Robotics

Robin designs and sells room dividers to defray college expenses. She is soon overwhelmed with orders and decides to build a robot to spray paint her dividers. As in most engineering projects, Robin begins with a simplified model that she will eventually refine to be more realistic. However, Robin quickly discovers that robotics (the design and control of robots) involves a considerable amount of mathematics, some of which we will discuss in this module.

The Design Plan

Robin's plan is to develop a two-dimensional version of the robot arm in Figure 1. As shown in Figure 2, Robin's robot arm will consist of two links of fixed length, each of which will rotate independently about a pivot point. A paint sprayer will be attached to the end of the second link, and a computer will vary the angles θ_1 and θ_2, thereby allowing the robot to paint a region of the xy-plane.

The Mathematical Analysis

To analyze the motion of the robot arm, Robin denotes the coordinates of the paint sprayer by (x, y), as in Figure 3, and she derives the following equations that express x and y in terms of the angles θ_1 and θ_2 and the lengths l_1 and l_2 of the links:

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{aligned} \tag{1}$$

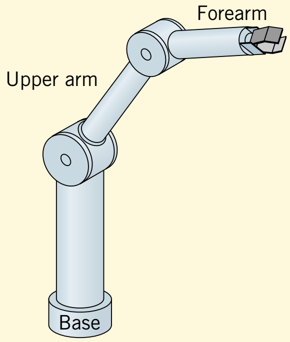

Figure 1

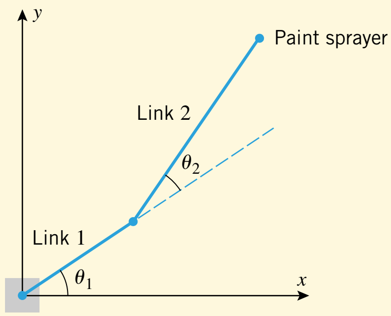

Figure 2

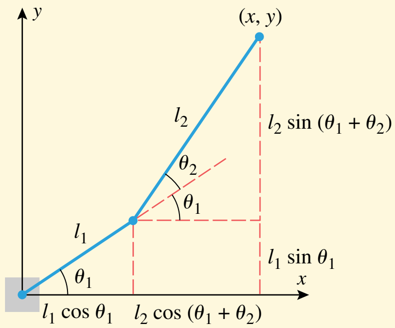

Figure 3

Exercise 1 Use Figure 3 to confirm the equations in (1).

In the language of robotics, θ_1 and θ_2 are called the ***control angles***, the point (x, y) is called the ***end effector***, and the equations in (1) are called the ***forward kinematic equations*** (from the Greek word *kinema*, meaning "motion").

Exercise 2 What is the region of the plane that can be reached by the end effector if: (a) $l_1 = l_2$, (b) $l_1 > l_2$, and (c) $l_1 < l_2$?

Exercise 3 What are the coordinates of the end effector if $l_1 = 2$, $l_2 = 3$, $\theta_1 = \pi/4$, and $\theta_2 = \pi/6$?

Simulating Paint Patterns

Robin recognizes that if θ_1 and θ_2 are regarded as functions of time, then the forward kinematic equations can be expressed as

$$x = l_1 \cos \theta_1(t) + l_2 \cos(\theta_1(t) + \theta_2(t))$$
$$y = l_1 \sin \theta_1(t) + l_2 \sin(\theta_1(t) + \theta_2(t))$$

which are parametric equations for the curve traced by the end effector. For example, if the arms extend horizontally along the positive x-axis at time $t = 0$, and if links 1 and 2 rotate at the constant rates of ω_1 and ω_2 radians per second (rad/s), respectively, then

$$\theta_1(t) = \omega_1 t \quad \text{and} \quad \theta_2(t) = \omega_2 t$$

and the parametric equations of motion for the end effector become

$$x = l_1 \cos \omega_1 t + l_2 \cos(\omega_1 t + \omega_2 t)$$
$$y = l_1 \sin \omega_1 t + l_2 \sin(\omega_1 t + \omega_2 t)$$

Exercise 4 Show that if $l_1 = l_2 = 1$, and if $\omega_1 = 2$ rad/s and $\omega_2 = 3$ rad/s, then the parametric equations of motion are

$$x = \cos 2t + \cos 5t$$
$$y = \sin 2t + \sin 5t$$

Use a graphing utility to show that the curve traced by the end effector over the time interval $0 \leq t \leq 2\pi$ is as shown in Figure 4. This would be the painting pattern of Robin's paint sprayer.

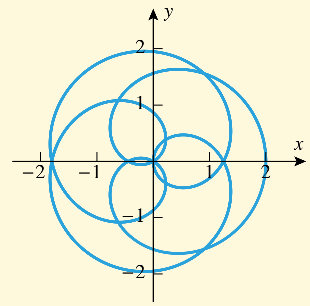

Figure 4

Exercise 5 Use a graphing utility to explore how the rotation rates of the links affect the spray patterns of a robot arm for which $l_1 = l_2 = 1$.

Exercise 6 Suppose that $l_1 = l_2 = 1$, and a malfunction in the robot arm causes the second link to lock at $\theta_2 = 0$, while the first link rotates at a constant rate of 1 rad/s. Make a conjecture about the path of the end effector, and confirm your conjecture by finding parametric equations for its motion.

Controlling the Position of the End Effector

Robin's plan is to make the robot paint the dividers in vertical strips, sweeping from the bottom up. After a strip is painted, she will have the arm return to the bottom of the divider and then move horizontally to position itself for the next upward sweep. Since the sections of her dividers will be 3 ft wide by 5 ft high, Robin decides on a robot with two 3-ft links whose base is positioned near the lower left corner of a divider section, as in Figure 5*a*. Since the fully extended links span a radius of 6 ft, she feels that this arrangement will work.

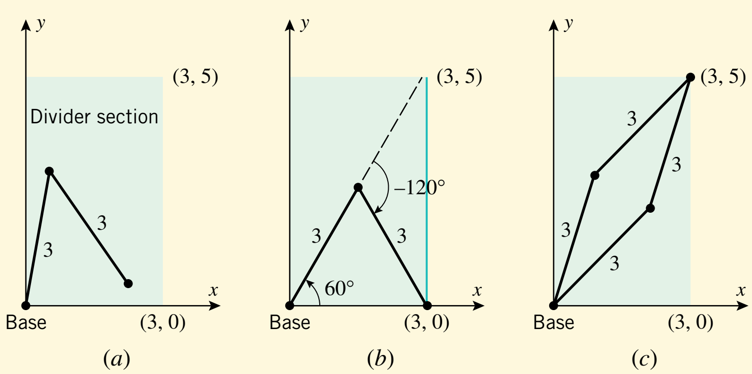

Figure 5

Robin starts with the problem of painting the far right edge from (3, 0) to (3, 5). With the help of some basic geometry (Figure 5*b*), she determines that the end effector can be placed at the point (3, 0) by taking the control angles to be $\theta_1 = \pi/3$ $(= 60^\circ)$ and $\theta_2 = -2\pi/3$ $(= -120^\circ)$ (verify). However, the problem of finding the control angles that correspond to the point (3, 5) is more complicated, so she starts by substituting the link lengths $l_1 = l_2 = 3$ into the forward kinematic equations in (1) to obtain

$$\begin{aligned} x &= 3\cos\theta_1 + 3\cos(\theta_1 + \theta_2) \\ y &= 3\sin\theta_1 + 3\sin(\theta_1 + \theta_2) \end{aligned} \tag{2}$$

Thus, to put the end effector at the point (3, 5), the control angles must satisfy the equations

$$\begin{aligned} \cos\theta_1 + \cos(\theta_1+\theta_2) &= 1 \\ 3\sin\theta_1 + 3\sin(\theta_1+\theta_2) &= 5 \end{aligned} \tag{3}$$

Solving these equations for θ_1 and θ_2 challenges Robin's algebra and trigonometry skills, but she manages to do it using the procedure in the following exercise.

Exercise 7

(a) Use the equations in (3) and the identity

$$\sin^2(\theta_1+\theta_2) + \cos^2(\theta_1+\theta_2) = 1$$

to show that

$$15\sin\theta_1 + 9\cos\theta_1 = 17$$

(b) Solve the last equation for $\sin\theta_1$ in terms of $\cos\theta_1$ and substitute in the identity

$$\sin^2\theta_1 + \cos^2\theta_1 = 1$$

to obtain

$$153\cos^2\theta_1 - 153\cos\theta_1 + 32 = 0$$

(c) Treat this as a quadratic equation in $\cos\theta_1$, and use the quadratic formula to obtain

$$\cos\theta_1 = \frac{1}{2} \pm \frac{5\sqrt{17}}{102}$$

(d) Use the arccosine (inverse cosine) operation of a calculating utility to solve the equations in part (c) to obtain

$$\theta_1 \approx 0.792436 \text{ rad} \approx 45.4032^\circ \quad \text{and} \quad \theta_1 \approx 1.26832 \text{ rad} \approx 72.6693^\circ$$

(e) Substitute each of these angles into the first equation in (3), and solve for the corresponding values of θ_2.

At first, Robin was surprised that the solutions for θ_1 and θ_2 were not unique, but her sketch in Figure 5*c* quickly made it clear that there will ordinarily be two ways of positioning the links to put the end effector at a specified point.

Controlling the Motion of the End Effector

Now that Robin has figured out how to place the end effector at the points (3, 0) and (3, 5), she turns to the problem of making the robot paint the vertical line segment between those points. She recognizes that not only must she make the end effector move on a vertical line, but she must control its velocity—if the end effector moves too quickly, the paint will be too thin, and if it moves too slowly, the paint will be too thick.

After some experimentation, she decides that the end effector should have a constant velocity of 1 ft/s. Thus, Robin's mathematical problem is to determine the rotation rates $d\theta_1/dt$ and $d\theta_2/dt$ (in rad/s) that will make $dx/dt = 0$ and $dy/dt = 1$. The first condition will ensure that the end effector moves vertically (no horizontal velocity), and the second condition will ensure that it moves upward at 1 ft/s.

To find formulas for dx/dt and dy/dt, Robin uses the chain rule to differentiate the forward kinematic equations in (2) and obtains

$$\frac{dx}{dt} = -3\sin\theta_1\frac{d\theta_1}{dt} - [3\sin(\theta_1+\theta_2)]\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right)$$

$$\frac{dy}{dt} = 3\cos\theta_1\frac{d\theta_1}{dt} + [3\cos(\theta_1+\theta_2)]\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right)$$

She uses the forward kinematic equations again to simplify these formulas and she then substitutes $dx/dt = 0$ and $dy/dt = 1$ to obtain

$$\begin{aligned} -y\frac{d\theta_1}{dt} - 3\sin(\theta_1+\theta_2)\frac{d\theta_2}{dt} &= 0 \\ x\frac{d\theta_1}{dt} + 3\cos(\theta_1+\theta_2)\frac{d\theta_2}{dt} &= 1 \end{aligned} \tag{4}$$

Exercise 8 Confirm Robin's computations.

The equations in (4) will be used in the following way: At a given time t, the robot will report the control angles θ_1 and θ_2 of its links to the computer, the computer will use the forward kinematic equations in (2) to calculate the x- and y-coordinates of the end effector, and then the values of θ_1, θ_2, x, and y will be substituted into (4) to produce two equations in the two unknowns $d\theta_1/dt$ and $d\theta_2/dt$. The computer will solve these equations to determine the required rotation rates for the links.

Exercise 9 In each part,
use the given information to sketch the position of the links, and then calculate the rotation rates for the links in rad/s that will make the end effector of Robin's robot move upward with a velocity of 1 ft/s from that position.

(a) $\theta_1 = \pi/3,\ \theta_2 = -2\pi/3$ (b) $\theta_1 = \pi/2,\ \theta_2 = -\pi/2$

Module by Mary Ann Connors, USMA, West Point, and Howard Anton, Drexel University, and based on the article "Moving a Planar Robot Arm" by Walter Meyer, MAA Notes Number 29, The Mathematical Association of America, 1993.