CHAPTER 01

FUNCTIONS AND GRAPHS

* Function : A function is a correspondence between a first set, called the domain, and a second set, called the range, such that each member of the domain corresponds to exactly one member of the range.

→ General Notation : $y = f(x)$.

* Identifying Functions in Real Life :-

Ex 1) LP-1 A car starts with an initial velocity of 10 m/s and accelerates at a constant rate of 2 m/s². Identify the parameters involved in this scenario & express the situation as a mathematical function. Then, represent this function in 4 different ways.

Solⁿ: The parameters involved are, velocity = v
acceleration = a.

→ Given, initially $t=0$ ∴ $v = 10\,m/s$ & $a = 2\,ms^{-2}$.

→ Because it is accelerating constantly, after each second we have

$t = 1$, $v = 10 + 2(1) = 12$

$t = 2$, $v = 10 + 2(2) = 14$

$t = 3$, $v = 10 + 2(3) = 16$

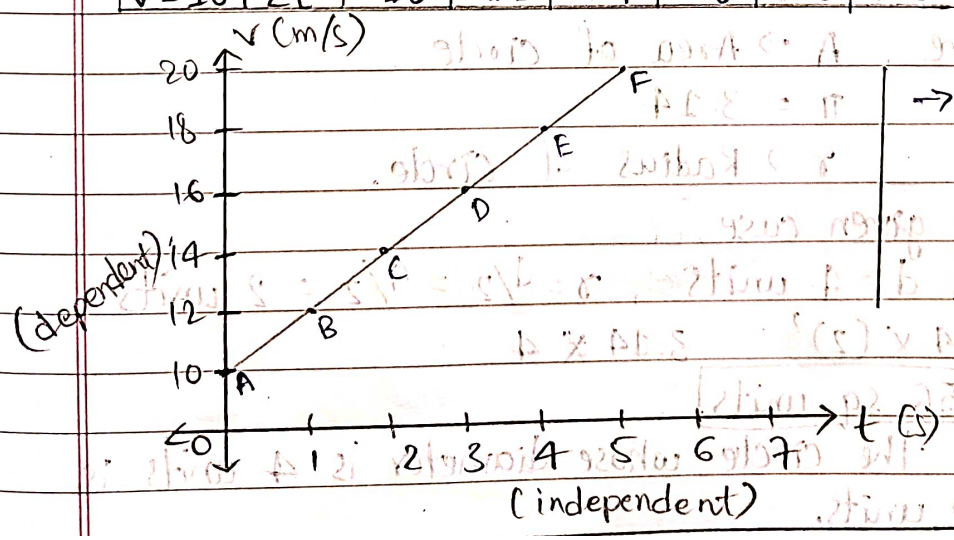
⋮

$t = t$, $\boxed{v = 10 + 2t}$

∴ ~~v(t) =~~ ∴ Velocity = f (time).

→

t	0	1	2	3	4	5
v=10+2t	10	12	14	16	18	20



→ In general this relation can be expressed as Dependent variable = f (Independent variable).

=> Representation of a Function.

(i) Verbally (by description in words).
→ The velocity of a constantly accelerating car can be expre as a function of time with a given initial velocity.
(ii) Numerically (by a table of values).
(iii) Graphically (by plotting a graph).
(iv) Algebraically (by an explicit formula).
→ $v(t) = v_0 + at$.

* Domain: The domain of a function is a set of all possible input values (independent variable) for which the function is defined.
* Range: The range of a function is a set of all poss output values (dependent variable) that the function can produce.

Ex 2) Identify the parameter that the area of a circle depends o
LP-2 Write a formula for the area A of a circle based on th parameter. Then, use the formula to find the area of a circle with a diameter of 4 units.

Sol^n: The parameter that the area of circle depends on is radiu of the circle.

→ It is given by $A = \pi r^2$
where, $A \rightarrow$ Area of circle.
$\pi = 3.14$
$r \rightarrow$ Radius of circle.

→ Considering the given case,
when diameter, $d = 4$ units, $r = d/2 = 4/2 = 2$ units.

→ $A = \pi r^2 = 3.14 \times (2)^2 = 3.14 \times 4$

$\boxed{A = 12.56 \text{ sq. units}}$

∴ The area of the circle whose diameter is 4 units is 12.56 square units.

Solⁿ: To find area, the parameters involved are radius (r), diameter (d) & circumferance (c).

i) Area $= \pi r^2$

$A(r) = \pi r^2$

Area $= f$ (radius).

ii) Area $= \frac{\pi d^2}{4}$

$A(d) = \frac{\pi d^2}{4}$

Area $= f$(diameter).

iii) Area $= \pi r^2$

$A(c) = \pi (c^2/4\pi^2) = c^2/4\pi$

$c = 2\pi r$

$r = c/2\pi$

Area $= f$(circumference)

Ex 3) LP-3 A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner & then folding up the sides as in the figure. Express the volume V of the box as a function of x.

Solⁿ: Let side of square cutout $= x$.

Length of box $= 20-2x$

Width of box $= 12-2x$

→ Volume,

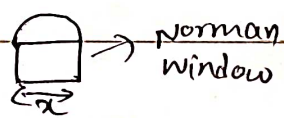
$$\boxed{V(x) = (20-2x)(12-2x)(x)}$$

Volume $= f$(side of cutout).

→ The domain for this function is $(0, 6)$.

Ex 4) LP-4 A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30ft, express the area A of the window as a function of the width 'x' of the window.

Solⁿ:



→ Width of window = Diameter of semicircle = x.
Let height = h.

→ Given perimeter = 30 ft.

$$30 = 2h + x + \pi x/2$$
$$2h = 30 - x - \pi x/2$$
$$h = 1/2\,[30 - x - \pi x/2]. \quad -(1)$$

$C = \frac{1}{2} 2\pi r = \pi r$

$C = \frac{\pi x}{2}$

→ Area = Area of rectangle + Area of semicircle.

$$A = hx + 1/2\,\pi (x/2)^2$$

→ From (1),

$$A(x) = x/2\,[30 - x - \pi x/2] + \pi/2\,(x^2/4).$$
$$\boxed{A(x) = x/4\,[60 - 2x - \pi x] + \pi x^2/8}$$

Area = f(width).

→ To find domain, let $A = 0$.

Then, $x/4\,[60 - 2x - \pi x] = -\pi x^2/8$

$$60 - 2x - \pi x = -\pi x/2$$
$$120 - 4x - 2\pi x = -\pi x$$
$$4x + 2\pi x - 120 = \pi x$$
$$4x + \pi x = 120$$
$$4x + 22/7\,x = 120$$
$$28x + 22x = 840 \Rightarrow \boxed{x = 16.8}$$

→ But if we substitute $x = 16.8$ in (1), $h = -6.588$ which is not possible.

∴ To find domain, $h \geq 0$.

i.e. $h = 0$, $30 - x - \pi x/2 = 0$.

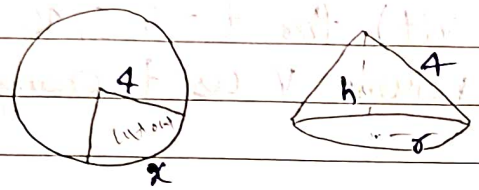
$$x + \pi x/2 = 30$$
$$2x + \pi x = 60$$
$$x = 60/2 + \pi = 60/5.14$$
$$\boxed{x = 11.67}$$

∴ The domain of the function is $(0, 11.67)$ i.e. $0 < x < 11.$

Ex 5) LP-5 Begin with a circular piece of paper with a 4 inch radius. Cut out a sector with an arc length of 'x'. Join the 2 edges of the remaining portion to form a cone with radius r & height h as shown in the figure.

a) Explain why circumferance of the base of cone is $8\pi - x$.
b) Explain the radius r as a function of x.
c) Express the height h as a function of x.
d) Express volume 'V' of the cone as a function of x.

Fig:



Sol^n: Since the original circle has a circumference of $2\pi(4) = 8\pi$.
We are cutting out a length of x from it & making a cone. Hence the circumference of the circle from the base of the cone is $\boxed{8\pi - x}$.

b) As circumference $= 2\pi r$.

$$8\pi - x = 2\pi r$$

$$r = \frac{8\pi - x}{2\pi}$$

$$\boxed{r(x) = 4 - x/2\pi}$$

c) Using Pythagoras Thm,

$$16 = h^2 + r^2$$

$$h = \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}$$

$$\boxed{h(x) = \sqrt{16 - \left(\frac{8\pi - x}{2\pi}\right)^2}}$$

d) Volume $= \frac{1}{3}\pi r^2 h$

$$\boxed{V(x) = \frac{1}{3}\pi \left(8\pi - x/2\pi\right)^2 \sqrt{16 - \left(8\pi - x/2\pi\right)^2}}$$

$\therefore$ Radius $= f(x)$.
Height $= f(x)$.
Volume $= f(x)$.

Ex 6) A tank initially holds 50 gallons of water. As the water drains through a leak at the bottom, the tank eventually becomes empty. The drain is faster when it is nearly full because the pressure on the leak is greater. According to Torricelli's Law, the volume of water remaining in the tank after t minutes is given by the function: $V(t) = 50(1 - t/20)^2$.

LP-6

a) Identify the domain & range of function $V(t)$.

b) Find $V(0)$ & $V(20)$.

c) Make a table of values of $V(t)$ for $t = 0, 5, 10, 15, 20$.

d) Find the net change in the volume V as t changes from 0 to 20.

Sol^n: $V(t) = 50(1 - t/20)^2$

$V(t) \rightarrow$ Volume of water.

$t \rightarrow$ Time in minutes.

a) Domain: We observe that $V(t)$ becomes zero when $t = 20$ & $V(t)$ is maximum i.e. 50 when $t = 0$.

$\therefore$ Domain is $[0, 20]$ $\quad 0 \le t \le 20$.

& Range is $[0, 50]$ $\quad 0 \le V(t) \le 50$.

b) $V(0) = 50$ & $V(20) = 0$.

c)

t	0	5	10	15	20
V	50	28.125	12.5	3.125	0

d) Net change in volume is $V(20) - V(0) = 0 - 50 = \boxed{-50 \text{ gallons}}$

Ex 7) The following graph approximates the incidence of lung & bronchus cancer L, per 100000 males as a function of t, the no of yrs since 1940. The eq^n is given as: $L(t) = -0.00054t^3 + 0.02917t^2 + 1.2329t + 8$.

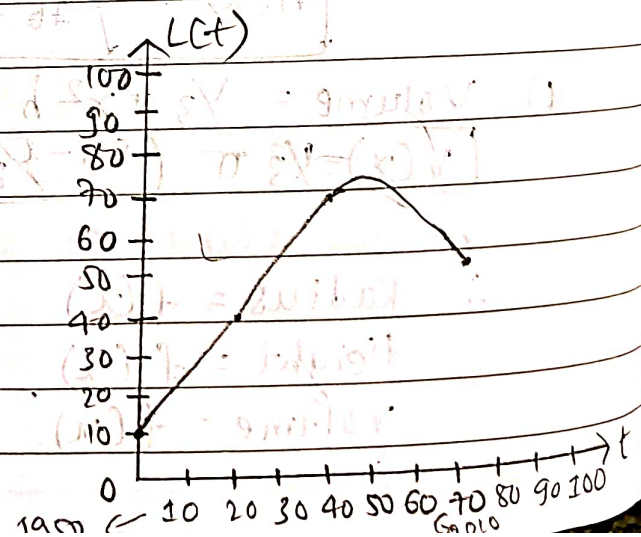
LP-7

a) Find domain using graph

b) Find range using graph.

c) Use the graph to approx. t-value where the rate is 40 per 100000 & verify it analytically.

d) Use the formula to approximate the cancer rate in 2010 & verify it.



Functions have only 1 output per input.

Solⁿ: $L(t) = -0.00054t^3 + 0.02917t^2 + 1.2329t + 8$.

a) Graphically we observe that the domain is $[0, 70]$ i.e. $[1940, 2010]$.

b) From the graph we observe that range is $[8, 75]$

c) From graph we observe that $L(t) = 40$ when $t = 20$. i.e. 1960.

Verification: $L(20) = -0.00054(20)^3 + 0.02917(20)^2 + 1.2329(20) + 8$.

$\boxed{L(20) = 40.006}$

d) The year 2010 is taken as 70 ∴ $L(t)$ at 70,

$L(70) = -0.00054(70)^3 + 0.02917(70)^2 + 1.2329(70) + 8$

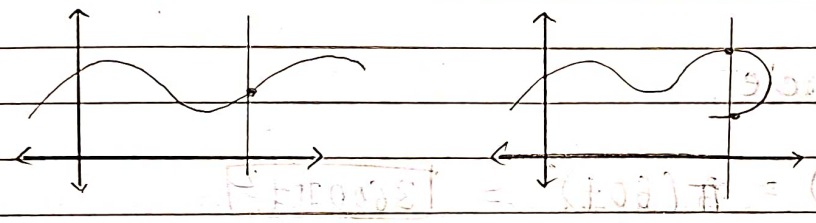
$= -185.22 + 142.933 + 86.303 + 8$

$\boxed{L(70) = 52.016}$

* Graph of a function :-

Defⁿ: The graph of a function is drawing that represents all the input-output pairs $(x, f(x))$.

⇒ Vertical Line Test : A curve in the XY Plane is the graph of a function iff no vertical lines intersect the curve more than once.



A function Not a function

* Composite Functions : Given 2 functions f & g, the composite function fog is defined as $(fog)(x) = f(g(x))$.

Ex 9) Find i) fog ii) gof iii) fof.

Solⁿ: (a) $f(x) = x^2 - 1$, $g(x) = 2x + 1$

LP-9

i) $fog = f(g(x)) = f(2x+1) = (2x+1)^2 - 1$

$= 4x^2 + 1 + 4x - 1 = \boxed{4x^2 + 4x}$.

ii) $gof = g(f(x)) = g(x^2 - 1) = 2(x^2 - 1) + 1$

$= 2x^2 - 2 + 1 = \boxed{2x^2 - 1}$

iii) $f \circ f = f(f(x)) = f(x^2-1) = (x^2-1)^2 - 1$
$= x^4 - 2x^2 + 1 - 1 = \boxed{x^4 - 2x^2}$

(b) $f(x) = x - 2$ & $g(x) = x^2 + 3x + 4$.

i) $f \circ g = f(g(x)) = f(x^2+3x+4) = x^2+3x+4-2$
$= \boxed{x^2+3x+2}$

ii) $g \circ f = g(f(x)) = g(x-2) = (x-2)^2 + 3(x-2) + 4$
$= x^2 + 4 + 4x + 3x - 6 + 4 = \boxed{x^2 + 7x + 2}$

iii) $f \circ f = f(f(x)) = f(x-2) = (x-2) - 2$
$= x - 2 - 2 = \boxed{x - 4}$

Ex 10) A stone is dropped into a lake, creating a circular ripple
LP 10 that travels outwards at a speed of 60 cm/s.

a) Express radius r as a function of t.

b) If A is the area of the circle as a function of the radius find $A \circ r$ & interpret it.

Sol^n: Since circular ripples are generated which travels at a speed of 60cm/s.

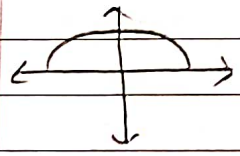
a) Distance = Speed × Time.
$\boxed{r(t) = 60t}$

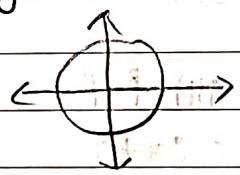
b) Area of the circle,
$A = \pi r^2$

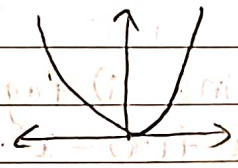
→ $A \circ r = A(r(t)) = \pi(60t)^2 = \boxed{3600\pi t^2}$

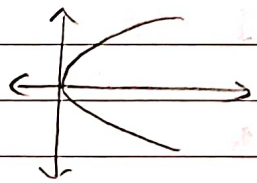
This formula gives us the extent of the rippled area (cm²) at any given time 't'.

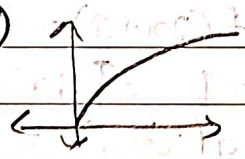
Ex 8) Determine whether they are functions or not.
LP-8

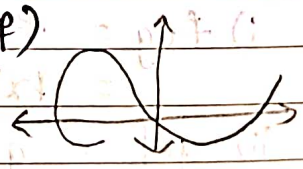
a)  Yes.

c)  No.

e)  Yes

b)  No.

d)  Yes.

f)  No.

Ex 11) A spherical ballon is being inflated. The radius of the ballon is increasing at 1cm/s. (a) Find function 'f' that models the radius as a function of time. (b) Find a function 'g' that models the volume as a function of radius. (c) Find $g \circ f$ & its representation.

Sol^n: Radius of the balloon, $r = x$cm/s, Speed $= 1$cm/s.

a) Distance = Speed × Time.

$r = 1 \times t \Rightarrow r = t$

∴ The function that models the radius as the function of time is $\boxed{f(t) = t}$

b) Volume, $V = 4/3\,\pi r^3$

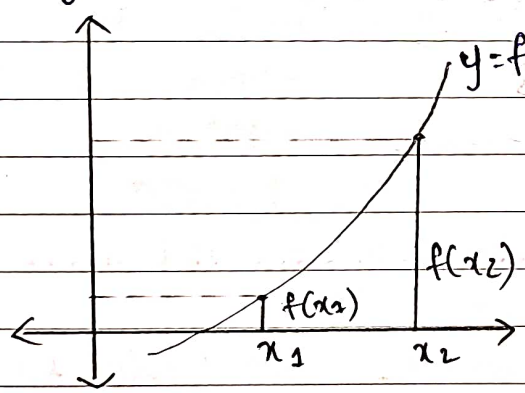
∴ The function that models the volume as a function of time is $\boxed{g(r) = 4/3\,\pi r^3}$

c) $g \circ f = g(f(t)) = g(t)$

$\boxed{g(t) = 4/3\,\pi t^3}$

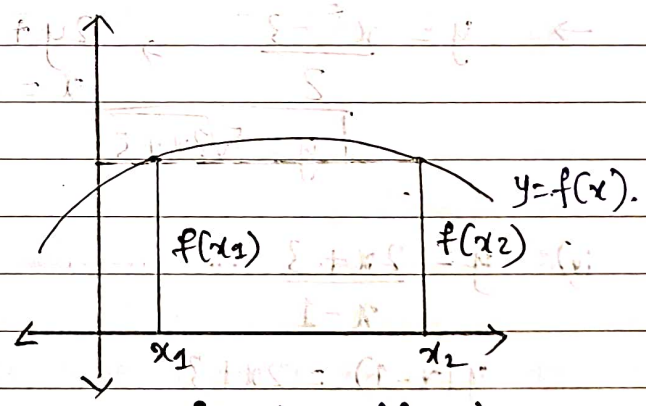
* Inverse Functions :-

- One-to-one function: A function with domain A is called a one-to-one function if no two elements of A have the same image, that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.



$f(x_1) \neq f(x_2)$

* One-to-one



$f(x_1) = f(x_2)$

Not one-to-one.

Note: Horizontal Line Test: A function is one-to-one iff no horizontal line intersects its graph more than once.

- **Inverse of a Function**: Let 'f' be a one-to-one function with domain A & range B. Then its inverse function f^{-1} has domain B & range A & is defined by,

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \quad \text{for any } y \text{ in } B.$$

Ex: i) $f(x) = x^2 + 2x + 1$

→ Let $y = x^2 + 2x + 1$

$y = (x+1)^2$

$x + 1 = \sqrt{y} \Rightarrow x = \sqrt{y} - 1$

Interchange x & y.

$y + 1 = \sqrt{x} \Rightarrow \boxed{y = \sqrt{x} - 1}$

ii) $y = x^2 + 4x$.

→ $y = x^2 + 4x + 4 - 4$

$y = (x+2)^2 - 4$

$y + 4 = (x+2)^2$

$x + 2 = \sqrt{y+4}$

$x = \sqrt{y+4} - 2 \Rightarrow \boxed{y = \sqrt{x+4} - 2}$

iii) $f(x) = \dfrac{x^5 - 3}{2}$

→ $y = \dfrac{x^5 - 3}{2} \Rightarrow 2y + 3 = x^5$

$x = \sqrt[5]{2y+3}$

$\boxed{y = \sqrt[5]{2x+3}}$

iv) $y = \dfrac{2x+3}{x-1}$

$y(x-1) = 2x + 3$

$yx - y = 2x + 3$

$\cancel{y(x-1)}$ $yx - 2x = y + 3$

$x(y-2) = y + 3$

$x = \dfrac{y+3}{y-2} \Rightarrow \boxed{y = \dfrac{x+3}{x-2}}$

v) $f(x) = \log_2 x$.

→ $y = \log_2 x$

$2^y = 2^{\log_2 x}$

$2^y = x \Rightarrow \boxed{y = 2^x}$

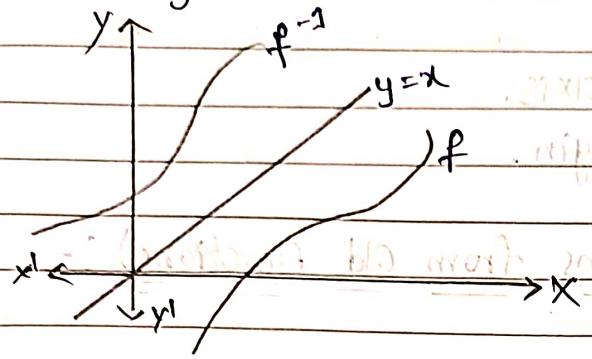
vi) $y = \sqrt{-1-x}$ (LP 11)

→ $y^2 = -1-x$

$x = -1-y^2$

$\Rightarrow \boxed{y = -1-x^2}$

Note: The graph of f^{-1} is obtained by reflecting the graph of 'f' in the line $y=x$.



Q: At a local pizza parlor the daily special is ₹100 for plain pizza plus ₹50 for each addition topping. The function that models price of pizza with 'n' toppings is $f(n) = 100 + 50n$. (a) Find the inverse of f. What does it represent? (b) If a pizza costs ₹350, how many toppings does it have?

S: $f(n) = 100 + 50n$.

→ The price 'P' of a pizza is a function of number of toppings 'n'.

a) To find inverse.

Let $P = 100 + 50n$

$50n = P - 100$

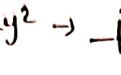
$\boxed{n = \frac{P-100}{50}} \Rightarrow f^{-1}(n) = \frac{n-100}{50}$

Imp → Thus this inverse function gives the number of toppings for a pizza of price 'P'.

b) If pizza costs ₹350 i.e. $P = 350$

$n = \frac{350-100}{50} = \frac{250}{50} \Rightarrow \boxed{n = 5}$

Imp → Hence pizza has 5 toppings.

Note: $4ay = 4ax^2$ → y is powerless (arms) →
(Parabola) $4ay = -4ax^2$ → $4ax = 4ay^2$ →  $4ax = -4ay^2$

* Symmetry of Functions :-

- **Even Function** : If a function 'f' satisfies $f(-x) = f(x)$ every number 'x' in this domain, then 'f' is called an function. Ex: $\cos x$.
- **Odd Function** : If a function 'f' satisfies $f(-x) = -f(x)$ every number 'x' in its domain, then 'f' is called odd fun Ex: $\sin x$.

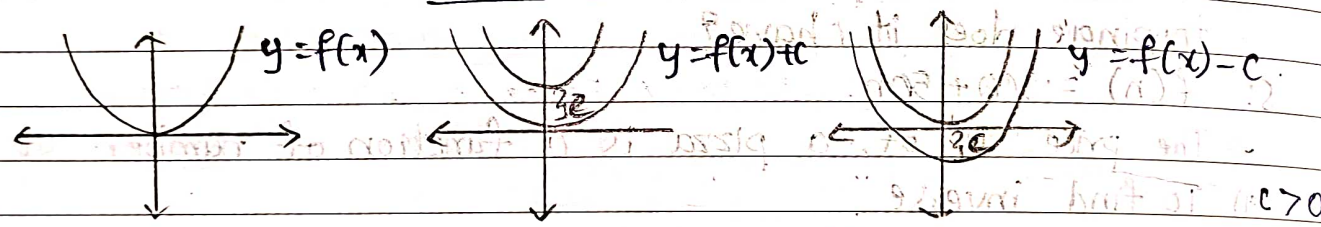
Note: Even : Symmetric about Y-axis.
Odd : Symmetric about origin.

* Transformation (New Functions from Old Functions) :-

- **Vertical Shift** :-

→ To obtain graph $y = f(x) + c$, shift the graph of $y = f(x)$ distance 'c' units upwards.

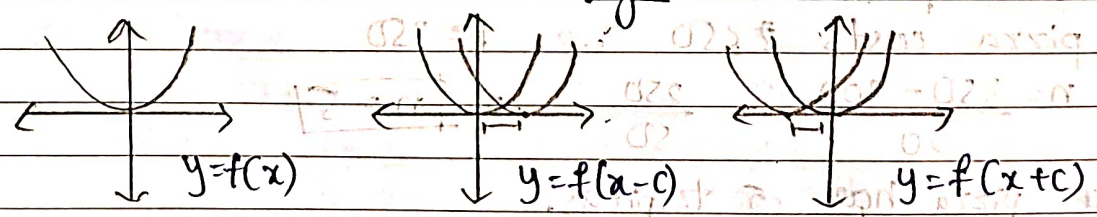
→ To obtain graph $y = f(x) - c$, shift the graph of $y = f(x)$ distance 'c' units downwards.

→ 

- **Horizontal Shift** :-

→ To obtain the graph of $y = f(x+c)$, shift the graph of y= a distance 'c' units to the left.

→ To obtain the graph of $y = f(x-c)$, shift the graph of y= a distance 'c' units to the right.

→ 

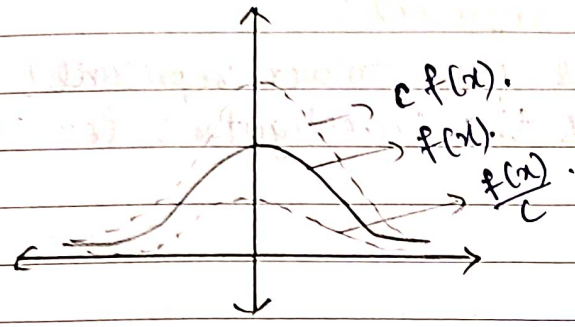
Outside → vertical
Inside → Horizontal

- **Vertical Stretch and Compression :-**

→ To obtain the graph of $y = c \cdot f(x)$, stretch the graph $y = f(x)$ vertically by a factor of c.

→ To obtain the graph of $y = \frac{1}{c} f(x)$, compress the graph $y = f(x)$ vertically by a factor of c.

→

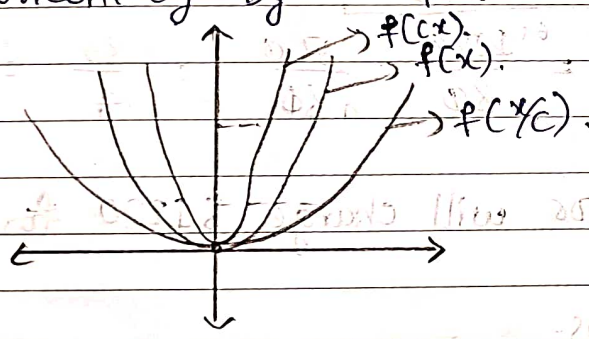


- **Horizontal Stretch and Compression :-**

→ To obtain the graph of $y = f(cx)$, compress the graph $y = f(x)$ horizontally by a factor of c.

→ To obtain the graph of $y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c.

→

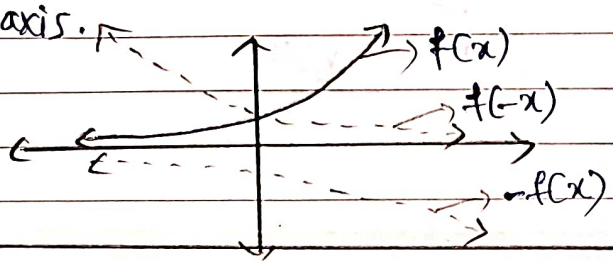


- **Reflection :-**

→ To obtain the graph of $y = -f(x)$, reflect the graph $y = f(x)$ about x-axis.

→ To obtain the graph of $y = f(-x)$, reflect the graph $y = f(x)$ about y-axis.

→



Q: For his services, a private investigator requires a \$500 retainer fee plus \$80 per hour. Let 'x' represent the no. of hrs the investigator spends working on a case.

(a) Find a function 'f' that models the investigator's fee as a function of x.

(b) Find f^{-1}. What does f^{-1} represent?

(c) Find $f^{-1}(1220)$. What does your answer represent?

S: (a) The function that models the investigator's fee is,

$$\boxed{f(x) = 500 + 80x}$$

(b) Let $y = f(x)$

$$y = 500 + 80x$$

$$80x = y - 500$$

$$x = \frac{y-500}{80}$$

→ Interchange x & y.

$$y = \frac{x-500}{80} \Rightarrow \boxed{f^{-1}(x) = \frac{x-500}{80}}$$

It represents the no. of hours worked.

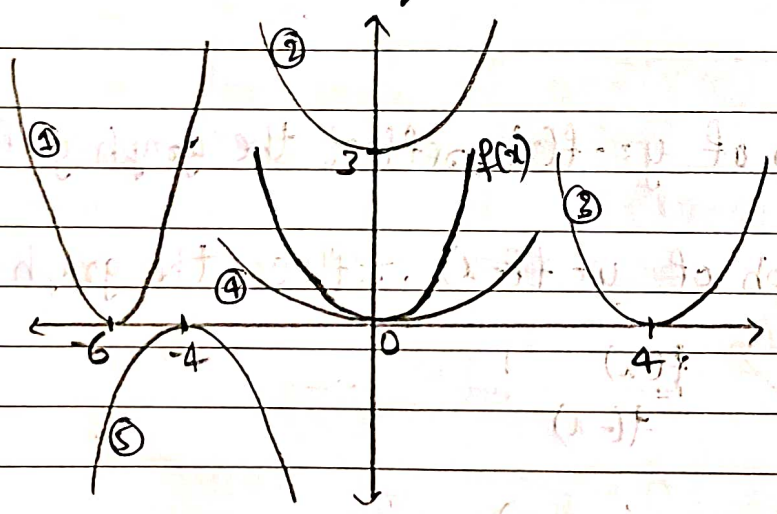
(c) $$f^{-1}(1220) = \frac{1220-500}{80} = \frac{\cancel{1220}^{6}}{\cancel{80}_{4}} - \frac{\cancel{500}^{25}}{\cancel{80}_{4}} = \frac{36}{4} = 9$$

$$\boxed{f^{-1}(1220) = 9}$$

→ It means that investigator will charge \$1220 for 9 hours.

Q: Match with the equations.

LP 129 →



① $2f(x+6)$.

② $f(x)+3$.

③ $f(x-4)$.

④ $\frac{1}{3}f(x)$.

⑤ $-f(x+4)$.

LP-12 Q: Explain how each graph is obtained from the graph $y=f(x)$.

i) $y = f(x) + 8$.

→ The graph will vertically shifts upwards by 8 units.

ii) $y = f(x+8)$.

→ The graph will horizontally shifts towards left by 8 units.

iii) $y = 8f(x)$.

→ The graph will vertically stretches by a factor of 8.

iv) $y = f(8x)$.

→ The graph will horizontally compresses by a factor of 8.

v) $y = -f(x) - 1$. $= -(f(x)+1)$.

→ The graph will shifts vertically ~~upwards~~ downwards by 1 unit & also reflects about the x-axis.

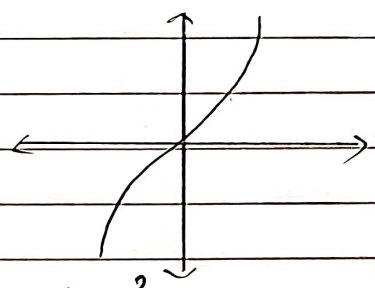
vi) $y = 8f(x/8)$.

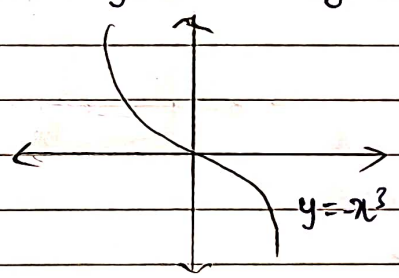
→ The graph will horizontally stretches by a factor of 8 & also vertically stretches by a factor of 8.

LP Ex 14. Graph the following function, by identifying its parent function & then applying the appropriate transformations. Also find domain & range.
LP 14

i) $y = -x^3$

→ Consider $y = x^3$ → By reflecting wrt x-axis.

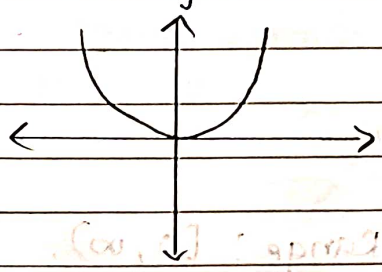


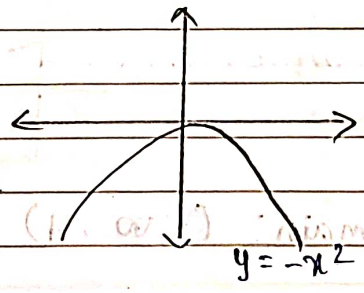


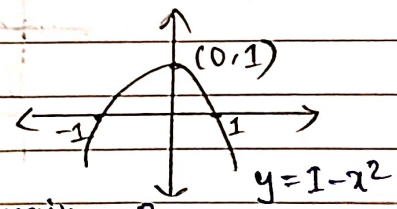
Domain = R
Range = R

ii) $y = 1 - x^2$

→ Consider $y = x^2$ → Reflecting wrt x-axis → Shifting vertically upwards by 1 units



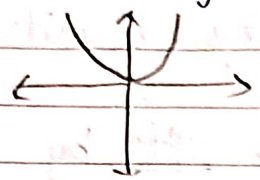




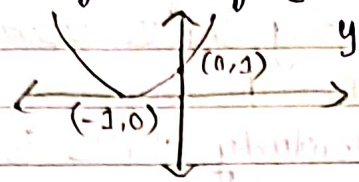
Domain = R
Range = $(-\infty, 1]$

iii) $y = (x+1)^2$.

→ Consider $y = x^2$ → By shifting graph by 1 unit to the left.



$y = (x+1)^2$



Domain : R
Range : $[0, \infty)$.

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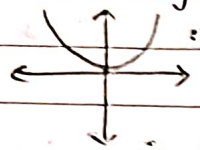
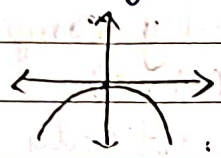
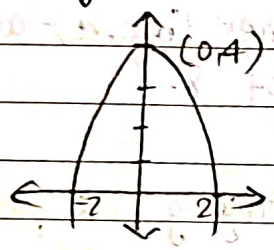
iv) $y = |(4-x^2)|$

→ Consider $y = x^2$ → Reflecting about X-axis, $y = -x^2$.

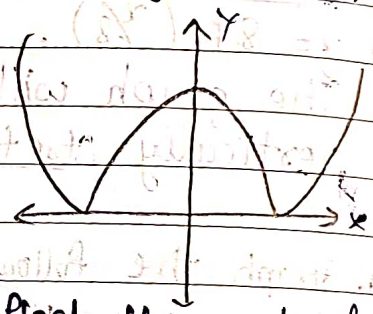




→ Shifting the graph vertically upwards by 4 units, $y = 4 - x^2$



→ For $y = |(4-x^2)|$.



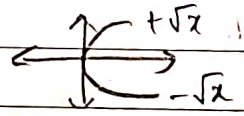
Reflect the part of the graph below X-axis to above X-axis.

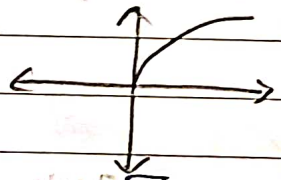
Domain: R.
Range: $[0, \infty)$.

---

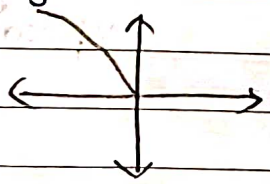
v) $y = \sqrt{4-x}$.

→ $y = \sqrt{x}$ → $y^2 = x$ →



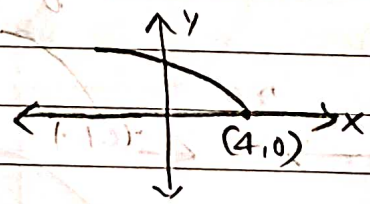


→ $y = \sqrt{-x}$



Imp → $y = \sqrt{4-x} = \sqrt{-(x-4)}$

→ whatever add/subtract, add to 'x' only.

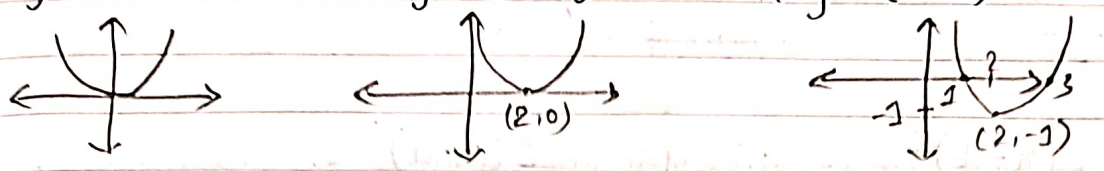


Domain: $(-\infty, 4)$ Range: $[0, \infty)$.

vi) $y = x^2 - 4x + 3$.

→ $y = x^2 - 4x + 3 + 1 - 1 = (x-2)^2 - 1$.

→ $y = x^2$ → $y = (x-2)^2$ → $y = (x-2)^2 - 1$

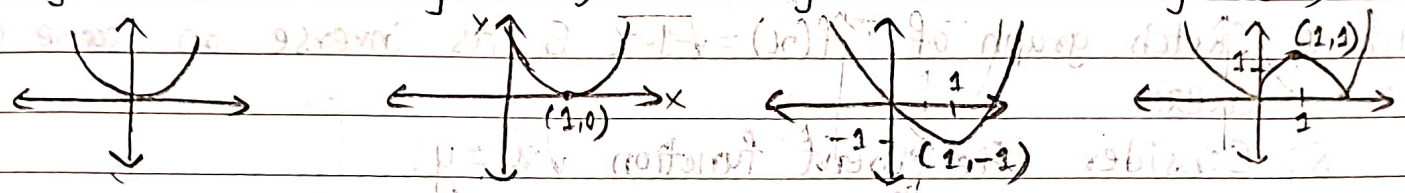


Domain : R Range : $[-1, \infty)$.

vii) $y = |x^2 - 2x|$.

→ $y = |x^2 - 2x + 1 - 1| = |(x-1)^2 - 1|$

→ $y = x^2$ → $y = (x-1)^2$ → $y = (x-1)^2 - 1$ → $y = |(x-1)^2 - 1|$

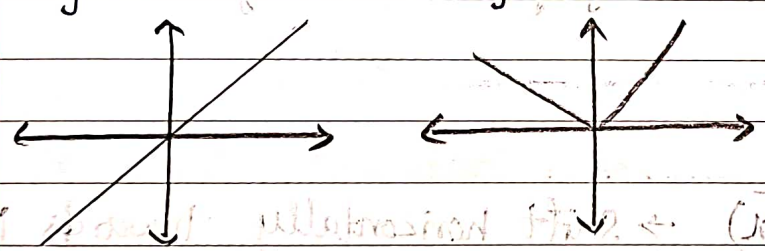


Domain : R Range : $[0, \infty)$.

viii) $y = 2 - |x|$

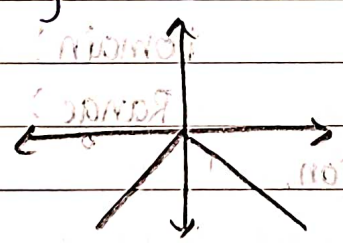
part below comes above

→ $y = x$ → $y = |x|$



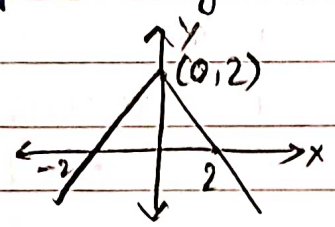
→ By reflecting wrt x-axis

$y = -|x|$



→ Vertically upwards by 2 units.

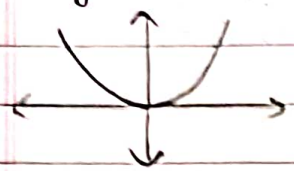
$y = 2 - |x|$



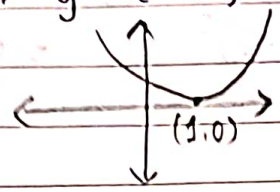
Domain : R
Range : $(-\infty, 2]$

ix) $y = 3 - 2(x-1)^2$.

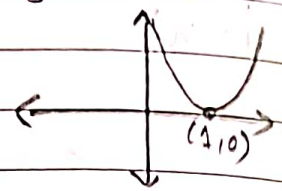
→ $y = x^2$



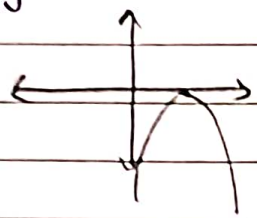
→ $y = (x-1)^2$



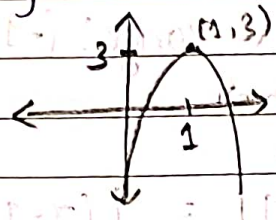
→ $y = 2(x-1)^2$



→ $y = -2(x-1)^2$



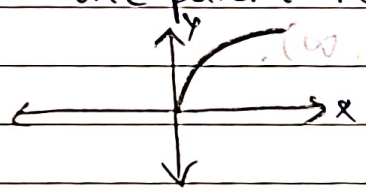
→ $y = 3 - 2(x-1)^2$



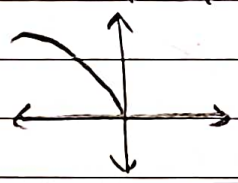
Domain: R.
Range: $(-\infty, 3]$.

HW) Q: Sketch graph of $f(x) = \sqrt{-1-x}$ & its inverse on same coordinate axes.

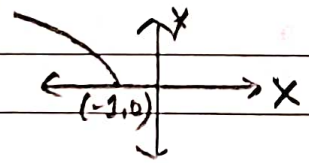
S: Consider the parent function $\sqrt{x} = y$.



→ To get $y = \sqrt{-x}$, reflect the graph ~~ab~~ wrt y-axis.



→ $y = \sqrt{-1-x} = \sqrt{-(1+x)}$ → Shift horizontally towards left by 1 unit.



Domain: $(\infty, -1]$
Range: $[0, \infty)$.

→ Finding inverse of the function.

Take $y = \sqrt{-1-x}$

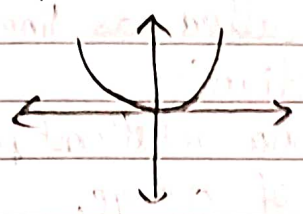
$y^2 = -1 - x$

Interchange x & y.

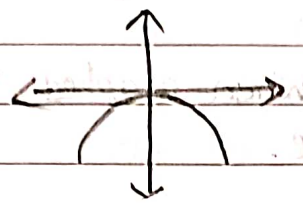
$x^2 = -1 - y$

$\boxed{y = -1 - x^2}$

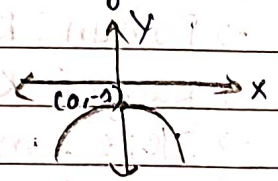
→ Consider parent function, $y = x^2$



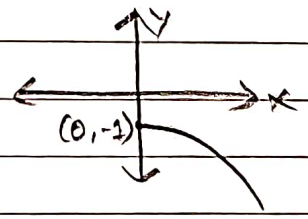
→ Reflect wrt X-axis to get $y = -x^2$.



→ Shift vertically downwards by 1 unit to get $y = -1 - x^2$.

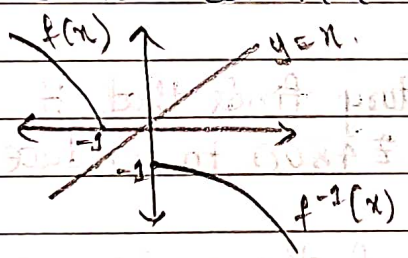


→ Since the range of given function becomes domain of its inverse the graph will look like, (x should be +ve).



Domain: $[0, \infty)$.
Range: $(\infty, -1]$.

→ On same coordinate axes,



* <u>Basic Types of Functions</u> :-

1. Linear Function.
2. Polynomial Function.
3. Exponential Function.
4. Trigonometric Functions.
5. Logarithmic Functions.

1) Linear Functions :-

→ A function with degree 1 is called as linear function.

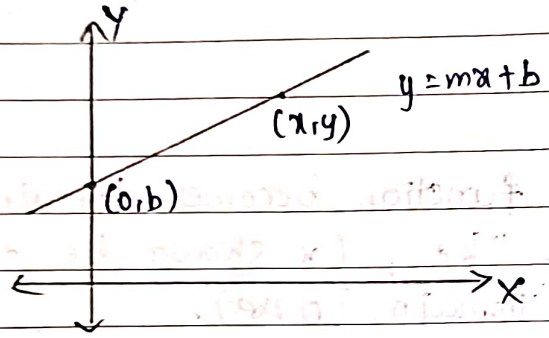
⇒ Key Steps to identify linear function :-

i) Constant Rate of change : A linear relationship has a constant slope, indicating a constant rate of change.

ii) Examine the mathematical form : The equations are of the form $y = mx + c$.

iii) Analyse the Data by plotting : When graphed, the data points or the eq^n form a straight line.

- A linear function is given by $y = mx + b$.



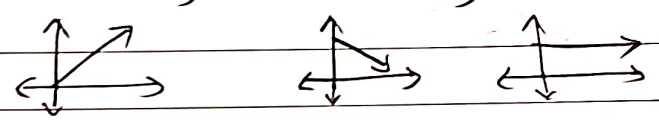
→ Domain : R. $(-\infty, \infty)$

→ Range : R. $(-\infty, \infty)$

→ Slope : m.

→ b is the y intercept, the p where line crosses Y-axis.

→ Increasing Decreasing Constant



$m > 0$ $m < 0$ $m = 0$

Q-56 Q: The manager of a furniture factory finds that it costs ₹2200 to manufacture 100 chairs & ₹4800 to produce 300 chairs in one day.

a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.

b) What is the slope of the graph & what does it represent?

c) What is the Y-intercept of the graph & what does it represent

d) Estimate the cost to produce 400 chairs.

Sol^n: a) Let y = cost (dependent variable).
x = number of chairs produced (independent variable).

→ By assumption, cost as a linear function of no. of chairs produced, we have

Model: $y = mx + b$ — (1)

Cost = m (no. of chairs produced) + b

where m = slope (rate of change of cost per unit producing chair)

→ It costs ₹2200 to manufacture 100 chairs.

Put $y = 2200$, $x = 100$ in (1)

$2200 = m(100) + b$ — (2)

→ It costs ₹4800 to manufacture 300 chairs

Put $y = 4800$, $x = 300$ in (1)

$4800 = m(300) + b$ — (3)

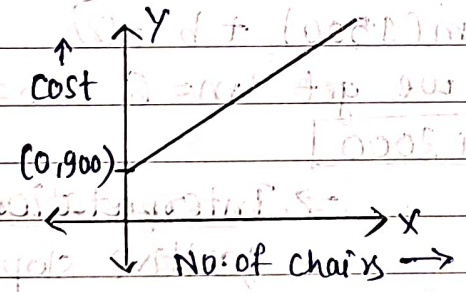
→ Solving (2) & (3) we get $m = 13$ & $b = 900$.

→ Substituting in the model we get,

$\boxed{y = 13x + 900}$

Cost = 13 (no. of chairs produced) + 900

→ Graph:



→ Interpretation: Graph is a straight line & positive slope indicates that as the number of chairs increases the cost also increases.

b) The slope 'm' is 13. This represents the additional cost per chair produced. For every chair produced, the cost increases by ₹13.

c) Y intercept, $b = 900$. This represents the fixed cost or the cost when no chairs are produced. Even if 0 chairs are produced, the cost is ₹900.

d) $x = 400 \rightarrow y = 13(400) + 900 = 5200 + 900 = \boxed{₹6100}$

LP-17 Q: A small-appliance manufacturer finds that it costs ₹9000 to produce 1000 toaster ovens a week & ₹12000 to produce 150 ovens per week.

i) Express the cost as a function of no. of ovens produced, assu that it is linear. Then sketch the graph.

ii) What is the slope of graph & what it represents?

iii) What is Y-intercept & what it represents?

S: i) Here, x = No. of ovens produced (Independent).

y = Cost (Dependent).

→ By assumption it is linear.

Model: $\therefore$ $y = mx + b$

Cost = m (No. of ovens) + b

$m \rightarrow$ Slope ; $b \rightarrow$ Y-intercept.

→ To produce 1000 ovens, it costs ₹9000.

$\therefore$ $y = 9000$ & $x = 1000$

$9000 = m(1000) + b$ — (1)

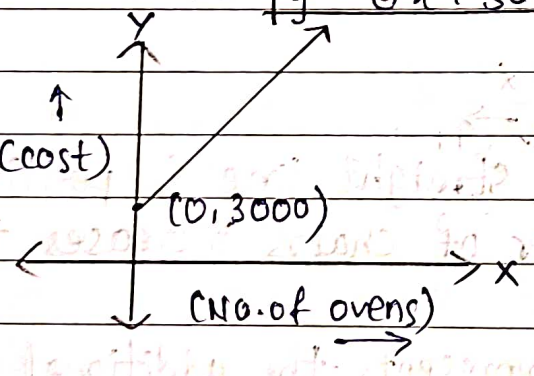
→ To produce 1500 ovens, it costs ₹12000.

$\therefore$ $y = 12000$, $x = 1500$

$12000 = m(1500) + b$ — (2)

→ Solving (1) & (2), we get $m = 6$, $b = 3000$.

$\therefore$ $\boxed{y = 6x + 3000}$



→ Interpretation: The graph has positive slope & straight line. Therefore as no. of ovens produced increases, the cost also increases.

ii) The slope of the graph is 6. This tells us that the additional charge for every ~~oven~~ oven produced is ₹6.

iii) The Y-intercept is 3000. It tells that it costs ₹3000 even i the manufacturer doesn't produce any oven (i.e. $x = 0$).

Q: a) The data in the table relate study time & test scores:
LP 15 b) Create a linear function to the data.
c) Make a scatterpot of data & superimpose the linear function on the scatterplot.
d) Use the linear model to predict the test score received when one has studied for 11 hr.

Study Time (hrs)	Test Grade (%)
7	82
8	85
9	88
10	91
11	9

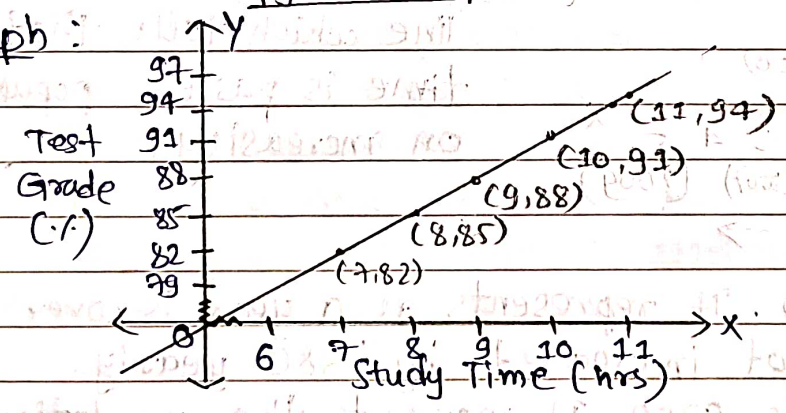
S: Let x = study time (independent)
y = test grade (dependent)

→ Given, need to create a linear function.
∴ $y = mx + b$.
test grade = m (study time) + b

→ When $x = 7$, $y = 83$ → When $x = 8$, $y = 85$
∴ $82 = 7m + b$ ∴ $85 = 8m + b$

→ By solving the above equations, we get $m = 3$ & $b = 61$
∴ $\boxed{y = 3x + 61}$

→ Graph:



→ Interpretation:
The graph has positive slope & straight line. Therefore as no. of study hrs increases, the test grades also increase.

→ When on has studied for 11 hr (i.e. $x = 11$)
$y = 3(11) + 61 = 33 + 61 = \boxed{94}$ → Grade (%)

→ The slope is 3 which tells that the additional test grade is 3% for every hour one studies. Y intercept is 61 which tells even if not studied the score is 61%.

Q: A town's population has been growing linearly. In 2004 the population was 6200. By 2009 the population had grown to 8100. Assume this trend continues.

a) Express population as a function of years, assuming it is line. Sketch the graph.

b) What is the slope & what does it represent?

c) What is Y-intercept & what it represent?

d) Predict the population in 2013.

e) Identify the year in which the population will reach 15000.

S: a) x = Year (Independent).

y = Population (Dependent).

→ Model: $y = mx + c$. (As the function is assumed as linear)

$m \to$ Slope, $c \to$ Y-intercept.

→ Let 2004 = 0 & 2005 = 1 & so on.

→ Given that in 2004 i.e. when $x = 0$, $y = 6200$

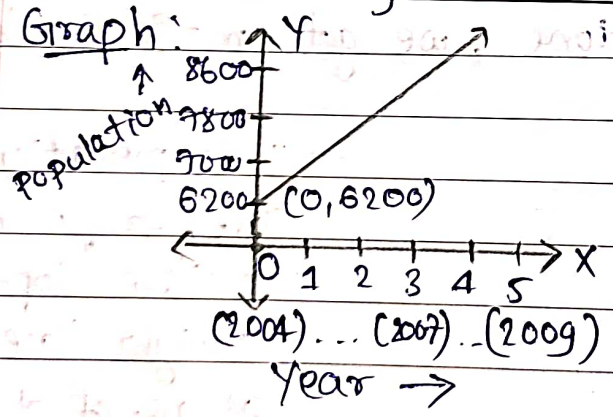
$\therefore 6200 = m(0) + c \Rightarrow \boxed{c = 6200}$

→ Also when $x = 5$ (i.e. in 2009), $y = 8100$.

$\therefore 8100 = m(5) + 6200 \Rightarrow 5m = 1900 \Rightarrow \boxed{m = 380}$

$\therefore \boxed{y = 380x + 6200}$ — (1)

→ Graph:



* Interpretation: The graph has positive slope & is a straight line which tells that as the time is passed population goes on increasing.

b) The slope is 380. It represents as a year is over after 200 the population got increased by 380 yearly.

c) The Y-intercept is 6200. It represents the population in the year 2004 i.e. when the study started.

d) Put $x = 9$ (i.e. 2013) in (1), $y = 380(9) + 6200 = \boxed{9620}$ → Population in 201

e) Put $y = 15000$ in (1), $15000 = 380x + 6200 \Rightarrow 8800 = 380x$

$\therefore \boxed{x = 23.15}$ ∴ The population will reach 15000 in 2028.

vertex $(x, y) = (-b/2a, y)$.

* Non-Linear Functions :-

- **Quadratic Functions :-**

→ Key signs to identify quadratic pattern :-

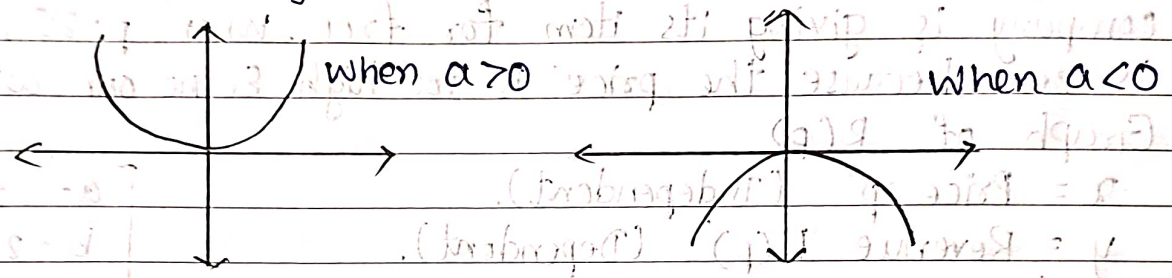
i) Examine the mathematical form : $y = ax^2 + bx + c$.

ii) Analyse the data by plotting : When graphed, the data points form a U-shaped curve (upward or downward).

iii) Axis of Symmetry : A parabola has an axis of symmetry, which is a vertical line that divides graph into 2 parts.

→ Characteristics of Quadratic Equations :-

i) General Eq^n : $\boxed{y = ax^2 + bx + c}$, $a \neq 0$.



Domain : R, Range : R

Q: A company wants to predict how much money it will make based on the price it charges for an item. They came up with the following quadratic function to model the revenue R(p) (in thousands of rupees) based on the price per item p:

LP-18

$$R(p) = -1.04p^2 + 26p \quad \text{for } p \in [0, 25].$$

a) Predict the revenue if the item is sold at $p = 5$ & $p = 17$.

b) Find the prices where this function gives a revenue of zero & explain what these prices mean.

c) Draw a graph of R(p).

d) Use the graph to find the price that gives the highest revenue. Calculate this maximum revenue.

→ To find vertex of a parabola (x, y)

$\boxed{x = -b/2a}$ → Put this in y to get y.

S: Revenue function, $\boxed{R(p) = -1.04p^2 + 26p}$ for $0 \le p \le$

a) At $p = 5$,

$R(5) = -1.04(25) + 26(5) = 104$ (so Revenue = ₹104.00)

→ At $p = 17$,

$R(17) = -1.04(289) + 26(17) = 141.44$ (so revenue = ₹141.44)

b) Zeroes of the functions means roots of the function.

i.e. when $R(p) = 0$.

$\therefore \quad 0 = -1.04p^2 + 26p$

$26p = 1.04p^2$

$p = 26/1.04 = 25$

$\therefore$ We get $\boxed{p=0}$ & $\boxed{p=25}$ ⇒ when $y=0$.

→ Interpretation: When $p=0$, the revenue is zero because the company is giving its item for free. When $p=25$, the revenue is zero because the price is too high & no one will buy it.

c) Graph of $R(p)$

→ x = Price 'p' (Independent).

y = Revenue 'R(p)' (Dependent).

$\boxed{\begin{array}{l} a = -1.04 \\ b = 26 \\ c = 0 \end{array}}$

→ $R(p) = 0$ when $p = 0$ & $p = 25$.

Vertex $= (-b/2a, y) = (x, y)$

$x = \frac{-b}{2a} = \frac{-26}{2(-1.04)} = 12.5 \Rightarrow \boxed{x = 12.5}$

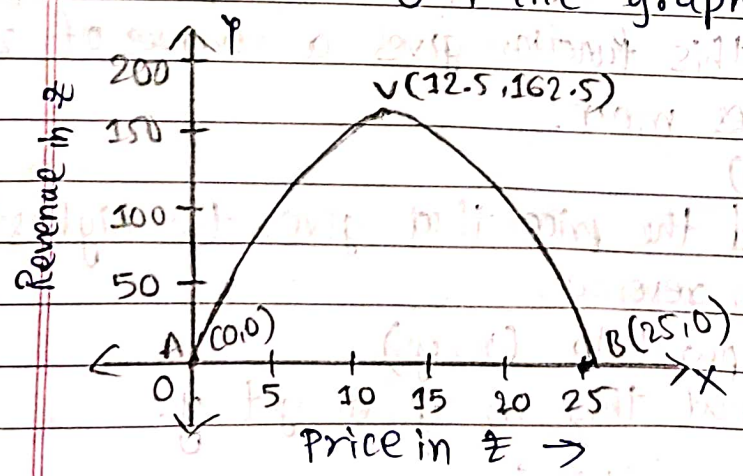
When $x = 12.5$, $y = -1.04(12.5)^2 + 26(12.5)$

$\boxed{y = 162.5}$ → Maximum Revenue

$\therefore$ The vertex is $(12.5, 162.5)$.

Since $a < 0$; the graph will be downwards.

→ Also when $y = 0$, $x = 0$



* Interpretation: When the price of item is ₹12.5, the revenue is maximum (₹162.5)

$\therefore$ Domain: $[0, 25]$

Range: $[0, 162.5]$

MOCK TEST

i) $f(x) = \frac{4x-1}{2x+3}$

Let $y = \frac{4x-1}{2x+3}$

$y(2x+3) = 4x-1$

$2xy + 3y = 4x - 1$

$2xy - 4x = -1 - 3y$

$x(2y-4) = -1-3y$

$x = \frac{-1-3y}{2y-4} = \frac{1+3y}{4-2y}$

Interchange x & y.

$\boxed{y = \frac{1+3x}{4-2x}}$

ii) $f(x) = \ln(x) + 3$

Let $y = \ln(x) + 3$

$y - 3 = \ln(x)$

$e^{(y-3)} = x$

Interchange x & y

$\boxed{y = e^{(x-3)}}$

* Exponential Functions :- Not defined for -ve values.

→ A function whose exponent is a variable is called as exponential function.

⇒ Key Signs to identify an Exponential Growth & Decay :-

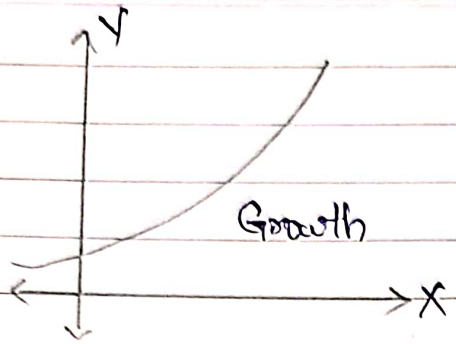
i) Constant Percent Change: Increases or decreases by constant percent over equal interval of time.

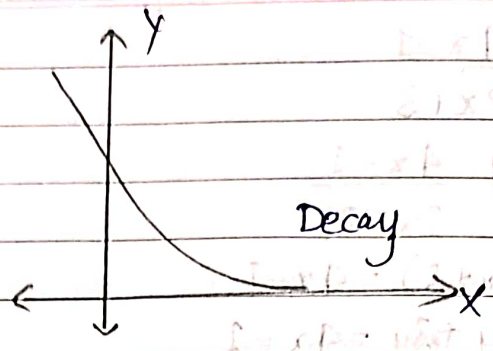
ii) Mathematical Form : $\boxed{y = ab^x}$ a → co-efficient, b → base.

iii) Analyse the data : By plotting. If the curve increases or decreases rapidly, then it is exponential in nature.

iv) Horizontal Asymptote : X-Axis.

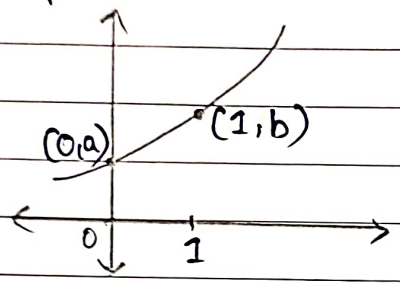
↓ Not touches but close to that axis.



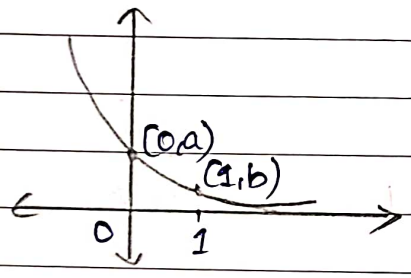


⇒ Characteristics :-

→ An exponential function is written in the form $\boxed{y = ab^x}$

→ 

→ Here $b > 1$
⇓
Exponential Growth.

→ 

→ Here $b < 1$ & $b > 0$.
(fraction)
⇓
Exponential Decay.

→ Domain : R.
→ Range : $(0, \infty)$.
→ Y Intercept : $(0, a)$.
→ Key Point : $(1, b)$.

* Logarithmic Functions :-

⇒ Key Signs to identify logarithmic pattern :-

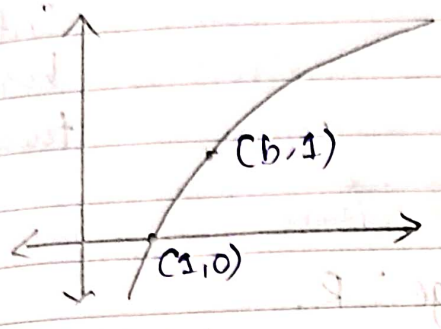
i) Slow Growth or Decay : As 'x' increases, the rate of increase of 'y' decreases.

ii) Mathematical Form : $\boxed{y = \log_b x}$

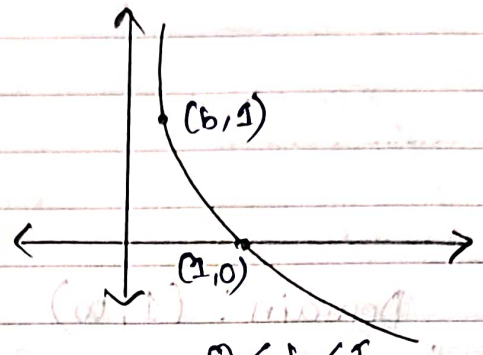
iii) Vertical Asymptote : The graph has vertical asymptote at $x=0$ (y-axis).

→ Characteristics :-

→ It is given by $\boxed{y = \log_b x}$.



$b > 1$



$0 < b < 1$.

→ Increasing if $b > 1$.

→ Decreasing if $0 < b < 1$.

→ Domain : $(0, \infty)$.

→ Range : $(-\infty, \infty)$.

→ X-intercept : $(1, 0)$

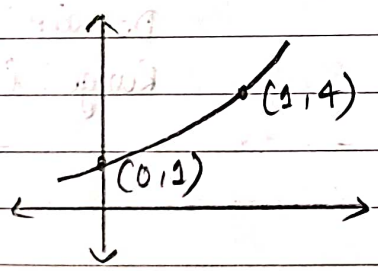
→ Key Point : $(b, 1)$

Note: Logarithmic & Exponential Functions are inverse of each other.

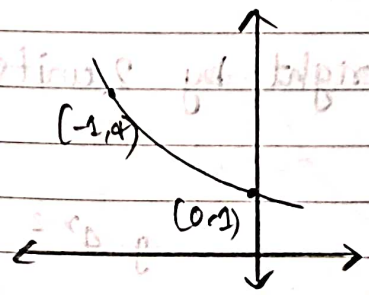
LP-29) Graph the function by applying the appropriate transformation. Also find domain & range.

S: i) $y = 4^{-x}$.

→ Consider $y = 4^x$ $(a = 1, b = 4)$



→ Reflect wrt Y-axis to get $y = 4^{-x}$.

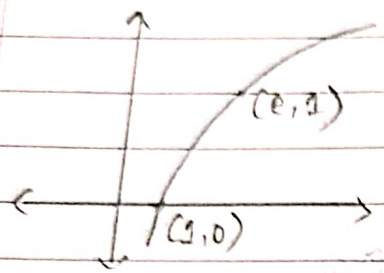


Domain : R

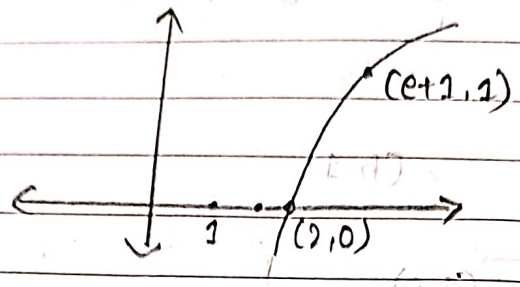
Range : $(0, \infty)$.

ii) $y = \ln(x-1)$

→ $y = \ln(x)$



$y = \ln(x-1)$

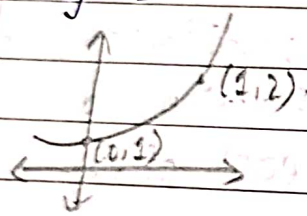


Shift horizontally towards right.

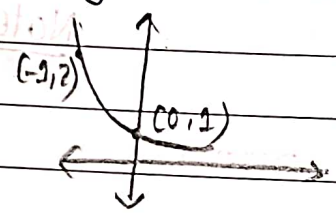
Domain : $(1, \infty)$, Range : R.

iii) $y = -2^{-x}$

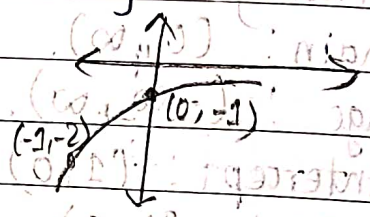
→ $y = 2^x$



$y = 2^{-x}$



$y = -2^{-x}$

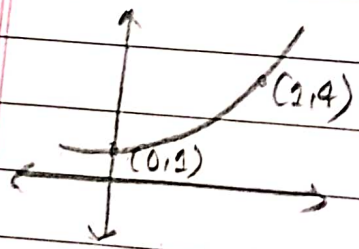


Reflet wrt Y-axis , Reflect wrt X-axis.

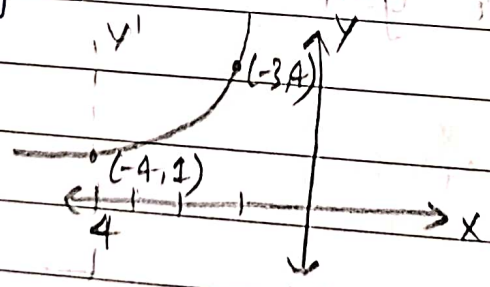
Domain : R , Range : $(-\infty, 0)$.

iv) $y = 4^{4+x}$.

→ $y = 4^x$



$= y = 4^{4+x}$

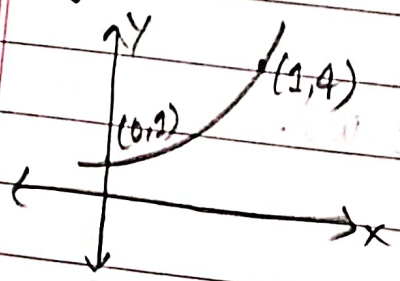


Domain : R

Range : $(0, \infty)$.

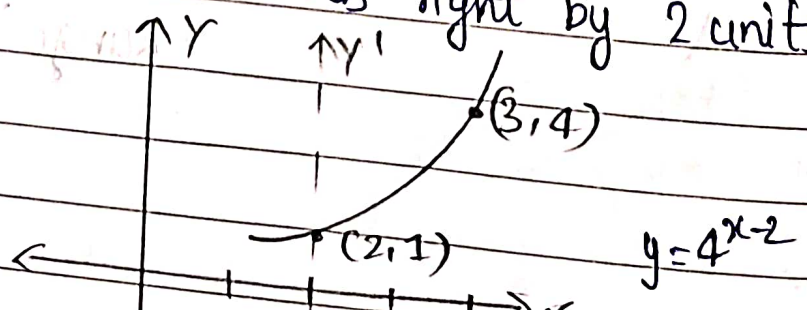
v) $y = 2 + 4^{x-2}$

→ $y = 4^x$



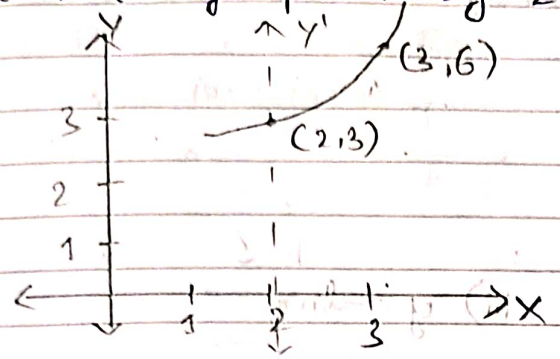
→ ~~Reflet wrt~~

→ Shift towards right by 2 units.



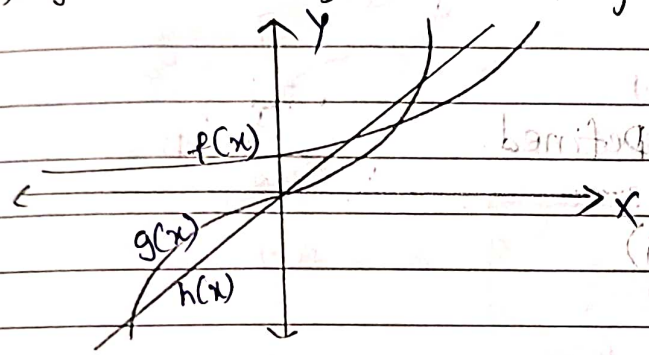
$y = 4^{x-2}$

→ Shift vertically upwards by 2 units to get $y = 2 + 4^{x-2}$.



LP 19) Match each equation with its graph & give reasons.
i) $y = 3x$ ii) $y = 3^x$ iii) $y = x^3$.

→



i) $h(x)$
ii) $f(x)$
iii) $g(x)$

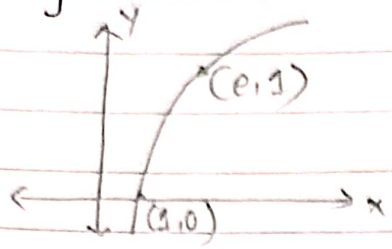
i) $y = 3x$. It is $h(x)$. Because it is a straight line & have constant slope.

ii) $y = 3^x$ is $f(x)$. It is an exponential function. Because it is not defined for negative values.

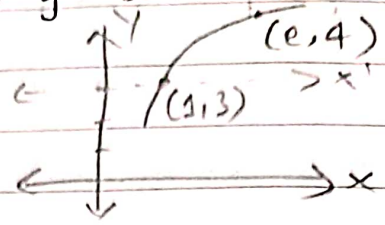
iii) $y = x^3$ is $g(x)$. It is cubic function, since it has variable slope.

LP 20) Starting with graph of $y = \ln x$, find the equation of the graph that results from i) shifting 3 units upwards.
ii) shifting 3 units to the left.
iii) Reflecting about X-axis.
iv) Reflecting about Y-axis.
v) Reflecting about $x = y$.
vi) Shifting 3 units left & then reflecting about $y = x$.

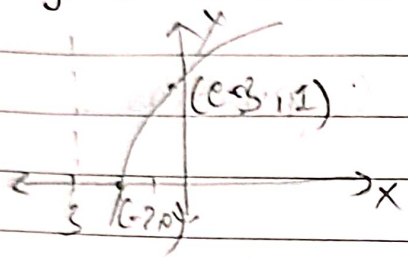
S: $y = \ln x$.



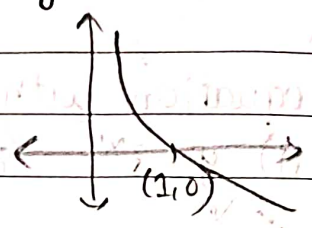
i) $y = 3 + \ln x$.



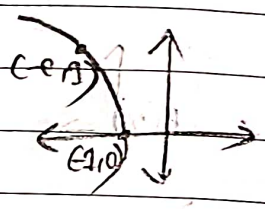
ii) $y = \ln(x+3)$.



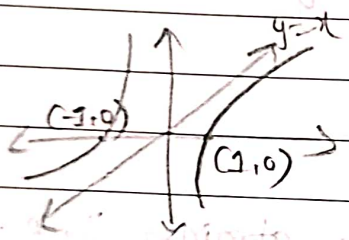
iii) $y = -\ln x$.



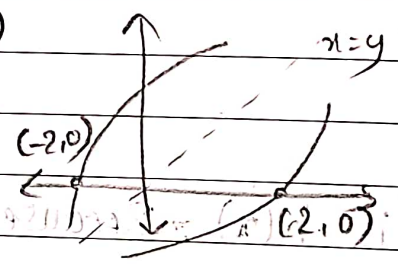
iv) $y = \ln(-x)$. → Not Defined.



v)



vi)



LP-22) Use the data in table & an exponential model to predict the population in the year 2010.

Year	1998	1999	2000	2001	2002	2003
Population (millions)	276.1	279.3	282.4	285.3	288.2	291.0

→ Since the problem is related to population growth, it is exponential in nature.

∴ Population at time 't' is given by,

$P(t) = ab^t$

a → Initial Population. (when $t=0$).

→ 1998 ⇒ $t=0$ & 2003 ⇒ $t=5$.

→ $P(0) = 276.1\,(b)^0 = 276.1$.

→ When $t=1$,

$P(1) = 279.3 = 276.1\,(b)^1$

→ By solving $b = 1.011$,

$\therefore \boxed{P(t) = 276.1\,(1.011)^t}$

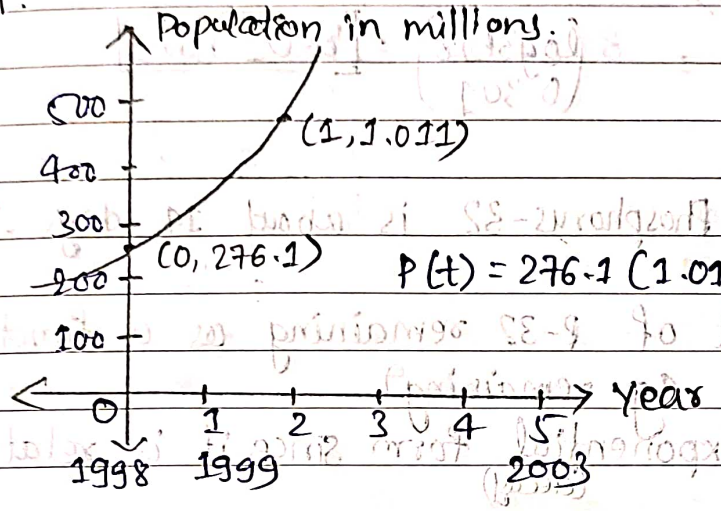
→ To find population in 2010, we take $t = 12$,

$\therefore P(12) = 276.1\,(1.011)^{12}$

$= 276.1 \times (1.011)^{12}$

$\boxed{P(12) = 311.1 \text{ million}}$

→ Graph.



P-23] If a bacteria population starts with 100 bacteria & doubles every 3 hours, then the number of bacteria after t hrs is $n = f(t) = 100(2)^{t/3}$

a) Find the inverse of this function & explain its meaning.

b) When will the population reach 50000?

S: Given: Population of bacteria,

$n = f(t) = \boxed{100 \cdot 2^{t/3}}$

a) Let $n = 100 \cdot 2^{t/3}$

$\frac{n}{100} = 2^{t/3}$

Take log on both sides

$\log(n/100) = \log(2)^{t/3} = t/3 \log 2$

$\frac{\log(n/100)}{\log 2} = \frac{t}{3}$

$\log_2(n/100) = t/3$

$\boxed{t = 3\log_2(n/100)}$

$\log A / \log B \neq \log(A-B)$
$\log(A/B) = \log(A \cdot B)$

→ Here the inverse function tells us the time 't' it will take to reach a specific size 'n'. In other words it tells us how long it took for the population to grow to that size.

b) When $n = 50000$

$$t = 3\log_2(n/100)$$
$$= 3\log_2(50000/100)$$
$$= 3\log_2(500)$$
$$t = 3\frac{\log 500}{\log 2} = 3\left(\frac{2.698}{0.301}\right) = \boxed{26.9 \text{ hours}}$$

LP-24) The half life of Phosphorus-32 is about 14 days. There are 6.6g present initially.

a) Express the amount of P-32 remaining as a function of time 't'.

b) When will there be 1g remaining?

S ⇒ a) It is of the exponential (decay) form since it is related to half life

→ $y = ab^t$

y → Amount of P-32 $= A(t)$, a → Initial amount.

→ At $t=0$, $A(t) = A(0) = a \Rightarrow A(0) = 6.6$

→ It is given that half life of P-32 is 14 days.

i.e. $A(14) = 3.3$

Also, $A(14) = ab^{14} = 6.6(b)^{14}$

$\therefore \cancel{3.3}^{2} = \cancel{6.6}(b)^{14}$

$b^{14} = 1/2 \Rightarrow \boxed{b = (1/2)^{1/14}}$

∴ The amount of P-32 remaining as a function of time 't' is,

$$\boxed{A(t) = 6.6\,(1/2)^{t/14}}$$

b) $A(t) = 1$, $t = ?$

$1 = 6.6\,(1/2)^{t/14}$

$2^{t/14} = 6.6$ → Taking log.

$\log(2)^{t/14} = \log(6.6)$

$t/14 \log(2) = \log(6.6) \Rightarrow t/14 = \frac{\log(6.6)}{\log(2)} = \frac{0.8195}{0.3010}$

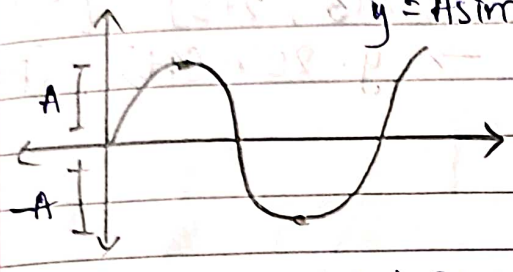
$t = 14 \times 2.7227$

$\boxed{t = 38.1 \text{ days}}$ ∴ 1g will be remaining after completion of 38.1 Days.

* Trigonometric Functions :-

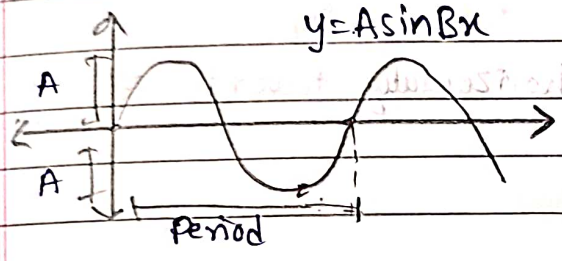
→ Most of the functions in real world are periodic in nature which can be easily expressed in terms of trigonometric functions like $\sin x$ & $\cos x$.

→ $y = A\sin x$



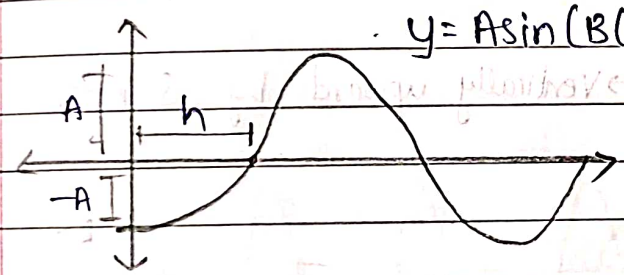
→ Characteristics
$|A|$ = Amplitude

→ $y = A\sin Bx$



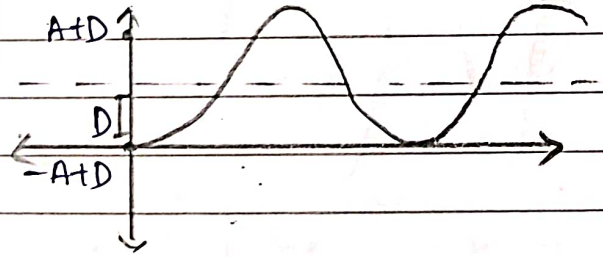
→ Characteristics
$T = 2\pi/|B|$ = period.
$|B|$ = Angular frequency.

→ $y = A\sin(B(x-h))$



→ Characteristics.
h = Phase shift.

→



→ Characteristics.
General sine function
$\boxed{y = A\sin(B(x-h)) + D}$
Domain : $(-\infty, \infty)$
Range : $[-A+D, A+D]$.

* Problems :-

LP-25) For the following functions determine.
a) Amplitude & Period.
b) Domain c) Range
d) Draw the graph.

i) $y = 2\sin(4t+\pi)+3.$

→ $y = 2\sin[4(t+\pi/4)]+3$

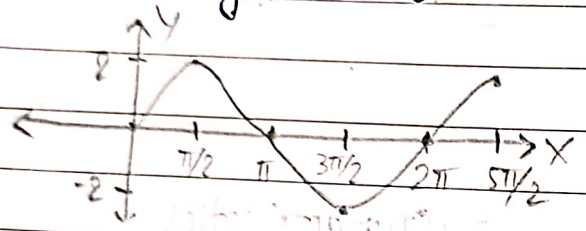
$|A| = 2$, $B = 4$

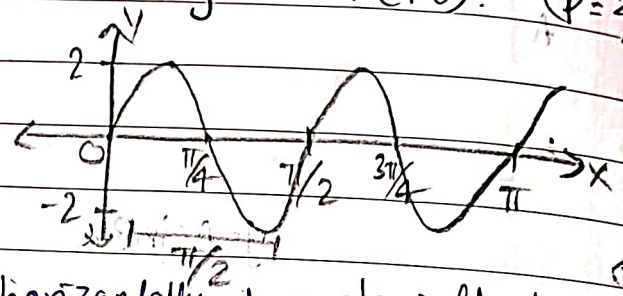
$T = \frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$ → Period.

$-A+D$, $A+D$

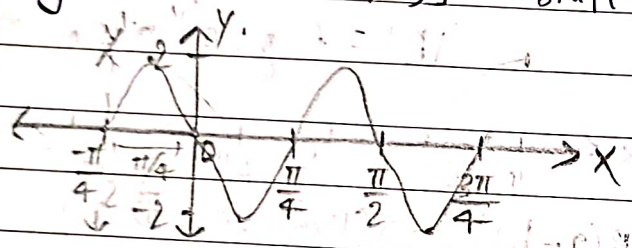
Domain : $(-\infty, \infty)$, Range : $[-2+3, 2+3] = [1, 5]$

→ Consider $y = 2\sin t$ → $y = 2\sin(4t)$. $(P = 2\pi/4 =$

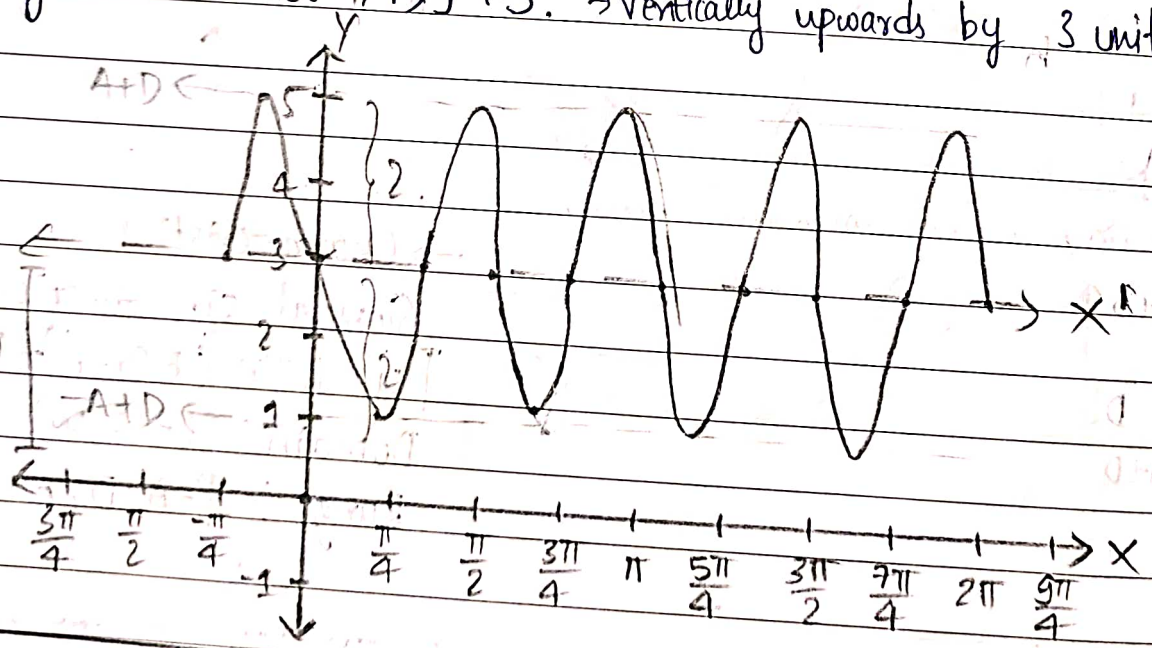




→ $y = 2\sin[4(t+\pi/4)]$ → Shift horizontally towards left by $\pi/4$ units



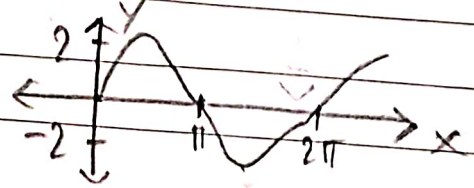
→ $y = 2\sin[4(t+\pi/4)]+3.$ → Vertically upwards by 3 units.



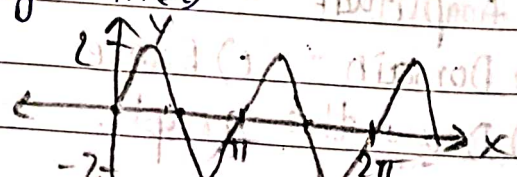
ii) $y = 2\sin(2t+\pi/3)$. $= 2\sin[2(t+\pi/6)]$.

→ $A = 2$, $B = 2$ ⇒ $T = \pi$, $D = 0$ ⇒ Domain : $(-\infty, \infty)$, Range : $[-2, 2]$.

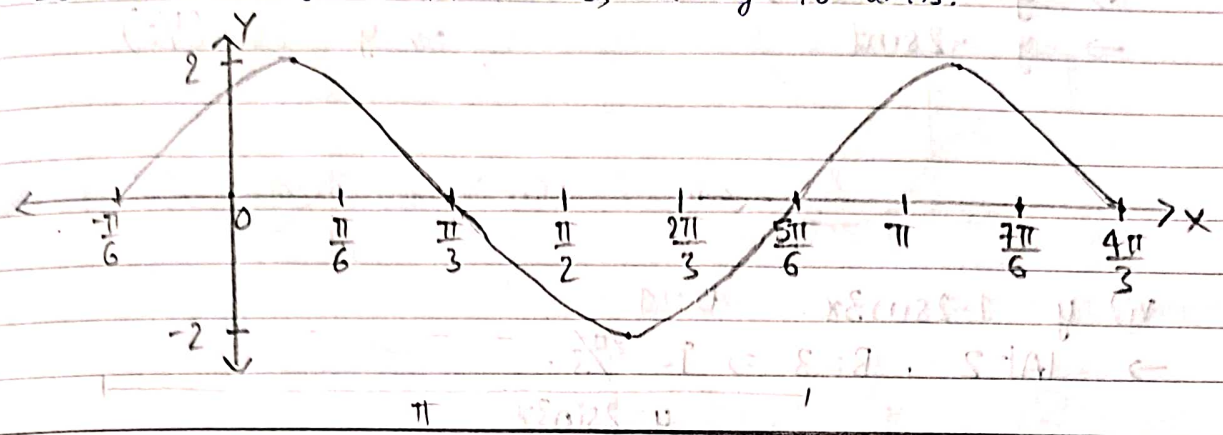
→ $y = 2\sin t$.



→ $y = 2\sin(2t)$



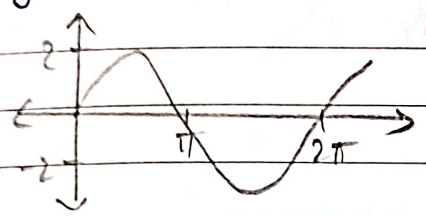
→ $y = 2\sin[2(t+\pi/6)]$. Shift towards left by $\pi/6$ units.



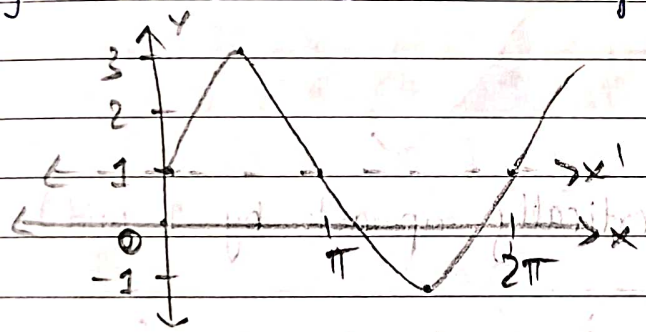
iii) $y = 1 + 2\sin t$. $A = 2$, $D = 1$

→ $y = 2\sin t$

Domain: $\mathbb{R}$
Range: $[-1, 3]$

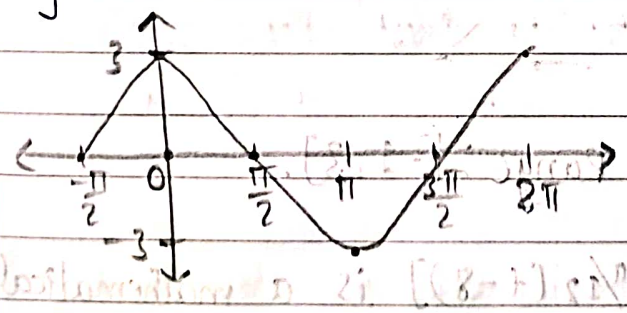


→ $y = 1 + 2\sin t$ (Shift vertically upwards by 1 unit).

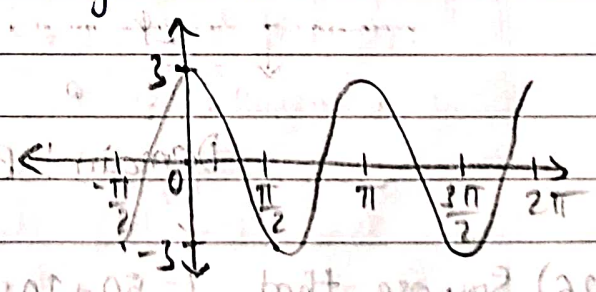


iv) $y = 3\cos 2t$

→ $y = 3\cos t$



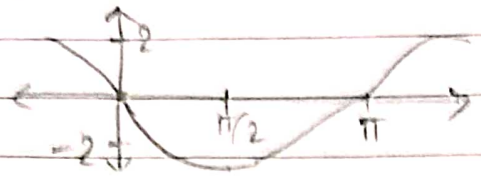
→ $y = 3\cos 2t$ $(P = \pi)$.

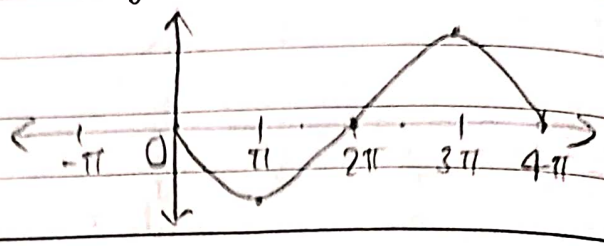


Domain: $\mathbb{R}$
Range: $[-3, 3]$.

v) $y = -2\sin(t/2)$. $|A| = 2$, $B = 1/2$, $P = 4\pi$. D: R

→ $y = -2\sin t$ → $y = -2\sin(t/2)$ R: $[-2, 2]$.

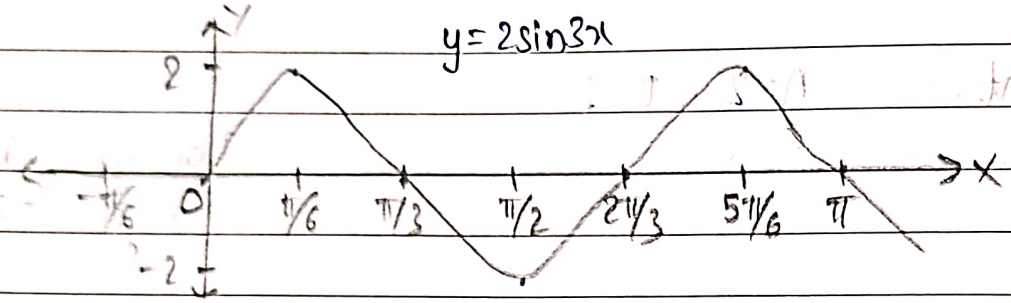




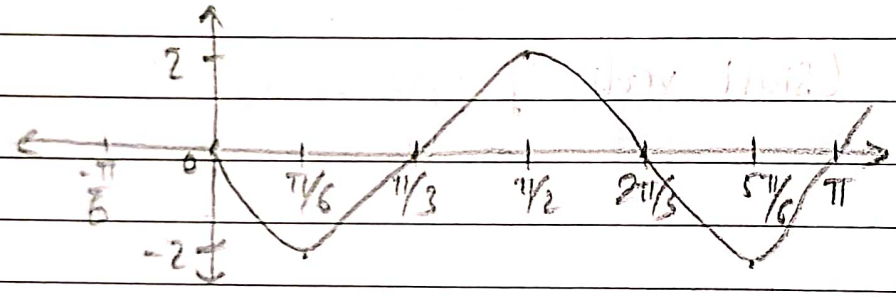
vi) $y = 1 - 2\sin 3x$. $D = 1$.

→ $|A| = 2$, $B = 3$ ⇒ $T = 2\pi/3$.

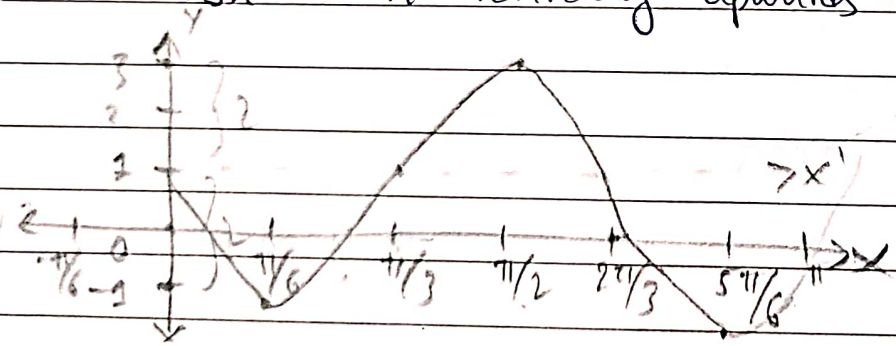
$y = 2\sin 3x$



→ $y = -2\sin 3x$ (Reflect wrt X-axis).



→ $y = 1 - 2\sin 3x$ (Shift vertically upwards by 1 unit).



Domain: R, Range: $[-1, 3]$.

P-26) Suppose that $T = 50 + 10\sin[\pi/12\,(t-8)]$ is a mathematical model of the temperature (in °F) at 't' hours after midnight on a certain day of the week.

a) Determine the amplitude & period.

b) Find the temperature 7 hrs after midnight.

c) At what time does $T = 60°$?

d) Sketch the graph of T over $0 \le t \le 24$.

S⇒ $T = 50 + 10\sin[\pi/12\,(t-8)]$

a) Given that ~~high~~ is the function. Compare it with general equation, $y = A\sin[B(t-h)] + D$

$\therefore \boxed{A = 10}$ $B = \pi/12 \Rightarrow \boxed{P = 2\pi \times 12/\pi = 24}$, $h = 8$, $D = 50$.

→ Amplitude is 10. It tells the maximum deviation of temperature from the average temperature 50°F.

→ Period is 24 hrs. This means the temperature completes one full cycle in 24 hrs.

→ Here $D = 50$, means avg. temp. is 50°F on any day.

b) Temperature 7hrs after midnight, $(t = 7)$.

$T(7) = 50 + 10\sin[\pi/12\,(7-8)]$

$= 50 + 10\sin[-\pi/12] = 50 - 2.58$

$\boxed{T(7) = 47.41°F}$ ∴ The temp. is 47.41°F at 7AM.

c) $T = 60°F$

$60 = 50 + 10\sin[\pi/12\,(t-8)]$

$10 = 10\sin[\pi/12\,(t-8)]$

$\sin[\pi/12\,(t-8)] = 1$

$\sin^{-1}(1) = \pi/12\,(t-8)$

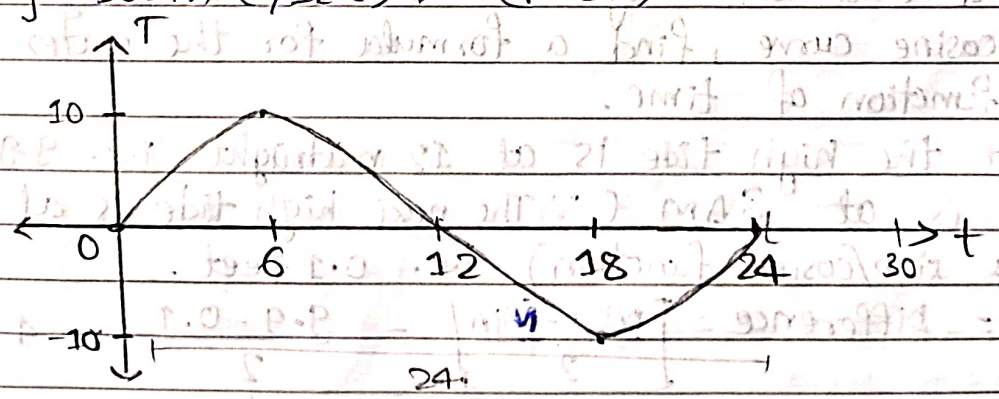
$\pi/2 = \pi/12\,(t-8)$

$6 = t - 8$ ∴ The temp. is 60°F when

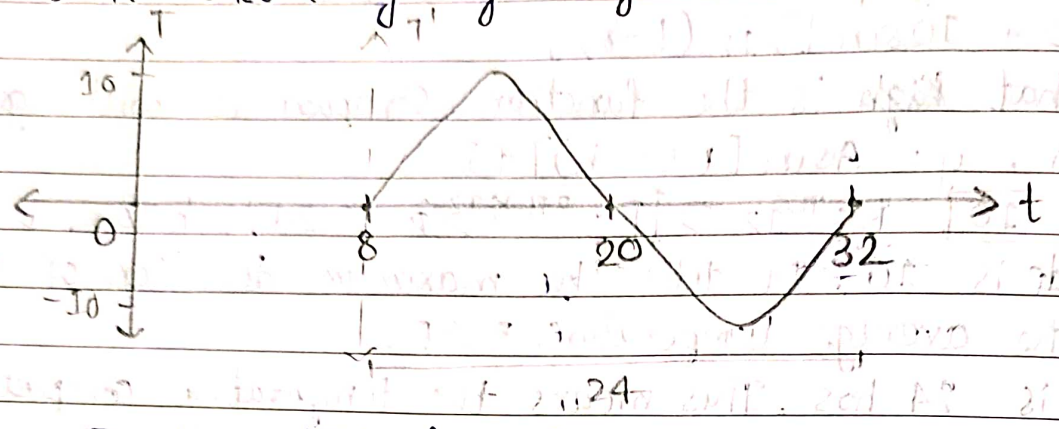
$\boxed{t = 14\text{ hrs}}$ $t = 14$ hrs i.e. at 2PM.

d) Graph for $0 \le t \le 24$. $(y \to T,\ x \to t)$.

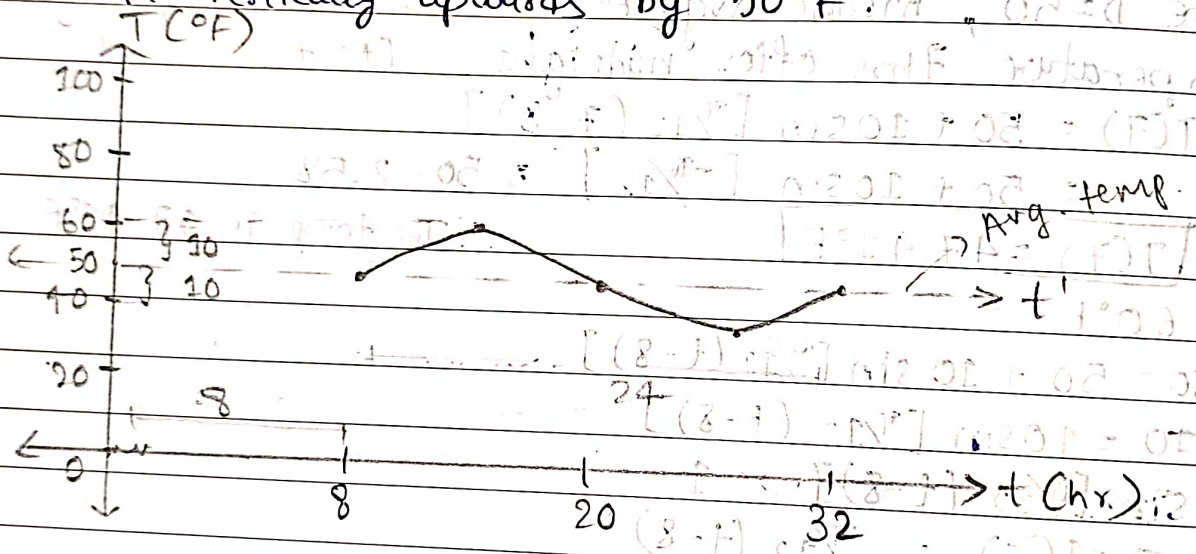
$T = y = 10\sin(\pi/12\,t)$. $(P = 24)$



→ Take $T = 10\sin[\pi/12\,(t-8)]$
Shift horizontally right by 8 units (hrs).



→ $y = T = 10\sin[\pi/12\,(t-8)] + 50$
Shift vertically upwards by 50°F.



LP 27) On Feb 10 2023, high tide in Goa was at midnight. The water level at high tide was 9.9 feet; at low tide, it was 0.1 feet. Assuming the next high tide is at exactly 12 noon & that the height of the water is given by a sine or cosine curve, find a formula for the water level in Goa as a function of time.

S⇒ Given that the high tide is at 12 midnight i.e. 9.9 feet & low tide is at 6AM (∵ The next high tide is at 12 noon & it is a sine/cosine function) i.e. 0.1 feet.

→ Amplitude = Difference = $\boxed{\dfrac{\text{Max}-\text{Min}}{2}} = \dfrac{9.9-0.1}{2} = 4.9$

Amplitude → Half of difference of max. & min.
D → Midline ⇒ Average of max. & min.

→ Average water level : Mean of max & min.

$\therefore D = \frac{9.9 + 0.1}{2} = \frac{10}{2} \Rightarrow \boxed{D = 5 ft}$

→ Period is 'T'.
Since the time between 2 high tides is 12 hrs,

$T = 12 hrs \Rightarrow T = \frac{2\pi}{B} \Rightarrow B = \frac{2\pi}{12_6} \Rightarrow \boxed{B = \frac{\pi}{6}}$

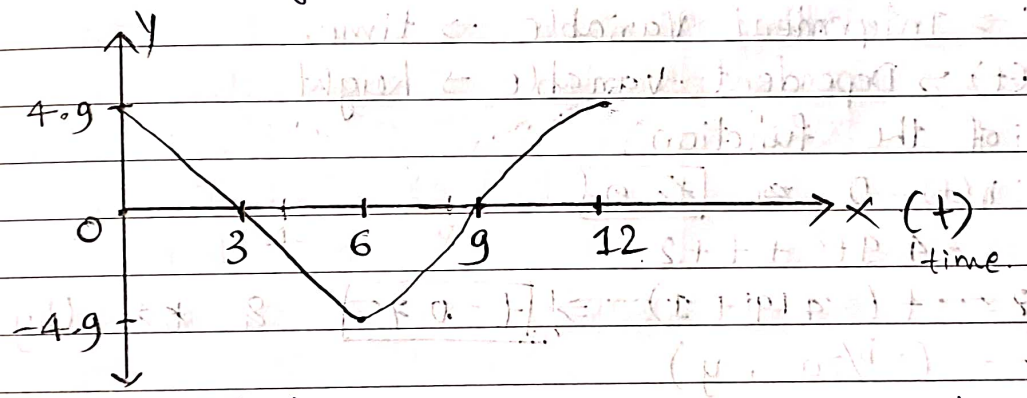
→ Since high tide happens at midnight (i.e. starting is max.) the suitable function is cosine.

$\therefore y = A\cos(B(x-h)) + D$

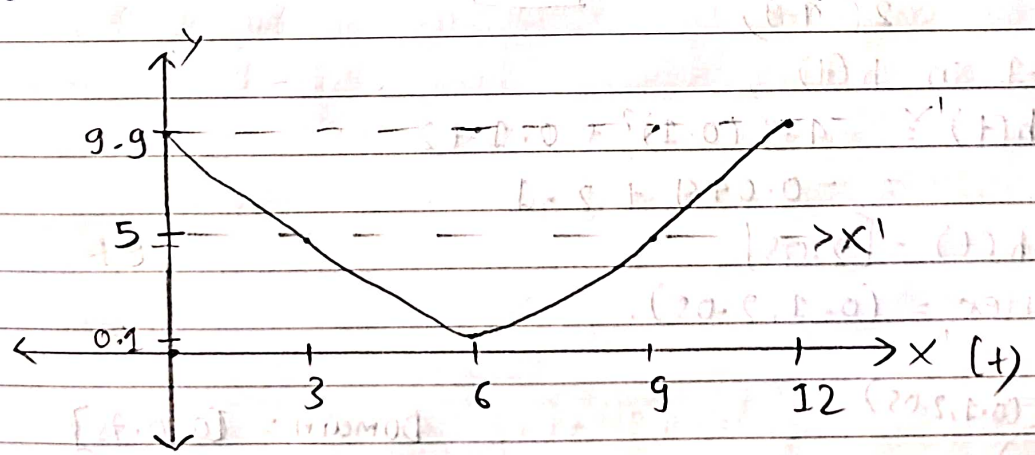
$\boxed{y = 4.9\cos(\pi/6\, t) + 5}$

⇒ Graph

→ Consider $y = 4.9\cos(\pi/6\, t)$. $T = 2\pi/\pi/6 = 12$.



→ $y = 4.9\cos(\pi/6\, t) + 5$ (Shift vertically upwards by 5 units).



CHAPTER 02

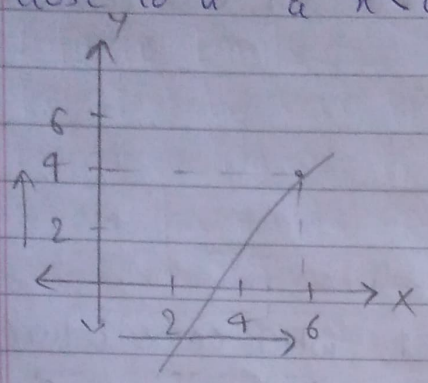
LIMITS AND DERIVATIVES

* Limits :-

- **Left Hand Limit :-**

→ We write $\lim_{x \to a^-} f(x) = L$ & say the LHL of $f(x)$ as 'x' approaches 'a' is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking 'x' to be sufficiently close to 'a' & $x < a$.

→



$y = \lim_{x \to 6^-} f(x) = 4$

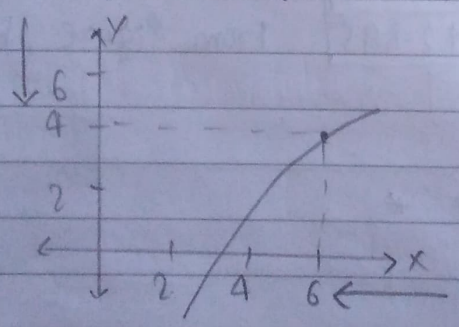
↓

LHL.

- **Right Hand Limit :-**

→ We write $\lim_{x \to a^+} f(x) = L$ & say the RHL of $f(x)$ as 'x' approaches 'a' is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking 'x' to be sufficiently close to 'a' & $x > a$.

→



$y = \lim_{x \to 6^+} f(x) = 4$

↓

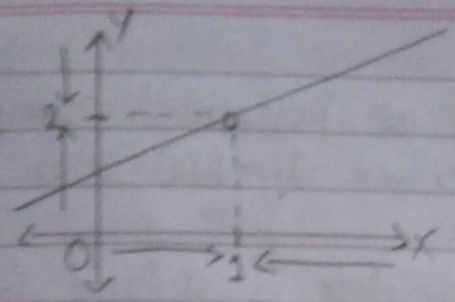
RHL

- **Limit of a Function :-**

→ We write $\lim_{x \to a} f(x) = L$ & say the limit of $f(x)$ as 'x' approaches 'a', equals 'L' if we can make the values of $f(x)$ arbitrarily close to L by taking 'x' to be sufficiently close to 'a' (on either side of a) but not equal to 'a'.

i.e. $\lim_{x \to a} f(x) = L$.

Ex →



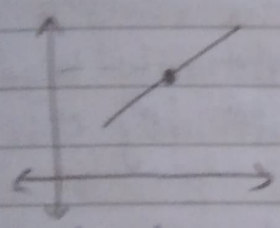
$$y = \frac{x^2-1}{x-1}$$

$y(1)$ = Not Defined

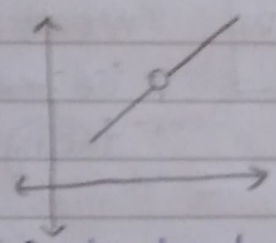
→ In other words, a function $f(x)$ is said to have limit if LHL = RHL.

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then only limit exists.

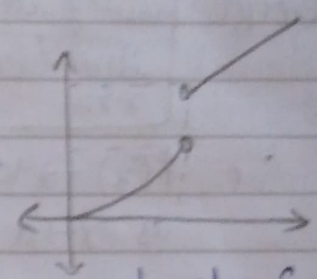
• Types of Limits :-



Exact value exists, Limit exists (Smooth Curve)



Exact value does not exist, limit exists. (Hole at $x=a$)



Exact value & limit both does not exist. (Jump at $x=a$)

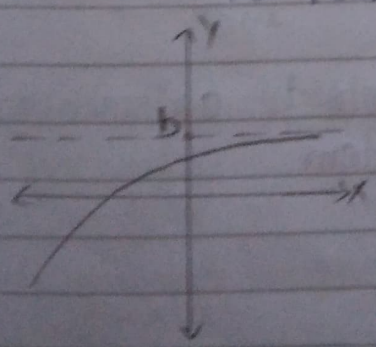
• Limits at Infinity :-

→ Let $f(x)$ be some function defined on an interval then $\boxed{\lim_{x \to \infty} f(x) = b}$

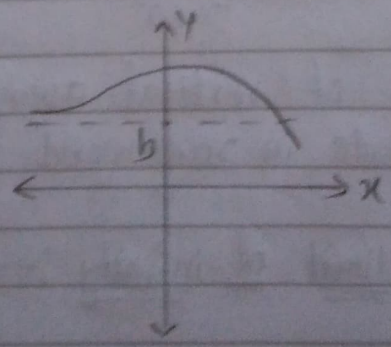
$$\boxed{\lim_{x \to \infty} f(x) = b}$$

means that values of $f(x)$ can be made arbitrarily close to 'b' by taking 'x' sufficiently large.

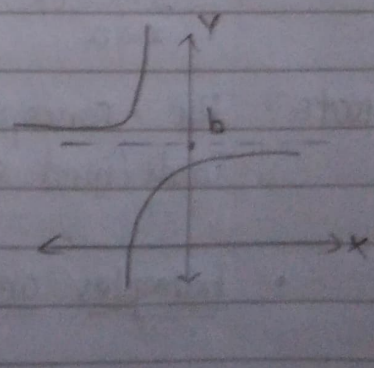
i.e. $f(x) \to b$ as $x \to \infty$



$y \to b$ as $x \to \infty$



$y \to b$ as $x \to -\infty$



Tangent touching at $\pm\infty$ $\Rightarrow$ Asymptote.

- **Horizontal Asymptote :-**

→ The line $y=b$ is called as horizontal asymptote of the curve $y=f(x)$ if either $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$.

Remove highest power term

Note: The concept of horizontal asymptote is a limit for long-term behaviour.

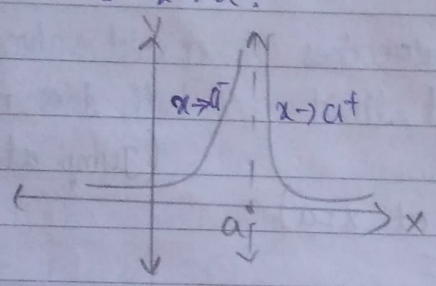
- **Infinite Limits :-**

→ Let $f(x)$ be some function defined on both sides of 'a', except possibly at 'a' itself, then,

$$\boxed{\lim_{x\to a} f(x)=\infty}$$

means that the values of $f(x)$ can be made arbitrarily large by taking 'x' sufficiently close to 'a'.

i.e. $f(x)\to\infty$, as $x\to a$.



Similarly,

$$\boxed{\lim_{x\to a} f(x)=-\infty}$$

- **Vertical Asymptote :-**

→ The line $x=a$ is called as vertical asymptote to the curve $y=f(x)$ if atleast one of the following statements are true

Denominator = zero

$\lim_{x\to a} f(x)=\infty$ $\quad$ $\lim_{x\to a^-} f(x)=\infty$ $\quad$ $\lim_{x\to a^+} f(x)=\infty$

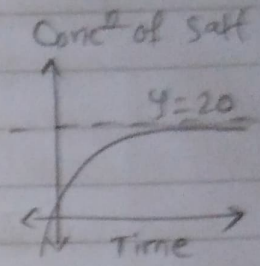
$\lim_{x\to a} f(x)=-\infty$ $\quad$ $\lim_{x\to a^-} f(x)=-\infty$ $\quad$ $\lim_{x\to a^+} f(x)=-\infty$

Note: The concept of vertical asymptote is an unmanageable or undefined state in real-world system.

- **Examples on limit of infinity :-**

1) A tank contains 10L of pure water. Salt water containing 20g of salt per litre is pumped into the tank at 2L/min & the salt conc^n C(t) after 't' minutes (in g/L) is 20t/5+t. What is the long-term conc^n of salt?

S⇒ $C(t) = \frac{20t}{5+t}$ → The long term conc^n of salt is,



→ $\lim_{t\to\infty} C(t) = \lim_{t\to\infty}\left(\frac{20t}{5+t}\right) = \lim_{t\to\infty}\left(\frac{20\cancel{t}}{\cancel{t}(5/t+1)}\right)$

$\lim_{t\to\infty} C(t) = \frac{20}{5/\infty+1} = \frac{20}{0+1} = \boxed{20}$

⇒ As the time increases, salt conc^n approaches a finite value & the limit represents the study state of concentration.

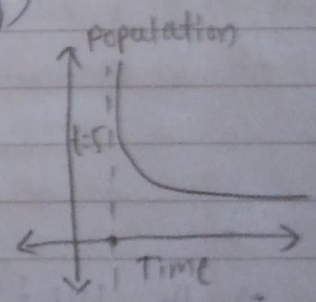
• Examples of Infinite Limits :-

1) Consider a town where the population P(t) of residents at time t is modeled by P(t) = 10t/t-5. What happens around t=5?

S⇒ $P(t) = \frac{10t}{t-5}$

→ $\lim_{t\to5} P(t) = \lim_{t\to5}\left(\frac{10t}{t-5}\right) = \lim_{t\to5}\left(\frac{10\cancel{t}}{\cancel{t}(1-5/t)}\right)$

$\lim_{t\to5} P(t) = \frac{10(5)}{5-5} = \frac{50}{0} = \boxed{\infty}$

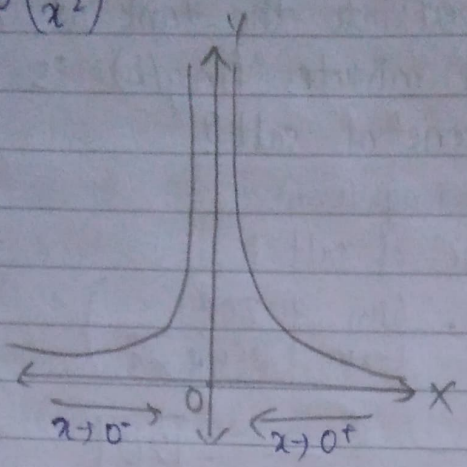


→ At t=5, population was not defined. But after t=5, the population is countable.

• Lesson Plan Questions :-

LP 1) Define limit of function & explain why graphically.
i) $\lim_{x\to0}\left(\frac{1}{x^2}\right)$ ii) $\lim_{x\to0}\left(\frac{|x|}{x}\right)$ does not exists.

S⇒ i) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)$.



→ As $x \to 0^+$ (from right), $1/x^2$ becomes larger & larger approaching ∞.

→ As $x \to 0^-$ (from left), $1/x^2$ also becomes larger & larger approaching ∞.

→ Graphically the function $1/x^2$ has vertical asymtote at $x=0$ i.e. $\boxed{\lim_{x\to 0}(1/x^2) = \infty}$

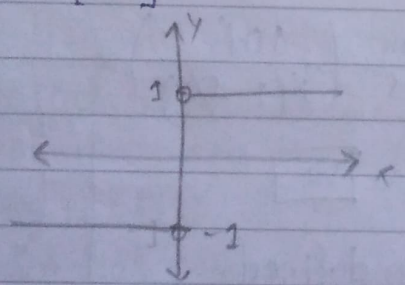
→ Though here LHL = RHL,

i.e. $\lim_{x\to 0^-}(1/x^2) = \infty = \lim_{x\to 0^+}(1/x^2)$

we say limit tends to infinity i.e. it doesn't exist. (i.e. limit is not defined).

→ But we require a finite limit then we say the limit does not exist.

ii) $\lim_{x\to 0}\left[\frac{|x|}{x}\right]$



→ The function $y = |x|/x$ behaves differently on either side of zero i.e. for $x>0$

$\lim_{x\to 0^+}\left(\frac{|x|}{x}\right) = 1$ (RHL)

for $x<0$,

$\lim_{x\to 0^-}\left(\frac{|x|}{x}\right) = -1$ (LHL)

→ As LHL ≠ RHL, we say limit <u>does not exist</u>.

LP 2) For the function f graphed in figure, find,

a) $\lim_{x\to 2^-} f(x) = 1$ b) $\lim_{x\to 2^+} f(x) = -2$ c) $\lim_{x\to 0^-} f(x) = 0$

d) $\lim_{x\to 0^+} f(x) = -\infty$ e) $\lim_{x\to 4^-} f(x) = ND$ f) $\lim_{x\to 4^+} f(x) = \infty$

LP 3) Determine $\lim_{x \to c^+} f(x)$ & $\lim_{x \to c^-} f(x)$. Does limit exists? If so, what is it? If not, explain.

i) $c = 2$, $f(x) = \begin{cases} 3-x, & x<2 \\ x/2 + 1, & x>2 \end{cases}$

ii) $c=2$, $f(x) = \begin{cases} 3-x, & x<2 \\ 2, & x=2 \\ x/2, & x>2 \end{cases}$

S→ i) LHL $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3-x) = 3-2 = 1$
$(x<2)$

$\boxed{LHL = 1}$

RHL $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{x}{2}+1\right) = \frac{2}{2}+1 = 2$
$(x>2)$

$\boxed{RHL = 2}$

→ Since LHL ≠ RHL, limit doesn't exist.

ii) LHL $= \lim_{x \to 2^-} (3-x) = 3-2 = 1$

$\boxed{LHL = 1}$

RHL $= \lim_{x \to 2^+} (x/2) = 2/2 = 1$

$\boxed{RHL = 1}$

→ Since LHL = RHL, limit exists. And it is 1.

LP 4) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of values of position of athlete versus time is given here, where x is the position in meters of the runner & t is time in seconds. What is $\lim_{x \to 2} x(t)$? What does it mean physically?

→ The table,

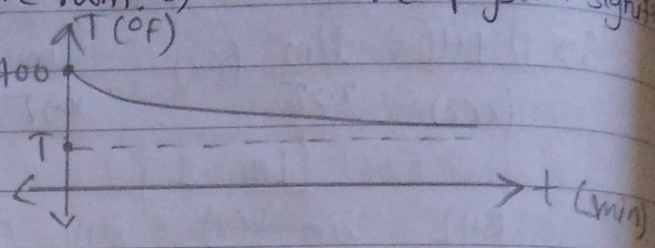
t (s)	1.75	1.95	1.99	2.01	2.05	2.25
x (m)	4.5	6.1	6.42	6.58	6.9	8.5

S⇒ Here 'x' represents distance covered by runner & 't' is the time.

i) as $t \to 2^-$ (from left) (1.99)
$x(t) = 6.42$ m
as $t \to 2^+$ (from right) (2.01)
$x(t) = 6.58$ m
∴ as $t \to 2$, $x(t) \to 6.50$ m.

ii) The limit represents the position of the runner at exactly 2 seconds which is 6.5 m approx.

LP 6) Let $T = f(t)$ denote the temperature of a baked potato 't' minutes after it has been removed from a hot oven. The accompanying figure shows the temperature versus time curve for potato, where 'r' is the temperature of the room. a) What is the physical significance of $\lim_{x \to 0^+} f(t)$?



b) What is the physical significance of $\lim_{x \to \infty} f(t)$?

S:) a) $\lim_{t \to 0^+} f(t)$ refers to the limiting value of temperature $f(t)$ as $t \to 0^+$ (from right).

→ From graph we have $\lim_{t \to 0^+} f(t) = 400°F$.

→ This limit signifies initial temperature of baked potato at the moment taken out of oven which is at 400°F.

b) From the graph $\lim_{t \to +\infty} f(t) = r$.

→ This limit represents the long-term behaviour of temperature of potato as time increases. Physically it means no matter how high the initial temperature of potato be, it will cool down to the room temperature which is 'r'.

LP 7) Find the horizontal & vertical asymptotes of each curve.

a) $y = \dfrac{x^2+1}{2x^2-3x-2}$

→ $2x^2 - 3x - 2 = (x-2)(2x+1)$

$\Rightarrow \boxed{x = 2}$ & $\boxed{x = -1/2}$

$-4x^2$

$-4x \quad x$

$-3x$

i) Vertical Asymptote,

As $\lim_{\substack{x \to 2 \\ x \to -1/2}} \left[\dfrac{x^2+1}{2x^2-3x-2} \right] = \infty$

$\therefore x = 2$ & $x = -1/2$ are the vertical asymptotes.

ii) Horizontal Asymptote,

$$\lim_{x\to\infty}\left[\frac{x^2+1}{2x^2-3x-2}\right] = \lim_{x\to\infty}\frac{\cancel{x^2}\,[1+1/x^2]}{\cancel{x^2}\,[2-3/x-2/x^2]} = \frac{1+0}{2-0} = \boxed{\frac{1}{2}}$$

$\therefore$ $y = 1/2$ is the horizontal asymptote.

b) $y = \dfrac{x^3-x}{x^2-6x+5}$

→ i) Vertical Asymptote,

$x^2-6x+5 = x^2-5x-x+5$

$= x(x-5)-1(x-5)$

$\Rightarrow (x-1)(x-5) \Rightarrow \boxed{x=1}$ & $\boxed{x=5}$

$+5x^2$

$-5x \quad -x$

$-6x$

$\therefore \lim_{x\to 5}\left[\dfrac{x^3-x}{x^2-6x+5}\right] = \infty$

If we take 'x' as 1, it will be zero/zero.

$\therefore$ $x=5$ is the vertical asymptote.

ii) Horizontal Asymptote,

$$\lim_{x\to\infty}\left[\frac{x^3-x}{x^2-6x+5}\right] = \lim_{x\to\infty}\left[\frac{x^3(1-1/x^2)}{x^2(1-6/x+5/x^2)}\right] = \infty$$

$\therefore$ It has no horizontal asymptotes.

c) $\dfrac{2x}{x-3} = y$.

→ i) Vertical Asymptote,

$x-3=0 \Rightarrow \boxed{x=3}$

$\therefore$ As $\lim_{x\to 3}\dfrac{2x}{x-3} = \infty$

$\therefore$ $x=3$ is the vertical asymptote.

ii) Horizontal Asymptote,

$$\lim_{x\to\infty}\frac{2x}{x-3} = \lim_{x\to\infty}\frac{\cancel{x}(2)}{\cancel{x}(1-3/x)} = \frac{2}{1-0} = \boxed{2}$$

$\therefore$ $y=2$ is the horizontal asymptote.

LP 8) i) $\lim_{x\to 10} f(x) = 0$ True.

→ From graph, as $x\to 10^-$, $f(x)=0=$ LHL.

as $x\to 10^+$, $f(x)=0=$ RHL

$\therefore$ RHL = LHL, limit exists.

ii) $\lim_{x \to -2^+} f(x) = 3$. False.

→ as $x \to -2$ (from right),
$f(x) \to \infty$ $\therefore \lim_{x \to -2^+} f(x) = \infty$.

iii) $\lim_{x \to -8^-} f(x) = f(-8)$. False.

→ As $x \to -8$ (from left).
$f(x) \to -6$
& $f(-8) = -3$ $\therefore \lim_{x \to -8^-} f(x) \neq f(-8)$.

iv) $\lim_{x \to 6^-} f(x) = 5$. False

→ As $x \to 6$ (from left)
$f(x) \to 2$ $\therefore \lim_{x \to 6^-} f(x) \neq 5$

LP 5) i) $\lim_{x \to 1} \frac{x^3+3x^2+5}{4-7x} = \frac{1+3+5}{4-7} = \frac{9}{-3} = \boxed{-3}$

ii) $\lim_{x \to -2} \sqrt{x^2-6x+3} = \sqrt{4+12+3} = \boxed{\sqrt{19}}$

iii) $\lim_{x \to 0} \frac{1}{1+\sin x} = \frac{1}{1+\sin 0} = \frac{1}{1+0} = \boxed{1}$

iv) $\lim_{x \to 2} e^{2x-x^2} = e^{4-4} = e^0 = \boxed{1}$

HW] i) $y = \frac{2x^2}{1+x^2}$ → VA: Make denominator zero.
→ HA: Take common the largest powered term.

→ VA: Take $1+x^2 = 0 \Rightarrow x^2 = -1$ → Not possible.
∴ vertical Asymptote donot exist.

→ HA: $\lim_{x \to \infty} \left[\frac{2x^2}{1+x^2}\right] = \lim_{x \to \infty} \left[\frac{x^2 2}{x^2(1+1/x^2)}\right] = \frac{2}{1+0} = \boxed{2}$

∴ $y=2$ is the horizontal asymptote.

ii) $y = \dfrac{x^2+4}{x^2-1}$.

→ VA: $x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow \boxed{x = \pm 1}$

$\therefore \lim\limits_{\substack{x \to 1 \\ x \to -1}} \left[\dfrac{x^2+4}{x^2-1}\right] = \infty$ $\therefore x = 1$ & $x = -1$ are the vertical asymptotes.

→ HA: $\lim\limits_{x \to \infty} \left[\dfrac{\cancel{x^2}\,[1 + 4/x^2]}{\cancel{x^2}\,[1 - 1/x^2]}\right] = \dfrac{1+0}{1-0} = \boxed{1}$

$\therefore y = 1$ is the horizontal asymptote.

iii) $y = \dfrac{x}{x+4}$.

→ VA: $x + 4 = 0 \Rightarrow \boxed{x = -4}$

$\therefore \lim\limits_{x \to -4} \left[\dfrac{x}{x+4}\right] = \infty$ $\therefore x = -4$ is the vertical asymptote.

→ HA: $\lim\limits_{x \to \infty} \left[\dfrac{x}{x+4}\right] = \lim\limits_{x \to \infty} \left[\dfrac{\cancel{x}}{\cancel{x}[1 + 4/x]}\right] = \dfrac{1}{1+0} = \boxed{1}$

$\therefore y = 1$ is the horizontal asymptote.

iv) $y = \dfrac{x^2-1}{x^2+1}$

→ VA: $x^2 + 1 = 0 \Rightarrow x^2 = -1$ → Not Possible.

$\therefore$ Vertical Asymptote does not exist.

→ HA: $\lim\limits_{x \to \infty} \left[\dfrac{x^2-1}{x^2+1}\right] = \lim\limits_{x \to \infty} \left[\dfrac{\cancel{x^2}\,[1 - 1/x^2]}{\cancel{x^2}\,[1 + 1/x^2]}\right] = \dfrac{1-0}{1+0} = \boxed{1}$

$\therefore y = 1$ is the horizontal asymptote.

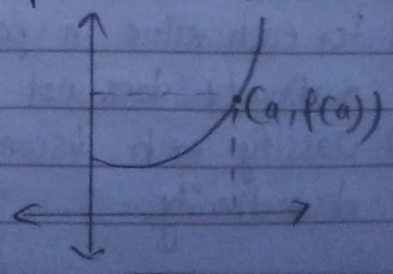
* <u>Continuity</u> :-

→ A function 'f' is continuous at a point 'a' if,

i) Limit exists at 'a'. (LHL = RHL).

ii) $f(a)$ is defined.

iii) $\lim\limits_{x \to a} f(x) = f(a)$. (Smooth curve).

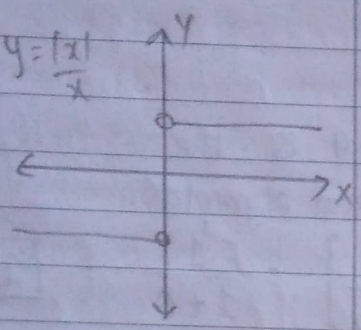
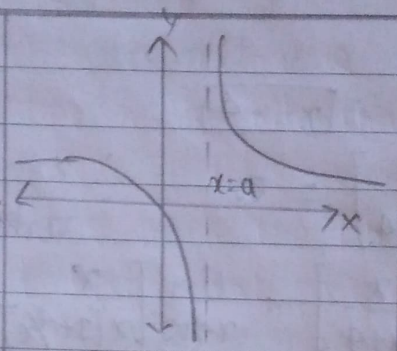
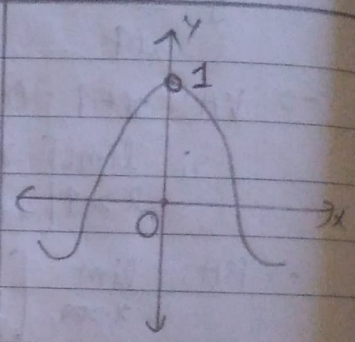


Note: i) If 'f' is not continuous at a point then it is said to be discontinuous at that point.

ii) If 'f' is continuous on an interval then we say it is continuous at every point of interval.

* Discontinuity :-

→ A function 'f' is discontinuous at a number 'a' if 'f' is not continuous at 'a'.

		
Jump Discontinuity (Sudden change) ↓ A jump in the graph where the function has sudden change in value. LHL ≠ RHL	Infinite Discontinuity (Vertical Asymptote) ↓ The function approaches infinity as x approaches a certain value. (vertical asymptote) $y=\infty$	Removable Discontinuity (Redefinable function) ↓ A hole in the graph where the function could be redefined to make it continuous. LHL = RHL ≠ f(a)

LP 13) Consider the graph of the function $y=f(x)$ shown in the following graph,

i) Find all values of x for which the function is discontinuous.

ii) For each value in part a, state why the formal definition of continuity does not apply.

iii) Classify each discontinuity as either jump, removable, or infinite discontinuity.

S⇒ From the graph we observe the function is discontinuous at $x=-1$, $x=0$, $x=1$.

→ At $x=-1$, limit does not exist, it is a jump (LHL ≠ RHL).

→ At $x=0$, limit does not exist & $f(0)$ is not defined.
∴ It is both jump & infinite disontinuity.

→ At $x=1$, limit exists i.e. LHL = RHL but not equal to $f(1)$.
∴ It is removable discontinuity.

LP 9) Define continuity. For what value of the constant is function
$$f(x) = \begin{cases} x^2-c^2, & x<4 \\ cx+20, & x \geq 4 \end{cases}$$
continuous on $(-\infty, \infty)$.

S⇒ $LHL = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x^2-c^2) = 16-c^2$.

$RHL = \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (cx+20) = 4c+20$.

$f(4) = cx+20 = 4c+20$.

→ For continuity, LHL = RHL = $f(4)$

$$16-c^2 = 4c+20$$
$$c^2+4c = -4$$
$$c^2+4c+4=0$$
$$(c+2)^2=0$$
$$\boxed{c=-2}$$

LP 11) A lab technician controls the temperature T inside a kiln. From an initial temperature of 0°C, he allows the kiln to increase by 2°C per minute for the next 60 minutes. After the 60th minute, he allows the kiln to cool at the rate of 3°C per minute. The temperature function T is, defined by,
$$T(t) = \begin{cases} 2t, & t \leq 60. \\ k-3t, & t > 60. \end{cases}$$

a) Find k such that T is continuous at t=60min.

b) Explain why T must be continuous at t=60min.

S⇒ $T(t) = \begin{cases} 2t, & t \le 60 \\ k-3t, & t > 60 \end{cases}$

→ To ensure $T(t)$ is continuous at $t=60$, LHL = RHL = $f(60)$.

$$LHL = \lim_{x \to 60^-} T(t) = \lim_{x \to 60^-} 2t = 2(60) = 120.$$

$$RHL = \lim_{x \to 60^+} T(t) = \lim_{x \to 60^+} k-3t = k-3(60) = k-180.$$

LHL = RHL

$120 = k - 180 \Rightarrow \boxed{k = 300}$

→ This means the function is continuous when $k=300$.

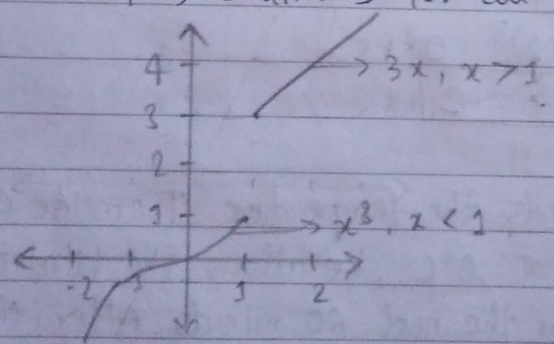
→ Here $T(t)$ i.e. temperature inside kiln should change smoothly without any sudden jumps. If there is a discontinuity at 60, it means the temperature inside the kiln jumps at different values which means it doesn't work properly.

LP 10) Let $f(x) = \begin{cases} 3x, & x > 1 \\ x^3, & x < 1 \end{cases}$

i) Sketch the graph of f.

ii) Is it possible to find a value k such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Breifly explain.

S⇒ i)



ii) $$LHL = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = 1^3 = \boxed{1}$$

$$RHL = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x = 3(1) = \boxed{3}$$

→ Since LHL ≠ RHL, it is not possible to find a value k such that $f(1) = k$. ∴ It is not possible to make the function continuous.

Tangent → Touches 1 Point
Secant → Touches 2 Points.

Derivative → It is always for a function "AT A POINT".

LP 12) Rick's Rocks sells decorative landscape rocks in bulk quantities. For quantities up to & including 500lb, he charges 2.5 per pound. For quantities above 500lb, he charges 2 per pound. The price function can be stated as a piecewise function,

$$p(x) = \begin{cases} 2.5x, & 0<x\leq 500. \\ 2x, & x>500. \end{cases}$$

→ Where p is the price in rupees & x is the quantity in pounds. Is the price function continuous at x=500 why or why not?

S⇒ $p(x) = \begin{cases} 2.5x, & 0<x\leq 500. \\ 2x, & x>500. \end{cases}$

→ To check p(x) is continuous at x=500, it should have LHL = RHL = p(x).

→ $LHL = \lim_{x\to 500^-} f(x) = \lim_{x\to 500^-} (2.5x) = 2.5(500) = 1250.$

→ $RHL = \lim_{x\to 500^+} f(x) = \lim_{x\to 500^+} (2x) = 2(500) = 1000.$

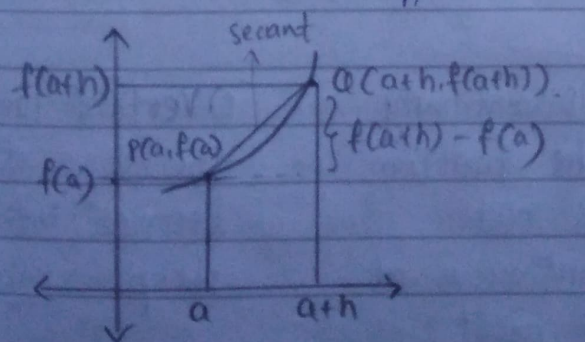
→ At x=500, p(500) = 2.5(500) = 1250.

→ As LHL ≠ RHL, the function is discontinuous at x=500.

* **Derivatives and Rate of Change :-**

→ Let y=f(x) be a function which changes from 'a' to 'a+h', then, average rate of change of 'f' over the interval a to a+h is equal to $\frac{f(a+h)-f(a)}{h}$.

→



h → very small

→ But if we want to consider the instantaneous rate of change of a function at a point, then we apply limit by taking h→0.

A B
1 = f(A1)
2
3
} GeoGebra

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

for the average rate of change.

⇒ **Derivatives :-**

→ The derivative of 'f' at 'a' written as $f'(a)$ is defined as,

Rate of change of 'f' at 'a' $= f'(a) = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right]$.

Note: i) Average rate of change = Slope of secant PQ.

ii) Instantaneous rate of change = slope of tangent at 'P'.

iii) Interpretation of derivative at a point,

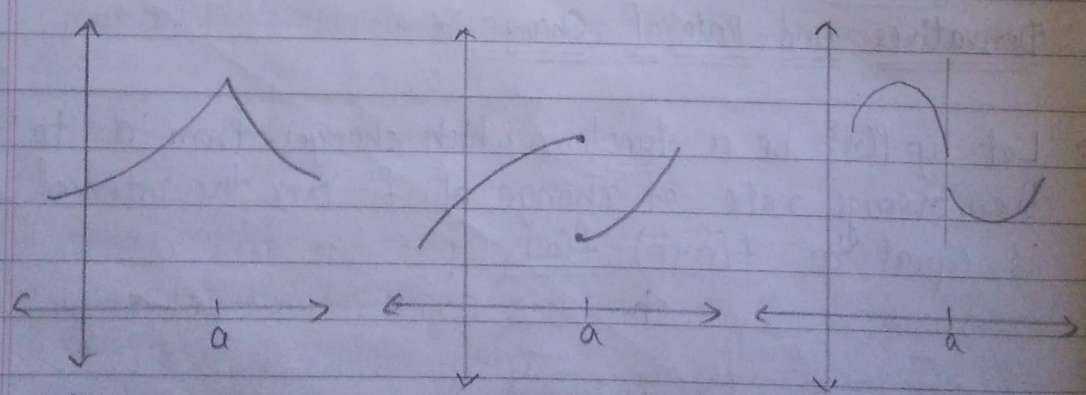
a) $f'(a)$ is the instantaneous rate of change of 'y' wrt 'x' at the point 'a'.

b) $f'(a)$ is the slope of the curve at a point $x = a$, i.e slope of the tangent line to the curve.

- **Condition for function to be differentiable :-**

→ A function is said to be differentiable if its derivative $\lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right]$ exists.

- **When a function fails to be differentiable :-**



a) Sharp corner or cusp.
The function has a sudden change in direction.

b) Discontinuity
The function is not continuous at a point.

c) Vertical Tangent.
The slope of function becomes infinitely steep.

LP 15) Consider an athlete running a 40m dash. The position is given by $d(t) = (t^3/6) + 4t$, where 'd' is the position in meters & 't' is time elapsed, measured in seconds. i) Compute the average velocity of the runner over the given time intervals. a) [1.95, 2.05], b) [1.995, 2.005], c) [1.9995, 2.0005], d) [2, 2.00001]. ii) Compute the instantaneous velocity of the runner at $t = 2$sec.

S⇒ i) a) [1.95, 2.05]

$$\text{Avg. velocity} = \frac{d(2.05) - d(1.95)}{2.05 - 1.95} = \boxed{6.00041667} \text{ m/s}$$

b) [1.995, 2.005]

$$\text{Avg. velocity} = \frac{d(2.005) - d(1.995)}{2.005 - 1.995} = \boxed{6.000004167} \text{ m/s}$$

c) [1.9995, 2.0005]

$$\text{Avg. velocity} = \frac{d(2.0005) - d(1.9995)}{2.0005 - 1.9995} = \boxed{6.000000042} \text{ m/s}$$

d) [2, 2.00001]

$$\text{Avg. velocity} = \frac{d(2.00001) - d(2)}{2.00001 - 2} = \boxed{6.00001} \text{ m/s.}$$

ii) Instantaneous velocity at $t = 2$s.

M1] $d(t) = \frac{t^3}{6} + 4t$

M2] $\lim_{t \to 2} \frac{d(t) - d(2)}{t - 2}$.

Differentiate wrt t,

$$d'(t) = \frac{3t^2}{6} + 4 = \frac{t^2}{2} + 4$$

At $t = 2$, $d'(2) = 4/2 + 4 = 2 + 4 = \boxed{6 \text{ m/s}}$

LP 16) The table shows the estimated percentage of the population of Europe that use cell phones.

a) Find the average rate of cell phone growth from 2000 to 2002, from 2000 to 2001 & from 1999 to 2000.

b) Estimate the instantaneous rate of growth in 2000 by taking the average of two average rates of change. What are its units?

c) Estimate the instantaneous rate of growth in 2000 using slope of the tangent.

S⇒

Year	1998	1999	2000	2001	2002	2003
P	28	39	55	68	77	83

a) i) [2000, 2002]

→ Avg. Rate $= \frac{77-55}{2002-2000} = \frac{22}{2} = \boxed{11}$ $\left(\frac{P(2002)-P(2000)}{2}\right)$

ii) [2000, 2001]

→ Avg. Rate $= \frac{68-55}{2001-2000} = \boxed{13}$

iii) [1999, 2000]

→ Avg. Rate $= \frac{55-39}{2000-1999} = \boxed{16}$

b) To find the instantaneous rate in 2000, we take the average of ii) & iii) i.e, $16+13/2 = 29/2 = \boxed{14.5}$

c) To find the instantaneous rate of growth in 2000, we use the slope of tangent method i.e.

Slope of Tangent $= \frac{y_2-y_1}{x_2-x_1} = \frac{68-39}{2001-1999} = \frac{29}{2}$

$(x_1, y_1) = (1999, 39)$

$(x_2, y_2) = (2001, 68)$

Slope of Tangent $= \boxed{14.5}$

∴ The instantaneous growth rate in 2000 is 14.5 % per year.

LP 14) Use the following graph to find average rate of change in energy consumption from 1970 to 1980, 1980 to 1990, 2000 to 2009.

S⇒ From 1970 to 1980,

$\frac{78.29 - 67.84}{1980 - 1970} = \frac{10.45}{10} = \boxed{1.045 \text{ quadrillion of BTU/year}}$

→ From 1980 to 1990,

$\frac{84.67 - 78.29}{1990 - 1980} = \frac{6.38}{10} = \boxed{0.638}$

→ From 2000 to 2009,

$\frac{95.60 - 98.90}{2009 - 2000} = \frac{-3.3}{9} = \boxed{-0.366}$

Ex: The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where 't' is in seconds. a) Find the average velocity over i) [3,4] ii) [3.5,4], iii) [4,5], iv) [4,4.5] b) Find the instantaneous velocity when $t = 4s$ c) Draw the graph of 's' as a function of 't' & draw the secant lines whose slopes are the average velocities in part a) & the tangent line whose slope is the instantaneous velocity in part b).

S: $s = t^2 - 8t + 18$

a) i) [3,4]

$$\text{Avg. velocity} = \frac{s(4) - s(3)}{4 - 3} = \frac{16 - 32 + 18 - 9 + 24 - 18}{1} = \frac{-1}{1} = \boxed{-1 \text{ m/s}}$$

ii) [3.5,4]

$$\text{Avg. velocity} = \frac{s(4) - s(3.5)}{4 - 3.5} = \frac{16 - 32 + 18 - 12.25 + 28 - 18}{0.5} = \frac{-0.25}{0.5} = \boxed{-0.5 \text{ m/s}}$$

iii) [4,5]

$$\text{Avg. velocity} = \frac{s(5) - s(4)}{5 - 4} = \frac{25 - 40 + 18 - 16 + 32 - 18}{1} = \frac{1}{1} = \boxed{1 \text{ m/s}}$$

iv) [4,4.5]

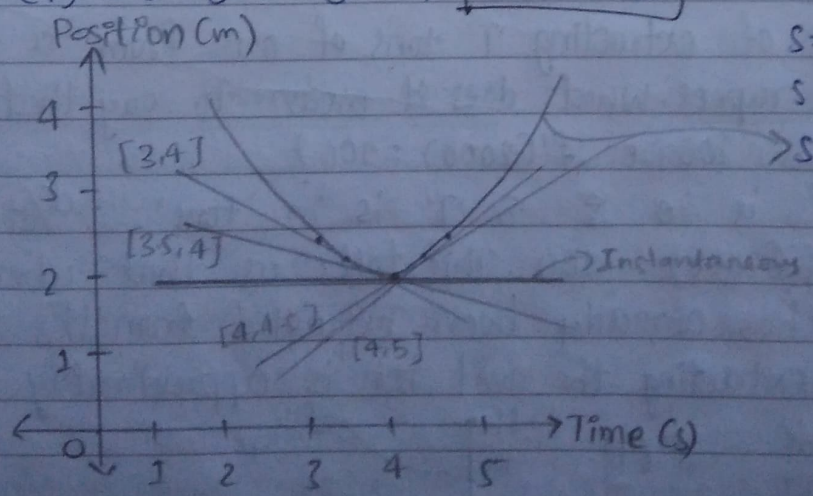
$$\text{Avg. velocity} = \frac{s(4.5) - s(4)}{4.5 - 4} = \frac{20.25 - 36 + 18 - 16 + 32 - 18}{0.5} = \frac{0.25}{0.5} = \boxed{0.5 \text{ m/s}}$$

b) Instantaneous Velocity at 4s,

$s'(t) = 2t - 8$

$s'(4) = 8 - 8 = 0 \Rightarrow \boxed{IV = 0 \text{ m/s}}$

c)



$s = t^2 - 8t + 18$

$s = t^2 - 8t + 16 + 2$

$\Rightarrow s = (t-4)^2 + 2$

Ex: The population P of Belgium is shown in table.

Year	1992	1994	1996	1998	2000
P	10036	10109	10152	10175	10186

a) Find avg. rate of growth.
b) Estimate instantaneous growth in 1996. c) Using slope of tangent

S ⇒ a) i) 1992 to 1996.

$$\text{Avg.} = \frac{10152 - 10036}{4} = \boxed{29}$$

ii) 1994 to 1996.

$$\text{Avg.} = \frac{10152 - 10109}{2} = \boxed{21.5}$$

iii) 1996 to 1998.

$$\text{Avg.} = \frac{10175 - 10152}{2} = \boxed{11.5}$$

b) Instantaneous Rate $= \frac{21.5 + 11.5}{2} = \boxed{16.5}$

c) $(x_1, y_1) = (1994, 10109)$
$(x_2, y_2) = (1998, 10175)$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10175 - 10109}{1998 - 1994} = \boxed{16.5}$$

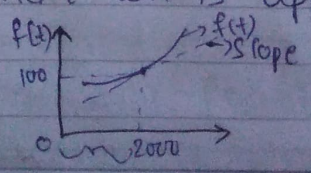
* Problems on Interpretation of Derivatives :-

Ex 1) The cost of extracting 'T' tons of ore from a mine is $C = f(T)$ rupees. What does it mean to say that $f'(x)$ at $x = 2000$ is 100 i.e. $f'(2000) = 100$?

S ⇒ Here 'C' is in '₹' & 'T' is in 'tons' ∴ dC/dT is in ₹/tons.

→ Given $f'(2000) = 100$. This tells us that when 2000 tons of ore has already been extracted from the mine. The cost of extracting the next ton is approximately ₹100.

→ Slope $= \frac{dC}{dT} = f'(T)$.



Ex 2) Suppose $P = f(t)$ is a population of a city in millions where 't' is the number of years since 1980. Explain meaning of the statements i) $f'(6) = 2$ & ii) $f^{-1}(95.5) = 16$.

S⇒ Here time 't' is measured in years & 'P' is in millions.
∴ dP/dt is in millions per year.

i) $f'(6) = 2$. It tells us that at $t=6$ i.e. 1986, the population of city was increasing at 2 millions per year.

ii) $f^{-1}(95.5) = 16$. It tells us that in 1996, the population of city was 95.5 million.

LP 17) A manufacturer produces bolts of fabric. The cost of producing 'x' yards of this fabric is $C = f(x)$ rupees. a) What is the meaning of derivative? What are its units? b) In practical terms, what does it mean to say that? c) Which do you think is greater $f'(50)$ or $f'(500)$? What about $f'(5000)$?

S⇒ a) x is yards of fabrics & $f(x)$ is cost in rupees.
∴ dC/dx is in rupees per yards.

b) $f'(1000) = 9$, it means that, after the production of 1000 yards of fabrics, the rate at which the production cost is increased is ₹9 per yard. i.e. The cost of producing the 1000th yard is about ₹9.

c) $f'(50) > f'(500)$ ∵ starting is time-consuming & more difficult.
$f'(5000) > f'(50) > f'(500)$ very large ⇒ very difficult.

---

* <u>Sketching a Derivative Curve</u> :-

⇒ <u>Procedure</u> :-

→ i) For a given curve, $y = f(x)$, first find pts where the tangent is parallel to x-axis (slope = 0). These are the pts at which the curve of $f'(x)$ cuts x-axis. We denote them as $a_1, a_2, \ldots$

ii) Next we discuss the nature of the slope of the tangent in the intervals formed by $a_1, a_2, \ldots$

iii) If slope in any of the intervals is positive, it means that the curve of $f'(x)$ lies above X-axis.

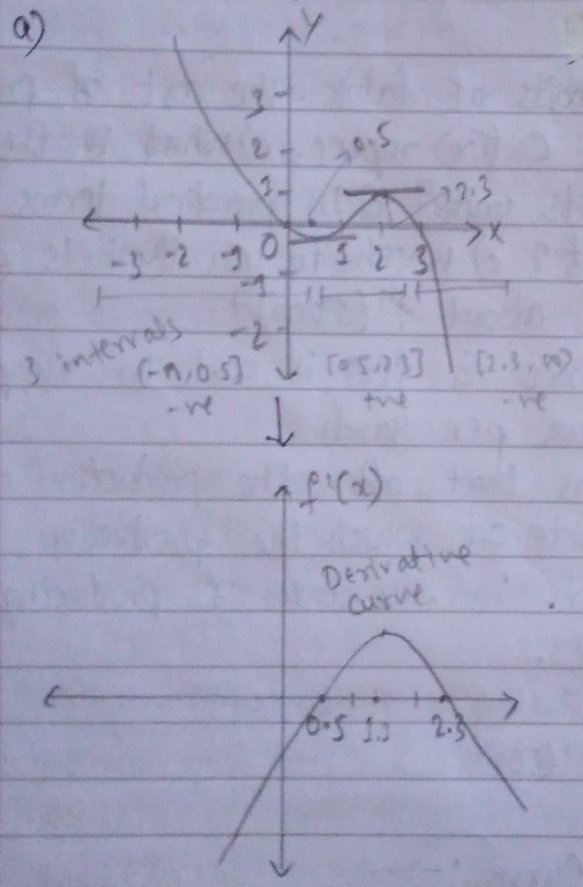
iv) If the slope is negative in any intervals then the curve of $f'(x)$ lies below X-axis.

v) The point where the tangent flips on the given curve gives it is the point of x' of which $f'(x)$ has max. or min value

V. Imp

Ex: Use the graph of $y=f(x)$ to sketch the graph of its derivative $f'(x)$.

a)

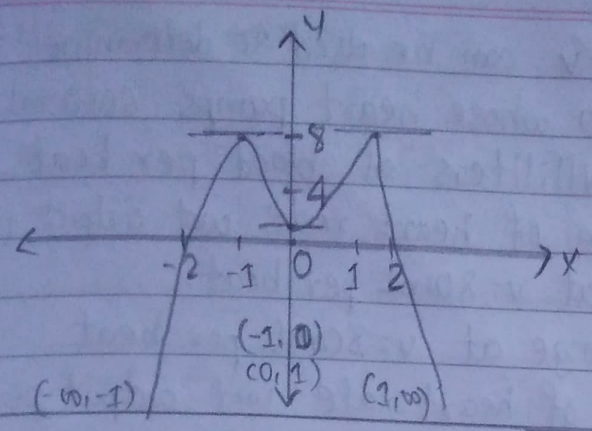


→ In the given graph tangent is parallel to X-axis at $x=0.5$ & $x=2.3$. It means that graph of $f'(x)$ cuts X-axis at these points.

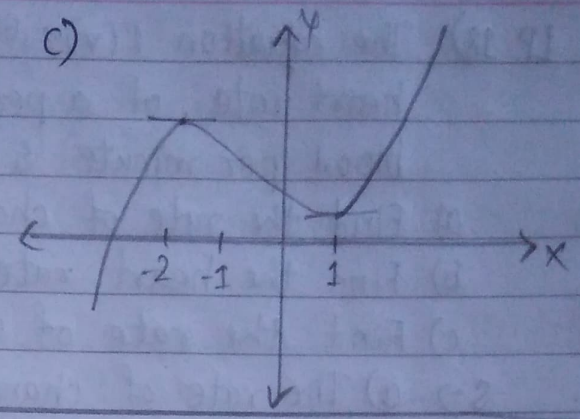
→ We divide the interval in 3 parts: $(-\infty, 0.5]$, $[0.5, 2.3]$, $[2.3, \infty)$

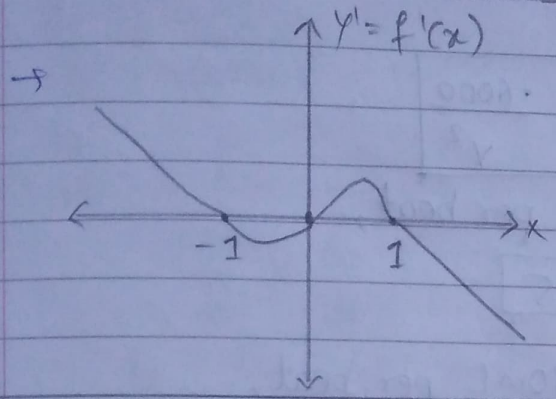
→ We observe that in $(-\infty, 0.5)$, the slope of tangent to the given curve is negative meaning that the graph of $f'(x)$ is below X-axis. In $(0.5, 2.3)$, the slope is positive, $f'(x)$ is above X-axis. In $[2.3, \infty)$, the slope is -ve, $f'(x)$ is below

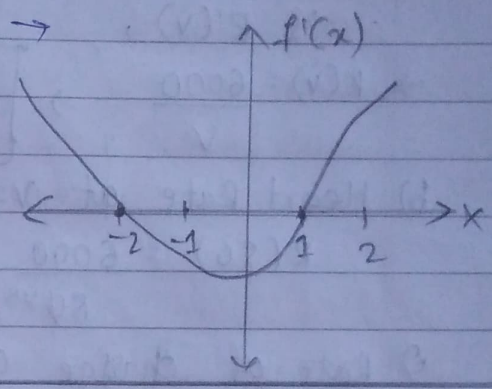
b)



c)







b) In the given graph tangent is parallel to x-axis at $x=-1$, $x=0$ & $x=1$. It means that $f'(x)$ cuts x axis at these points. The intervals will be $(-\infty,-1)$, $(-1,0)$, $(0,1)$ & $(1,\infty)$.

→ In $(-\infty,-1)$, the slope of tangent is positive which means the graph of $f'(x)$ is above x-axis. Similarly in $(-1,0)$ it is -ve, so graph is below x-axis, in $(0,1)$ it is +ve so the graph is above x-axis & in $(1,\infty)$ it is -ve, the graph of $f'(x)$ is below x-axis.

c) In the given graph, tangent is parallel to x-axis at $x=-2$ & $x=1$, so $f'(x)$ will be zero at these points i.e. the graph cuts x-axis here.

→ The intervals will be $(-\infty,-2)$, $(-2,1)$ & $(1,\infty)$.

→ In $(-\infty,-2)$, the tangent is positive which means the graph of $f'(x)$ is above x-axis.

→ In $(-2,1)$, the tangent is negative so the graph is below x-axis

→ In $(1,\infty)$, the tangent is positive so the graph is above the x-axis again.

$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ $\quad (uv)' = uv' + vu'$

LP 18) The equation $R(v) = 6000/v$ can be used to determine the heart rate, of a person whose heart pumps 6000 mL of blood per minute & milliliters of blood per beat.

a) Find the rate of change of heart rate wrt output per beat.

b) Find the heart rate at $v = 80$ mL per beat.

c) Find the rate of change at $v = 80$ mL per beat.

S⇒ a) The rate of change of heart rate wrt output per beat is $R'(v)$,

→ $R(v) = \frac{6000}{v}$ ⇒ $\boxed{R'(v) = \frac{-6000}{v^2}}$

b) Heart Rate at $v = 80$ mL per beat,

$R(80) = \frac{6000}{80} = \boxed{75}$

c) Rate of change at $v = 80$ mL per beat,

$R'(80) = \frac{-6000}{80 \times 80} = \frac{-60}{64} = \boxed{-0.93}$

LP 20) Differentiate the following.

i) $y = (2x+1)^2$

→ $y' = 2(2x+1)\,2 = 4(2x+1) = \boxed{8x+4}$

ii) $y = (x-4)^8 (2x+3)$

→ $y' = (x-4)^8 (2) + (2x+3)\,8\,(x-4)^7$

$\boxed{y' = 2(x-4)^8 + (16x+24)(x-4)^7}$

iii) $y = \frac{1}{(3x+8)^2}$

→ $y' = -2(3x+8)^{-3}(3) = \boxed{\frac{-6}{(3x+8)^3}}$

iv) $y = \frac{4x^2}{(7-5x)^3}$

→ $y' = \frac{(7-5x)^3(8x) - (4x^2)\,3(7-5x)^2(-5)}{(7-5x)^6}$

$y' = \frac{(56x - 40x^2) + 60x^2}{(7-5x)^4} = \boxed{\frac{56x + 20x^2}{(7-5x)^4}}$

v) $y = \sqrt[3]{2x-1} + (4-x)^2$.

→ $y = (2x-1)^{2/3} + (4-x)^2$

$y' = \frac{1}{3}(2x-1)^{-2/3}(2) + 2(4-x)(-1)$

$$\boxed{y' = \frac{2}{3(2x-1)^{2/3}} - 8 - 2x}$$

LP 21) i) Find dy/dt if $y = 1/u^2+u$ & $u = 5+3t$.

→ $\frac{dy}{du} = \frac{-1}{(u^2+u)^2}(2u+1) = \frac{-2u-1}{(u^2+u)^2}$

→ $\frac{du}{dt} = 3$

→ $\frac{dy}{du} \times \frac{du}{dt} = \frac{3(-2u-1)}{(u^2+u)^2}$

$\frac{dy}{dt} = \frac{-3[2(5+3t)+1]}{[(5+3t)^2+(5+3t)]^2} = \frac{-3[6t+11]}{[25+9t^2+30t+3t+5]^2}$

$$\boxed{\frac{dy}{dt} = \frac{-3[6t+11]}{[9t^2+33t+30]^2}}$$

ii) Find dy/dx if $y = \sqrt[3]{u}$ & $u = x^3-x/1+x$.

→ $\frac{dy}{du} = \frac{1}{3}u^{-2/3} = \frac{1}{3u^{2/3}}$

→ $\frac{dy}{dx} = \frac{(1+x)(3x^2-1)-(x^3-x)}{(1+x)^2} = \frac{3x^2+3x^3-1-x-x^3+x}{(1+x)^2}$

or $u = \frac{x^3-x}{1+x} = \frac{x(x^2-1)}{1+x} = \frac{x(x-1)\cancel{(x+1)}}{\cancel{x+1}}$

$\frac{du}{dx} = x + x - 1 = 2x-1$.

→ $\frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{dx} = \frac{2x-1}{3u^{2/3}}$

$$\boxed{\frac{dy}{dx} = \frac{2x-1}{3[x(x-1)]^{2/3}}}$$

iii) Find dy/dx if $x^2+y^2=25$ at $(3,4)$.

S⇒ $x^2+y^2=25$

Differentiate wrt x,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\cancel{2}y\frac{dy}{dx} = -\cancel{2}x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

At $(3,4)$ $\boxed{\frac{dy}{dx} = \frac{-3}{4}}$

iv) Find dy/dx if $x^3+y^3=6xy$.

S⇒ $x^3+y^3=6xy$

$$3x^2+3y^2\frac{dy}{dx} = 6\left[x\frac{dy}{dx}+y\right]$$

$$\frac{dy}{dx}\left[3y^2-6x\right] = 6y-3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}}$$

v) Find dy/dx if $y=\sec(4x^5+2x)$.

S⇒ $y=\sec(4x^5+2x)$

$y=\sec f(x)$ where $f(x)=4x^5+2x$

→ Differentiate wrt x,

$$\frac{dy}{dx} = \sec f(x)\tan f(x)\cdot f'(x) \quad - (1)$$

→ $f(x)=4x^5+2$, $f'(x)=20x^4+2$ - (2)

$$\boxed{\frac{dy}{dx} = \sec[4x^5+2x]\tan[4x^5+2x][20x^4+2]}$$

* * * * *

CHAPTER 03

APPLICATIONS OF DIFFERENTIATION

* Related Rates :-

→ In a related rates problem, the idea is to compute the rate of change of one quantity in terms of rate of change of other quantity.

→ In this method, we usually use chain rule to evaluate.

Ex 1) A spherical balloon is filled with air at the constant rate of $2 cm^3/s$. How fast is the radius increasing when radius is 3cm?

S⇒ Let the volume of the sphere be $V = \frac{4}{3}\pi r^3$.

→ Since volume & radius are changing quantities, they are functions of time.

$$V(t) = \frac{4}{3}\pi (r(t))^3$$

Differentiate w.r.t. 't',

$$V'(t) = \frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

Given

$$\frac{dV}{dt} = 2$$

$$r = 3$$

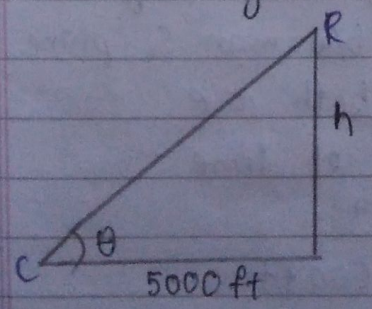
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$2 = 4\pi(9)\frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{18\pi} \text{ cm/s}}$$

Ex 2) A rocket is launched so that it rises vertically. A camera is positioned 5000ft from the launch pad. When the rocket is 1000ft above, its velocity is 600 ft/s. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused.

S⇒



→ Given: Let h = Height of rocket above ground

θ → Angle of camera.

→ Given :-

Distance b/w camera & point at which rocket is launched = 5000

Velocity when $h = 1000ft$ → $600 ft/s$.

→ To find :-

$$\frac{d\theta}{dt}$$

→ θ & h are the functions of time.

→ From the Δ^{le},

$$\tan\theta = \frac{opp}{adj} = \frac{h}{5000}$$

Differentiate w.r.t. t,

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{5000}\frac{dh}{dt} \qquad \left(\sec\theta = \frac{hyp}{adj} = \frac{5039}{5000} = 1.039\right)$$

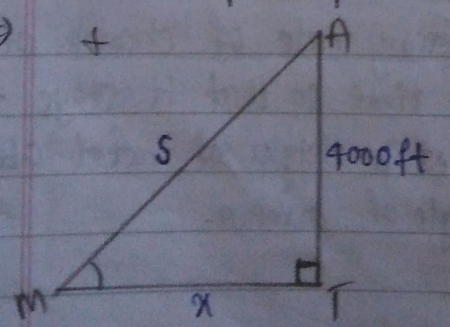
$$1.039 \frac{d\theta}{dt} = \frac{600}{5000} \quad (\text{at } h = 1000ft)$$

$$\boxed{\frac{d\theta}{dt} = 0.11 \text{ rad/s}} = 6.13 \text{ degrees/seconds.}$$

LP 1) An Airplane Flying at a Constant Elevation an airplane is flying overhead at a constant elevation of 4000ft. A man is viewing the plane from a position 3000ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man & the plane increasing when the plane passes over the radio tower?

→ At page 85

S⇒



x → Distance between man & position on the ground directly be airplane.

S → Distance b/w man & plane.

→ x & S both are functions of time.

→ $dx/dt = 0$

→ $ds/dt = 600 ft/s$.

$s^2 = x^2 + (4000)^2$

Differentiate wrt t,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + (4000)^2$$

$x = 3000\,ft$

$s = \sqrt{3000^2 + 4000^2}$

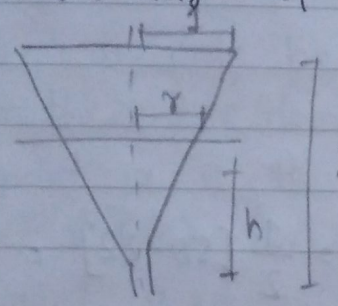
$s = 5000.$

$2(5000)(600) = 2 \times 3000 \times dx/dt + 16000000$

$8000000 = 600\, dx/dt + 16000000$ ×

LP 2) Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ ft^3/sec$. The height of the funnel is 2ft & the radius at the top is 1ft. At what rate is the height of water changing when the height of water is $1/2$ ft?

S⇒



Given: $\frac{dv}{dt} = -0.03\ ft^3/s$ (Decreasing).

$r = 1\,ft$, $h = 2\,ft$.

To find: $\frac{dh}{dt} = ?$ at $h = 1/2\,ft$.

→ As volume of cone,

$V = \frac{1}{3}\pi r^2 h$ — (1)

Here V & h are functions of time. (we take).

→ From the figure, we have 2 similar triangle.

From properties of Δ^{le}, $r/h = 1/2 \Rightarrow r = h/2$.

$\therefore V = \frac{1}{3}\pi \frac{h^3}{4} = \frac{\pi h^3}{12}$

Differentiate wrt t,

$$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$$

$$-0.03 = \frac{\pi}{4\,12} 3(1/2)^2 \frac{dh}{dt} = \frac{\pi}{16}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16(-0.03)}{\pi}$$

$$\boxed{\frac{dh}{dt} = -0.1528\ ft/s}$$

$s^2 = x^2 + (4000)^2$

Differentiate wrt t,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + (4000)^2$$

$$2(5000)(600) = 2 \times 3000 \times dx/dt + 16000000$$

$$8000000 = 600\, dx/dt + 16000000$$

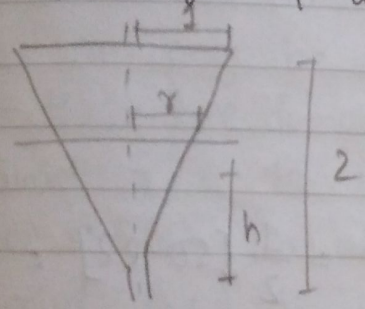
$x = 3000\,ft$

$s = \sqrt{3000^2 + 4000^2}$

$s = 5000.$

2) Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ ft^3/sec$. The height of the funnel is 2ft & the radius at the top is 1ft. At what rate is the height of water changing when the height of water is $1/2$ ft?

S⇒



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→ As volume of cone,

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Here v & h are functions of time. (we take).

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$\therefore V = \frac{1}{3}\pi \frac{h^3}{4} = \frac{\pi h^3}{12}$

Differentiate wrt t,

$$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt}$$

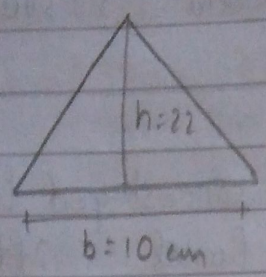
$$-0.03 = \frac{\pi}{\cancel{12}_{4}} \cancel{3}(1/2)^2 \frac{dh}{dt} = \frac{\pi}{16}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{16(-0.03)}{\pi}$$

$$\boxed{\frac{dh}{dt} = -0.1528\ ft/s}$$

LP 3) The base of a triangle is shrinking at a rate of 1cm/min the height of the triangle is increasing at a rate of 5cm/min. Find the rate at which the area of triangle changes when the height is 22cm & the base is 10cm.

S ⇒



$h = 22cm$, $b = 10cm$.

$\frac{dh}{dt} = 5cm/min$ $\quad \frac{db}{dt} = -1 cm/min$

$dA/dt = ?$

→ h & b both are functions of time.

WKT, $A = \frac{1}{2} bh$.

Differentiate wrt t,

$$\frac{dA}{dt} = \frac{1}{2}\left[b\frac{dh}{dt} + h\frac{db}{dt}\right]$$

$$\frac{dA}{dt} = \frac{1}{2}\left[10(5) + 22(-1)\right] = \frac{1}{2}[50-22] = 25-11$$

$$\boxed{\frac{dA}{dt} = 14\ cm^2/min}$$

* **Maxima and Minima :-**

→ There are 2 types,

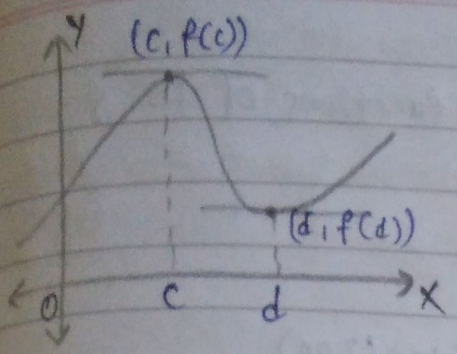
i) Local or Relative Maxima & Minima.

ii) Absolute or Global Maxima & Minima.

⇒ **Local Maxima & Minima.**

→ These are the maxima & minima of the function which arise in a particular interval.

→ A function has a local maximum at $x = c$ if $f(c) \geq f(x)$ when x is in the neighbourhood of c. Similarly, has a local minimum at $x = d$ if $f(d) \leq f(x)$ when x is in the neighbourhood of d.



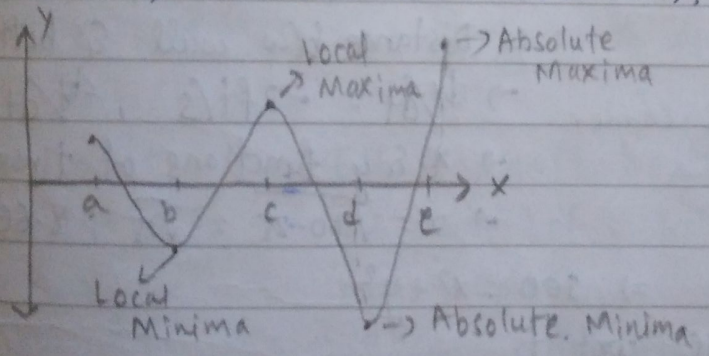
Absolute Maxima & Minima.

→ The highest point of a function within the entire domain is known as the absolute maxima of the function whereas the lowest point of a function within the entire domain is known as the absolute minima of the function.

→ There can only be one absolute maxima, absolute minima value over the entire domain.

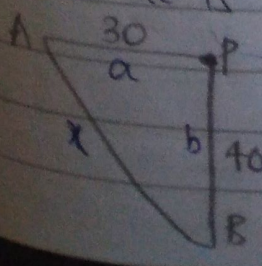
⇒ A point $x = e$ is a point of global maximum for $f(x)$ if $f(x) \leq f(e)$ for all $x \in D$ (Domain).

⇒ A point $x = d$ is a point of global minimum for $f(x)$ if $f(x) \geq f(d)$ for all $x \in D$ (Domain).



Q 3) Two airplanes are flying in the air at the same height: airplane A is flying east at 250 miles/hr & airplane B is flying north at 300 miles/hr. If they are both heading to the same airport, located 30 miles east of airplane A & 40 miles north of airplane B, at what rate is the distance between the airplanes changing?

Sol)



$dx/dt = ?$

$da/dt = -250$ miles/hr , $a = 30$ miles.

$db/dt = -300$ miles/hr , $b = 40$ miles.

$x = \sqrt{900+1600} = \sqrt{2500} = 50$ miles.

→ Using Pythagoras Theorem,

$x^2 = a^2 + b^2$ (All are functions of time).

Differentiate wrt t,

$$2x\frac{dx}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$$

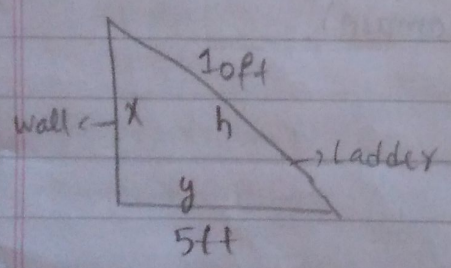
$$2(50)\frac{dx}{dt} = 2(30)(-250) + 2(40)(-300)$$

$$\frac{dx}{dt} = 2(3)(-25) + 2(4)(-30) = -150 - 240$$

$$\boxed{\frac{dx}{dt} = -390 \text{ miles/hour}}$$

Ex 4) A 10ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2ft/s, how fast is the bottom moving along the ground when the bottom of the ladder is 5ft from the wall?

S⇒



→ Ladder, h = 10ft, x ⇒ wall.

Distance b/w wall & bottom, y = 5ft.

→ $dx/dt = -2$ ft/s, $dy/dt = ?$

→ x & y functions of time.

→ $x = \sqrt{100-25} = \sqrt{75} = 8.66$ ft.

→ $h^2 = x^2 + y^2 \Rightarrow 100 = x^2 + y^2$

Differentiate wrt t,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

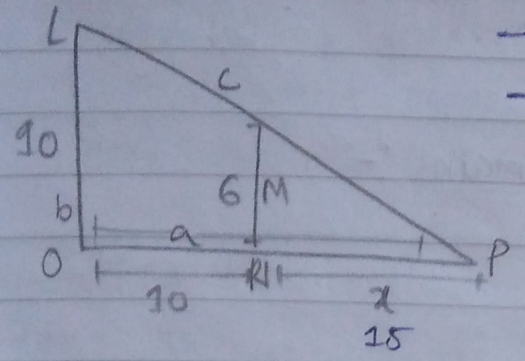
$$2(8.66)(-2) + 2(5)\frac{dy}{dt} = 0 \Rightarrow -34.64 + 10\frac{dy}{dt} = 0$$

$$10\, dy/dt = 34.64$$

$$\boxed{\frac{dy}{dt} = 3.464 \text{ ft/s}}$$

Ex 5) A 6-ft tall person walks away from a 10ft lamppost at a constant rate of 3ft/s. What is the rate that the tip of the shadow moves away from the pole when the person is 10ft away from the pole?

S→



→ $dx/dt = 3$ ft/s. → $da/dt = ?$

→ From the Δ^{le}s.

$$\frac{x}{6} = \frac{x+10}{10}$$

$$10x = 6x + 60$$

$$4x = 60 \Rightarrow \boxed{x = 15 \text{ ft}}$$

→ Since ΔLOP & ΔCRP are similar. | $a = 25$ ft.

$$\frac{a}{10} = \frac{x}{6} \Rightarrow 6a = 10x$$

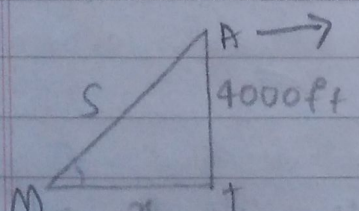
$$10x = 6a$$

Differentiate wrt t,

$$10\frac{dx}{dt} = 6\frac{da}{dt}$$

$$\cancel{10}^{5}(\cancel{3}) = \cancel{6}^{2}\, da/dt \Rightarrow \boxed{da/dt = 5 \text{ ft/s.}}$$

LP 1)



→ Let $y = 4000$ft be the vertical distance of the plane. Let 's' be the distance b/w the man & plane.

→ $dx/dt = 600$ ft/s, $x = 3000$ ft, $y = 4000$ft.

→ $ds/dt = ?$

→ $s^2 = x^2 + y^2$

$s^2 = x^2 + (4000)^2$ → Differentiate wrt t,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 0$$

$s = \sqrt{(3000)^2 + (4000)^2} = 5000$ft.

$$2(5000)\, ds/dt = 2(3000)(600)$$

$$\frac{ds}{dt} = 2 \times 3 \times 60$$

$$\boxed{\frac{ds}{dt} = 360 \text{ ft/s}}$$

* Critical Point :-

→ Let 'c' be an interior point in the domain of 'f', we say that 'c' is the critical point of 'f' when $f'(c)=0$ or $f'(c)$ is undefined.

• Find the critical pts in domains :-

LP8) 1) $y = 4x^3 - 3x$

→ Domain : R.

→ To find critical pts, $dy/dx = 0$,

$$\frac{dy}{dx} = 12x^2 - 3$$

$$12x^2 - 3 = 0$$

$$12x^2 = 3$$

$$x^2 = 1/4$$

$$\boxed{x = \pm 1/2}$$

∴ The critical points are $1/2$ & $-1/2$.

2) $y = 4x - x^2$

→ Domain : R.

$$\frac{dy}{dx} = 4 - 2x$$

$0 = 4 - 2x \Rightarrow \boxed{x = 2}$ ∴ 2 is the critical point.

3) $y = \ln(x-2)$.

→ Domain : $(2, \infty)$.

$$\frac{dy}{dx} = \frac{1}{x-2}$$

As dy/dx is not defined, (Denominator = 0)

$x - 2 = 0$

$x = 2$ → Not in domain ∴ Function is not defined at 2

∴ There are no critical points.

4) $y = \tan x$

Domain : $\mathbb{R}$. $(-\pi/2, \pi/2)$. → For 1 period.

$\frac{dy}{dx} = \sec^2 x$

$\sec^2 x = 0 \Rightarrow \sec x = 0 \Rightarrow$ But $\sec x$ can't be zero.

∴ The derivative curve doesn't touch the x-axis.

$\sec x = 0 \Rightarrow \cos x = \infty \Rightarrow$ Not Possible.

5) $y = x^{3/5}(4-x)$

→ $\frac{dy}{dx} = x^{3/5}(-1) + (4-x)\ 3/5\ x^{-2/5}$

$0 = -x^{3/5} + (4-x)\ 3/5\ x^{-2/5}$

$x^{3/5} = \frac{3(4-x)}{5x^{2/5}}$

$x = 3/5\,(4-x) \Rightarrow 5x = 12 - 3x$

$12 = 8x$

$\boxed{x = 3/2}$

∴ $x = 0$ & $x = 3/2$ are the critical pts.

6) $y = xe^{-x}$.

→ $\frac{dy}{dx} = x(-e^{-x}) + e^{-x}$

$0 = -e^{-x}x + e^{-x} = e^{-x}(-x+1)$

$-x + 1 = 0$

$\boxed{x = 1}$

7) $y = x + 1/x$.

→ $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

$0 = 1 - 1/x^2$

$1/x^2 = 1$

$\boxed{x = \pm 1}$

8) $y = x \ln x$

→ $\frac{dy}{dx} = x\frac{1}{x} + \ln x$

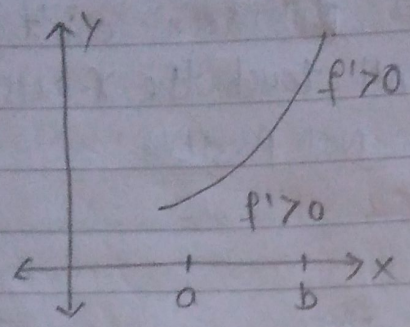
$0 = \ln x + 1$

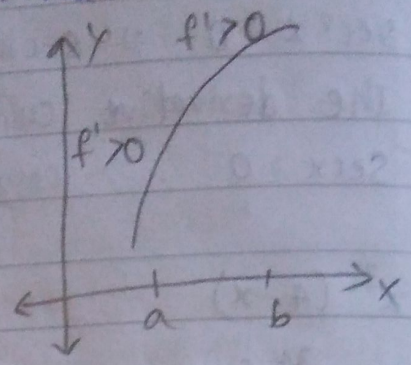
$\ln x = -1 \Rightarrow 2.303 \log x = -1$

$\boxed{x = 0.369}$

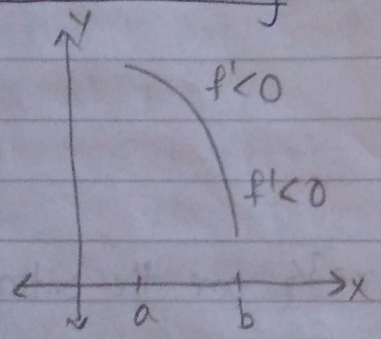
* What does first derivative f' tells about f?

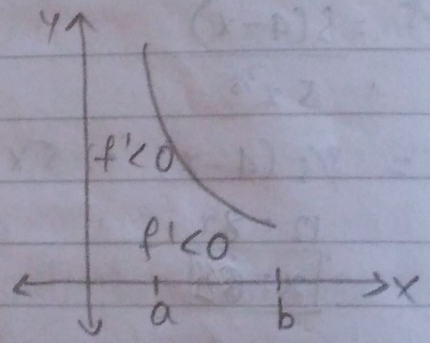
→ If $f'(x)$ is greater than zero on an interval. This means f is increasing in that interval.





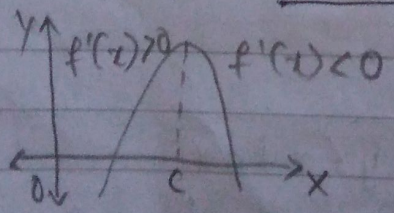
→ If $f'(x)$ is lesser than zero on an interval. This means f is decreasing in that interval.



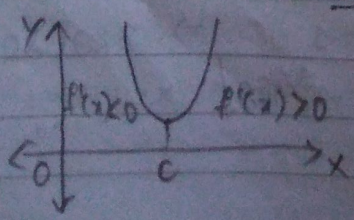


* First Derivative Test (To find Local Minima & Maxima):-

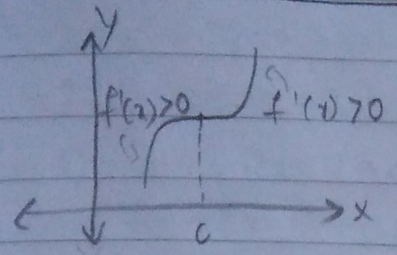
i) If f' changes sign from positive to negative at $x=c$, Then $f(c)$ is a local maximum of f. ($c \to$ critical point)

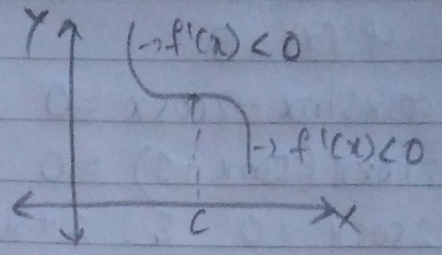


ii) If f' changes sign from negative to positive at $x=c$, Then $f(c)$ is a local minimum of f.



iii) If f' does not changes its sign at $x=c$, Then it is neither maxima nor minima at c.





- Use first derivative test to find local maxima for the following functions :-

6M 1) $f(x) = x^3 - 3x^2 - 9x - 1$

→ $f'(x) = 3x^2 - 6x - 9$

To find critical pts, Take $f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$3x^2 - 9x + 3x - 9 = 0$

$3x(x-3) + 3(x-3) = 0$

$3x+3 = 0$ & $x-3 = 0$

$\boxed{x=-1}$ & $\boxed{x=3}$

$-27x^2$
$-9x \quad 3x$
$-6x$

→ Intervals : $(-\infty, -1)$, $(-1, 3)$, $(3, \infty)$.

Intervals	Sign of $f'(x)$	Conclusion
$(-\infty, -1)$	+	f is increasing
$(-1, 3)$	−	f is decreasing
$(3, \infty)$	+	f is increasing

∴ By first derivative test, since f' switches from +ve to −ve as x increases through −1, f has local maximum at $x=-1$.
And f' switches from −ve to +ve as increases through 3, f has local minimum at $x=3$.

→ At $x=-1$, $f(x) = -1-3+9-1 = 4$

∴ 4 is the maximum value of $f(x)$.

→ At $x=3$, $f(x) = 27-27-27-1 = -28$

∴ −28 is the minimum value of $f(x)$.

LP10) 2) $f(x) = \cos^2 x - 2\sin x$, $[0, 2\pi]$.

→ $f'(x) = 2\cos x(-\sin x) - 2\cos x$

→ Take $f'(x) = 0$

$-2\cos x \sin x - 2\cos x = 0$

$-2\cos x(\sin x + 1) = 0$

$-2\cos x = 0$ & $1 + \sin x = 0$

$\cos x = 0$ & $\sin x = -1$

$\boxed{x = \pi/2}$ & $\boxed{x = 3\pi/2}$

→

Intervals	Sign of $f'(x)$	Conclusion
$[0, \pi/2)$	-ve	$f(x)$ is decreasing
$(\pi/2, 3\pi/2)$	+ve	$f(x)$ is increasing
$(3\pi/2, 2\pi]$	-ve	$f(x)$ is decreasing

∴ $f(x)$ has local minima at $x = \pi/2$ & local maxima at $x = 3\pi/2$.

→ At $x = \pi/2$, $f(x) = -2$ → Minimal value.

At $x = 3\pi/2$, $f(x) = 2$ → Maximum value.

3) $f(x) = x + 2\sin x$, $0 \le x \le 2\pi$.

→ $f'(x) = 1 + 2\cos x$

→ $f'(x) = 0$

$2\cos x = -1$

$\cos x = -1/2$ ⇒ $\boxed{x = 2\pi/3}$ & $\boxed{x = 4\pi/3}$

→

Intervals	Sign of $f'(x)$	Conclusion
$[0, 2\pi/3)$	+ve	Incresing
$(2\pi/3, 4\pi/3)$	-ve	Decreasing
$(4\pi/3, 2\pi]$	+ve	Increasing

→ $x = 2\pi/3$ → Local Maximum

$f(2\pi/3) = 2.7$

→ $x = 4\pi/3$ → Local Minimum

$f(4\pi/3) = 2.45$

4) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

→ $f'(x) = 12x^3 - 12x^2 - 24x$

Take $f'(x) = 0$

$12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$12x(x^2 - 2x + x - 2) = 0$

$12x(x(x-2) + 1(x-2)) = 0$

$12x(x+1)(x-2) = 0$

$-2x^2$
$-2x \quad x$
$-x$

$\boxed{x=0}$, $\boxed{x=-1}$, $\boxed{x=2}$

→

Intervals	$f'(x)$	$f(x)$
$(-\infty, -1)$	-ve	Decreasing
$(-1, 0)$	+ve	Increasing
$(0, 2)$	-ve	Decreasing
$(2, \infty)$	+ve	Increasing

Local Minima: $x = -1$
Local Maxima: $x = 0$
Local Minima: $x = 2$

5) $f(x) = 5 - 3x^2 + x^3$

→ Differentiate wrt x,

$f'(x) = 0 - 6x + 3x^2 = 3x^2 - 6x$

→ Take $f'(x) = 0$ to find critical points.

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$\boxed{x=0}$ & $\boxed{x=2}$

Intervals	$f'(x)$	$f(x)$
$(-\infty, 0)$	+ve	Increasing
$(0, 2)$	-ve	Decreasing
$(2, \infty)$	+ve	Increasing

Local Maxima: $x = 0$.
Local Minima: $x = 2$.
$f(0) = 5$, $f(2) = 1$

* <u>Concavity and Points of Inflection</u> :- only for $f(x)$

→ Let f' be a function that is differentiable over an open interval I. If 'f'' is increasing over I, we say f' is <u>concave up over I</u>. If 'f'' is decreasing over I, we say 'f'' is <u>concave down over I</u>.

$f''(x) = 0 \Rightarrow x$ is point of inflection.
where concavity changes.

⇒ **Test for Concavity :-**

→ Let f be a function that is twice differentiable over an interval I.

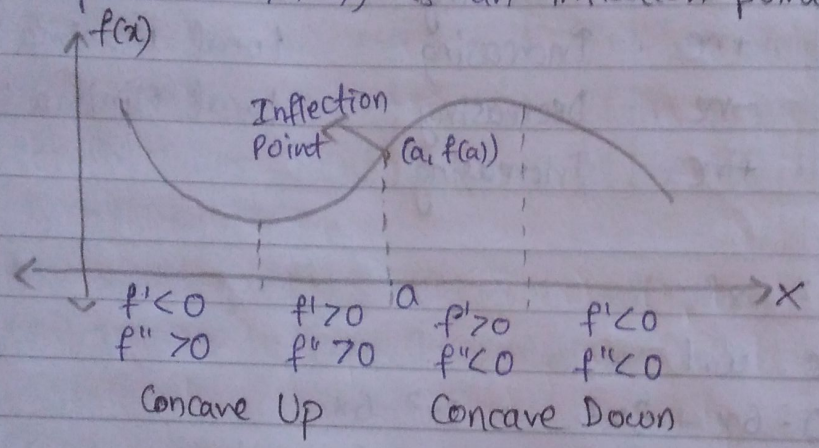
i) If $f''(x) > 0$, for all $x \in I$, then f is concave up over

ii) If $f''(x) < 0$, for all $x \in I$, then f is concave down over

* **Inflection Point :-**

→ If f' is continuous at a & f changes concavity at a, the point $(a, f(a))$ is an inflection point of f.

→



Ex: For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up & all intervals where f is concave down. List all inflection pts for f.

S ⇒ $f(x) = x^3 - 6x^2 + 9x + 30$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

→ To find inflection pts, take $f''(x) = 0$

$6x - 12 = 0 \Rightarrow 6x = 12 \Rightarrow \boxed{x = 2}$

→

Intervals	Sign of $f''(x)$	Concavity of $f(x)$
$(-\infty, 2)$	-ve	Concave Down
$(2, \infty)$	+ve	Concave Up

Imp. pts.

→ f is concave down over the interval $(-\infty, 2)$.

→ f is concave up over the interval $(2, \infty)$.

→ The point $(2, f(2)) = (2, 32)$ is an inflection point.

Point ⇒ (x, y)

* Second Derivative Test :-

→ Suppose $f'(c) = 0$, f'' is continuous over interval 'I' containing 'c'.

i) If $f''(c) > 0$, then 'f' has a local minimum at c.

ii) If $f''(c) < 0$, then 'f' has a local maximum at c.

iii) If $f''(c) = 0$, then the test is inconclusive.

Ex: Use the second derivative test to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

→ $f(x) = x^5 - 5x^3$

$f'(x) = 5x^4 - 15x^2$. ⇒ (If $f'(x) = 0$, $x = 0$ & $x = \pm\sqrt{3}$).

$f''(x) = 20x^3 - 30x$.

→ Take $f''(x) = 0$

$20x^3 - 30x = 0$

$10x(2x^2 - 3) = 0$

$10x = 0$ & $2x^2 - 3 = 0$ ⇒ $2x^2 = 3$ ⇒ $x^2 = 3/2$

$\boxed{x = 0}$ & $\boxed{x = \pm\sqrt{1.5}}$ → Pts of inflection.

→

Intervals	Sign of $f''(x)$	Concavity of $f(x)$
$(-\infty, -\sqrt{1.5})$	-ve	Concave Down
$(-\sqrt{1.5}, 0)$	+ve	Concave Up
$(0, \sqrt{1.5})$	-ve	Concave Down
$(\sqrt{1.5}, \infty)$	+ve	Concave Up.

→ The critical points,

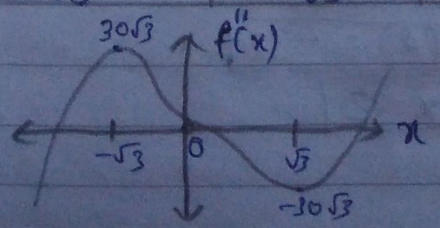
take $f'(x) = 0$ ⇒ $x = 0$, $x = -\sqrt{3}$, $x = +\sqrt{3}$.

→ $f''(0) = 0$, $f''(-\sqrt{3}) = -30\sqrt{3}$, $f''(\sqrt{3}) = 30\sqrt{3}$

→

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local Maximum
0	0	Inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local Minimum

→



LP 9) For the following functions, determine (a) intervals where 'f' is increasing or decreasing, (b) local minima & local maxima of f(x), intervals where 'f' is concave up and concave down, (d) the inflection points of 'f'.

i) $f(x) = x^2 - 6x$

→ $f'(x) = 2x - 6$

$f''(x) = 2$.

a) For critical pts, $2x - 6 = 0 \Rightarrow 2x = 6 \Rightarrow \boxed{x = 3}$

→

Intervals	Sign of $f'(x)$	Nature of $f(x)$
$(-\infty, 3)$	-ve	Decreasing
$(3, \infty)$	+ve	Increasing.

b) Local Minima → $x = 3$

$\boxed{f(3) = -9}$

c) For inflection pts, take $f''(x) = 0$, But $f''(x) = 2$.

But it is not possible ∴ No inflection points.

→

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, \infty)$	+ve	Concave Up

ii) $f(x) = x^3 - 6x^2$.

→ $f'(x) = 3x^2 - 12x$

$f''(x) = 6x - 12$

a) Critical pts, $3x^2 - 12x = 0$

$3x(x - 4) = 0 \Rightarrow \boxed{x = 0}$ & $\boxed{x = 4}$

→

Intervals	Sign of $f'(x)$	Nature of $f(x)$
$(-\infty, 0)$	+ve	Increasing
$(0, 4)$	-ve	Decreasing
$(4, \infty)$	+ve	Increasing

b) Local Maxima → $x = 0$ → $\boxed{f(0) = 0}$

Local Minima → $x = 4$ → $\boxed{f(4) = -32}$

c) Inflection pts, $6x - 12 = 0 \Rightarrow \boxed{x = 2}$

→

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 2)$	-ve	Concave Down
$(2, \infty)$	+ve	Concave Up

iii) $f(x) = x + x^2 - x^3$

→ $f'(x) = 1 + 2x - 3x^2$.

$f''(x) = 2 - 6x$.

a) Critical pts, $-3x^2 + 2x + 1 = 0$

$-3x^2 + 3x - x + 1 = 0$

$3x(-x+1) + 1(-x+1) = 0$

$3x + 1 = 0$ & $-x + 1 = 0$

$\boxed{x = -1/3}$ & $\boxed{x = 1}$

$-3x^2$, $3x - x$, $2x$

→

Intervals	Sign of $f'(x)$	Nature of $f(x)$
$(-\infty, -1/3)$	-ve	Decreasing
$(-1/3, 1)$	+ve	Increasing
$(1, \infty)$	-ve	Decreasing

b) Local Minima → $x = -1/3$ → $\boxed{f(-1/3) = -7/27}$

Local Maxima → $x = 1$ → $\boxed{f(1) = 1}$

c) Inflection pts, $2 - 6x = 0$ → $6x = 2$ ⇒ $\boxed{x = 1/3}$

→

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 1/3)$	+ve	Concave up
$(1/3, \infty)$	-ve	Concave down

iv) $f(x) = x^4 + x^3$

→ $f'(x) = 4x^3 + 3x^2$.

$f''(x) = 12x^2 + 6x$.

a) Critical pts, $4x^3 + 3x^2 = 0$

$x^2(4x + 3) = 0$

$\boxed{x = 0}$ & $\boxed{x = -3/4}$

→

Intervals	Sign of $f'(x)$	Nature of $f(x)$
$(-\infty, 0)$	-ve	Decreasing
$(0, -3/4)$	-ve	Decreasing
$(-3/4, \infty)$	+ve	Increasing

b) Local Minima

→ $x = -3/4$

$\boxed{f(-3/4) = -27/256}$

c) Inflection pts, Take $f''(x) = 0$

$$12x^2 - 6x = 0$$
$$6x(2x-1) = 0$$
$$\boxed{x=0} \ \& \ \boxed{x = 1/2}$$

→

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 0)$	+ve	Concave Up
$(0, 1/2)$	-ve	Concave Down
$(1/2, \infty)$	+ve	Concave Up.

v) $f(x) = (x-2)^2(x-4)^2$

→ $f'(x) = 2(x-2)(x-4)^2 + 2(x-2)^2(x-4)$

$= 2(x-2)(x-4)[(x-4)+(x-2)]$

$f'(x) = 2(x-2)(x-4)(2x-6) = 4(x^2-6x+8)(x-3)$

→ $f''(x) = 4[x^2-6x+8 + (x-3)(2x-6)]$

$= 4[x^2-6x+8+2x^2-6x-6x+18]$

$f''(x) = 4[3x^2-18x+26]$

a) Critical pts, $f'(x) = 0$

$$\boxed{x=2} \ \& \ \boxed{x=4} \ \& \ \boxed{x=3}$$

→

Intervals	Sign of $f'(x)$	Nature of $f(x)$
$(-\infty, 2)$	-ve	Decreasing
$(2, 3)$	+ve	Increasing
$(3, 4)$	-ve	Decreasing
$(4, \infty)$	+ve	Increasing.

b) Local Minima → $x=2$ & $x=4$ → $\boxed{f(2)=0}$ & $\boxed{f(4)=0}$

Local Maxima → $x=3$ → $\boxed{f(3)=1}$

c) Inflection pts, $f''(x) = 0$ ⇒ $\boxed{x=3.5}$ & $\boxed{x=2.4}$ → use calculator

→

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 2.4)$	+ve	Concave Up
$(2.4, 3.5)$	-ve	Concave Down
$(3.5, \infty)$	+ve	Concave Up

LP 10) As solved earlier we have, $f'(x) = -2\sin x\cos x - 2\cos x$
$[0, 2\pi]$

$f''(x) = -2(-\sin^2 x + \cos^2 x) + 2\sin x$

$\Rightarrow f''(x) = 2\sin^2 x - 2\cos^2 x + 2\sin x$

$\rightarrow$ Take $f''(x) = 0$,

$2(\sin^2 x - \cos^2 x + \sin x) = 0$

$\sin^2 x - (1 - \sin^2 x) + \sin x = 0$

$\sin^2 x - 1 + \sin^2 x + \sin x = 0$

$2\sin^2 x + \sin x - 1 = 0$

$2\sin^2 x + 2\sin x - \sin x - 1 = 0$ $\quad -2x$, $2x - x$, x

$2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$

$2\sin x = 1$ & $\sin x = -1$

$\boxed{x = 5\pi/6}$ & $\boxed{x = \pi/6}$ & $\boxed{x = 3\pi/2}$ $\rightarrow$ Inflection pts.

$\rightarrow$

Intervals	Sign of $f''(x)$	Concavity
$[0, \pi/6)$	$-ve$	Down
$(\pi/6, 5\pi/6)$	$+ve$	Up
$(5\pi/6, 3\pi/2)$	$-ve$	Down
$(3\pi/2, 2\pi]$	$-ve$	Down

The curve is concave up in & concave down in

P 13) Using second derivative test classify the critical pts as local maxima & minima.

i) $f(x) = 3x^5 - 20x^3$.

$\rightarrow f'(x) = 15x^4 - 60x^2$

$f''(x) = 60x^3 - 120x$

$\rightarrow$ Take $f'(x) = 0$

$15x^2(x^2 - 4) = 0$

$\boxed{x = 0}$ & $\boxed{x = \pm 2}$ $\rightarrow$ critical points.

$\rightarrow$ At $x = 0$, $f''(x) = 0$

$\therefore$ At $x = 0$, it has neither maxima nor minima.

$\rightarrow$ At $x = 2$, $f''(x) = +240$

$\therefore$ At $x = 2$, it has local minima which is -64 i.e. $(2, -64)$.

$\rightarrow$ At $x = -2$, $f''(x) = -240$

$\therefore$ At $x = -2$, it has local maxima which is 64 i.e. $(-2, 64)$.

LP 14) Determine whether $P(x) = 3x^4 + 4x^3$ has any absolute extrema in $[-1.5, 1]$. If so, find them.

* **Closed Interval Method** :- To find absolute minima & maxima.

Step 1] Find the values of 'f' at the critical points of 'f'.
Step 2] Find the values of 'f' at the end points of the interval.
Step 3] The largest of the values from above steps is absolute maximum value & the smallest is the absolute minimum value.

LP 14) $p(x) = 3x^4 + 4x^3$, $[-1.5, 1]$.

$p'(x) = 12x^3 + 12x^2$

→ Take $p'(x) = 0$

$12x^2(x+1) = 0$

$\boxed{x=0}$ & $\boxed{x=-1}$

→ $f(0) = 0$

$f(-1) = -1$ → Absolute Minimum.

$f(-1.5) = 1.68$

$f(1) = 7$ → Absolute Maximum.

∴ The absolute minimum is -1 at $x=-1$ & the absolute maximum value is 7 at $x=1$.

Ex: $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

→ $f'(x) = 6x^2 - 6x - 12$

→ Take $f'(x) = 0$ to find critical pts.

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$\boxed{x=2}$ & $\boxed{x=-1}$ → Critical pts.

→ $f(2) = -19$, $f(-1) = 8$ } Critical Pts. → Absolute Maximum Value = 8 at $x=1$.

→ $f(-2) = -3$, $f(3) = -8$ } End Pts → Absolute Minimum Value = -19 at $x=2$.

LP 11) $h(t) = -6.2t^2 + 80t$. A cricketer hits a ball into the air & its position is given as $h(t)$. A fielder catches the ball.

a) How long does it take to downfall happen?

b) What is the height at which the ball starts falling?

c) How much time did the ball take to reach ground?

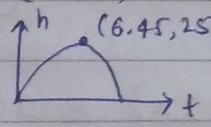
S⇒ $h'(t) = -12.4t + 80$

→ $h'(t) = 0$

$-12.4t = 80$

$\boxed{t = 6.45}$

→ $h''(6.45) = -12.4$. (Maxima)



a) After 6.45 s, the downfall started happening.

b) $h(6.45) = 258.1$

The height at which the ball starts falling is 258.1 ft.

c) To find time to reach ground, $h(t) = 0$

$-6.2t^2 + 80t = 0$

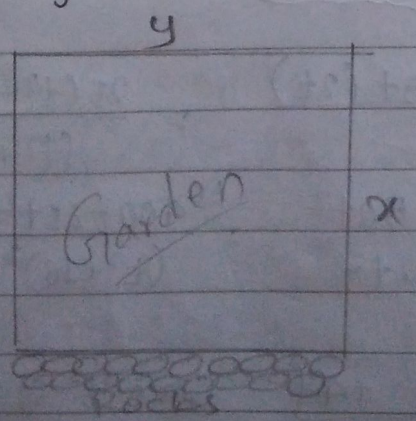
$t = 0$ & $\boxed{t = 12.9}$ ∴ After 12.9 s, ball reaches ground.

* **Optimization** :- Create a f^n & find maxima & minima

→ Optimization is the process of finding the maximum or minimum value of a function.

LP 16) A rectangular garden is to be constructed using a rock wall as one side of the garden & wire fencing for the other 3 sides. Given 100ft of wire fencing. Determine the dimensions that would create a garden of maximum area. What is the maximum area?

→



→ Given value is parameter
→ which is not given is the main f^n.

S⇒ x = Length of garden
y = Width of garden
Area, $A = xy$.

→ Given, perimeter of garden to be fenced by length of wire 100ft is,

$$2x + y = 100$$
$$y = 100 - 2x \quad -(1)$$

→ $A = xy$

$$A = x(100 - 2x) = 100x - 2x^2$$

$\boxed{A = 100x - 2x^2}$ → Main function.

→ $A'(x) = 100 - 4x$

Take $A'(x) = 0$

$$100 = 4x \Rightarrow \boxed{x = 25} \Rightarrow \boxed{y = 50}$$

→ $A''(x) = -4 \Rightarrow$ -ve ⇒ Local Maxima.

It means $x = 25$ gives maximum area.

→ ∴ The maximum area, $(x = 25, y = 50)$.

$$A_{max} = 25(50)$$

$\boxed{A_{max} = 1250 \text{ sqft}}$

LP 12) Consider the production of gold for 40 yrs from 1948 to 1988. The production of gold is modeled by $G(t) = 25t/t^2+16$, where $t \in [0, 40]$ is the no. of years & G is in millions. Find local & global maxima & minima.

S⇒ $G(t) = \dfrac{25t}{t^2+16}$, $[0, 40]$. ⇒ Closed Interval Method

→ Critical pts ⇒ $G'(t) = 0$

$$G'(t) = \frac{25(t^2+16) - 25t(2t)}{(t^2+16)^2} = \frac{25(t^2+16) - 50t^2}{(t^2+16)^2}$$

$$G'(t) = \frac{25t^2 + 400 - 50t^2}{t^4 + 256 + 32t^2} = \frac{400 - 25t^2}{(t^2+16)^2}$$

When $G'(t) = 0$

$$400 = 25t^2 \rightarrow \boxed{t = \pm 4}$$

Since 't' is +ve, $\boxed{t = 4}$

→ $\boxed{G(4) = 3.125}$
$\boxed{G(0) = 0}$
$G(40) = 0.61$

∴ The production of gold is maximum in the year 1952 i.e. 3.125 million ounces & minimum in 1948 i.e. 0.

LP 13) (ii) $f(x) = x/x^2+4$.

→ $f'(x) = \dfrac{x^2+4 - x(2x)}{(x^2+4)^2} = \dfrac{x^2+4-2x^2}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$

→ $f''(x) = \dfrac{(x^2+4)^2(-2x) - (4-x^2)(2(x^2+4)(2x))}{(x^2+4)^4}$

→ Take $f'(x) = 0 \Rightarrow x^2 = 4 \Rightarrow \boxed{x = \pm 2}$

→ At $x = -2$, $f''(-2) = +ve$

∴ It has local minima at $x = -2$ i.e. $\boxed{f(-2) = -1/4}$

→ At $x = 2$, $f''(2) = -ve$

∴ It has local maxima at $x = 2$ i.e. $\boxed{f(2) = 1/4}$

LP 15) Absolute Maxima & Minima.

→ $f(x) = 2x^3 - 15x^2 + 36x$ on $[1,5]$.

S⇒ $f'(x) = 6x^2 - 30x + 36$

$f'(x) = 6(x^2 - 5x + 6)$

→ Take $f'(x) = 0 \Rightarrow x^2 - 5x + 6 = 0$

$\boxed{x = 2}$ & $\boxed{x = 3}$

→ $f(2) = 28$ → $f(1) = 37$

$f(3) = 27$ → $f(5) = 55$

∴ Absolute minimum = 27 at $x = 3$.

Absolute maximum = 55 at $x = 5$.

Ex: i) $f(x) = \ln x/x$ on $[1,3]$.

→ $f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = \dfrac{1}{x^2} - \dfrac{\ln x}{x^2}$

→ Take $f'(x) = 0 \Rightarrow \ln x = 1 \Rightarrow \boxed{x = 2.71}$

→ $f(2.71) = 0.369$.

$f(1) = 0$

$f(3) = 0.366$.

∴ The maximum value is 0.369 at $x = 2.71$ & minimum is 0 at $x = 1$.

ii) $f(x) = \sin x + \cos x$ on $[0, \pi/3]$.

→ $f'(x) = \cos x - \sin x$.

Take $f'(x) = 0$

$\cos x = \sin x \Rightarrow \boxed{x = \pi/4}$

→ $f(\pi/4) = \sqrt{2} = 1.41$ → Maximum.

$f(0) = 1$ → Minimum.

$f(\pi/3) = 1.36$.

iii) $f(x) = x/x^2+1$ on $[0, 2]$.

→ $f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{x^2+1} = \dfrac{-x^2+1}{x^2+1}$

$f'(x) = \dfrac{1-x^2}{x^2+1} = \dfrac{(x+1)(1-x)}{x^2+1}$

→ Take $f'(x) = 0$

$(x+1)(1-x) = 0$

Not in Domain. ← $x = -1$ & $\boxed{x = 1}$

→ $f(1) = 1/2 = 0.5$ → Maximum.

$f(0) = 0$ → Minimum.

$f(2) = 2/5 = 0.4$

iv) $xe^{-x} = f(x)$ on $[0, 2]$.

→ $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(-x+1)$

→ Take $f'(x) = 0$

$e^{-x} = 0$ & $-x+1 = 0$.

(Not possible) $\boxed{x = 1}$

→ $f(1) = 0.367$

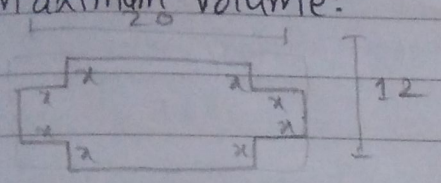
$f(0) = 0$

$f(2) = 0.270$

∴ The absolute maximum value is 0.367 which is at $x = 1$ & absolute minimum value is 0 at $x = 0$.

LP 17) A box with an open top is to be constructed from rectangular box of 12 in. by 20 in. by cutting it as in figure. What size should be exactly be cutout of each corner to get the maximum volume?

S→



Side cutout $= x$.

Length $= 20-2x$.

Breadth $= 12-2x$.

→ volume, $V = (20-2x)(12-2x)(x)$ Domain : $(0,6)$.

→ Differentiate the volume wrt x,

$$V(x) = (240-40x-24x+4x^2)(x)$$
$$= (240-64x+4x^2)(x)$$
$$= 240x-64x^2+4x^3$$
$$V(x) = 4x^3-64x^2+240x.$$

→ $V'(x) = 12x^2-128x+240$; $V''(x) = 24x-128$.

$$V'(x) = 4(3x^2-32x+60)$$

→ Take $V'(x)=0$ to find critical pts,

$3x^2-32x+60=0$ $\qquad 60\times 3 = 180x^2$

$x = 8.2$ & $\boxed{x=2.4}$ $\qquad -32x$

Because Domain is $(0,6)$.

→ When $x=2.4$, $V''(x) = -70.6$.

Since $V''(x) < 0$, $x=2.4$ gives maximum volume.

→ And that is,

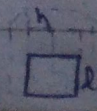
$$V_{max} = (20-2(2.4))(12-2(2.4))(2.4)$$
$$= (20-4.8)(12-4.8)(2.4) = (15.2)(7.2)(2.4)$$

$\boxed{V_{max} = 262.656 \text{ cubic inches.}}$

∴ When the cutout is 2.4 inches, we get the maximum volume.

LP 18) A rectangular box with a square base, an open top, and a volume of 216 inch cube is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

→



S⇒ It has a square base $\therefore l = b$.

$\therefore V = l^2 h = 216$.

→ To find: Dimensions when area = min.

$SA = l^2 + 4lh$

From (1), $h = 216/l^2$

$\therefore S = l^2 + \frac{4(216)}{l}$

→ Differentiate wrt l,

$$\frac{dS}{dl} = 2l - \frac{864}{l^2}$$

→ Take $dS/dl = 0$ for critical points.

$$2l = \frac{864}{l^2} \Rightarrow 2l^3 = 864$$

$$l^3 = 432$$

$$\boxed{l = 7.55 \text{ inches}}$$

→ Again differentiate wrt l,

$$\frac{d^2S}{dl^2} = 2 + \frac{2(864)}{l^3} = +ve$$

→ Since $d^2S/dl^2 > 0$, surface are is minimum at $l = 7.55$.

$\therefore h = 216/l^2 = 216/(7.5)^2$

$$\boxed{h = 3.78 \text{ inches}}$$

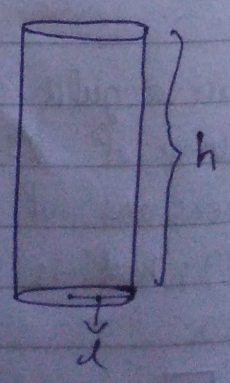
→ $S_{min} = l^2 + 4lh$

$= (7.55)^2 + 4(7.55)(3.78)$

$$\boxed{S_{min} = 171.15 \text{ sq. inches}}$$

$\therefore$ When length is 7.55 inches & height is 3.78 inches, the surface area is minimum which is 171.15 sq. inches

LP 20) A manufacturer of food storage containers makes a cylindrical can with a volume of 500 mL. What dimensions (height & radius) will minimize the material needed to produce each can, that is, minimise the surface area?



to find roots we use
Bisection & NR method.

S⇒ Given: $V = 500$ mL.

To find: h & r when surface area = min.

→ $V = \pi r^2 h = 500$ mL → $h = 500/\pi r^2$.

$S = 2\pi r h + 2\pi r^2$

$$S = \frac{2\pi r(500)}{\pi r^2} + 2\pi r^2 = \frac{1000}{r} + 2\pi r^2$$

→ Differentiate wrt r,

$$S' = \frac{-1000}{r^2} + 4\pi r$$

When $S' = 0$, $\boxed{r = 4.3}$

→ $$S'' = \frac{+2000}{r^3} + 4\pi = +ve$$

∴ The surface area is minimum at $\boxed{r = 4.3}$

∴ $h = 500/\pi(4.3)^2$ ⇒ $\boxed{h = 8.61}$

→ $S_{min} = 2\pi r h + 2\pi r^2 = 2\pi(4.3)(8.61) + 2\pi(4.3)^2$

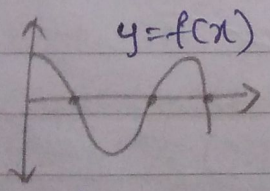
$\boxed{S_{min} = 348.62 \text{ sq. units}}$

---

* Applications of Derivatives to find Roots/Zeros of a function :-

→ Roots of a function $f(x)$ are the solutions to the eq^n $f(x) = 0$.

→ Graphically, roots are the points of intersection of the function with X-axis.

→ 

It has 3 roots.

→ When it is complicated to calculate roots of any equation (transcendental), we use numerical methods.

→ The 2 numerical methods to find one real root of an equation are,

i] Bisection Method.

ii] Newton-Raphson Method.

- Intermediate Value Theorem :-

→ In an interval $[a,b]$, $f(a)$ & $f(b)$ are of opposite sign, it means there exists a point 'c' in $[a,b]$ such that $f(c)=0$ which means there is a root exist in the interval.

LP 19) A hobby store has 20ft fencing to fence off a rectangular area for an electric train in one corner of its display room. The two sides up against the wall require no fence. What dimensions of the rectangle will maximize the area? What is the maximum area?

S⇒ Perimeter of fence = 20 ft.

→ Let length $= x$ & bredth will be $20-x$.

→ Area, $A = lb = x(20-x) = 20x - x^2$.

Differentiate wrt x,

$$\frac{dA}{dx} = 20 - 2x$$

→ For critical pts, $dA/dx = 0$

$20 = 2x \Rightarrow \boxed{x = 10 \text{ ft}} \Rightarrow \boxed{20 - x = 10 \text{ ft}}$

→ Again differentiate wrt x,

$$\frac{d^2A}{dx^2} = -2 \rightarrow -ve$$

∴ At $x = 10$ ft, the area is maximum.

i.e. $A_{max} = x(20-x) = 10 \times 10$

$\boxed{A_{max} = 100 \text{ sq. ft.}}$

* Bisection Method :-

→ Let the given function be $f(x)=0$ & let us suppose the root lies in the interval $(\bar{a}, \overset{+}{b})$ & let $f(a) < 0$ & $f(b) > 0$.

→ 1st approximate root : $x_1 = \frac{1}{2}(a+b)$.

→ If $f(x_1) > 0 \Rightarrow$ root lies in $(\bar{a}, \overset{+}{x_1})$.

→ If $f(x_1) < 0 \Rightarrow$ root lies in $(\bar{x_1}, \overset{+}{b})$.

→ If $f(x_1) = 0 \Rightarrow x_1$ is the root.

→ 2nd approximate root : $x_2 = \frac{1}{2}(a + x_1)$ or $x_2 = \frac{1}{2}(x_1 + b)$.

→ And again repeat the same process as above & continue this iterative process till you get root of desired accuracy.

LP 22) Find a root to $x^2 - x - 3 = 0$ using Bisection method. Perform 6 iterations.

S⇒ $f(x) = x^2 - x - 3$.

→ To find the interval,

$f(0) = -3$, $f(1) = -3$, $\boxed{f(2) = -1, f(3) = 3}$ (−ve, +ve)

→ By intermediate value theorem root lies in $(\bar{2}, \overset{+}{3})$.

→ 1st approximate root,

$x_1 = \frac{1}{2}(2+3) = \underline{2.5}$

$f(2.5) = 0.75$

∴ Root lies in $(\bar{2}, \overset{+}{2.5})$.

→ 2nd approximate root,

$x_2 = \frac{1}{2}(2+2.5) = \underline{2.25}$

$f(2.25) = -0.1875$

∴ Root lies in $(\overset{-}{2.25}, \overset{+}{2.5})$.

→ 3rd approximate root,

$x_3 = \frac{1}{2}(2.25+2.5) = \underline{2.375}$

$f(2.375) = 0.265$

∴ Root lies in $(\overset{-}{2.25}, \overset{+}{2.375})$

→ 4th approximate root,

$x_4 = \frac{1}{2}(2.25+2.375) = \underline{2.3125}$

$f(2.3125) = 0.0351$

∴ Root lies in $(\overset{-}{2.25}, \overset{+}{2.3125})$

→ 5th approximate root,

$x_5 = \frac{1}{2}(2.25+2.3125) = \underline{2.28125}$

$f(2.28125) = -0.07$

∴ Root lies in $(\overset{-}{2.28125}, \overset{+}{2.3125})$

→ 6th approximate root, $x_6 = \frac{1}{2}(2.28125+2.3125) = \underline{2.296875}$

$f(2.296875) = -0.02$ ∴ The approx. root is $\boxed{2.296875}$

LP 23) Find a root to $\sin x + 3^x$ by bisection method in the interval $(-1, 0)$. Perform 6 iterations.

S⇒ $f(x) = \sin x + 3^x$, $(\overset{-}{-1}, \overset{+}{0})$

→ 1st approximate root,

$x_1 = \frac{1}{2}(-1+0) = \underline{-0.5}$

$f(-0.5) = 0.09$

∴ Root lies in $(\overset{-}{-1}, \overset{+}{-0.5})$

→ 2nd approximate root,

$x_2 = \frac{1}{2}(-1-0.5) = \underline{-0.75}$

$f(-0.75) = -0.24$

∴ Root lies in $(\overset{-}{-0.75}, \overset{+}{-0.5})$

→ 3rd approximate root,

$x_3 = \frac{1}{2}(-0.75-0.5) = \underline{-0.625}$

$f(-0.625) = -0.08$

∴ Root lies in $(\overset{-}{-0.625}, \overset{+}{-0.5})$

→ 4th approximate root,

$x_4 = \frac{1}{2}(-0.625-0.5) = \underline{-0.5625}$

$f(-0.5625) = 5.7 \times 10^{-3}$

∴ Root lies in $(\overset{-}{-0.625}, \overset{+}{-0.5625})$

→ 5th approximate root,

$x_5 = \frac{1}{2}(-0.625-0.5625) = \underline{-0.59375}$

$f(-0.59375) = -0.038$

∴ Root lies in $(\overset{-}{-0.59375}, \overset{+}{-0.5625})$

→ 6th approximate root,

$x_6 = \frac{1}{2}(-0.59375-0.5625) = \underline{-0.578125}$ → $f(x_6) = -0.01$

∴ The approximate root is $\boxed{-0.578125}$.

LP 24) Find the real roots of the following equations by using (a) Bisection Method.

i) $xe^x = 1$.

ii) $x\sin x = \cos x$.

iii) $x^3 - x - 1 = 0$.

i) $f(x) = xe^x - 1$.

→ To find the interval, we use intermediate value theorem.

$f(0) = -1$
$f(1) = 1.71$ } ∴ The root lies in $(\bar{0}, \overset{+}{1})$

→

Iteration (r)	$\ominus$ a	$\oplus$ b	x_r	$f(x_r)$
1	0	1	0.5	-ve
2	0.5	1	0.75	+ve
3	0.5	0.75	0.625	+ve
4	0.5	0.625	0.5625	-ve
5	0.5625	0.625	0.59375	+ve
6	0.5625	0.59375	0.578125	+ve

∴ The approximate root is $\boxed{0.578125}$

ii) $f(x) = x\sin x - \cos x$.

→ $f(0) = -1$, $f(1) = 0.3$ ⇒ Root lies in $(\bar{0}, \overset{+}{1})$.

r	$\ominus$ a	$\oplus$ b	x_r	$f(x_r)$
1	0	1	0.5	-ve
2	0.5	1	0.75	-ve
3	0.75	1	0.875	+ve
4	0.75	0.875	0.8125	-ve
5	0.8125	0.875	0.84375	-ve
6	0.84375	0.875	0.859375	-ve

∴ The approximate root is $\boxed{0.859375}$

iii) $f(x) = x^3 - x - 1$.

→ $f(0) = -1$, $f(1) = -1$, $f(2) = 5$ ⇒ Root lies in $(\bar{1}, \overset{+}{2})$

r	$\ominus$ a	$\oplus$ b	x_r	$f(x_r)$
1	1	2	1.5	+ve
2	1	1.5	1.25	-ve
3	1.25	1.5	1.375	+ve
4	1.25	1.375	1.3125	-ve
5	1.3125	1.375	1.34375	+ve
6	1.3125	1.34375	1.328125	+ve

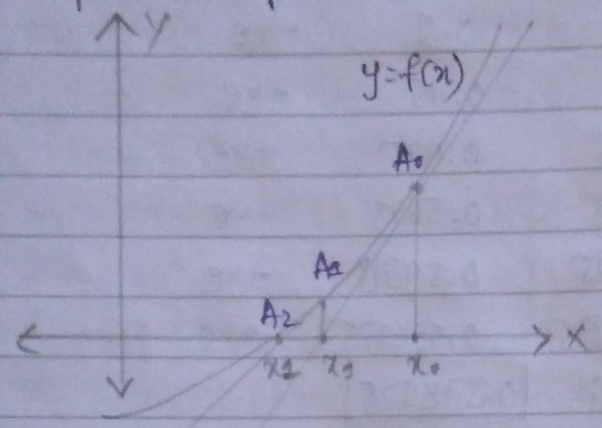
∴ The approximate root is $\boxed{1.328125}$

Tangent Formula $\rightarrow$ $y - y_0 = \underset{\downarrow \atop f'(x_0)}{m}(x - x_0)$

* Newton - Raphson Method or Method of Tangent :-

$\rightarrow$ Sometimes Bisection Method takes long iterations to find the exact value, but NR-Method converges faster.

$\Rightarrow$ Graphical Representation :-



$\rightarrow$ Let x_0 be initial approximate root. Let $A_0(x_0, f(x_0))$ be a point on curve.

$\rightarrow$ Eq^n of tangent at A_0 is,

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$\rightarrow$ The tangent cuts x axis at $(x_1, 0)$

$$0 - f(x_0) = f(x_0)[x_1 - x_0]$$

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

lly, $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Note: i) NR-Method converges faster than Bisection Method.

ii) The initial approximate root x_0 is chosen for which $f(x_0)$ & $f''(x_0)$ are greater than zero.

iii) NR-Method converges slow if f' is small & it fails when f' is zero.

LP 24) Find real root of following eq^{n}s by (b) Newton-Raphson Method.

i) $xe^x = 1$.

ii) $x \sin x = \cos x$.

iii) $x^3 - x - 1 = 0$.

i) $f(x) = xe^x - 1$.

→ $f'(x) = xe^x + e^x = e^x(x+1)$.

→ $f(0) = -1$ & $f(1) = 1.71$

∴ The root lies in $(0,1)$

→ As -1 is nearer to zero as compared to 1.71 so we take $\boxed{x_0 = 0}$.

→ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 + \frac{1}{1} = 0 + 1 = 1$

$\boxed{x_1 = 1}$

→ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1.71}{5.43} = 0.685$

$\boxed{x_2 = 0.685}$

→ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.577$

$\boxed{x_3 = 0.577}$

→ $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.56722$

$\boxed{x_4 = 0.56722}$

→ $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.567143$

$\boxed{x_5 = 0.567143}$

→ $x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 0.567143$

$\boxed{x_6 = 0.567143}$

∴ The perfect root is 0.567143. Since $x_5 = x_6$.

ii) $f(x) = x\sin x - \cos x$.

→ $f'(x) = x\cos x + \sin x + \sin x = x\cos x + 2\sin x$.

→ $f(0) = -1$ & $f(1) = 0.301$ (nearer to zero).

∴ $x_0 = 1$

→ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.8645$, $x_2 = 0.8603$, $x_3 = 0.8603$

∴ The root is $\boxed{0.8603}$

Set Ans as x_0 in calculator
& calculate x_1 as $Ans - \frac{f(Ans)}{f'(Ans)}$
& Press '='

iii) $f(x) = x^3 - x - 1$.

→ $f'(x) = 3x^2 - 1$.

→ $f(0) = -1$, $f(1) = -1$, $f(2) = 5$.

∴ -1 is nearer to zero.

∴ Initial root, $\boxed{x_0 = 1}$

→ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 + \frac{1}{2} = 1.5$

$\boxed{x_1 = 1.5}$

→ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{0.875}{5.75} = 1.3478$

$\boxed{x_2 = 1.3478}$

→ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.3252$

→ $x_4 = 1.3247$, $\boxed{x_5 = 1.3247}$ $(x_4 = x_5)$

∴ The root is 1.324717957

* Indeterminate Forms :-

→ Some of the indeterminate forms are $0/0$, ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0.

→ To evaluate the limits which take the indeterminate forms $0/0$ or ∞/∞, we apply L-Hospital's Rule.

* L-Hospital's Rule :-

→ Let $f(x)$ & $g(x)$ be 2 functions such that,

i) $\lim_{x \to a} f(x) = 0$ & $\lim_{x \to a} g(x) = 0$

ii) $f'(a)$ & $g'(a)$ exists and $g'(a) \neq 0$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Note : L-Hospital's rule is only applied for the forms $0/0$ & ∞/∞
All other forms should be converted to either of these forms

$\frac{1}{0} = \infty$

LP 21) Identify the following indeterminate forms & evaluate using L-Hospital's rule.

i) $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$

→ Put $x = 0$, we get $0/0$. form.

→ By LH Rule,

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = \lim_{x \to 0} \frac{2\cos x - 2\cos 2x}{3x^2} = \frac{0}{0} \text{ form}$$

$$= \lim_{x \to 0} \frac{-2\sin x + 4\sin 2x}{6x} = \frac{0}{0} \text{ form.}$$

$$= \lim_{x \to 0} \frac{-2\cos x + 8\cos 2x}{6} = \frac{6}{6} = \boxed{1}$$

* Find the value of 'a' such that,

$\lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3}$ is finite.

→ Put $x = 0$, $0/0$ form.

∴ By LH Rule, Since limit is constant.

we get $\lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^2} = \frac{2+a}{0} = \frac{0}{0}$ form

→ Since the limit is constant, $\frac{2+a}{0} = \frac{0}{0} \Rightarrow 2+a = 0$

$\boxed{a = -2}$

→ ∴ $\lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3x^2} = \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x} = \frac{0}{0}$ form

$$= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = \frac{-6}{6} = \boxed{-1}$$

∴ The value of 'a' is -2 & the limit is -1.

* $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{0}{0}$ form.

By LH Rule,

$\rightarrow \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \frac{0}{0}$ form

$= \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{0}{0}$ form.

$= \lim_{x \to 0} \frac{2[\sec^2 x \sec^2 x + \tan x \, 2\sec^2 x \tan x]}{6} \neq \frac{0}{0}$ form.

$= \lim_{x \to 0} \frac{\sec^4 x + 2\sec^2 x \tan^2 x}{3} = \frac{1+0}{3} = \boxed{\frac{1}{3}}$

ii) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \frac{1}{0} - \frac{1}{0}$

$\rightarrow = \lim_{x \to 0} \left(\frac{x - \sin x}{x \sin x}\right) = \frac{0}{0}$ form

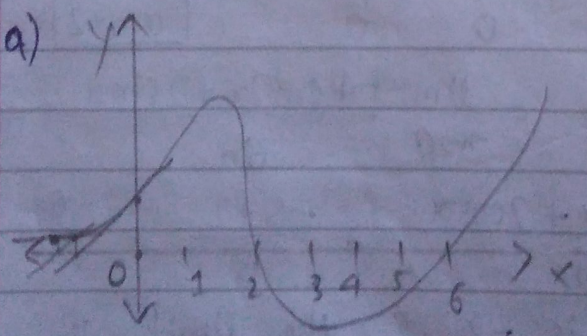
By LH Rule,

$= \lim_{x \to 0} \left(\frac{1 - \cos x}{x\cos x + \sin x}\right) = \frac{0}{0}$ form

$= \lim_{x \to 0} \left(\frac{\sin x}{-x\sin x + \cos x + \cos x}\right) = \frac{0}{0+1+1} = \frac{0}{2} = \boxed{0}$

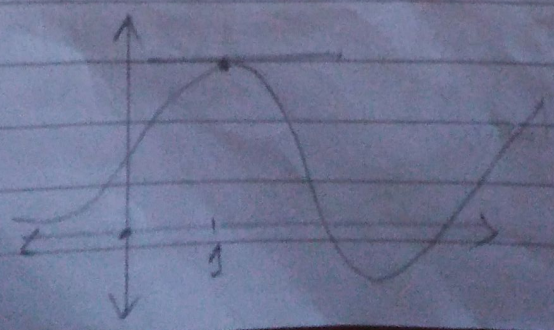
LP 25) For each initial approximation, determine graphically what happens if NR method is used for the function whose graph is shown. a) 0, b) 1, c) 3, d) 4, e) 5.

S⇒ a)



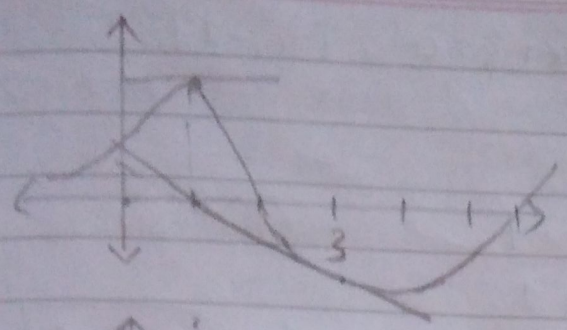
→ When $x = 0$, after few iterations the tangent becomes parallel to x-axis where NR method fails.

b)



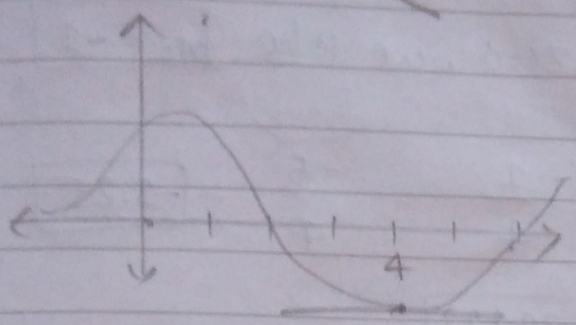
→ When $x = 1$, the tangent is parallel to x-axis ∴ NR Method fails.

c)



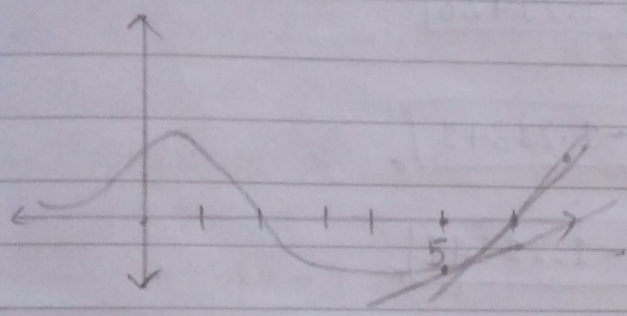
→ when $x=3$, the tangent cuts x axis at $x=1$, where the tangent is parallel to X-axis where NR Method fails.

d)



→ When $x=4$, the tangent is parallel to x-axis
∴ NR Method fails

e)



→ When $x=5$, after some iterations, the tangent cuts X-axis which is the root of the function i.e. $f(x)=0$. i.e. $x=6$

LP 26) $x^4 - x - 1 = 0 \rightarrow [1, 2]$.

→ $f(1) = -1$ & $f(2) = 13$. ; $f'(x) = 4x^3 - 1$.

Since -1 is nearer to zero, $\boxed{x_0 = 1}$

→ $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1 - \left(\dfrac{1-1-1}{3}\right) = 1 + \dfrac{1}{3} = \boxed{\dfrac{4}{3}}$ ⇒ $\boxed{x_1 = 1.33}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \boxed{1.2358}$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = \boxed{1.2210}$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = \boxed{1.2207}$

$x_5 = x_4 - \dfrac{f(x_4)}{f'(x_4)} = \boxed{1.2207}$

→ Since $x_4 = x_5$, the perfect root of the equation $x^4 - x - 1$ is $\boxed{1.2207}$

LP 27) Usind NR Method, $x^3+x+3=0$, $[-2,-1]$.

S⇒ $f(x) = x^3+x+3$.

$f'(x) = 3x^2+1$

→ $f(-2) = -8-2+3 = -10+3 = -7$.

$f(-1) = -1-1+3 = -2+3 = 1$.

→ Since $f(-1)$ is closer to zero, we take $\boxed{x_0 = -1}$

→ By NR Method

$$x_1 = x_0 - \frac{f(-1)}{f'(-1)} = -1 - \frac{1}{4} = \frac{-5}{4} = \boxed{-1.25}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \boxed{-1.21428}$$

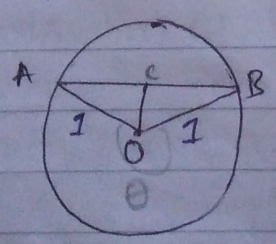
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \boxed{-1.21341},$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \boxed{-1.21341},$$

→ Since $x_3 = x_4$, the real root of equation is $\boxed{-1.21341}$

LP 28) The circle below has a radius of 1, & the longer circular arc joining A & B is twice as long as the chord AB. Use NR Method to find the length of chord AB, correct to 4 decimal places.

→



Let chord AB be x.
Then arc AB is 2x

$A\triangle ABC = \frac{\sin x}{\text{radius}} = \sin x$

$x = 2\pi - \theta$

$\sin(2\pi - \theta) = \sin\theta$

$AB = \frac{2\sin\theta}{\text{radius}}$

S⇒ WKT, arc length = radius × θ

$2x = r\theta$. $(r=1)$

$2x = \theta$ — (1)

→ Also chord length $= 2\sin(\theta/2)$ × radius

$x = 2\sin\theta/2$

→ ∴ (1) becomes,

$2(2\sin\theta/2) = \theta \Rightarrow f(\theta) = 4\sin(\theta/2) - \theta$

$f'(\theta) = 2\cos\theta/2 - 1$

→ $f(0) = 0$, $f(1) = 1.41$, $f(2) = 2.36$, $f(3) = 0.98$.

→ Since $f(3)$ is closer to zero, $\therefore \boxed{\theta_0 = 3}$

→ $\theta_1 = \theta_0 - \dfrac{f(\theta_0)}{f'(\theta_0)} = \boxed{4.1530}$

$\theta_2 = \theta_1 - \dfrac{f(\theta_1)}{f'(\theta_1)} = \boxed{3.8210}$

$\theta_3 = \theta_2 - \dfrac{f(\theta_2)}{f'(\theta_2)} = \boxed{3.7912}$

$\theta_4 = \theta_3 - \dfrac{f(\theta_3)}{f'(\theta_3)} = \boxed{3.7909}$

$\theta_5 = \theta_4 - \dfrac{f(\theta_4)}{f'(\theta_4)} = \boxed{3.7909}$

$\therefore$ The value of θ is 3.7909.

→ Since $2x = \theta \Rightarrow x = \theta/2 = 3.7909/2$

$\boxed{x = 1.89545}$ → length of the chord.

* <u>Some Problems on Optimization</u> :-

1) If 1200 sqcm of material is available to make a box with a square base & an open top, find the largest possible volume of the box.

S→ Area = 1200 sqcm, Let 'l' be the side of square, 'b' be height

$A = l^2 + 4lb$

$1200 = l^2 + 4lb \Rightarrow b = (1200 - l^2)/4l$

→ volume $= l^2 b \Rightarrow V = l^2 b$

$V = l^2(1200 - l^2)/4l = \dfrac{\overset{300}{\cancel{1200}}l}{\cancel{4}} - \dfrac{l^3}{4}$

$\boxed{V = 300l - l^3/4}$

→ $\dfrac{dV}{dl} = 300 - \dfrac{3}{4}l^2$; $\dfrac{d^2V}{dl^2} = -\dfrac{3}{2}l$

Take $dV/dx = 0 \Rightarrow \overset{100}{\cancel{300}} = \cancel{3}/4\, l^2 \Rightarrow 400 = l^2 \Rightarrow \boxed{l = 20\text{cm}} \Rightarrow \boxed{b = 10\text{cm}}$

→ At $l = 20$cm, $d^2V/dl^2 = -30 \Rightarrow$ At $l = 20$, the volume is maximum.

$\therefore$ Maximum volume is $\boxed{V_{max} = 4000 \text{ cubic cm}}$

2) A box with sq. base & open top must have a volume of 32000 cubic cm. Find the dimension of the box that minimize the amount of material used.

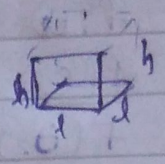
S⇒ Let length of sq. base be 'l' & height of box be 'h'.

→ Given that ; $V = 32000$.

$V = l^2 h$

$32000 = l^2 h$

$h = 32000/l^2$.



→ WKT, $A = l^2 + 4lh$.

$A = l^2 + 4l(32000/l^2)$

$A = l^2 + 128000/l$

→ $\frac{dA}{dl} = 2l + \left(\frac{-128000}{l^2}\right)$ & $\frac{d^2A}{dl^2} = 2 + \frac{256000}{l^3}$.

Put $dA/dl = 0$

$2l = \frac{128000}{l^2}$

$2l^3 = 128000 \Rightarrow l^3 = 64000$

$\boxed{l = 40 \text{ cm}} \Rightarrow \boxed{h = 20\text{cm}}$

→ When $l = 40$, d^2A/dl^2 = +ve ⇒ It will have minimum area

$\boxed{A_{min} = 4800 \text{ sq. cubes}}$

3) A farmer has 1200 m of fencing wire & wants to fence a rectangular field that borders a straight river. He needs no fence along river, What are the dimensions of fields so that it has largest area?

S⇒ Farmer has 1200 m fencing that is needed to cover 3 sides.

$\therefore 2l + b = 1200 \Rightarrow b = 1200 - 2l$.

→ Area, $A = lb = l(1200 - 2l) = 1200l - 2l^2$

$A' = 1200 - 4l$ & $A'' = -4$.

→ When $A' = 0 \Rightarrow 4l = 1200 \Rightarrow \boxed{l = 300\text{m}} \Rightarrow \boxed{b = 600\text{m}}$

→ When $l = 300$, $A'' = -4 \Rightarrow$ Maximum area.

$\therefore \boxed{A_{max} = 180000\text{m}^2}$

4) A piece of wire 10cm long is cut into 2 pieces. One piece is bent into a square & other into an equilateral triangle. How should the wire be cut so that the total area enclosed is maximum & minimum.

S⇒ Let x is the length of wire used to the square
We have $10-x$ as length used for Δ^{le}.

→ ∴ Square has its side as $x/4$.
Δ^{le} has its side as $(10-x)/3$.

→ Area of square $= (x/4)^2 = x^2/16$.

→ Area of eq. $\Delta^{le} = \sqrt{3}/4 \left((10-x)^2/9\right) = \sqrt{3}/36\,(100-20x+x^2)$.

→ Total Area, $A = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(100-20x+x^2)$

$$A' = \frac{2x}{16} + \frac{\sqrt{3}}{36}(-20+2x)$$

$$A' = \frac{x}{8} - \frac{20\sqrt{3}}{36} + \frac{2\sqrt{3}x}{36}$$

$$A' = \frac{x}{8} - \frac{5\sqrt{3}}{9} + \frac{\sqrt{3}x}{18}$$

→ Take $A'=0 \Rightarrow \boxed{x=4.34}$

$A'' = \frac{1}{8} + \frac{\sqrt{3}}{18}$ ⇒ when $x=4.34$, A'' is +ve. ∴ Minimum area is possible when $x = 4.34$ cm.

→ When we cut 4.34 cm for square & 5.64 cm for the Δ^{le}, we have the minimum area. When all 10cm is used for square we get a maximum area.

* <u>Indeterminate Forms Examples</u> :-

→ If $\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$ find a & b.

→ Put $x=0$,
we get $\lim_{x\to 0} f(x) = \frac{0}{0}$

→ By LH Rule,

$$\lim_{x\to 0} \frac{x(-a\sin x) + (1+a\cos x) - b\cos x}{3x^2} = \frac{0+1+a-b}{0} = \frac{0}{0}$$

$\therefore\ 1+a-b = 0 \Rightarrow \boxed{a-b=-1}$ — ①

→ Again by LH Rule,

$$= \lim_{x\to 0} \frac{x(-a\cos x) + (-a\sin x) + (-a\sin x) + b\sin x}{6x} = \frac{0}{0} \text{ form}$$

$$= \lim_{x\to a} \frac{-xa\cos x - 2a\sin x + b\sin x}{6x}$$

By LH Rule.

$$= \lim_{x\to a} \frac{x(a\sin x) - a\cos x - 2a\cos x + b\cos x}{6} = \frac{0-a-2a+b}{6} = \frac{0}{0}$$

Given limit value is 1,

$\boxed{-3a+b=6}$ — ②

→ Solve ① & ②,

$a-b=-1$

$-3a+b=6$

$-2a=5 \Rightarrow \boxed{a=-5/2}$ & $\boxed{b=-3/2}$

Type 2 • Indeterminate Forms of the type $\infty-\infty$ & $0\times\infty$.

i) Evaluate $\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right] = \frac{1}{0} - \frac{1}{0} = \infty-\infty$ form

→ $$= \lim_{x\to 2}\left[\frac{\log(x-1) - (x-2)}{(x-2)(\log(x-1))}\right] = \frac{0}{0} \text{ form.}$$

By LH Rule,

$$= \lim_{x\to 2}\left[\frac{1/(x-1) - 1}{\frac{x-2}{x-1} + \log(x-1)}\right] = \frac{0}{0} \text{ form.}$$

$$= \lim_{x\to 2}\left[\frac{-1/(x-1)^2}{\frac{(x-1)-(x-2)}{(x-1)^2} + \frac{1}{x-1}}\right] = \frac{-1}{1/1 + 1/1} = \boxed{-1/2}$$

$\therefore$ The value of the limit is $-1/2$.

ii) $\lim_{x \to \pi/4} \left[(1-\tan x) \sec 2x \right]$.

$\rightarrow \lim_{x \to \pi/4} [(1-\tan x)\sec 2x] = 0 \times \infty$ form.

$= \lim_{x \to \pi/4} \left[\frac{1-\tan x}{\cos 2x} \right] = \frac{0}{0}$ form

By LH Rule,

$= \lim_{x \to \pi/4} \left[\frac{-\sec^2 x}{-2\sin 2x} \right] = \frac{2}{2(1)} = \boxed{1}$

Type 3] 1^∞, 0^∞ & ∞^0.

$\rightarrow$ In these forms we evaluate by taking 'log' on both sides of the given equation & then convert it into 0/0 or ∞/∞ form.

i) $\lim_{x \to 0} (1+x)^{1/x} = 1^\infty$ form.

$\rightarrow$ Let $y = \lim_{x \to 0} (1+x)^{1/x}$

Take log on both sides,

$\log y = \lim_{x \to 0} \log (1+x)^{1/x} = \lim_{x \to 0} \frac{1}{x} \log(1+x) = \frac{0}{0}$ form

$\rightarrow$ By LH Rule $\Rightarrow \lim_{x \to 0} \frac{1/(1+x)}{1} = \log y$

$\log y = \lim_{x \to 0} \frac{1}{1+x} = 1$

$\log y = 1$

$\boxed{y = e}$

ii) $\lim_{x \to 1} x^{\frac{1}{1-x}} = 1^\infty$ form

$\rightarrow$ Take $y = x^{\frac{1}{1-x}}$ $\rightarrow$ Apply log,

$\log y = \lim_{x \to 1} \frac{1}{1-x} \log x = \lim_{x \to 1} \frac{1/x}{-1} = \frac{-1}{1} = -1 \Rightarrow \boxed{y = \frac{1}{e}}$

(By LH Rule)

iii) $\lim_{x\to\infty}\left[\frac{ax+1}{ax-1}\right]^x = \left(\frac{\infty}{\infty}\right)^\infty$ form. → Not a standard form

→ $\lim_{x\to\infty}\left[\frac{x[a+1/x]}{x[a-1/x]}\right]^x = \left(\frac{a+0}{a-0}\right)^\infty = 1^\infty$ form.

→ Apply log on both sides, Put $1/x = t$, $x\to\infty$, $t\to 0$.

$$\log y = \lim_{x\to\infty} x \log\left[\frac{a+1/x}{a-1/x}\right] = \lim_{t\to 0}\left[\frac{a+t}{a-t}\right]^{1/t}$$

→ By LH Rule,

$$\log y = \lim_{t\to 0}\frac{1}{t}\log\left[\frac{a+t}{a-t}\right]$$

$$\log y = \lim_{t\to 0}\left[\frac{\frac{a-t}{a+t}\,\frac{(a-t-(a+t))}{(a-t)^2}}{1}\right]$$

$$= \lim_{t\to 0}\left[a-t\,(-2t)\right.$$

→ Put $1/x = t$, $x\to\infty$, $t\to 0$.

$$\therefore \lim_{t\to 0}\left[\frac{a+t}{a-t}\right]^{1/t} = 1^\infty \text{ form.}$$

→ Apply log,

$$\log y = \lim_{t\to 0}\frac{1}{t}\log\left[\frac{a+t}{a-t}\right]$$

$$\log y = \lim_{t\to 0}\frac{\log(a+t)-\log(a-t)}{t} = \frac{0}{0} \text{ form}$$

→ By LH Rule,

$$\log y = \lim_{t\to 0}\frac{1/a+t + 1/a-t}{1} = \frac{1}{a}+\frac{1}{a} = \frac{2}{a}$$

$$\log y = 2/a \Rightarrow \boxed{y = e^{2/a}}$$

iv) $\lim_{x\to 0}(\cos x)^{1/x^2} = 1^\infty$ form

→ $\log y = \lim_{x\to 0}\frac{1}{x^2}\log(\cos x) = \lim_{x\to 0}\frac{1/\cos x\,(-\sin x)}{2x}$

$$= \lim_{x\to 0}\frac{-\tan x}{2x} = \frac{-1}{2} \Rightarrow \boxed{y = e^{-1/2}}$$

v) $\lim_{x\to 0}\left[\frac{\tan x}{x}\right]^{1/x} = 1^{\infty}$ form.

$\rightarrow \log y = \lim_{x\to 0}\frac{1}{x}\log\left[\frac{\tan x}{x}\right] = \lim_{x\to 0}\frac{\log[\tan x/x]}{x} = \frac{0}{0}$ form

$\rightarrow$ By LH Rule,

$$\log y = \lim_{x\to 0}\frac{1}{\tan x/x}\left[\frac{x\sec^2 x - \tan x}{x^2}\right]$$

$$\log y = \lim_{x\to 0}\frac{1}{(1)}\left[\frac{x\sec^2 x - \tan x}{x^2}\right] = \frac{0}{0} \text{ form}$$

$\rightarrow$ By LH Rule,

$$\log y = \lim_{x\to 0}\frac{\cancel{x}\,2\sec^2 x\tan x + \cancel{\sec^2 x} - \cancel{\sec^2 x}}{2\cancel{x}} = \lim_{x\to 0}\sec^2 x\tan x$$

$$\log y = 0 \Rightarrow y = e^0 \Rightarrow \boxed{y = 1}$$

21) iii) $\lim_{x\to\infty}\left[\frac{2x-3}{2x+5}\right]^{2x+1} = \left(\frac{\infty}{\infty}\right)^{\infty}$ form.

$\rightarrow = \lim_{x\to\infty}\left[\frac{\cancel{x}(2-3/x)}{\cancel{x}(2+5/x)}\right]^{2x+1} = \left(\frac{2-0}{2+0}\right)^{\infty} = 1^{\infty}$ form.

$\rightarrow$ Put $1/x = t$, $x\to\infty$, $t\to 0$

$\therefore\ y = \lim_{t\to 0}\left[\frac{2-3t}{2+5t}\right]^{2/t+1}$ $\rightarrow$ Apply log.

$$\log y = \lim_{t\to 0}\left[\frac{2}{t}+1\right]\log\left[\frac{2-3t}{2+5t}\right]$$

$$\log y = \lim_{t\to 0}\left[\frac{2+t}{t}\right]\left[\log(2-3t) - \log(2+5t)\right] = \lim_{t\to 0}\frac{\log(2-3t)-\log(2+5t)}{(t/2+t)}$$

$\rightarrow$ Apply LH Rule, $\log y = \lim_{t\to 0}\frac{\frac{-3}{2-3t} - \frac{5}{2+5t}}{(2+t) - \frac{t}{(2+t)^2}} = \lim_{t\to 0}\frac{\frac{-3}{2-3t} - \frac{5}{2+5t}}{\frac{2}{(2+t)^2}}$

$$\log y = \frac{-3/2 - 5/2}{2/4} = \frac{-8/2}{1/2} = -8$$

$$\boxed{y = e^{-8}}$$

iv) $\lim_{x\to 0}(\tan x)^{\tan 2x} = 0^0$ form.

→ Let $y = \lim_{x\to 0} (\tan x)^{\tan 2x}$

$$\log y = \lim_{x\to 0} \tan 2x \log(\tan x) = 0 \times \infty \text{ form}$$

$$\log y = \lim_{x\to 0} \frac{\log(\tan x)}{\cot 2x} = \frac{\infty}{\infty} \text{ form.}$$

→ By LH Rule, $\log y = \lim_{x\to 0} \dfrac{\frac{1}{\tan x}(\sec^2 x)}{-\operatorname{cosec}^2 2x\,(2)}$

$$\log y = \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}\left(\frac{1}{\cos^2 x}\right)}{-2/\sin^2 2x}$$

$$\log y = \lim_{x\to 0} \frac{-1}{2} \frac{\sin^2 2x}{\sin x \cos x} = \frac{0}{0} \text{ form}$$

→ By LH Rule, $\log y = \dfrac{-1}{2} \lim_{x\to 0} \dfrac{2\sin 2x \cos 2x}{-\sin^2 x + \cos^2 x}$

$$\log y = \frac{-1}{2}\left(\frac{0}{0+1}\right) = 0$$

$$\boxed{y = 1}$$

v) $\lim_{x\to 0} (\cot x)^{1/\log x} = \infty^0$ form.

→ Take $y = \lim_{x\to 0} (\cot x)^{1/\log x}$.

$$\log y = \lim_{x\to 0} \frac{1}{\log x} \log(\cot x) = \lim_{x\to 0} \frac{\log(\cot x)}{\log x} = \frac{\infty}{\infty} \text{ form}$$

→ By LH Rule, $\log y = \lim_{x\to 0} \dfrac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{1/x} = \dfrac{\infty}{\infty}$ form

$$\log y = \lim_{x\to 0} \frac{-x \operatorname{cosec}^2 x}{\cot x} = \lim_{x\to 0} \frac{-x\left(\frac{1}{\sin^2 x}\right)}{\cos x/\sin x}$$

$$\log y = \lim_{x\to 0} \frac{-x}{\sin x \cos x}$$

→ By LH Rule, $\log y = \lim_{x\to 0} \dfrac{-1}{\cos^2 x - \sin^2 x} = \dfrac{-1}{1} = -1$

$$\boxed{y = 1/e}$$

vi) $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]^{1/x^2} = 1^\infty$ form.

$\rightarrow \log y = \lim_{x\to 0}\frac{1}{x^2}\left[\log\left(\frac{\sin x}{x}\right)\right] = \lim_{x\to 0}\frac{\log(\sin x) - \log x}{x^2}$

$\log y = \lim_{x\to 0}\frac{\log(\sin x/x)}{x^2} = \lim_{x\to 0}\frac{x^{-2}}{1/\log(\sin x/x)} = \frac{0}{0}$ form

$\rightarrow$ By LH Rule, $\log y = \lim_{x\to 0}\frac{-2x^{-3}}{x\sin x/x\cos x - \sin x} = \frac{0}{0}$ form

$\log y = \lim_{x\to 0}\frac{-2x^{-3}(x\cos x - \sin x)}{x\sin x}$

$\rightarrow$ By LH Rule, $\log y = \lim_{x\to 0}\frac{-2x^{-3}(-x\sin x + \cancel{\cos x} - \cancel{\cos x})}{x\cos x + \sin x} = \frac{0}{0}$ form

$\log y = \lim_{x\to 0}\frac{2x^{-2}\sin x}{x\cos x + \sin x}$

$\rightarrow$ By LH Rule, $\log y = \lim_{x\to 0}\frac{(2x^{-2})(\cos x) + (\sin x)(-4x^{-3})}{-x\sin x + \cos x + \cos x} = \frac{0}{2}$

$\therefore \boxed{y = 1}$

vii) $\lim_{x\to 0}(\sin x)^{\sqrt{x}} = 0^0$ form.

$\rightarrow \log y = \lim_{x\to 0}\sqrt{x}\log(\sin x) = \lim_{x\to 0}\frac{\sqrt{x}}{1/\log(\sin x)} = \frac{0}{0}$ form.

$\rightarrow$ By LH Rule, $\log y = \lim_{x\to 0}\frac{1/2\sqrt{x}}{\sin x/\cos x} = \lim_{x\to 0}\frac{1/2\sqrt{x}}{\tan x}$

$\log y = \lim_{x\to 0}\frac{1}{2\sqrt{x}\tan x}$

$\log y = \frac{1}{0} = \infty$

$\log y = \infty$

$y = e^\infty$

$\boxed{y = 0}$

viii) $\lim_{x\to 0} (1+\sin 4x)^{\cot x} = 1^{\infty}$ form

$\rightarrow \log y = \lim_{x\to 0} (\cot x) \log(1+\sin 4x) = \lim_{x\to 0} \frac{\log(1+\sin 4x)}{\tan x} = \frac{0}{0}$ f

$\rightarrow$ By LH Rule, $\log y = \lim_{x\to 0} \frac{\frac{1}{1+\sin 4x}(4\cos 4x)}{\sec^2 x} = \frac{1/1\,(4)}{1} = 4$

$\boxed{y = e^4}$

$\Rightarrow$ Find a, b, c.

i) $\lim_{x\to 0} \frac{x(1-a\cos x) + b\sin x}{x^3} = \frac{1}{3}$

$\rightarrow$ Put $x=0$, we get $0/0$ form.

$\therefore$ By LH Rule,

$= \lim_{x\to 0} \frac{x(+a\sin x) + (1-a\cos x) + b\cos x}{3x^2} = \frac{1}{3}$

$\Rightarrow 0+1-a+b = 0 \Rightarrow \boxed{a-b=1}$ — (1)

$\rightarrow$ Again by LH Rule.

$= \lim_{x\to 0} \frac{x(a\cos x) + a\sin x + a\sin x - b\sin x}{6x} = \frac{0+0}{0} = \frac{0}{0}$ form

$\rightarrow = \lim_{x\to 0} \frac{x(-a\sin x) + a\cos x + 2a\cos x - b\cos x}{6} = \frac{0+a+2a-b}{6}$

Given the limit is $1/3$,

we get $\boxed{3a-b=2} \Rightarrow$ (2)

$\rightarrow$ Solve (1) & (2)

$a-b = 1$

$3a-b = 2$

$-2a = -1 \Rightarrow \boxed{a = 1/2}$ & $\boxed{b = -1/2}$

ii) $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$

$\rightarrow$ put $x=0$, we get $\boxed{a-b+c=0}$ — (1)

→ By LH Rule,

$$\Rightarrow \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{x\cos x + \sin x} = \frac{a+0-c}{0} = \frac{0}{0} \text{ form}$$

$$\therefore \boxed{a-c=0} \quad -(2)$$

→ Again by LH Rule,

$$\Rightarrow \lim_{x\to 0} \frac{ae^x + b\cos x + ce^x}{-x\sin x + 2\cos x} = \frac{a+b+c}{2} = 2 \text{ (Given)}$$

$$\therefore \boxed{a+b+c = 4}$$

→ Solving (1), (2), (3),

$$a+b+c = 4$$
$$a-b+c = 0$$
$$2a+2c = 4 \Rightarrow \boxed{a+c=2}$$
$$a-c=0$$
$$2a = 2 \Rightarrow \boxed{a=1}, \boxed{c=1}, \boxed{b=2}$$

* $\lim_{x\to 0} \left[\frac{1}{x^2} - \cot^2 x\right]$. $= \infty - \infty$ form.

→ $= \lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\tan^2 x}\right] = \lim_{x\to 0} \left[\frac{\tan^2 x - x^2}{x^2(\tan^2 x)}\right] = \frac{0}{0}$ form.

→ By LH Rule,

$$= \lim_{x\to 0} \left[\frac{2\tan x\sec^2 x - 2x}{x^2(2\tan x\sec^2 x) + \tan^2 x(2x)}\right] = \lim_{x\to 0} \left[\frac{\cancel{2}(\tan x\sec^2 x - x)}{\cancel{2}(x^2\tan x\sec^2 x + x\tan^2 x)}\right]$$

$$= \lim_{x\to 0} \left[\frac{\tan x\sec^2 x - x}{x^2\tan x\sec^2 x + x\tan^2 x}\right]$$

$$= \lim_{x\to 0} \left[\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right] = \lim_{x\to 0} \left[\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}\right] = \frac{0}{0} \text{ form}$$

Multiply & Divide by x^2 in denominator.

$$= \lim_{x\to 0} \left[\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x\,(x^2/x^2)}\right] = \lim_{x\to 0} \left(\frac{1}{\frac{\sin^2 x}{x^2}}\right) 1 \left(\frac{\sin^2 x - x^2\cos^2 x}{x^4}\right)$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2\cos^2 x}{x^4} = \frac{0}{0} \text{ form}$$

→ Apply LH Rule,

$$\rightarrow = \lim_{x\to 0} \frac{2\sin x\cos x - x^2(-2\sin x\cos x) - \cos^2 x\,(2x)}{4x^3}$$

$$= \lim_{x\to 0} \frac{\sin 2x + x^2\sin 2x - \cos^2 x\,(2x)}{4x^3} = \frac{0}{0} \text{ form}$$

$$\rightarrow = \lim_{x\to 0} \frac{2\cos 2x + x^2(2\cos 2x) + \sin 2x\,(2x) - \cos^2 x\,(2) + 2x\,(\sin 2x)}{12x^2}$$

$$= \lim_{x\to 0} \frac{2\cos 2x + 2x^2\cos 2x + 4x\sin 2x - 2\cos^2 x}{12x^2} = \frac{0}{0} \text{ form}$$

$$\rightarrow = \lim_{x\to 0} \frac{-4\sin 2x + 2x^2(-2\sin 2x) + 4x\cos 2x + 4x(2\cos 2x) + 4\sin 2x + 4\cos x\sin x}{24x}$$

$$= \lim_{x\to 0} \frac{-4\sin 2x - 4x^2\sin 2x + 4x\cos 2x + 8x\cos 2x + 4\sin 2x + 2\sin 2x}{24x}$$

$$= \lim_{x\to 0} \frac{+2\sin 2x + 12x\cos 2x - 4x^2\sin 2x}{24x} = \frac{0}{0} \text{ form}$$

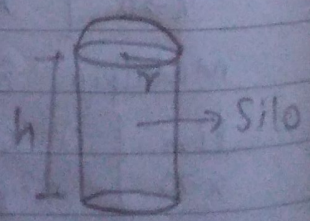
$$\rightarrow = \lim_{x\to 0} \frac{4\cos 2x + 12x(-2\sin 2x) + \cos 2x\,(12) - 4x^2(2\cos 2x) - \sin 2x\,(8x)}{24}$$

$$= \lim_{x\to 0} \frac{4\cos 2x - 24x\sin 2x + 12\cos 2x - 8x^2\cos 2x - 8x\sin 2x}{24}$$

$$= \frac{4 - 0 + 12 - 0 - 0}{24} = \frac{16}{24} = \boxed{\frac{2}{3}}$$

* A grain silo consists of a cylindrical main section, with height 30ft, & a hemispherical roof. In order to achieve a total volume of 15000 ft^3 (including part inside roof section), what would the radius of silo have to be? Use NR Method.

→ Let the height of silo be $h = 30ft$.
Let the radius of hemisphere be r.



→ Volume,

$$V = \pi r^2 h + \tfrac{2}{3}\pi r^3$$

$$15000 = 94.2r^2 + 2.09r^3$$

$$2.09r^3 + 94.2r^2 - 15000 = 0.$$

→ Let $f(r) = 2.09r^3 + 94.2r^2 - 15000$.

$$f'(r) = 6.28r^2 + 188.4r$$

→ For finding initial approximate root x_0,

$f(x) \rightarrow f(0) = -15000$, $f(1) = -14903$, $f(10) = -3490$.

$f(11) = -820.01$, $f(12) = 2176.32$.

$\therefore \boxed{x_0 = 11}$

→ 1st approximate root,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 11.2895$$

→ 2nd approximate root,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 11.2849$$

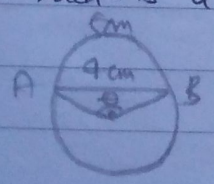
→ 3rd approximate root,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 11.2849$$

→ Since $x_2 = x_3$, the value of radius is $\boxed{11.2849 \text{ ft}}$ when the volume is 15000 ft^3.

* The length of chord AB is 4cm & arc AB is 5cm, Find θ in radians & degress.

→



Let chord AB, $x = 4$cm & arc lenght, $y = 5$cm.

→ WKT, arc length = radius × θ. — ①

chord length = radius × $2\sin(\theta/2)$ — ②

→ ① → $5 = r\theta$ & ② → $4 = 2r\sin(\theta/2)$

Dividing both, $5/4 = \theta / 2\sin(\theta/2)$

$1.25 \times 2\sin(\theta/2) = \theta$

$2.5\sin(\theta/2) = \theta$

→ $\therefore f(\theta) = 2.5\sin(\theta/2) - \theta$.

$f'(\theta) = 2.5\cos(\theta/2)(1/2) - 1$.

$f(0) = 0$, $f(1) = 0.69$

$f(2) = 0.10$, $f(3) = -0.5$

$\Rightarrow x_0 = \boxed{\theta_0 = 2}$

→ $\theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)} = 2.31$, $\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)} = 2.2640$,

$\theta_3 = \theta_4 = \boxed{2.2622}$ $\therefore \boxed{\theta = 2.2622 \text{ rad} = 50.35^\circ}$

Ex: Volume of a cone is $27 cm^3$. Consider its slant height 'l' & radius 'r'. Find the minimum area.

S ⇒ Volume of cone $= 27 cm^3$

$27 = \frac{1}{3}(\pi r^2 h) \Rightarrow h = 81/\pi r^2$

→ Area, $A = \pi r l$ — (1)

→ By pythagoras thoorm,

$l^2 = r^2 + h^2$

$l = \sqrt{r^2 + h^2}$ → Put in (1)

→ $A = \pi r (\sqrt{r^2 + h^2})$

$$\boxed{A = \pi r \left(\sqrt{r^2 + (81/\pi r^2)^2}\right)}$$

CHAPTER 04

INTEGRALS

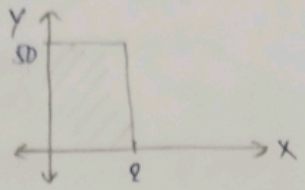
* Distance as Area :-

→ A vehicle travels at 50 miles per hour for 2 hrs. How far has it travelled?

→ Area = $2 \times 50 = \boxed{100}$

Distance = Time × velocity

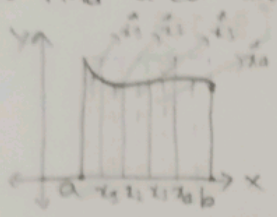
$= 2 \times 50 = \boxed{100 \text{ miles}}$



* Area = $\Delta x\, f(x_1) + \Delta x\, f(x_2) + \dots + \Delta x\, f(x_n)$

$$A = \sum_{i=1}^{n} \Delta x\, f(x_n) = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \Delta x\, f(x_n) \Rightarrow \boxed{\text{Area} = \int f(x)\,dx}$$

⇒ To find area under the curve $y = f(x)$ above x-axis.



→ First we divide the area into 'n' strips.

Width = $\boxed{\Delta x = \frac{b-a}{n}}$

Area of 1st rectangle = $f(x_1^*)\Delta x$

Area of 2nd rectangle = $f(x_2^*)\Delta x$

⋮

Area of nth rectangle = $f(x_n^*)\Delta x$.

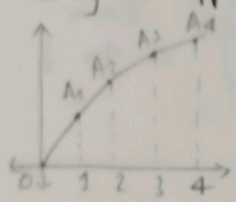
$$\boxed{\therefore \text{Area} = \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x}$$

This sum is known as $\boxed{\text{Reimann Sum}}$.

→ As $n \to \infty$, $\Delta x \to 0$ ⇒ $\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \boxed{\int_a^b f(x)\,dx}$

LP 1) Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x=0$ to $x=4$ using 4 approximating rectangles.

S ⇒



→ From 0 to 4 we need 4 rectangles

Width = $\frac{b-a}{n} = \frac{4-0}{4} = \boxed{1}$

→ Area = Width × Height

→ Mid pts are 0.5, 1.5, 2.5, 3.5.

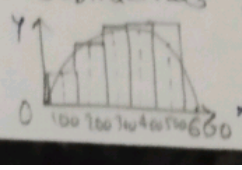
→ Area under curve = $A_1 + A_2 + A_3 + A_4 = \Sigma \text{width} \times f(\text{mid pt})$

$A = 1\,f(0.5) + 1\,f(1.5) + 1\,f(2.5) + 1\,f(3.5)$

$A = \sqrt{0.5} + \sqrt{1.5} + \sqrt{2.5} + \sqrt{3.5} = \boxed{5.3838 \text{ sq. units}}$

LP 2) Estimate area under the graph $f(x) = 600 - x^2$ over $[0, 600]$ using 6 subintervals & 12 subintervals.

S ⇒



⇒ 6 subintervals ⇒ $\Delta x = \frac{600-0}{6} = 100$.

⇒ 12 subintervals ⇒ $\Delta x = \frac{600-0}{12} = 50$.

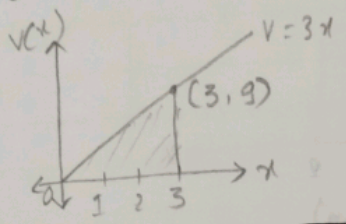
⇒ For 6, Area = Σ width × f (mid pts) = 36500000 sq. units.

⇒ For 12, Area = Σ width × f (mid pts) = 36125000 sq. units.

LP 3) The velocity of a moving object is $v(x) = 3x$. Find area under it which is the distance travelled by it a) During 1st 3hrs b) [3,5]

S⇒ a) Area under the graph (Distance covered during 1st 3 hrs).

→

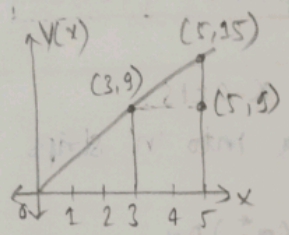


Distance Covered $= \frac{1}{2} \times$ Base × Height

$D = \frac{1}{2} \times 3 \times 9 = 27/2$

$\boxed{\therefore \text{Distance Covered} = 13.5 \text{ miles}}$

b)



Distance Covered = Area of trapezium

$= \frac{1}{2} \times 2 \times (9 + 15)$

$\therefore \boxed{D = 24 \text{ miles}}$

Note: i) Differentiation is a process of determining the rate of change in one variable with respect to another variable.

ii) Integration is the process of bringing smaller components into a single unit that acts as one single component.

iii) The Definite Integral of f(x) is a Number & represents area under the curve f(x) from x = a to x = b.

iv)

$$\int_a^b f(x)\,dx$$

Integral sign ← ∫; b → Upper Limit; a → Lower Limit; f(x) → Integrand; dx → variable of integration.

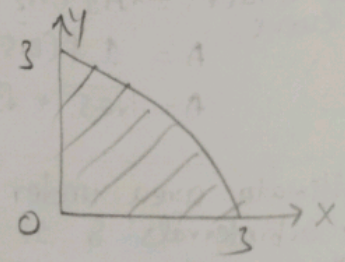
Integral of f from a to b.

Ex: $f(x) = \int_0^3 \sqrt{9 - x^2}\,dx$

→ Given $f(x) = \int_0^3 \sqrt{9 - x^2}\,dx$

Let $y = \sqrt{9 - x^2}$

$x^2 + y^2 = 9 \Rightarrow \boxed{x^2 + y^2 = 3^2}$



Area $= \frac{1}{4}\pi r^2 = \frac{1}{4} \times \pi \times 3^2 = \frac{9\pi}{4}$

$\boxed{\text{Area} = 9\pi/4 \text{ sq. units}}$

→ Let $f(x)$ be an integrable function defined on an interval $[a,b]$. Let $A1$ represent the area between $f(x)$ & the X-axis that lies above the axis & let $A2$ represent the area between $f(x)$ & the x-axis that lies below the axis.

→ Net Signed Area : $\int_a^b f(x)\,dx = A_1 - A_2$.

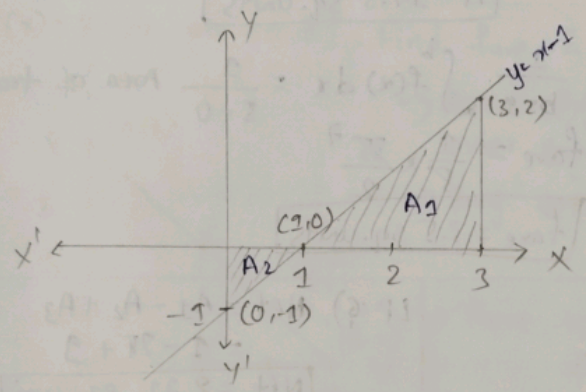
→ Total Area : $\int_a^b |f(x)|\,dx = A_1 + A_2$.

1.6) Evaluate the integrals by interpreting each in terms of areas.

→ $\int_0^3 (x-1)\,dx$.

$y = x-1 \Rightarrow x - y = 1$

$m = 1,\ c = -1$.



→ From figure,

Area of $A_1 = \frac{1}{2} \times b \times h$

$A_1 = \frac{1}{2}(2)(2)$

$\boxed{A_1 = 2 \text{ sq. units}}$

Area of $A_2 = \frac{1}{2} \times b \times h$

$A_2 = \frac{1}{2} \times 1 \times 1$

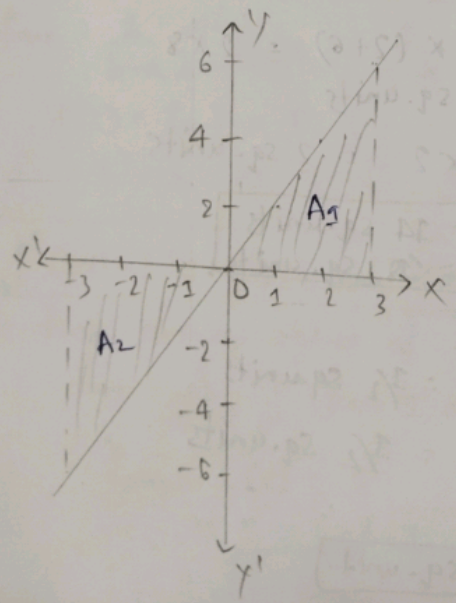
$\boxed{A_2 = \frac{1}{2} \text{ sq. units}}$

∴ Net Signed Area $= A_1 - A_2 = 2 - \frac{1}{2} = \boxed{\frac{3}{2} \text{ sq. units}}$

Total Area $= A_1 + A_2 = 2 + \frac{1}{2} = \boxed{\frac{5}{2} \text{ sq. units}}$

• Consider $f(x) = 2x$, $[-3,3]$. Find Net Signed area of curve wrt X-axis.

→



→ From figure,

$A_1 = \frac{1}{2} \times 3 \times 6 = \underline{9 \text{ sq. units}}$

$A_2 = \frac{1}{2} \times 3 \times 6 = \underline{9 \text{ sq. units}}$

∴ Net Signed Area,

$= A_1 - A_2 = \boxed{0 \text{ sq. units.}}$

→ Total Area $= \underline{18 \text{ sq. units}}$

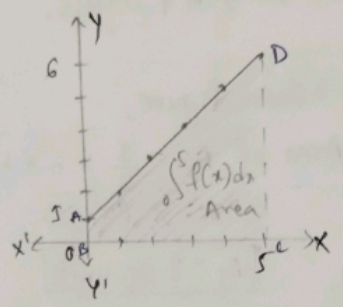
* Average Value of a function :-

→ Let $f(x)$ be continuous over the interval $[a,b]$. Then average value of $f(x)$ (f_{ave}) on $[a,b]$ is given by,

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Ex: Consider $f(x) = x+1$ over $(0,5)$. Find f_{ave}.

→



Area of Trapezium $= \frac{1}{2}$ (Distance b/w Parallel Sides) (Height)

$= \frac{1}{2}(5)(1+6) = \frac{35}{2}$

$\boxed{A = 17.5 \text{ sq. units}}$

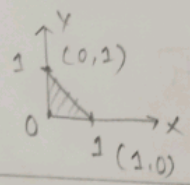
$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{5-0}$ Area of trapezium

$f_{ave} = \frac{1}{\cancel{5}} \times \frac{\cancel{35}^{7}}{2}$

$\boxed{f_{ave} = 3.5 \text{ sq. units}}$

LP 4) $\int_0^1 \sqrt{1-x^2}\,dx$

→

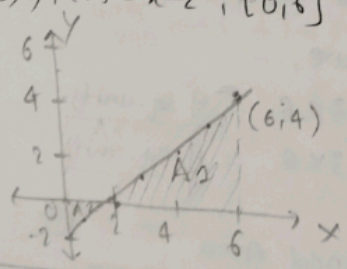


Area $= \frac{1}{2} \times 1 \times 1$

$\boxed{\text{Area} = \frac{1}{2} \text{ sq. units}}$

LP 5) i) $f(x) = x-2$, $[0,6]$

→



$A_1 = \frac{1}{2} \times 4 \times 4$

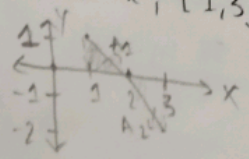
$A_1 = 8$ sq. units

$A_2 = \frac{1}{2} \times 2 \times 2 = 2$ sq. units

$\boxed{\text{Net Area} = 6 \text{ sq. units}}$
$\boxed{\text{Total Area} = 10 \text{ sq. units}}$

ii) $f(x) = 2-x$, $[1,3]$.

→



$A_1 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ sq. units

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ sq. units

$\boxed{\text{Net Area} = 0}$

Total Area $= \boxed{1 \text{ sq. unit}}$

LP 7) Net $= A_1 - A_2 + A_3$

$= 1 - 2\pi + 9$

$\boxed{\text{Net} = 3.72 \text{ sq. units}}$

→ Total $= A_1 + A_2 + A_3$

$= 1 + 2\pi + 9$

$\boxed{\text{Total} = 16.28 \text{ sq. units}}$

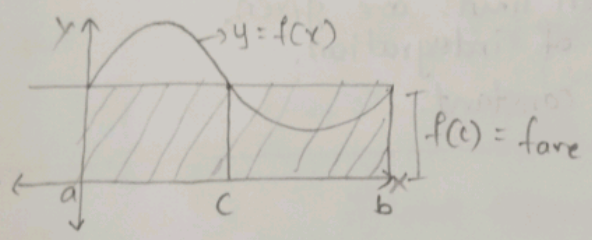
* Mean Value Theorem for Integral :-

→ If $f(x)$ is continuous over an interval $[a,b]$, then there is atleast one point $c \in [a,b]$ such that $\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$

→ This formula is also stated as,

$$\boxed{\int_a^b f(x)\,dx = (b-a)f(c)}$$

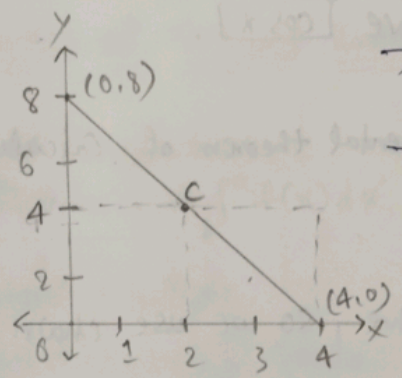
⇒ Geometric Interpretation



$\boxed{\text{Area of rectangle} = \text{Area under the curve}}$

LP10) $f(x) = 8-2x$ on $[0,4]$. Find fave & c such that $f(c) = f_{ave}$.

S⇒



→ Area under graph $= \frac{1}{2} \times 4 \times 8 = \boxed{16 \text{ sq. units}}$

→ Avg. area $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4}(16)$

$= \boxed{4 \text{ sq. units}}$

→ There is a point such that $f(c) = 4 \Rightarrow \boxed{c=2}$

→ Area of rectangle $= l \times b = 4 \times 4 = \boxed{16 \text{ sq. units}} \Rightarrow \boxed{\text{Area of Rectangle} = \text{Area of under graph}}$

LP8) $\int_0^3 x^2\,dx = 9$ over $[0,3]$. Find c such that $f(c) = f_{ave}$.

S⇒ $f_{ave} = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{3-0}(9) = \frac{9}{3} = 3$ sq. units

$f(c) = f_{ave} = 3$

$x^2 = 3 \Rightarrow x = \sqrt{3} \Rightarrow \boxed{c = \sqrt{3}}$

LP9) $\int_0^3 (2x^2-1)\,dx = 15$ over $[0,3]$.

S⇒ To find average value of $f(x) = 2x^2-1$ over $[0,3]$

$f_{ave} = \frac{1}{3-0}\int_0^3 (2x^2-1)\,dx = \frac{1}{3}(15) = 5$ sq. units.

$f_{ave} = f(c) = 5$

$\Rightarrow 5 = 2c^2 - 1 \Rightarrow \cancel{6}^{3} = \cancel{2}c^2$

$\boxed{c = \sqrt{3}}$

* Fundamental Theorem of Calculus :-

→ There are 2 fundamental theorems of calculus.

Theorem 1) If f is continuous on $[a,b]$, then the function $F(x)$ is defined by $F(x) = \int_a^x f(t)\,dt$ then

$$\boxed{\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)} \text{ over } [a,b].$$

⇒ Derivative of an integral when limits are given.
⇒ Derivative matches upper limit of integration.
⇒ Lower limit of integration is constant.

Ex: $\frac{d}{dx}\int_{\pi}^{x} \cos t\,dt = \cos x$.

→ Here upper limit & derivative variable are same (x)

∴ By 1st fundamental theorem we have $\boxed{\cos x}$.

Ex: $\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \boxed{\frac{1}{1+x^2}}$. (By 1st Fundamental theorem of Calculus).

Ex: $\frac{d}{dx}\int_0^{x^2} \cos t\,dt = \cos x^2 (2x) = \boxed{2x\cos x^2}$

→ Here upper limit doesnot match derivative, so we use chain rule.

$\therefore \frac{d}{dx}\int_0^{x^2} \cos t\,dt = \cos(x^2)\frac{d}{dx}x^2 = 2x\cos x^2$.

Ex: $\frac{d}{dx}\int_x^5 3t\sin t\,dt$.

→ Here upper limit is constant so reverse the limits.

$\therefore \frac{d}{dx}\int_x^5 3t\sin t\,dt = -\frac{d}{dx}\int_5^x 3t\sin t\,dt = \boxed{-3x\sin x}$

Theorem 2) If f is continuous at every point of $[a,b]$, & if F is any anti-derivative of f on $[a,b]$, then,

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

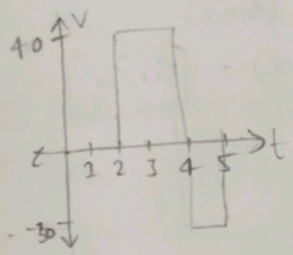
⇒ This theorem is also called Integral Evaluation Theorem.

Ex: $\int_1^2 (x^3+6)\,dx$

$\rightarrow \left[\frac{x^4}{4}+6x\right]_1^2 = \left[\frac{16}{4}+12-\frac{1}{4}-6\right] = \frac{15}{4}+6 = \boxed{\frac{39}{4}}$

Ex: A car is moving due North at 40mph b/w 2PM & 4PM. Then the car moves South at 30mph b/w 4PM & 5PM. Find net displacement, total distance travelled.

$\rightarrow$



i) Net Displacement $= \int_2^4 40\,dt + \int_4^5 -30\,dt$

$= [40t]_2^4 - [30t]_4^5$

$= 80 - 30 = \boxed{50 \text{ miles}}$

ii) Total distance travelled $= 80 + 30 = \boxed{110 \text{ miles}}$

* Properties of Definite Integral :-

1) Order of Integration.

$\rightarrow \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2) Zero width Interval.

$\rightarrow \int_a^a f(x)\,dx = 0.$

3) Constant Multiple.

$\rightarrow \int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx.$

4) Sum and Difference.

$\rightarrow \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

$\int \log x\,dx = x\log x - x, \; x \neq 0.$

5) Additivity.

$\rightarrow \int_a^b f(x)\,dx \quad \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$

* Bernoulli's Rule of Integration :-

$\rightarrow \boxed{\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots} \text{ until zero.}$

where, $u, u', u'', u''', \ldots \rightarrow$ Derivatives of u.

$v_1, v_2, v_3, v_4, \ldots \rightarrow$ Integrals of v.

Ex: $\int x^3 e^{2x}\,dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right) + 0$

* Integration By Parts :-

$\rightarrow \boxed{\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx}$

Arrangement of Functions $\boxed{\text{ILATE}}$ Rule.

LP 11) Using Fundamental Theorem of Calculus find $F'(x)$.

i) $F(x) = \int_1^{\sqrt{x}} \sin t \, dt$

$\rightarrow F'(x) = \sin\sqrt{x} \frac{d}{dx}\sqrt{x} = \boxed{\frac{\sin\sqrt{x}}{2\sqrt{x}}}$

ii) $F(x) = \int_1^{x} e^{\cos t} dt$

$\rightarrow F'(x) = \boxed{e^{\cos x}}$

iii) $F(x) = \int_x^{2x} t \, dt = -\int_0^x t\,dt + \int_0^{2x} t\,dt$

$= -x + 2x(2) = 4x - x = 3x.$

$\rightarrow F(x) = \left[\frac{t^2}{2}\right]_x^{2x} = \frac{4x^2}{2} - \frac{x^2}{2}$

$F(x) = \frac{3x^2}{2}$ & $F'(x) = \boxed{3x}$

iv) $F(x) = \int_{\cos x}^{1} \sqrt{1-t^2}\, dt = -\int_1^{\cos x} \sqrt{1-t^2}\, dt$

$\rightarrow F'(x) = -\sqrt{1-\cos^2 x}\,(-\sin x)$

$= \sqrt{\sin^2 x}\,(\sin x) = \boxed{\sin^2 x}$

LP 12) Evaluate the integrals.

i) $\int_{-2}^{2} (t^2-4)\,dt = \left[\frac{t^3}{3} - 4t\right]_{-2}^{2} = \left[\frac{8}{3} - 8 + \frac{8}{3} - 8\right] = \frac{16}{3} - 16 = \boxed{\frac{-32}{3}}$

ii) $\int_1^2 x^{-4}\,dx = \left[\frac{-x^{-3}}{3}\right]_1^2 = \left[\frac{-1}{3x^3}\right]_1^2 = \left[\frac{-1}{24} + \frac{1}{3}\right] = \boxed{\frac{7}{24}}$

iii) $\int_{-2}^{3} (x^2+3x-5)\,dx = \left[\frac{x^3}{3} + \frac{3x^2}{2} - 5x\right]_{-2}^{3} = \left[9 + \frac{27}{2} - 15 + \frac{8}{3} - 6 - 10\right]$

$= \left[-6 - 6 - 10 + \frac{97}{6}\right] = -22 + \frac{97}{6} = \boxed{\frac{-35}{6}}$

iv) $\int_0^{\pi/2} \sin\theta\, d\theta = \left[-\cos\theta\right]_0^{\pi/2} = 0 + 1 = \boxed{1}$

LP 13) Given velocity function $v(t) = 3t - 5$ (in m/s) for a particle in motion from time $t=0$ to time $t=3$, find net displacement.

S=) $V = 3t - 5$

Displacement $= \int_0^3 (3t-5)\,dt$

$= \left[\frac{3t^2}{2} - 5t\right]_0^3 = \frac{27}{2} - 15 = \frac{-3}{2}$

$\boxed{\text{Displacement} = -1.5 \text{ m}}$

* Net Change Theorem :-

→ This theorem considers the integral of a rate of change, it says when a quantity changes, the new value equals (the initial value) + (the integral of the rate of change of that quantity).

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

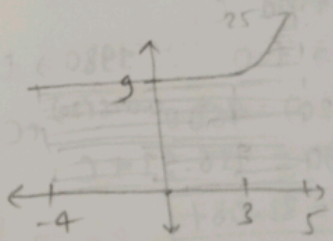
$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

LP 15) Find area under the graph $y = x^2+1$ over $[-1, 2]$.

S⇒ $\text{Area} = \int_{-1}^{2} (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^{2} = \frac{8}{3} + 2 + \frac{1}{3} + 1 = 3 + 3 = \boxed{6 \text{ sq. units}}$

LP 16) Find area under $y = f(x)$ from $[-4, 5]$.

$$f(x) = \begin{cases} 9, & x < 3 \\ x^2, & x \geq 3 \end{cases}$$



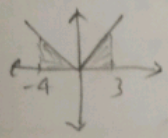
S⇒ $\text{Area} = \int_{-4}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$

$= \int_{-4}^{3} 9\,dx + \int_{3}^{5} x^2\,dx = [9x]_{-4}^{3} + \left[\frac{x^3}{3}\right]_{3}^{5}$

$= [27+36] + \left[\frac{125}{3} - \frac{27}{3}\right] = 63 + \frac{98}{3} = \frac{189+98}{3} = \boxed{\frac{287}{3}}$

LP 17) Evaluate each of the definite integrals.

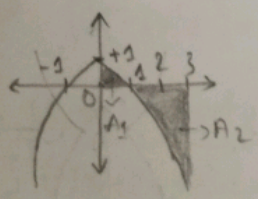
i) $\int_{-4}^{3} |x|\,dx = \int_{-4}^{0} -x\,dx + \int_{0}^{3} x\,dx = \left[\frac{-x^2}{2}\right]_{-4}^{0} + \left[\frac{x^2}{2}\right]_{0}^{3}$



$= \frac{16}{2} + \frac{9}{2} = \boxed{\frac{25}{2}}$

Mod Function
⇒ Calculate Total Area.

ii) $\int_{0}^{3} |1-x^2|\,dx \Rightarrow A_1 = \int_{0}^{1} (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{0}^{1} = 1 - \frac{1}{3} = \frac{2}{3}$

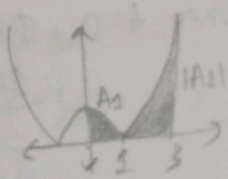


$A_2 = \int_{1}^{3} (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{1}^{3} = 3 - 9 - 1 + \frac{1}{3} = -7 + \frac{1}{3}$

$= \frac{-20}{3}$

$\therefore \int_{0}^{3} |1-x^2|\,dx = \left|\frac{2}{3}\right| + \left|\frac{-20}{3}\right|$

$= \frac{2}{3} + \frac{20}{3} = \boxed{\frac{22}{3}}$



Take original f^n & calculate

18-12-24

LP Ex: The rate of change of population is modelled by exponential func $P'(t) = 11.7e^{0.026t}$ where t is no. of years since 1960 & $P'(t)$ in thousands of people per year. In 1980, it was 790.

a) Find $P(t)$ b) Find $P(t)$ in 2012.

S⇒ Rate of change of population, $P'(t) = 11.7e^{0.026t}$.

$$P(t) = \int P'(t)\,dt$$

$$= \int 11.7e^{0.026t}\,dt$$

$$= \frac{11.7e^{0.026t}}{0.026} + C$$

$$P(t) = 450\,e^{0.026t} + C \quad - (1)$$

→ To find C,

$1960 \Rightarrow t=0$ ∴ $1980 \Rightarrow t=20$. ; At $t=0$, $P(0) = 450 + C$

$$P(20) = 450e^{0.026(20)} + C$$

$$790 = 756.91 + C$$

$$\boxed{C = 33.087}$$

a) ∴ $\boxed{P(t) = 450e^{0.026t} + 33.087}$

b) $2012 \Rightarrow t=52$

$$P(52) = 450e^{0.026(52)} + 33.087.$$

$$\boxed{P(52) = 1772.4}$$

LP 19) $P'(t) = 45$, $Q'(t) = 105e^{0.03t}$

In 1990, $P(t) = 5000$, $Q(t) = 3500$. → Find population models, Population in 2000, draw graph.

S⇒ Rate of change of population in Alphaville, Betaburgh.

$P'(t) = 45$, $Q'(t) = 105e^{0.03t}$

∴ Population, $P(t) = \int P'(t)\,dt = \int 45\,dt = 45t + C_1 \quad - (1)$

$Q(t) = \int Q'(t)\,dt = \int 105e^{0.03t}\,dt = \frac{105e^{0.03t}}{0.03} + C_2 \quad - (2)$

→ When $t=0$, $P(t) = 5000$

$45t + C_1 = 5000 \Rightarrow \boxed{C_1 = 5000}$

→ When $t=0$, $Q(t) = 3500$

$$\frac{105e^{0.03(0)}}{0.03} + C_2 = 3500 \Rightarrow \frac{105}{0.03} + C_2 = 3500$$

$$\boxed{C_2 = 0}$$

$\therefore \boxed{\begin{array}{l} P(t) = 45t + 5000 \\ Q(t) = 3500e^{0.03t} \end{array}}$

$\rightarrow \boxed{P(10) = 5450}$ & $Q(10) = 4724.5$ $\rightarrow$ In 2000.

$\rightarrow$ Since both cities are expected to have same population, we solve for 't',

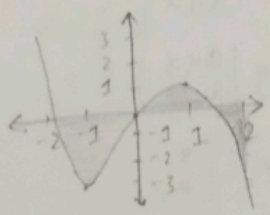
$P(t) = Q(t)$

$45t + 5000 = 3500e^{0.03t}$ $\Rightarrow \boxed{t = 16.5} \Rightarrow$ In 2006

Common Population ≈ 5742.77.

LP 20) Consider $\int_{-1}^{2} (-x^3 + 3x - 1)\,dx$. Predict sign of the result by examining the graph & then evaluate it.

S$\Rightarrow$



$\rightarrow$ The total area below X-axis is more than the area above X-axis. The sign of net area is negative.

$\rightarrow$ By Solving,

$$\rightarrow \int_{-1}^{2} (-x^3 + 3x - 1)\,dx = \left[\frac{-x^4}{4} + \frac{3x^2}{2} - x\right]_{-1}^{2}$$

$$= \left[-4 + 6 - 2 + \frac{1}{4} - \frac{3}{2} - 1\right] = -1 - \frac{5}{4} = \boxed{\frac{-9}{4}}$$

Negative.

LP 18) Integrate $\int_{-2}^{2} (3x^8 - 2)\,dx$ & verify that the integration formula for even function holds.

S$\Rightarrow$ <u>Normal Integration</u>

$$\rightarrow \int_{-2}^{2} (3x^8 - 2)\,dx = \left[\frac{3x^9}{9} - 2x\right]_{-2}^{2} = \left[\frac{1536}{9} - 4 + \frac{1536}{9} - 4\right] = \boxed{333.33}$$

$\Rightarrow$ <u>Using Formula</u>

Since $3x^8 - 2$ is even function ($\because 3(-x)^8 - 2 = 3x^8 - 2$)

$$\therefore \int_{-2}^{2} (3x^8 - 2)\,dx = 2\int_{0}^{2} (3x^8 - 2)\,dx = 2\left[\frac{3x^9}{9} - 2x\right]_{0}^{2} = 2\left[\frac{1536}{9} - 4\right] = \boxed{333.33}$$

LP 22) Use the properties of definite integrals to express $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx$ as sum of 3 definite integrals.

S$\Rightarrow$ $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx = \left[\frac{-3x^4}{4} + \frac{2x^2}{2} + 2x\right]_{-2}^{1} = \left[\frac{-3}{4} + 1 + 2 + 12 - 4 + 4\right]$

$= \boxed{57/4}$

$\rightarrow$ Using Property,

$$\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx = \int_{-2}^{-1} (-3x^3 + 2x + 2) + \int_{-1}^{0} (-3x^3 + 2x + 2) + \int_{0}^{1} (-3x^3 + 2x + 2)$$

$$= \left[-3/4 + 1 - 2 + 12 - 4 - 4\right] + \left[3/4 - 1 + 2\right] + \left[-3/4 + 1 + 2\right]$$

$$= \left[41/4\right] + \left[7/4\right] + \left[9/4\right] = \boxed{57/4}$$

INTEGRATION FORMULAE

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x} dx = \log x$$

Add 'c' constant for all indefinite integrals

$$\int k\,dx = kx$$

$$\int \log x\,dx = x\log x - x$$

$$\int a^x dx = \frac{a^x}{\log a}$$

$$\int e^x dx = e^x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \tan x\,dx = \log(\sec x)$$

$$\int \cot x\,dx = \log(\sin x)$$

$$\int \sec x\,dx = \log(\sec x + \tan x)$$

$$\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \sec x \tan x\,dx = \sec x$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right|$$

$$\int \frac{1}{\sqrt{a^2+x^2}} dx = \log\left|x+\sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{f'(x)}{f(x)} dx = \log(f(x))$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even function.} \\ 0, & \text{if } f(x) \text{ is odd function.} \end{cases}$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x). \\ 0, & \text{if } f(2a-x) = -f(x). \end{cases}$$

$$\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx.$$

LP 21) Evaluate each of the following.

a) $\int_{-1}^{2}(x^2-x)\,dx = \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{2} = \left[\frac{8}{3}-\frac{4}{2}+\frac{1}{3}+\frac{1}{2}\right] = 3-\frac{3}{2} = \boxed{\frac{3}{2}}$

b) $\int_{0}^{3} e^x\,dx = [e^x]_0^3 = e^3-e^0 = \boxed{e^3-1}$

c) $\int_{1}^{e}\left(1+2x-\frac{1}{x}\right)dx = [x+x^2-\log x]_1^e = [e+e^2-\cancel{\log e}^{\,1}-1-1+0]$

$= \boxed{e(1+e)-3}$

LP 24) i) $\int x^2\sqrt{x^3+1}\,dx$

$\rightarrow$ Take $x^3+1=t \rightarrow 3x^2dx=dt \rightarrow x^2dx=\frac{dt}{3}$

$= \int x^2\sqrt{t}\,dx = \int \frac{\sqrt{t}}{3}\,dt$

$= \frac{1}{3}\left(\frac{2t^{3/2}}{3}\right)+c = \boxed{\frac{2(x^3+1)^{3/2}}{9}+c}$

ii) $\int \cos^3\theta\sin\theta\,d\theta$.

$\rightarrow$ Take $\cos\theta = t \Rightarrow -\sin\theta\,d\theta = dt$

$\Rightarrow \int\cos^3\theta\sin\theta\,d\theta = -\int t^3\,dt = -\left[\frac{t^4}{4}\right] = \boxed{-\frac{\cos^4\theta}{4}}$

iii) $\int_0^\infty \frac{\sin(\tan^{-1}t)}{1+t^2}\,dt$

$\rightarrow$ Take $\tan^{-1}t = y \Rightarrow \frac{1}{1+t^2} = \frac{dy}{dt}$ when $t=0$, $y=0$. When $t=\infty$, $y=\pi/2$.

$= \int \frac{\sin(\tan^{-1}t)}{1+t^2}\,dt = \int \sin y\,dy = -\cos y + c = -\cos(\tan^{-1}t)+c.$

$\rightarrow \int_0^\infty \frac{\sin(\tan^{-1}t)}{1+t^2}\,dt = [-\cos(\tan^{-1}t)]_0^\infty = [-0+1] = \boxed{1}$

iv) $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

$\rightarrow$ Take $1/x = t \Rightarrow \frac{-1}{x^2} = \frac{dt}{dx}$

$\rightarrow \int \frac{e^{1/x}}{x^2}\,dx = -\int e^t\,dt = -e^{1/x}+c$

$\rightarrow \int_1^2 \frac{e^{1/x}}{x^2}\,dx = -[e^{1/x}]_1^2 = -[e^{1/2}-e] = \boxed{-\sqrt{e}+e}$

v) $\int_0^1 x e^{-x^2} dx$.

→ Take $x^2 = t \Rightarrow 2x\,dx = dt$

→ $\int x e^{-x^2} dx = \frac{1}{2}\int e^{-t} dt = \frac{1}{2}\left[-e^{-t} + c\right] = \frac{1}{2}\left[-e^{-x^2} + c\right]$

→ $\int_0^1 x e^{-x^2} dx = \frac{1}{2}\left[-e^{-x^2}\right]_0^1 = \frac{1}{2}\left[-e^{-1} + 1\right] = \frac{1}{2} - \frac{e^{-1}}{2} = \frac{1-e^{-1}}{2} = \boxed{\frac{1}{2}}$

vi) $\int_1^2 \frac{\cos(\ln(2x))}{x} dx$.

→ Take $\ln(2x) = t \Rightarrow \frac{1}{x} dx = dt$

→ $\int \frac{\cos(\ln 2x)}{x} dx = \int \cos t\, dt = \sin t + c = \sin(\ln 2x) + c.$

→ $\int_1^2 \frac{\cos(\ln 2x)}{x} dx = \left[\sin(\ln 2x)\right]_1^2 = \sin(1.38) - \sin(0.69)$

$= 0.98 - 0.63 = \boxed{0.34}$

vii) $\int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx$

→ Take $1 + \cos x = t \Rightarrow -\sin x\, dx = dt$.

→ $\int \frac{\sin x}{1+\cos x} dx = -\int \frac{dt}{t} = -\left[\log t + c\right] = -\left[\log(1+\cos x) + c\right]$

→ $\int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx = -\left[\log(1+\cos x)\right]_0^{\pi/2} = -\left[\log 1 - \log 2\right] = -[0-0.3$

$= \boxed{0.3010}$

LP 23) i) $\int (x^2 + x^{-2}) dx$

→ $\boxed{\frac{x^3}{3} - \frac{1}{x} + c}$

ii) $\int (u+4)(2u+1) du$

→ $= \int (2u^2 + u + 8u + 4) du$

$= \boxed{\frac{2u^3}{3} + \frac{u^2}{2} + 4u^2 + 4u + c}$

iii) $\int_0^1 (5x - x^5) dx = \left[\frac{5x^2}{2} - \frac{x^6}{6}\right]_0^1 = \left[\frac{5}{2} - \frac{1}{6}\right] = \boxed{\frac{14}{6}}$

iv) $\int_0^{\pi} (\sin x - \cos x) dx = \left[-\cos x - \sin x\right]_0^{\pi} = \left[1 - 0 + 1\right] = \boxed{2}$

v) $\int_0^{\pi/4} \frac{1+\cos^2 x}{\cos^2 x} dx = \int_0^{\pi/4} (\sec^2 x + 1) dx = \left[\tan x + x\right]_0^{\pi/4}$

$= \boxed{1 + \pi/4}$

$\boxed{\int uv = u\int v - \int du \int v}$

LP 25) Integration by Parts.

i) $\int \ln x \, dx$

$\rightarrow \int \ln x \, dx = \int \overset{\rightarrow u}{\ln x} \cdot \overset{\rightarrow v}{1} \, dx$

$= \ln x \int 1 \, dx - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$

$= x \ln x - x + c$

$\boxed{\int \ln x \, dx = x \ln x - x + C}$

ii) $\int_1^2 x^3 \ln x \, dx$.

$\rightarrow \int \ln x \cdot x^3 dx = \ln x \int x^3 dx - \int \frac{1}{x} \cdot \frac{x^4}{4} dx$

$= \ln x \left(\frac{x^4}{4}\right) - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$

$= \frac{x^4}{4}\left[\ln x - \frac{1}{4}\right] + c$

$\rightarrow \int_1^2 \ln x \cdot x^3 dx = \left[\frac{x^4}{4}\left(\ln x - \frac{1}{4}\right)\right]_1^2 = \left[4(0.69 - 0.25) - \frac{1}{4}(-0.25)\right]$

$= 1.76 + 0.0625 = \boxed{1.8225}$

iii) $\int_0^1 z^2 e^z dz$.

$\rightarrow \int z^2 e^z dz = z^2 \int e^z dz - \int 2z e^z dz$

$= z^2 e^z - 2\int z e^z dz$.

$= z^2 e^z - 2\left[z e^z - e^z\right]$

$= z^2 e^z - 2z e^z + 2e^z + c$

$\rightarrow \int_0^1 z^2 e^z dz = \left[z^2 e^z - 2z e^z + 2e^z\right]_0^1$

$= \left[e - 2e + 2 - 2\right] = \boxed{-e}$

$\rightarrow \int_0^1 z^3 e^z dz$

$\rightarrow$ Bernoulli's Theorem.

$= \left[z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z\right]_0^1$

$= e - 3e + 6e - 6e + 6$

$= \boxed{6 - 2e}$

iv) $\int_0^{2\pi} t^2 \sin 2t \, dt$.

$\rightarrow = \left[t^2\left(\frac{-\cos 2t}{2}\right) - 2t\left(\frac{-\sin 2t}{4}\right) + 2\left(\frac{\cos 2t}{8}\right)\right]_0^{2\pi}$

$= \left[4\pi^2\left(\frac{-1}{2}\right) + \frac{1}{4} - \frac{1}{4}\right] = \boxed{-2\pi^2}$

* Numerical Integration :-

$\rightarrow$ Sometimes it is difficult to evaluate an integral analytically. In such case, we use numerical integration techniques.

① Trapezoidal Rule :-

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left[\underbrace{(y_0+y_n)}_{1^{st}+last} + 2\underbrace{(y_1+y_2+\cdots+y_{n-1})}_{\text{middle terms}}\right]}$$

$h \to$ Step-length , $\boxed{h = \frac{b-a}{n}}$

$n \to$ No. of sub-divisions.

② Simpson's $1/3^{rd}$ Rule :-

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[\underbrace{(y_0+y_n)}_{(1st+last)} + 2\underbrace{(y_2+y_4+\cdots+y_{n-2})}_{(Even)} + 4\underbrace{(y_1+y_3+\cdots+y_{n-1})}_{(odd)}\right]}$$

Note : Simpson's $1/3^{rd}$ Rule is applied only when 'n' is even.

LP 26) Use trapezoidal rule to evaluate $\int_0^{1/2} \sin t\,dt$, $n=8$

S ⇒ $h = \frac{b-a}{n} = \frac{1/2 - 0}{8} = \boxed{\frac{1}{16}}$

t	0	$1/16$	$2/16$	$3/16$	$4/16$	$5/16$	$6/16$	$7/16$	$8/16 \to 1/2$
$\sin t$	0	0.0624	0.1246	0.1864	0.2474	0.3074	0.3662	0.4236	0.4794
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

→ By Trapezoidal Rule,

$$\int_0^{1/2} \sin t\,dt = \frac{1/16}{2}\left[(0+0.4794)+2(1.718)\right]$$

$$= \frac{1}{32}\left[0.4794+3.436\right] = \frac{3.9154}{32} = \boxed{0.1224}$$

$$\to \int_0^{1/2} \sin t\,dt = \left[-\cos t\right]_0^{1/2} = -\cos 1/2 + \cos 0 = \boxed{0.1224}$$

LP 27) Using Simpson's $1/3^{rd}$ rule evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$, $n=6$. Hence find π.

S ⇒ $h = \frac{b-a}{h} = \frac{1-0}{6} = \frac{1}{6}$

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$
$1/1+x^2$	1	0.9729	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

→ By Simpson's Rule,

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/6}{3}\left[(1+0.5)+2(0.9+0.6923)+4(0.9729+0.8+0.5901)\right]$$

$$= \frac{1}{18}\left[1.5+3.1846+9.452\right] = \boxed{0.7853}$$

$$\rightarrow \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1)-\tan^{-1}(0) = \pi/4$$

$$\therefore \pi/4 = 0.7853 \Rightarrow \pi = 4\times 0.7853 \Rightarrow \boxed{\pi = 3.1414}$$

LP28) A river is 80m wide. The depth 'y' of the river at a distance 'x' from one bank is given in the table. Find area of cross-section of river using Simpson's 1/3rd Rule.

→

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

S⇒ Area of cross-section = width × Depth → day'

$$\boxed{A = \int_{x=0}^{80} y\,dx}$$

$a=0,\ b=80,\ n=8,\ h=10.$

→ By Simpson's rule,

$$A = \frac{10}{3}\left[(0+3)+2(7+12+14)+4(4+9+15+8)\right]$$

$$= \frac{10}{3}\left[3+66+144\right] = \boxed{710 \text{ sq. meters}}$$

Imp

LP 29) The velocity 'v' of a particle at distance 's' from a pt is given in table. Estimate time taken to travel 60ft by using Simpson's Rule.

→

S (ft)	0	10	20	30	40	50	60
v (ft/s)	47	58	64	65	61	52	38

S⇒ Time = $\frac{\text{Distance}}{\text{Velocity}}$ ⇒ $\boxed{t = \int_{s=0}^{60} \frac{1}{v}\,ds}$ $\quad a=0,\ b=60,\ n=6,\ h=10$

→ By Simpson's Rule,

$$t = \frac{10}{3}\left[\left(\frac{1}{47}+\frac{1}{38}\right)+\left(\frac{1}{64}+\frac{1}{61}\right)+\left(\frac{1}{58}+\frac{1}{65}+\frac{1}{52}\right)\right]$$

$$t = \frac{h}{3}\left[\left(\frac{1}{v_0}+\frac{1}{v_6}\right)+2\left(\frac{1}{v_2}+\frac{1}{v_4}\right)+4\left(\frac{1}{v_1}+\frac{1}{v_3}+\frac{1}{v_5}\right)\right]$$

$$t = \frac{10}{3}\left[0.0475+0.0640+0.2074\right]$$

$$\boxed{t = 1.063 \text{ seconds.}}$$

LP 30) The table shows values of force function $f(x)$. Use Simpson's to estimate the work done to move a distance of 18 m.

→

x	0	3	6	9	12	15	18
$f(x)$	9.8	9.1	8.5	8.0	7.7	7.5	7.4

S⇒ Work Done = Force × Distance

$$\boxed{W = \int_0^{18} f(x)\,dx}.$$

$a = 0$ $h = 3$
$b = 18$ $n = 6$

→ By Simpson's Rule,

$$W = \frac{3}{3}\left[(9.8+7.4) + 2(8.5+7.7) + 4(9.1+8.0+7.5)\right]$$

$$= [17.2 + 32.4 + 98.4] = \boxed{148\ Nm}$$

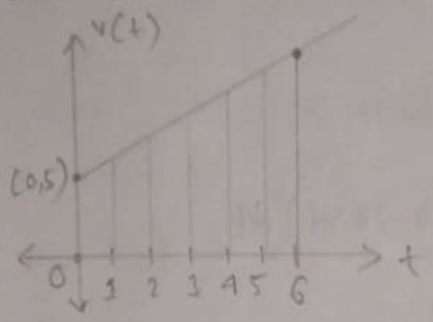
CHAPTER 05

APPLICATIONS OF INTEGRATION

* Area Under the Curve :-

LP 1) A vehicle is moving with velocity $v(t) = (3t+5)$. What is the distance travelled from $t=0$ to $t=6$?

S⇒ $v(t) = 3t+5$.



Distance Travelled = velocity × Time.

$$\therefore S = \int_0^6 v(t)\,dt$$
$$= \int_0^6 (3t+5)\,dt$$
$$= \left[\frac{3t^2}{2} + 5t\right]_0^6$$
$$= [54+30] = \boxed{84\,m}$$

LP 2) The speed of hare is given by $H(t) = \frac{1}{2} - \frac{1}{2}\cos(2\pi t)$, whereas speed of tortise is $T(t) = \sqrt{t}$. If race ends in 1 hr, who won & by how much?

S⇒ velocity of Hare, $H(t) = \frac{1}{2} - \frac{1}{2}\cos(2\pi t) = 0.5 - 0.5\cos(2\pi t)$.

∴ Distance travelled by hare in 1 hr is,

$$S_H = \int_0^1 H(t)\,dt$$
$$= \int_0^1 (0.5 - 0.5\cos(2\pi t))\,dt$$
$$= \left[0.5t - \frac{0.5\sin(2\pi t)}{2\pi}\right]_0^1$$
$$= 0.5 - 0$$
$$\boxed{S_H = 0.5\,km}$$

→ velocity of Tortois, $T(t) = \sqrt{t}$

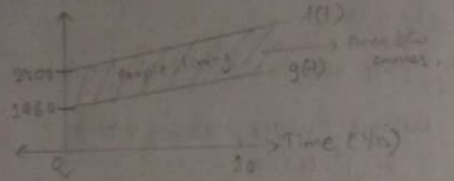
∴ Distance travelled by tortoise in 1 hr is,

$$S_T = \int_0^1 T(t)\,dt$$
$$= \int_0^1 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^1 = \frac{2}{3} - 0$$
$$\boxed{S_T = 0.67\,km}$$

∴ Distance covered by tortoise in 1 hr is more than hare. ∴ Tortoise wins the race by 0.17 km.

LP 5) If birth rate of a population of a city is modelled as $f(t) = 2200 + 52.3t$ people per year. The death rate of a same city is modelled as $g(t) = 1460 + 28.8t$ people per year. Find the area between these curves for $0 \le t \le 10$. What does this area represent?

$f(t) = 2200 + 52.3t$ → Birth Rate.
$g(t) = 1460 + 28.8t$ → Death Rate.



→ Area b/w curves $= \int_0^{10} (f(t) - g(t))\,dt$

$= \int_0^{10} (2200 + 52.3t - 1460 - 28.8t)\,dt$

$= \int_0^{10} (740 + 23.5t)\,dt$

$= \left[740t + \frac{23.5t^2}{2}\right]_0^{10}$

$= \left[7400 + \frac{23.5[100]}{2}\right] = 7400 + 1175$

$\boxed{\text{Area b/w curves} = 8575 \text{ people per year}}$

∴ 8575 represents increase in population over a 10-yr-period.

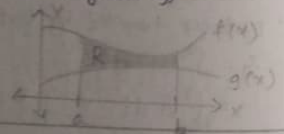
* Area of a Region between 2 curves :-

→ Let $f(x)$ & $g(x)$ be continous functions over an interval $[a,b]$, such that $f(x) \geq g(x)$ on $[a,b]$.

→ Then the area R b/w 2 curves is given by

$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$$

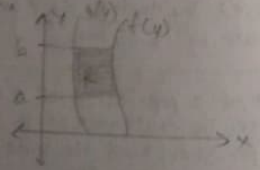
$$A = \int_a^b \left(\text{Upper Curve} - \text{Lower Curve}\right) dx.$$



→ Similarly area b/w curves $x = f(y)$ & $x = g(y)$, $f(y) > g(y)$ bounded b/w $y = a$ to $y = b$ is.

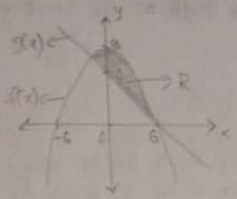
$$\boxed{A = \int_a^b [f(y) - g(y)]\,dy}$$

$$A = \int_a^b \left(\text{Right Curve} - \text{Lower Curve}\right) dy$$



LP 3) If R is the region bounded by graph of $f(x) = \frac{9}{4}(\frac{x}{2})^2$ & below by the graph of function $g(x) = 6 - x$. Find its area.

S⇒



$A = \int_a^b (f(x) - g(x))\,dx$

$A = \int_a^b [9 - x^2/4 - 6 + x]\,dx$

Here, $b = 6$

→ To find, a, we need find the intersection b/w both curves.

$f(x) = g(x)$

$9 - \frac{x^2}{4} = 6 - x \Rightarrow 3 - \frac{x^2}{4} + x = 0$

$x^2 - 4x - 12 = 0$

$x^2 - 6x + 2x - 12 = 0$

$x(x-6) + 2(x-6) = 0$

$\boxed{x = -2}$ & $x = 6$

$-12x^2$
$-6x \quad 2x$
$-4x$

∴ $a = -2$.

∴ $A = \int_{-2}^{6} [3 - x^2/4 + x] = \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]_{-2}^{6}$

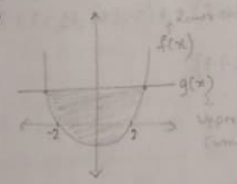
$= \left(18 - \frac{216}{12} + \frac{36}{2}\right) - \left(-6 + \frac{8}{12} + \frac{4}{2}\right) = 18 - \frac{216}{12} + 18 + 6 - \frac{8}{12} - 2$

$= 36 + 4 - \frac{224}{12} = 40 - \frac{\cancel{224}^{56}}{\cancel{12}_3} = 40 - \frac{56}{3} = \frac{120 - 56}{3} = \frac{64}{3}$

$\boxed{\text{Area} = 64/3 \text{ sq. units}}$

LP 4) If R is the region bounded by $y = x^2 - 3$ & $g(x) = 1$, find its area.

S⇒



→ Points of Intersection

$f(x) = g(x)$

$x^2 - 3 = 1$

$x^2 = 4 \Rightarrow \boxed{x = \pm 2}$

∴ Area $= \int_{-2}^{2} [g(x) - f(x)]\,dx$

$= \int_{-2}^{2} (1 - x^2 + 3) = \left[-\frac{x^3}{3} + 4x\right]_{-2}^{2}$

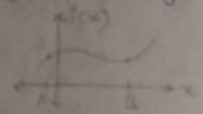
$= \frac{-8}{3} + 8 - \left(\frac{8}{3} - 8\right) = \frac{-16}{3} + 16 = \frac{-16 + 48}{3}$

$\boxed{\text{Area} = 32/3 \text{ sq. units}}$

* Arc Length of a Curve :-

→ If $f'(x)$ is continuous on $[a,b]$, then the length of the curve $y=f(x)$ is given by,

$$\boxed{l = \int_a^b \sqrt{1+(f'(x))^2}\,dx}$$



LP 10) A steady wind blows a kite west. The kite's height above ground from horizontal position $x=0$ to $x=80$ is given by $y = 150 - \frac{1}{40}(x-50)^2$. Find the distance travelled by the kite.

Sol Given $y = 150 - \frac{1}{40}(x-50)^2 = f(x)$

Distance travelled by the kite is the length of arc from 0 to 80

$$\therefore l = \int_0^{80} \sqrt{1+f'(x)^2}\,dx$$

$$\rightarrow f'(x) = \frac{1}{20}(x-50) = \frac{x-50}{20} = \frac{x}{20} - \frac{5}{2}$$

$$l = \int_0^{80} \sqrt{1+\left(\frac{x}{20}-\frac{5}{2}\right)^2}\,dx$$

$$= \int_0^{80} \sqrt{1+\frac{(x-50)^2}{400}}\,dx = \frac{1}{20}\int_0^{80} \sqrt{400+x^2+2500-100x}\,dx$$

→ Analytic Method fails, we solve this by Numerical Method i.e. Simpson's Rule.

x	0	10	20	30	40	50	60	70	80
y_i	2.7	2.23	1.80	1.41	1.11	1	1.11	1.41	1.80
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$n=8$
$h=10$
$a=0$
$b=80$

$$\int_a^b f(x)\,dx = \int_0^{80} \sqrt{1+\frac{(x-50)^2}{400}}\,dx$$

$$l = \frac{10}{3}\left[(2.7+1.80) + 2(1.80+1.11+1.11) + 4(2.23+1.41+1.41)\right]$$

$$l = \frac{10}{3}\left[4.5 + 8.04 + 24.1\right] = \frac{10}{3}[36.74]$$

$$\boxed{l = 122.46 \text{ units}}$$

LP 8) Find the arc length of the following curves.

i) $y = \ln x$, $1 \le x \le \sqrt{3}$.

ii) $y = 1 + 6x^{3/2}$, $0 \le x \le 1$

Sol i) $y = \ln x$, $1 \le x \le \sqrt{3}$ $\qquad f(x) = \ln x$, $f'(x) = \frac{1}{x}$.

$$l = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$l = \int_1^{\sqrt{3}} \sqrt{1+\frac{1}{x^2}}\,dx$$

$a=1$ $\quad n=6$
$b=\sqrt{3}$ $\quad h = \frac{\sqrt{3}-1}{6} = 0.122$

x	1	1.122	1.244	1.366	1.488	1.61	1.732
y_i	1.41	1.339	1.283	1.239	1.204	1.177	1.154

By Simpson's 1/3rd rule,

$$l = \int_1^{\sqrt{3}} \sqrt{1+\frac{1}{x^2}}\,dx = \frac{0.122}{3}\left[(1.41+1.154) + 2(1.283+1.204) + 4(1.339+1.239+1.177)\right]$$

$$= 0.04\left[2.564 + 4.974 + 15.02\right]$$

$$= 0.04 \times 22.558$$

$$\boxed{l = 0.9 \text{ units}}$$

ii) $f(x) = 1 + 6x^{3/2}$ $\quad a=0$, $b=1$.

$f'(x) = 9x^{1/2} = 9\sqrt{x}$

$$l = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+81x}\,dx$$

$n=10$
$h=0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y_i	1	3.01	4.14	5.02	5.79	6.44	7.04	7.59	8.15	8.59	9.055
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$\therefore l = \frac{0.1}{3}\left[(1+9.055) + 2(4.14+5.79+7.04+8.15) + 4(3.01+5.02+6.44+8.59)\right]$$

$$= 0.033\left[10.055 + 50.32 + 122.6\right]$$

$$= 0.033 \times 182.975$$

$$\boxed{l = 6.0376 \text{ units}}$$

LP 9) A hawk flying at 15m/s at altitude 180m accidently drops its prey. The parabolic trajectory of falling prey is given by $y = 180 - x^2/45$ until it hits ground, where y is its height above the ground & x is the horizontal distance in metres. Calculate the distance travelled from the time it is dropped until it hits the ground.

Sol $f(x) = 180 - \frac{x^2}{45}$ $\qquad$ Distance travelled by the prey is, $f'(x) = -\frac{2x}{45}$

$$\text{Dist} = \int_0^{180} \sqrt{1+\frac{4x^2}{2025}}\,dx$$

→ $a=0$, $b=180$, $n=10$, $h=18$

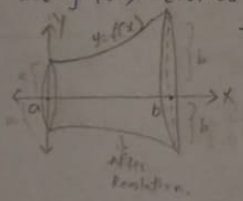
x	0	18	36	54	72	90	108	126	144	162	180
$\sqrt{1+\frac{4x^2}{2025}}$	1	1.28	1.88	2.6	3.35	4.12	4.9	5.68	6.47	7.26	8.06

$$\text{Dist} = \frac{18}{3}\left[9.06 + 33.2 + 83.76\right] = 6 \times 126.02$$

$$\boxed{\text{Dist} = 756.12 \text{ m}}$$

* Surface Area & Volume of Revolution :-

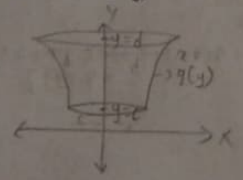
→ Let $y=f(x)$ revolved about x-axis.



→ Then area of revolution is,

$$\boxed{SA = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

→ Let $x=g(y)$ revolved above Y-axis.



→ Then area of revolution is

$$\boxed{SA = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}$$

→ The curve $y=f(x)$ bounded b/w $x=a$ & $x=b$ is revolved about x-axis then volume of solid of revolution is,

$$\boxed{V = \int_a^b \pi y^2\, dx}$$

→ The curve $x=g(y)$ bounded b/w $y=c$ & $y=d$ is revolved about Y-axis then volume of solid of revolution is,

$$\boxed{V = \int_c^d \pi x^2\, dy}$$

LP 12) Find the area of surface obtained by rotating $y=x^3$ b/w 0 to 2 about x-axis.

S⇒ $a=0$, $b=2$, $y=x^3$, $y'=3x^2$

$$SA = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 2\pi x^3 \sqrt{1+(3x^2)^2}\, dx$$

$$= \int_0^2 2\pi x^3 \sqrt{1+9x^4}\, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\, dx$$

→ Take $1+9x^4 = t$, $36x^3 = dt/dx$, $x^3\, dx = \frac{dt}{36}$; $x \to 0, t \to 1$; $x \to 2, t \to 145$

$$= \frac{2\pi}{36} \int_1^{145} \sqrt{t}\, dt$$

$$= \frac{\pi}{18}\left[\frac{2}{3} t^{3/2}\right]_1^{145} = \frac{\pi}{18} \times \frac{2}{3}\left[t^{3/2}\right]_1^{145}$$

$$SA = \frac{\pi}{18} \times \frac{2}{3}\left[1746.03 - 1\right]$$

$$\boxed{SA = 202.94 \text{ sq. units}}$$

LP 11) A lampshade is constructed by rotating $y = 1/x$ around Y-axis from $y=1$ to $y=2$. Determine how much material required to construct this lampshade.

S⇒ $SA = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ $c=1$, $d=2$, $y = 1/x \Rightarrow x = 1/y \Rightarrow \frac{dx}{dy} = \frac{-1}{y^2}$

$$SA = \int_1^2 2\pi \left(\frac{1}{y}\right) \sqrt{1+1/y^4}\, dy$$

→ Amount of material required. (w.r.t Y)

$a=1$, $b=2$, $n=10$, $h=0.1$

y	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
IF	8.88	7.40	6.37	5.61	5.03	4.58	4.21	3.90	3.65	3.43	3.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

→ By Simpson's $1/3^{rd}$ Rule.

$$SA = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+\dots+y_{n-2}) + 4(y_1+\dots+y_{n-1})\right]$$

$$= \frac{0.1}{3}\left[12.11 + 2(19.26) + 4(24.92)\right]$$

$$= \frac{0.1}{3}\left[12.11 + 38.52 + 99.68\right] = 0.033 \times 150.31$$

$$\boxed{SA = 5.0103 \text{ sq. units}}$$

LP 9 Redone

→ Since the prey hits ground, we take $y=0$

$$y = 180 - \frac{x^2}{45}$$

$$\frac{x^2}{45} = 180 \Rightarrow x^2 = 180 \times 45 \Rightarrow \boxed{x = 90}$$

$$\therefore \text{Dist.} = \int_0^{90} \sqrt{1 + \frac{4x^2}{2025}}\, dx \qquad \begin{matrix} a = 0 & n = 10 \\ b = 90 & h = 9 \end{matrix}$$

x	0	9	18	27	36	45	54	63	72	81	90
$\sqrt{1+\frac{4x^2}{2025}}$	1	1.07	1.28	1.56	1.88	2.23	2.6	2.97	3.35	3.73	4.12

$$\text{Dist} = \frac{9}{3}\left[5.12 + 19.22 + 46.24\right] \Rightarrow \boxed{\text{Distance} = 208.74\,m}$$