Solutions:

Question 1

(a) $\int 3x^2 + 5\cos(x) + 2x\sec^2(x^2)dx$

Substitute $z = x^2, dz = 2xdx$

y $= \int 3x^2 dx + 5\cos(x)dx + \sec^2(z)dz)$

Answer, y $= x^3 + 5\sin(x) + \tan(z)$

(b) $\int \frac{x+4}{x^2+8x+4}dx$

Substitute $z = x^2 + 8x + 4, dz = (2x+8)dx$

$$\frac{dz}{2} = (x+4)dx$$

y $= \int \frac{1}{2z}dz$

$= \frac{\ln(z)}{2}$

Answer, y $= \frac{\ln(x^2+8x+4)}{2}$

(c) $\int \frac{1}{x(x^2+1)}dx$

Substitute $z = x^2$,

$$\frac{dz}{2} = xdx$$

y $= \int \frac{x}{x^2(x^2+1)}dx$

$= \int \frac{1}{2z(z+1)}dz$

$= \int \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+1}\right)dz$

Answer, y $= \frac{\ln(x^2) - \ln(x^2+1)}{2}$

$$\textbf{(d)} \qquad \int \sin^4(x)\cos(x)\, dx$$

Substitute, $z = \sin(x), dz = \cos(x)dx$

y$= \int z^4 dz$

$= \dfrac{z^5}{5}$

Answer, y $= \dfrac{\sin^5(x)}{5}$

$$\textbf{(e)} \qquad \int e^{\sin(5x)} \cos(5x) dx$$

Substitute, $z = \sin(5x), dz = 5\cos(5x)dx$

$$\frac{dz}{5} = \cos(5x)dx$$

y$= \int \dfrac{e^z}{5} dz$

Answer, y $= \dfrac{e^{\sin(5x)}}{5}$

Question 2

$$\textbf{(a)} \qquad \int_0^1 xe^{x^2}\ dx$$

Substitute $z = x^2, dz = 2xdx$

$$\frac{dz}{2} = xdx$$

y$= \int_0^1 \dfrac{e^z}{2} dz$

y $= \left[\dfrac{e^z}{2}\right]_0^1$

Answer, y =0.8591

(b) $\int_0^{\pi/2} x^2 \cos x \, dx$

Integrating by parts,

$$y = \int_0^{\pi/2} x^2 \cos(x) dx$$

$$= \left[x^2 \int \cos(x) \right]_0^{\pi/2} - \int_0^{\pi/2} \left[\frac{d(x^2)}{dx} \int \cos(x) dx \right] dx$$

$$= \left[x^2 \sin(x) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{d(x^2)}{dx} \sin(x) dx$$

$$= \frac{\pi^2}{4} - \int_0^{\pi/2} \frac{d(x^2)}{dx} \sin(x) dx$$

$$= \frac{\pi^2}{4} - \int_0^{\pi/2} \frac{d(x^2)}{dx} \sin(x) dx$$

$$= \frac{\pi^2}{4} - \int_0^{\pi/2} 2x \sin(x) dx$$

$$= \frac{\pi^2}{4} - \left[\left[2x(-\cos x \right]_0^{\pi/2} - \int_0^{\pi/2} 2(-\cos(x)) dx \right]$$

$$= \frac{\pi^2}{4} - \left[0 + \left[2 \sin x \right]_0^{\pi/2} \right]$$

$$= \frac{\pi^2}{4} - 2$$

$$= \frac{\pi^2}{4} - 2$$

Answer, y $= 0.4674$

(c) $$\int_1^3 x\log(x)+\log(x)\ dx$$

Integrating by parts,

$$\text{y}=\int_1^3 \log(x)(x+1)dx$$

$$=\left[\log x\int_1^3(x+1)dx\right]-\int_0^{\pi/2}\left[\frac{1}{x}\int(x+1)dx\right]dx$$

$$=\left[\log x\left(\frac{x^2}{2}+x\right)\right]_0^3-\int_0^{\pi/2}\left[\frac{1}{x}\left(\frac{x^2}{2}+x\right)\right]dx$$

$$=\log(3)\left(\frac{3^2}{2}+3\right)-\int_0^{\pi/2}\left[\left(\frac{x}{2}+1\right)\right]dx$$

$$=\log(3)\left(\frac{15}{2}\right)-\left[\left(\frac{x^2}{4}+x\right)\right]_0^3$$

$$=\log(3)\left(\frac{15}{2}\right)-\left(\frac{9}{4}+3\right)$$

Answer, y $=2.9896$

(d) $$\int_1^2 \frac{4x+7}{(x+3)\left(x^2+1\right)}\ dx$$

Using partial fraction expansion,

$$\text{y}=\int_1^3\frac{1}{2}\left[\frac{x+5}{x^2+1}-\frac{1}{x+3}\right]dx$$

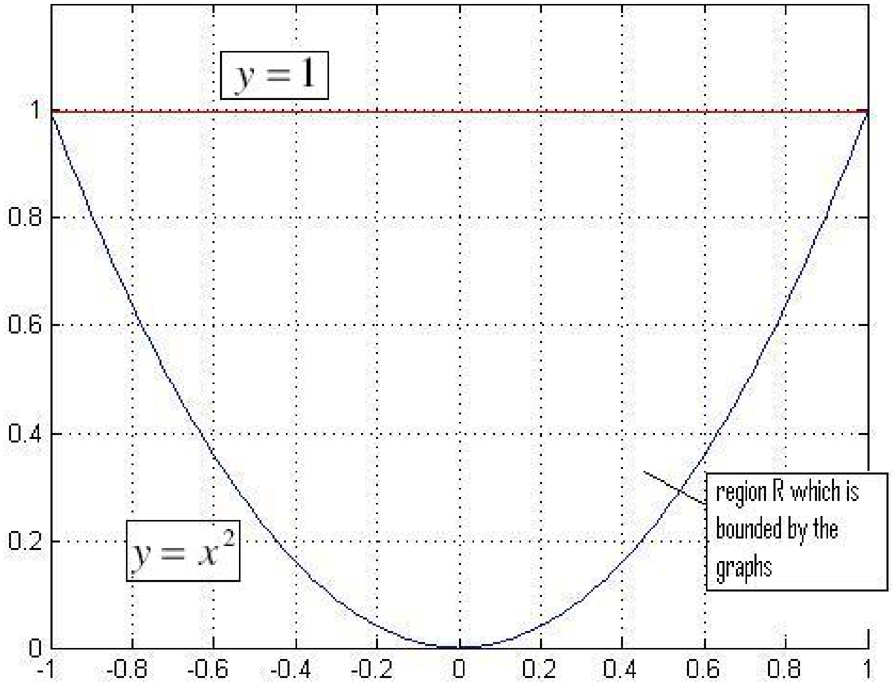
$$=\int_1^3\frac{1}{2}\left[\frac{x}{x^2+1}+\frac{5}{x^2+1}-\frac{1}{x+3}\right]dx$$

$$=\left[\frac{1}{2}\left(\frac{1}{2}\log(x^2+1)+5\tan^{-1}(x)-\log(x+3)\right)\right]_1^3$$

Answer, y $=1.3587$

Question 3

(a) Sketch the (finite) region *R* which is bounded by the graphs of $y = x^2$ and $y = 1$.



(b) Calculate the area of the region *R*.

The area of the region *R*

= (The area under the curve y=1) – (the area under the curve $y = x^2$)

$$= = \int_{-1}^{1} 1 dx - \int_{-1}^{1} x^2 dx$$

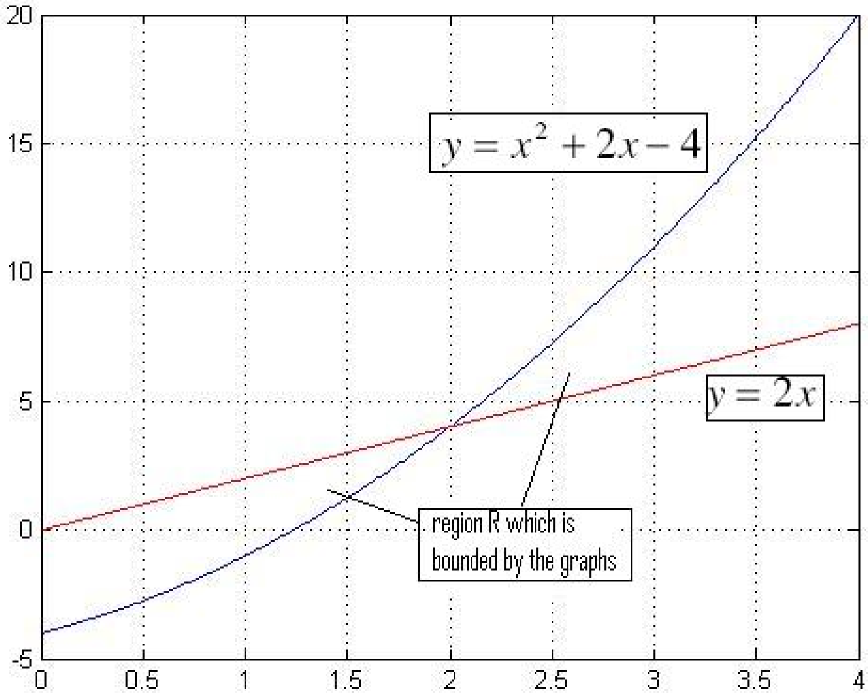
$$= \int_{-1}^{1} (1 - x^2) dx$$

$$= \left[x - \frac{x^2}{3} \right]_{-1}^{1}$$

Area= $\frac{4}{3} = 1.3333$

Question 4

(a) Sketch the (finite) region *R* which is bounded by the graphs of $y = x^2 + 2x - 4$ and $y = 2x$ between $x = 0$ and $x = 4$.



(c) Calculate the area of the region *R*.

The curve $y = x^2 + 2x - 4$ intersects $y = 2x$ at x=2 for x between 0 and 4.

Area of region R=

[(The area under the curve y=2*x*) – (the area under the curve $y = x^2 + 2x - 4$) for x between 0 and 2] + [(the area under the curve $y = x^2 + 2x - 4$) – (The area under the curve y=2*x*) for x between 2 and 4]

$$= \int_0^2 (2x - (x^2 + 2x - 4))dx + \int_2^4 ((x^2 + 2x - 4) - 2x)dx$$

$$= \int_0^2 (-x^2 + 4))dx + \int_2^4 (x^2 - 4)dx$$

$$= \left[-\frac{x^3}{3} + 4x \right]_0^2 + \left[\frac{x^3}{3} - 4x \right]_2^4$$

Area $= 16$

Question 5
A ball, initially at rest, is dropped from a height of 1000 meters. The acceleration due to gravity is $9.8m/\,s^2$.

(a) Derive the equation for the velocity of the ball with respect to time *t*.

Acceleration due to gravity g can be expressed as rate of change of velocity, v with respect to time, or time derivative of velocity

$$g = \frac{dv}{dt}$$

Let the initial velocity at time t=0 be u and final velocity at t=t be v,

$$\int_0^t g dt = \int_u^v dv$$

$$gt = v - u$$

Hence, equation for the velocity of the ball with respect to time *t,v*

$$v = u + gt$$

(b) Derive an equation for the displacement of the ball at time *t*.

Velocity of the ball v can be expressed as rate of change of displacement, s with respect to time, or time derivative of displacement

$$v = \frac{ds}{dt}$$

Let the initial displacement at time t=0 be 0 and final displacement at t=t be s,

$$\int_0^t v dt = \int_0^s ds$$

As velocity v is variable, substituting

$$v = u + gt$$

$$\int_0^t (u + gt) dt = s$$

$$\left[ut + \frac{1}{2} gt^2 \right]_0^t = s$$

Displacement of the ball at time *t*,

$$s = ut + \frac{1}{2} gt^2$$

(c) After how much time will the ball hit the ground?

As ball is initially at rest, so u=0, g=9.8, s=1000.

Substituting the values in $s = ut + \frac{1}{2}gt^2$, we get

$$1000 = \frac{1}{2}9.8t^2$$

$$t = \sqrt{\frac{2000}{9.8}} = 14.2857\text{s}$$

Hence, the ball will hit the ground after 14.2857 seconds.

(d) What is the velocity with which the ball hits the ground?

Substituting u=0, g=9.8, t=14.2857 in equation

$$v = u + gt$$

we get,

$$v = 140$$

Hence, the velocity with which the ball hits the ground is 140 m/s