N)}(t,s)\phi = -iH(t)U^{(N-1)}(t,s)\phi$$

Now, due to the strong continuity of $H(\cdot)$ and uniform convergence of $U^{(N-1)}(t,s)$,

$$\lim_{N\to\infty} -iH(t)U^{(N-1)}(t,s)\phi = -H(t)U(t,s)\phi$$

TO SHOW: $\frac{d}{dt}U^{(N-1)}(t,s)\phi = \frac{d\psi_t}{dt}$ as $N \to \infty$

$$\psi_t = U(t,s)\psi_s$$

$$\frac{d\psi_t}{dt} = \lim_{h\to 0}\sum_{n=0}^{\infty}\frac{U_n(t+h,s)\phi - U_n(t,s)\phi}{h}$$

So,

$$\begin{aligned}||\frac{d\psi_t}{dt} - \frac{dU^{(N)}(t,s)}{dt}|| &= \lim_{h\to\infty}||\sum_{n=0}^{\infty}\frac{U_n(t+h,s)\phi - U_n(t,s)\phi}{h} - \sum_{n=0}^{N}\frac{dU_n(t,s)\phi}{dt}|| \\ &\leq \lim_{h\to\infty}\left\{||\sum_{n=0}^{N}\frac{U_n(t+h,s)\phi - U_n(t,s)\phi}{h} - \frac{dU_n(t,s)\phi}{dt}||\right. \\ &\qquad + ||\sum_{n=N+1}^{\infty}\frac{U_n(t+h,s)\phi - U_n(t,s)\phi}{h}||\end{aligned}$$

The first term in the limit vanishes in the limit $h \to 0$ by the definition of the derivative. The second term is bounded by the previous estimate (16), so it is bounded by the tail of an exponential series, which decays to 0, as $N \to \infty$.

This completes the proof of theorem 3.2.

Interaction picture of Quantum Mechanics:

In most problems in modern quantum physics, one deals with time dependent interactions. To handle such problems, one uses the interaction picture of quantum mechanics. The fundamental equation of the interaction picture of quantum mechanics is obtained as follows:

Let ψ_S be the wave-function in the Schrodinger picture of quantum mechanics. Denote by ψ_I, the wave-function in the interaction picture, where:

$$\psi_I(t) = e^{iH_0t}\psi_S(t)$$

The Schrodinger equation is:

$$i\frac{\partial\psi_S(t)}{\partial t} = \hat{H}\psi_S(t) = (\hat{H}_0 + \hat{V})\psi_S(t)$$

where $\hbar$ has been set to 1. Using the concept of operator derivative, as described in detail in [11], we can show the following:

$$\frac{\partial\psi_S(t)}{\partial t} = -iH_0\psi_S(t) + e^{-iH_0t}\frac{\partial\psi_I(t)}{\partial t}$$

Thus the Schrodinger equation modifies to:

$$i\frac{\partial\psi_I(t)}{\partial t} = e^{iH_0t}V(t)e^{-iH_0t}\psi_I(t) = V_I(t)\psi_I(t) \tag{17}$$

This is the fundamental equation in the interaction picture. A similar expression for $U_n(t,s)$ and $U(t,s)$ as in Theorem 3.2 can be obtained for the interaction picture equation as well. Equation (14) can be rewritten as:

$$\psi(x,t) = \psi_S(x,t) = e^{-it\hat{H}}\phi = \sum_{n=0}^{\infty}(-i)^n \int_{0\leq t_1\leq t_2\leq ....\leq t_n\leq t} .... \int H(t_n)H(t_{n-1})....H(t_1)\phi\, dt_1dt_2....dt_n \tag{18}$$

From equation (17), we obtain:

$$\psi_I(x,t) = e^{-iV_I(t)t}\phi = \sum_{n=0}^{\infty}(-i)^n \int_{0\leq t_1\leq t_2\leq ....\leq t_n\leq t} .... \int e^{it_nH_0}V(t_n)e^{-it_nH_0} e^{it_{n-1}H_0}V(t_{n-1})e^{-it_{n-1}H_0}...e^{-t_1H_0}\phi\, dt_1dt_2....dt_n \tag{19}$$

From equation (19), we finally get the following expression:

$$\psi(x,t) = \sum_{n=0}^{\infty}(-i)^n \int_{0\leq t_1\leq t_2\leq ....\leq t_n\leq t} .... \int e^{-i(t-t_n)H_0}V(t_n)e^{-i(t_n-t_{n-1})H_0} V(t_{n-1})...V(t_1)e^{-t_1H_0}\phi\, dt_1dt_2....dt_n \tag{20}$$

4 Theory of fresnel integrals

In this section, we analyze oscillating integrals of the form $\int e^{ix^2/2}f(x)dx$, which are related to the Fresnel integrals used to describe near-field Fresnel diffraction in optics. This is of relevance to our current context here because the path integral given in equation (13) can be expanded as:

$$\begin{aligned}\int_{\gamma(t)=x} e^{iS[\gamma]}\phi(\gamma(0))D\gamma &= \int_{\gamma(t)=x} e^{i\int \frac{m}{2}(\frac{d\gamma}{d\tau})^2 - V(\gamma(\tau)d\tau}\phi(\gamma(0))D\gamma \\ &= \int_{\gamma(t)=x} e^{i\frac{\langle\gamma,\gamma\rangle}{2}}e^{-i\int V(\gamma(\tau)d\tau}\phi(\gamma(0))D\gamma\end{aligned} \tag{21}$$

under suitable definition of the inner product $\langle\ ,\ \rangle$. In order to make sense of the integral above, we first study such integrals on the finite-dimensional space $\mathbb{R}$ and then proceed to define such integrals in a Hilbert space.

As a first step, we consider $f(x) = e^{ixy}$, so that:

$$\int e^{ix^2/2}f(x)dx = \int e^{ix^2/2}e^{ixy}dx = (2\pi i)^{1/2}e^{-iy^2/2}$$

which can be shown using the residue theorem. This integral, which is the fourier transform, is interpreted in the sense of a *tempered distribution*, which is formalized in the next part of this section.

Definition: Schwartz space

The Schwartz space $\mathcal{S}(\mathbb{R})$ is the following set,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^{\infty}(\mathbb{R}) : ||f||_{\alpha,\beta} < \infty\}$$

along with the topology induced by the countable family of semi-norms $||f||_{\alpha,\beta}$, where

$$||f||_{\alpha,\beta} := \sup_{x\in R} |x^{\alpha}\frac{d^{\beta}f}{dx^{\beta}}|$$

Thus, $\mathcal{S}(\mathbb{R})$ is the space of rapidly decaying functions on $\mathbb{R}$. We shall denote $\mathcal{S}(\mathbb{R})$ by $\mathcal{S}$.

Definition: Tempered distribution

A tempered distribution T is a continuous linear functional on the Schwartz space $\mathcal{S}(\mathbb{R})$. The dual space $\mathcal{S}'$ of $\mathcal{S}(\mathbb{R})$ is the topological vector space of tempered distribution, with the weak* topology.

Remarks:

1. It can easily be shown that Schwartz functions are L^p functions $\forall p \in [1, \infty)$, i.e $\mathcal{S}(\mathbb{R}) \subseteq L^p(\mathbb{R})\ \forall p \in [1, \infty)$. Since, in particular, $\mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$, the fourier transform is well defined for any Schwartz function.

2. We can associate a tempered distribution with any bounded continuous function on $\mathbb{R}$.

Fourier transforms on $\mathcal{S}'$:

For any $\phi, \psi \in \mathcal{S}$, we get:

$$\int \phi\hat{\psi}dx = \int \phi(x)\frac{1}{2\pi}\int \psi(y)e^{-ixy}dydx$$

$$= \int \left\{\frac{1}{2\pi}\int \phi(x)e^{-ixy}dx\right\}\psi(y)dy = \int \hat{\phi}\psi dx$$

by Fubini's theorem. We also have that $\hat{\phi} \in \mathcal{S}$ if $\phi \in \mathcal{S}$, by the Riemann-Lebesgue lemma.

Given a tempered distribution T, we define its fourier transform $\hat{T}$ as follows:

$$\hat{T}\phi := T\hat{\phi} \quad \forall \phi \in \mathcal{S}.$$

It can be proved using the basic properties of fourier transform that $\hat{\phi} \in \mathcal{S}$ if $\phi \in \mathcal{S}$. Since the fourier transform operator and the tempered distribution T are continuous, we have that $\hat{T}$ is continuous and hence $\hat{T} \in \mathcal{S}'$.

Since $e^{ix^2/2}$ is bounded and continuous, it has an associated tempered distribution and hence its fourier transform is well defined in the sense of a tempered distribution as defined above. Thus,

$$\int e^{\frac{ix^2}{2}}e^{ixy}dy = (2\pi i)^{1/2}e^{-i\frac{y^2}{2}}$$

Defining $\widetilde{\int}$ as the scaled integral $(2\pi i)^{-1/2}\int$, we obtain:

$$\widetilde{\int} e^{i\frac{x^2}{2}}\hat{\varphi}(x)dx = \int e^{-i\frac{x^2}{2}}\varphi(x)dx \quad \forall \varphi \in \mathcal{S}.$$

Before we proceed further, we mention the following definition:

Definition: Complex measure

A complex measure μ is a complex valued function defined on a sigma algebra Σ, that is countably additive, i.e. $\mu : \Sigma \mapsto \mathbb{C}$ such that:

(i) $\mu(\Phi) = 0$

(ii) $\mu(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty}\mu(A_n) \qquad \forall \{A_n\} \subseteq \Sigma$

Now, consider the set $\mathcal{F}(\mathbb{R})$ of functions which are fourier transforms of complex measures, $f(x) = \int e^{ixy} d\mu(y)$, where μ is some complex measure. We extend the definition of the scaled integral above to this class of functions as follows:

$$\widetilde{\int} e^{i\frac{x^2}{2}} f(x)dx = \int e^{-i\frac{x^2}{2}} d\mu(x). \tag{22}$$

The rest of this section will be devoted to prove the well-definedness of this integral and to understand the properties of the space $\mathcal{F}(\mathbb{R})$.

Definition: Total variation

Given a complex measure μ, variation is the set function defined as:

$$|\mu|(E) = \underset{P}{sup} \sum_{A \in P} |\mu(A)| \quad \forall E \in \Sigma$$

The total variation of $\mu := |\mu|(X)$, where X is the measure space over which the sigma algebra Σ is defined.

We state the following theorems without proof:

Theorem 4.1:

If μ is a complex measure on a space X, then we have that $|\mu| < \infty$

Theorem 4.2:

If μ is a complex measure on a space X, then $|\mu|$ is a finite positive measure on X.

Theorem 4.3:

The space $\mathcal{M}(X)$ of complex measures (or in general, measures with bounded variation) defined on a measurable space (X, Σ), is a banach space in the total variation norm $||\mu||$.

Note that the space $\mathcal{M}(X)$ is banach for any measurable space (X, Σ); It doesn't depend on the properties of the space X.

For $X = \mathbb{R}$, we can define the convolution operation on complex measures as follows:

Let μ, ν be two complex measures defined on $\mathbb{R}$ with the usual Borel σ-algebra. Then, the convolution of μ and ν is the measure $\mu * \nu$ defined using the integral:

$$\int_{\mathbb{R}} f(x)d(\mu * \nu)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y)d\mu(x)d\nu(y)$$

for each measurable function f. Taking $f(x) = 1$ identically, one obtains:

$$||\mu * \nu|| \leq ||\mu|| ||\nu||$$

Thus, in particular, $\mathcal{M}(\mathbb{R})$ becomes a banach algebra under convolution; Now define a norm on the space $\mathcal{F}(\mathbb{R})$ as follows:

$$||f||_0 := ||\mu||, \;\; where\ f(x) = \int e^{ixy} d\mu(y)$$

It follows from the Theorem 4.3 that $\mathcal{F}(\mathbb{R})$ is a banach algebra under pointwise multiplication.

Define a linear functional $\mathcal{F}$ on $\mathcal{F}(\mathbb{R})$ as follows:

$$\mathcal{F}[f] = \widetilde{\int} e^{i\frac{x^2}{2}} f(x)dx = \int e^{-i\frac{x^2}{2}} d\mu(x)$$

$$| \mathcal{F}[f] | \leq \int |e^{-i\frac{x^2}{2}}| d|\mu|(x) = |\mu|(\mathbb{R}) = ||\mu|| = ||f||_0$$

So, the fresnel integral defined by equation (22) is a continuous linear functional on $\mathcal{F}(\mathbb{R})$ and thus the extension of the scaled integral $\widetilde{\int}$ is a continuous extension. If $f \in \mathcal{F}(\mathbb{R})$, then f is said to be fresnel integrable and $\mathcal{F}[f]$ is called the fresnel integral of f.

The fact that $\mathcal{M}(\mathbb{R})$ is a banach algebra yields the following theorem, which is one of the main result of this section:

Theorem 4.4:

$\mathcal{F}(\mathbb{R})$ is a banach-function-algebra under the norm $||f||_0$ and the fresnel integral $\mathcal{F}$ is a continuous linear functional on $\mathcal{F}(\mathbb{R})$. Sum and products of fresnel integrable functions are fresnel integrable. Composition of a fresnel integrable function with an entire function is fresnel integrable.

We now proceed to generalize the definition of the fresnel integral to a Hilbert space.

Fresnel integral on a Hilbert space:

Let $\mathcal{H}$ be a real separable Hilbert space with the inner product $< x, y >$ and norm $|x|$. Then $\mathcal{H}$ is a separable metric group under addition. As was shown before, the space $\mathcal{M}(\mathcal{H})$ of bounded complex measures on $\mathcal{H}$, is a banach space. Unlike in the finite dimensional case of $\mathbb{R}$, the definition of a convolution on $\mathcal{M}(\mathcal{H})$ isn't straightforward. A detailed description of the construction of convolutions is provided in [6]; In this report, we provide a sketch of this construction:

Let G be a metric group, and $C_0(G)$ be the banach space of continuous functions on G, that are compactly supported. Set $C_0^*(G)$ to be the dual space of $C_0(G)$, i.e. the space of bounded continuous linear functionals defined on $C_0(G)$.

For $f \in C_0(G)$, denote by ${}_xf$, the function obtained by left translation of f. Let M, L be linear functionals in $C_0^*(G)$. For a given $f \in C_0(G)$ and $M \in C_0^*(G)$, Define the function $\bar{M}f$ on G as follows:

$$\begin{aligned} \bar{M}f : G &\to \mathbb{C} \\ x &\mapsto M({}_xf) \end{aligned}$$

Now, define the convolution $L * M := L \circ \bar{M}$. Clearly, $L * M \in C_0^*(G)$. With this, we state the following theorem without proof.

Riesz representation theorem:

Any bounded continuous linear functional defined on $C_0(G)$ can be uniquely expressed as an integral over a complex measure.

Let μ, ν be the complex measures that represent L, M respectively. Then, the convolution of μ and ν, $\mu * \nu$ is defined to be the measure that represents the functional $L * M$. Since integration over any complex measure defines a linear functional, this is the definition of the convolution of two measures μ and ν.

Proposition 4.1:

Let L, M be linear functionals in $C_0^*(G)$, represented by μ and ν. Define $\tau : G \times G \to G$ to be the multiplication map $(x, y) \mapsto xy$. If $f \in L^1(G \times G, |\mu| \times |\nu|$, then $f \circ \tau$ is in $L^1(G \times G, |\mu \times \nu|)$ and

$$\begin{aligned}\int_G f d(\mu * \nu) &= \int_{G\times G} (f \circ \tau) d(\mu \times \nu) \\ &= \int_G \int_G f(xy) d\mu(x) d\nu(y) = \int_G \int_G f(xy) d\nu(y) d\mu(x)\end{aligned}$$

From the above proposition, it follows that:

$$\mu * \nu = \nu * \mu \ \ and \ \ ||\mu * \nu|| \leq ||\mu|| ||\nu||$$

Thus, we get that $\mathcal{M}(\mathcal{H})$ is a commutative banach algebra under the convolution operation. The following proposition gives an explicit definition of the convolution measure on Borel sets.

Proposition 4.2:

Let L, M, μ, and τ be as in Proposition 4.1. Then, for every $|\mu|*|\nu|$ measurable set A, $\tau^{-1}(A)$ is $|\mu \times \nu|$ measurable and

$$\mu * \nu(A) := \mu \times \nu(\tau^{-1}(A)) = \int_G \mu(A - y) d\nu(y)$$

With this, we now define $\mathcal{F}(\mathcal{H})$ to be the space of functions that are fourier transform of complex measures. For $f(x) = \int_{\mathcal{H}} e^{i<x,y>} d\mu(y) \in \mathcal{F}(\mathcal{H})$, we define $||f||_0 := ||\mu||$. With this definition, $\mathcal{F}(\mathcal{H})$ becomes a banach algebra under the operation of pointwise multiplication.

Theorem 4.5:

The space of complex measures $\mathcal{M}(\mathcal{H})$ is isomorphic (algebra isomorphism) to the space $\mathcal{F}(\mathcal{H})$.

Proof:

By our definition of the norm on $\mathcal{F}(\mathcal{H})$, we clearly have that $\mathcal{M}(\mathcal{H})$ is isometric to $\mathcal{F}(\mathcal{H})$. Also, the mapping $\mu \mapsto f$ is clearly linear. It remains to show that $f \equiv 0 \Rightarrow \mu \equiv 0$.

Suppose $f \equiv 0$. Then the μ-measure of any set of the form $S_{\alpha,x} = \{y : \langle x, y\rangle \geq \alpha\}$ is zero. Any closed convex set can be written in the form of $S_{\alpha,x}$ or a finite union of such sets. So, the μ-measure of any closed convex set is zero. Any open ball in $\mathcal{H}$ can be written as a countable union of closed convex sets, so the μ-measure of any open ball is zero. Since we define the complex measure μ on the Borel σ-algebra, it follows that $\mu \equiv 0$.

With the above theorem, we are now ready to define the scaled integral on the Hilbert space $\mathcal{H}$:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|x|^2} f(x)dx = \int_{\mathcal{H}} e^{-\frac{i}{2}|x|^2} d\mu(x) \tag{23}$$

For $f \in \mathcal{F}(\mathcal{H})$, we define the Fresnel integral to be:

$$F[f] = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|x|^2} f(x)dx \tag{24}$$

It can easily be seen that $\mid F[f] \mid \leq ||f||_0$, thus the functional defined by (24) is continuous and bounded. $\mathcal{F}(\mathcal{H})$ shall be called the space of Fresnel-integrable function. Similar to the finite dimensional case, the following theorem can be easily inferred from the fact that $\mathcal{F}(\mathcal{H})$ is a Banach algebra.

Theorem 4.6:

The space $\mathcal{F}(\mathcal{H})$ of fresnel integrable function is a Banach algebra under the $||f||_0$ norm. The fresnel integral $\mathcal{F}$ is a continuous linear functional defined on $\mathcal{F}(\mathcal{H})$. It follows from the fact that $\mathcal{F}(\mathcal{H})$ is a Banach algebra that sum and products of fresnel integrable functions are fresnel integrable and so are composition of fresnel integrable functions with entire functions.

With the necessary tool to handle oscillating integrals in infinite dimensional space, we proceed to the derivation of the Feynman integral in the next section.

5 A derivation for the Feynman path integral

In this section, we show that if the potential V and initial wavefunction ϕ are assumed to be fourier transform of complex measures, then an exact derivation of the Feynman path integral (13) is possible.

Suppose that $V(x) = \int_{\mathbb{R}} e^{i\alpha x} d\mu(\alpha)$ and $\phi(x) = \int_{\mathbb{R}} e^{i\alpha x} d\nu(\alpha)$ for some complex measures μ and ν.

Since the free Hamiltonian $\hat{H}_0$ is Hermitian, it can be shown that:

$$e^{-it\hat{H}_0} e^{i\alpha x} = e^{-it\frac{\alpha^2}{2m}} e^{i\alpha x} \tag{25}$$

Substituting (25) into (20), we get:

$$\begin{aligned}\psi(x,t) = \sum_{n=0}^{\infty} (-i)^n \int_{0\leq t_1 \leq t_2 \leq .... \leq t_n \leq t} \int \int .... \int \\ e^{-\frac{i}{2m}[(t-t_n)(\alpha_0+\alpha_1....\alpha_n)^2+(t_n-t_{n-1})(\alpha_0+\alpha_1...\alpha_{n-1})^2....(t_2-t_1)(\alpha_0+\alpha_1)^2+t_1\alpha_0^2]} \\ e^{i(\sum_{j=0}^{n}\alpha_j)x} d\nu(\alpha_0) \prod_{j=1}^{n} d\mu(\alpha_j)dt_j\end{aligned} \tag{26}$$

Setting $t_0 = 0$ and simplifying further we get:

$$\psi(x,t) = \sum_{n=0}^{\infty}(-i)^n \int_{0\le t_1\le t_2\le ....\le t_n\le t} \int\int .... \int e^{-\frac{i}{2m}\sum_{j,k=0}^{n}(t-t_j\vee t_k)\alpha_j\alpha_k} e^{i(\sum_{j=0}^{n}\alpha_j)x} d\nu(\alpha_0)\prod_{j=1}^{n} d\mu(\alpha_j)dt_j \tag{27}$$

Here, $\sigma \vee \tau = max\{\sigma, \tau\}$. Noting that the exponent in this equation $\sum_{j,k=0}^{n}(t - t_j \vee t_k)\alpha_j\alpha_k$ is symmetric in $\{t_0, t_1....t_n\}$, we can simplify this to:

$$\psi(x,t) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_0^t .... \int_0^t \int .... \int e^{-\frac{i}{2m}\sum_{j,k=0}^{n}(t-t_j\wedge t_k)\alpha_j\alpha_k} e^{i(\sum_{j=0}^{n}\alpha_j)x} d\nu(\alpha_0)\prod_{j=1}^{n} d\mu(\alpha_j)dt_j \tag{28}$$

Before simplifying this expression further, we make the following remark:

Remarks: Green's function.

Consider the following Sturm-Liouville problem:

$$-\frac{d^2u}{d\tau^2} = 0 \tag{29}$$

With the boundary conditions:

$$\frac{du}{d\tau}(0) = 0$$
$$u(t) = 0$$

To solve this Boundary value problem, one employs the *Green's function* approach. For a general Sturm-Liouville problem given by:

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} - q(x)y = f(x)$$

With the boundary conditions:

$$a_1y(a) + a_2y'(a) = 0$$
$$b_1y(b) + b_2y'(b) = 0,$$

the Green's function $G(x,s)$ satisfies the following properties:

(i) $G(\sigma,\tau)$ satisfies the given ODE along with the boundary conditions, in the s variable, except for $s = x$

(ii) $\lim_{s\to x_+} G(x,s) - \lim_{s\to x_-} G(x,s) = 0$

(iii) $\lim_{s\to x_+} \frac{\partial G(x,s)}{\partial s} - \lim_{s\to x_-} \frac{\partial G(x,s)}{\partial s} = \frac{-1}{p(x)}$

For the boundary value problem (29), the Green's function turns out to be:

$$G(\sigma,\tau) = (t - \sigma \vee \tau) \tag{30}$$

Substituting (30) into (28), we can simplify it to:

$$\psi(x,t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t .... \int_0^t \int .... \int e^{-\frac{i}{2m}\sum_{j,k=0}^{n} \alpha_j G(t_j,t_k)\alpha_k} \\ e^{i(\sum_{j=0}^{n}\alpha_j)x} d\nu(\alpha_0) \prod_{j=1}^{n} d\mu(\alpha_j) dt_j \tag{31}$$

Now, the rest of this section shall be devoted to show that the Feynman integral evaluates to this expression, under appropriate assumptions.

Path space:

Consider the following subset of $C([0,t])$:

$$\mathcal{H} := \left\{ \gamma : \gamma \in C([0,t]), \gamma(t) = 0 \ \& \ \frac{d\gamma}{d\tau} \in L^2([0,t]) \right\}$$

Define the inner product $\langle \ , \ \rangle$ on $\mathcal{H}$ as,

$$\langle \gamma_1, \gamma_2 \rangle = m \int_0^t \frac{d\gamma_1}{d\tau} . \frac{d\gamma_2}{d\tau} d\tau$$

The inner product space $\mathcal{H} \subseteq C([0,t])$ can be shown to be complete in the given norm, by following the arguments of section 2.8 in [8]. Thus, $\mathcal{H}$ is a Hilbert space, and hence the theory developed in section 4 can be used to define integrals on this space.

We revisit the Feynman integral (13) here:

$$\begin{aligned}\psi(x,t) &= \int_{\gamma(t)=x} e^{iS[\gamma]} \phi(\gamma(0)) D\gamma \\ &= \int_{\gamma(t)=x} e^{i\frac{m}{2}\int_0^t |\frac{d\gamma}{d\tau}|^2 d\tau} e^{-i\int_0^t V(\gamma(\tau))d\tau} \phi(\gamma(0)) D\gamma\end{aligned}$$

Remark:

Given a path $\gamma \in \mathcal{H}$, we define the path γ_σ as follows:

$$\gamma_\sigma(\tau) = G(\sigma,\tau)$$

By a simple integration, it can be shown that:

$$\gamma(\sigma) = \frac{\langle \gamma, \gamma_\sigma \rangle}{m}$$

Consider the function ϕ defined on $\mathcal{H}$ as:

$$\phi(\gamma(0) + x) = \int_{\mathbb{R}} e^{i\gamma(0)\alpha} e^{i\alpha x} d\nu(\alpha) \tag{32}$$

By the previous remark, this integral can be expressed as:

$$\phi(\gamma(0) + x) = \int_{\mathbb{R}} e^{\frac{i\langle \gamma, \gamma_0 \rangle \alpha}{m}} e^{i\alpha x} d\nu(\alpha)$$

Remark:

Consider the continuous map Φ:

$$\Phi : \mathbb{R} \to \mathcal{H}$$
$$\alpha \mapsto C\alpha\gamma$$

for some constant C and $\gamma \in \mathcal{H}$. The range of this map is the one dimensional subspace of $\mathcal{H}$ spanned by γ.

With this map Φ, we can define the pullback measure $\widetilde{\mu}$ as follows:

$$\widetilde{\mu}(A) = \mu(\Phi^{-1}(A))$$

where $A \subseteq \mathcal{H}$ is a Borel subset of $\mathcal{H}$ and μ is a complex measure defined on the Borel σ-algebra of $\mathbb{R}$. It can be easily seen that this pullback measure is a complex measure defined on the Borel σ-algebra of $\mathcal{H}$.

Using this remark, the integral in (32) can be rewritten as:

$$\phi(\gamma(0) + x) = \int_{\mathcal{H}} e^{i\langle\gamma,\widetilde{\gamma}\rangle} d\widetilde{\nu}(\widetilde{\gamma})$$

Thus, $\phi \in \mathcal{F}(\mathcal{H})$ as it is a fourier transform of the complex measure $\widetilde{\nu}$. Similarly, we have:

$$\int_0^t V(\gamma(\tau) + x)d\tau = \int_0^t \int_{\mathbb{R}} e^{i\gamma(\tau)\alpha} e^{i\alpha x} d\mu(\alpha)d\tau \tag{33}$$

$$V(\gamma(\tau) + x) = \int_{\mathbb{R}} e^{i\gamma(\tau)\alpha} e^{i\alpha x} d\mu(\alpha)$$

By similar arguments as above, we get:

$$V(\gamma(\tau) + x) = \hat{\widetilde{\mu}}(\gamma;\tau)$$
$$\int_0^t V(\gamma(\tau) + x)d\tau = \int_0^t \hat{\widetilde{\mu}}(\gamma;\tau)d\tau$$

The integral above can be considered as the limit of a sequence of Riemann sum. Each term in this sequence is a sum of fourier transforms of complex measures, and hence belongs to $\mathcal{H}$. Since $\mathcal{H}$ is a Banach algebra, the limit also belongs to this sequence. So, $\int_0^t V(\gamma(\tau) + x)d\tau \in \mathcal{F}(\mathcal{H})$, i.e. it is fresnel integrable.

From these two results, we have that the function f defined as:

$$f(\gamma) = e^{-i\int_0^t V(\gamma(\tau)+x)d\tau}\phi(\gamma(0) + x)$$

is fresnel integrable, by Theorem 4.6. So, $\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma$ is well defined.

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma = \widetilde{\int}_{\gamma(t)=0} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma = \widetilde{\int}_{\gamma(t)=x} e^{\frac{i}{2}|\gamma|^2} f(\gamma - x)d\gamma$$

where $\gamma - x$ is a path defined by $(\gamma - x)(\tau) = \gamma(\tau) - x$. This is true because the inner product $\langle \, , \, \rangle$ remains invariant under translation of paths. But,

$$\begin{aligned}\widetilde{\int}_{\gamma(t)=x} e^{\frac{i}{2}|\gamma|^2} f(\gamma - x)d\gamma &= \widetilde{\int}_{\gamma(t)=x} e^{i\frac{m}{2}\int_0^t |\frac{d\gamma}{d\tau}|^2 d\tau} e^{-i\int_0^t V(\gamma(\tau))d\tau} \phi(\gamma(0)) \\ &= \widetilde{\int}_{\gamma(t)=x} e^{iS[\gamma]}\phi(\gamma(0))d\gamma\end{aligned}$$

Thus, this normalized integral can be interpreted as the Feynman path integral. We now expand the integral: $\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma) d\gamma$

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma) d\gamma = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} \left(\int_0^t V(\gamma(\tau)+x) d\tau \right)^n \phi(\gamma(0)+x) d\gamma$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} \left[\int_0^t \dots\dots \int_0^t \int \dots\dots \int e^{i\left(\sum_{j=0}^{n} \gamma(t_j)\alpha_j\right)} e^{i(\sum_{j=0}^{n} \alpha_j)x} \right.$$
$$d\nu(\alpha_0) \prod_{j=1}^{n} e^{i\alpha_j x} d\mu(\alpha_j) dt_j d\gamma$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} \left[\int_0^t \dots\dots \int_0^t \int \dots\dots \int e^{\frac{i}{m}\left\langle \gamma, \sum_{j=0}^{n} \alpha_j \gamma_{t_j} \right\rangle} e^{i(\sum_{j=0}^{n} \alpha_j)x} \right.$$
$$d\nu(\alpha_0) \prod_{j=1}^{n} e^{i\alpha_j x} d\mu(\alpha_j) dt_j d\gamma$$

By defining the appropriate pullback complex measure as was done for (32) and then transferring back to the original complex measure on $\mathbb{R}$, we obtain the following expression:

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma) d\gamma = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[\int_0^t \dots\dots \int_0^t \int \dots\dots \int e^{-\frac{i}{2m^2}\left\langle \sum_{j=0}^{n} \alpha_j \gamma_{t_j}, \sum_{j=0}^{n} \alpha_j \gamma_{t_j} \right\rangle} \right.$$
$$e^{i(\sum_{j=0}^{n} \alpha_j)x} d\nu(\alpha_0) \prod_{j=1}^{n} e^{i\alpha_j x} d\mu(\alpha_j) dt_j$$

Now,

$$\left\langle \sum_{j=0}^{n} \alpha_j \gamma_{t_j}, \sum_{j=0}^{n} \alpha_j \gamma_{t_j} \right\rangle = \sum_{j,k=0}^{n} \alpha_j \left\langle \gamma_{t_j}, \gamma_{t_k} \right\rangle \alpha_k$$
$$\left\langle \gamma_{t_j}, \gamma_{t_k} \right\rangle = m\, \gamma_{t_j}(t_k) = m\, G(t_j, t_k)$$

Using this result, we obtain:

$$\widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma) d\gamma = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[\int_0^t \dots\dots \int_0^t \int \dots\dots \int e^{-\frac{i}{2m} \sum_{j,k=0}^{n} \alpha_j G(t_j,t_k) \alpha_k} \right.$$
$$e^{i(\sum_{j=0}^{n} \alpha_j)x} d\nu(\alpha_0) \prod_{j=1}^{n} e^{i\alpha_j x} d\mu(\alpha_j) dt_j$$

Combining this result with (31), we get:

$$\psi(x,t) = \widetilde{\int}_{\mathcal{H}} e^{\frac{i}{2}|\gamma|^2} f(\gamma) d\gamma = \widetilde{\int}_{\gamma(t)=x} e^{iS[\gamma]} \phi(\gamma(0)) d\gamma$$

This completes the derivation of the Feynman path integral. The whole derivation that was presented in this report can be summarized by the following theorem:

Theorem 5.1:

Let V and ϕ be fourier transforms of complex measures μ and ν respectively. Let $\mathcal{H}$ be the real Hilbert space of continuous paths γ from $[0,t]$ to $\mathbb{R}$ s.t. $\gamma(t)=0$ and $\frac{d\gamma}{d\tau}\in L^2([0,t];\mathbb{R})$ with the inner product $\langle\gamma_1,\gamma_2\rangle = m\int_0^t \frac{d\gamma_1}{d\tau}.\frac{d\gamma_2}{d\tau}d\tau$. Then,

$$\psi(x,t) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2}|\gamma|^2} f(\gamma)d\gamma$$

solves the Schrodinger equation with the initial condition $\psi(x,0)=\phi(x)$. Here, $f(\gamma) = e^{-i\int_0^t V(\gamma(\tau)+x)d\tau}\phi(\gamma(0)+x)$

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