CSCI 470: Dijkstra’s algorithm, Bellman-Ford algorithm

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Overview

1. Properties of shortest paths and relaxation
2. Correctness of Dijkstra's algorithm
3. Bellman-Ford algorithm
4. Single-source shortest paths in directed acyclic graphs

Properties of shortest paths and relaxation

Properties of shortest paths and relaxation

- **Triangle inequality** (Lemma 24.10)
  For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.
- **Upper-bound property** (Lemma 24.11)
  We always have $v.d \geq \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.
- **No-path property** (Lemma 24.12)
  If there is no path from s to v, then we always have $v.d = \delta(s, v) = \infty$.
- **Convergence property** (Lemma 24.14)
  If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $v.d = \delta(s, v)$ at all times afterward.

Properties of shortest paths and relaxation

- **Path-relaxation property** (Lemma 24.15)
  If $p = \langle v_0, v_1, ..., v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), ..., (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.
- **Predecessor-subgraph property** (Lemma 24.17)
  Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

Correctness of Dijkstra's algorithm

Dijkstra’s algorithm: pseudocode

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

Correctness of Dijkstra's algorithm

We use the following loop invariant:

At the start of each iteration of the ***while*** *loop of lines 4-8, $v.d = \delta(s, v)$ for each vertex $v \in S$.*

Initialization / Base case

Initially, $S = \emptyset$, and so the invariant is trivially true.

A quick recap

- We define the ***shortest-path weights*** $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} min\{w(p) : u \leadsto v\} & \exists \text{ a path } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

- A ***shortest path*** from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.

Maintenance/Inductive step

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- We must have $u \neq s$ because s is the first vertex added to set S and $s.d = \delta(s, s) = 0$ at that time.

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- Because $u \neq s$, we have that $S \neq \emptyset$ just before u is added to S.

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- We must have $u \neq s$ because s is the first vertex added to set S and $s.d = \delta(s, s) = 0$ at that time.
- Because $u \neq s$, we have that $S \neq \emptyset$ just before u is added to S.
- There must be some path from s to u, for otherwise $u.d = \delta(s, u)$. Because there is at least one path, there is a shortest path p from s to u. Prior to adding u to S, path p connects a vertex in S, namely s, to a vertex in $V - S$, namely u.

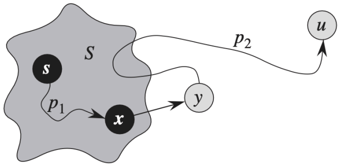

Figure 24.7 The proof of Theorem 24.6. Set S is nonempty just before vertex u is added to it. We decompose a shortest path p from source s to vertex u into $s \overset{p_1}{\leadsto} x \to y \overset{p_2}{\leadsto} u$, where y is the first vertex on the path that is not in S and $x \in S$ immediately precedes y. Vertices x and y are distinct, but we may have $s = x$ or $y = u$. Path p_2 may or may not reenter set S.

Maintenance (contd)

- Let us consider the first vertex y along p such that $y \in V - S$, and let $x \in S$ be y's predecessor along p. Thus, Figure 24.7 illustrates, we can decompose path p into $s \rightsquigarrow x \rightarrow y \rightsquigarrow u$. (Either of paths p_1 or p_2 may have no edges.)

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- Let us consider the first vertex y along p such that $y \in V - S$, and let $x \in S$ be y's predecessor along p. Thus, Figure 24.7 illustrates, we can decompose path p into $s \rightsquigarrow x \to y \rightsquigarrow u$. (Either of paths p_1 or p_2 may have no edges.)
- We claim that $y.d = \delta(s, y)$ when u is added to S. To prove this claim, observe that $x \in S$. Then, because we chose u as the first vertex for which $u.d \neq \delta(s, u)$ when it is added to S, we had $x.d = \delta(s, x)$ when x was added to S. Edge (x, y) was relaxed at that time, the claim follows from the convergence property.

Maintenance (contd)

- We can now obtain a contradiction to prove that $u.d = \delta(s, u)$. Because y appears before u on a shortest path from s to u and all edge weights are non-negative (notably those on path p_2), we have $\delta(s, y) \leq \delta(s, u)$, and thus

$$
\begin{aligned}
y.d &= \delta(s, y) && (1)\\
&\leq \delta(s, u) && (2)\\
&\leq u.d \quad \text{by the upper-bound property} && (3)
\end{aligned}
$$

Maintenance (contd)

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&\leq \delta(s, u) \tag{2}\\
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\end{align}
$$

- But because both vertices u and y were in $V - S$ when u was chosen in line 5, we have $u.d \leq y.d$. Thus, the two inequalities in (2) and (3) are in fact equalities, giving $y.d = \delta(s, y) = \delta(s, u) = u.d$.

Maintenance (contd)

- Consequently, $u.d = \delta(s, u)$, which contradicts our choice of u. We conclude that $u.d = \delta(s, u)$ when u is added to S, and that this equality is maintained at all times thereafter.

Termination

At termination, $Q = \emptyset$ which, along with our earlier invariant that $Q = V - S$, implies that $S = V$. Thus, $u.d = \delta(s, u)$ for all vertices $u \in V$.

Bellman-Ford algorithm

Bellman-Ford algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| − 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

Lemma 24.2

Lemma 24.2
Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, after the $|V| - 1$ iterations of the **for** loop of lines 2-4 of BELLMAN-FORD, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s.

Lemma 24.2: Proof

Proof We prove the lemma by appealing to the path-relaxation property.

Consider any vertex v that is reachable from s, and let $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v. Because shortest paths are simple, p has at most $|V| - 1$ edges, and so $k \leq |V| - 1$. Each of the $|V| - 1$ iterations of the **for** loop of lines 2-4 relaxes all $|E|$ edges. Among the edges relaxed in the i-th iteration, for $i = 1, 2, \ldots, k$, is (v_{i-1}, v_i). By the path-relaxation property, therefore, $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$. $\square$

Corollary 24.3

Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \rightarrow \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, for each vertex $v \in V$, there is a path from s to v **if and only if** BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G.

Proof is left as an exercise for HW 5.

Correctness of Bellman-Ford algorithm

Let BELLMAN-FORD be run on a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \to \mathbb{R}$. If G contains no negative-weight cycles that are reachable from s, then the algorithm returns TRUE, we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s. If G does contain a negative-weight cycles reachable from s, then the algorithm returns FALSE.

Proof: Case 1

Case 1:

Suppose that graph G contains no negative-weight cycles that are reachable from the source s. We first prove the claim that at termination, $v.d = \delta(s, v)$ for all vertices $v \in V$.

- If vertex v is reachable from s, then Lemma 24.2 proves this claim.
- If v is not reachable from s, then the claim follows from the no-path property. Thus, the claim is proven.

Proof: Case 1

- The predecessor-subgraph property, along with the claim, implies that G_π is a shortest-paths tree.
- Now we use the claim to show that Bellman-Ford returns TRUE. At termination, we have for all the edges $(u, v) \in E$,

$$\begin{aligned} v.d &= \delta(s, v) \\ &\leq \delta(s, u) + w(u, v) \\ &= u.d + w(u, v), \end{aligned}$$

and so none of the tests in line 6 causes Bellman-Ford to return FALSE. Therefore, it returns TRUE.

Case 2:

Now, suppose that graph G contains a negative-weight cycle that is reachable from the source s; let this cycle be $c = \langle v_0, v_1, ..., v_k \rangle$, where $v_0 = v_k$. Then,

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0. \tag{4}$$

Proof: Case 2

- Assume for the purpose of contradiction that the BELLMAN-FORD algorithm returns TRUE.
- Thus, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, 2, ..., k$. Summing the inequalities around cycle c gives us

$$\begin{aligned}\sum_{i=1}^{k} v_i.d &\leq \sum_{i=1}^{k}(v_{i-1}.d + w(v_{i-1}, v_i)) \\ &= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)\end{aligned}$$

Proof: Case 2

- Since $v_0 = v_k$, each vertex in c appears exactly once in each of the summations $\sum_{i=1}^{k} v_i.d$ and $\sum_{i=1}^{k} v_{i-1}.d$, and so

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d.$$

- Moreover, by Corollary 24.3, $v_i.d$ is finite for $i = 1, 2, ..., k$. Thus,

$$0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i),$$

which contradicts inequality (4).

Proof

We conclude that the BELLMAN-FORD algorithm returns TRUE if graph G contains no negative-weight cycles reachable from the source, and FALSE otherwise. $\square$

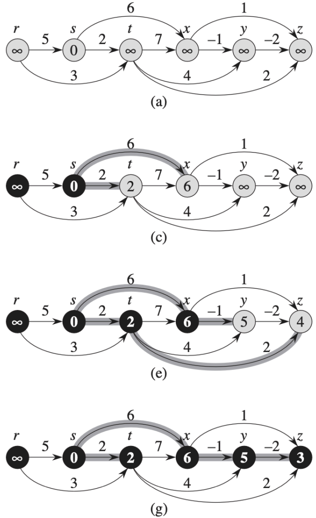
Single-source shortest paths in directed acyclic graphs

Shortest paths in DAGs

DAG-SHORTEST-PATHS(G, w, s)

```
topologically sort the vertices in G
INITIALIZE-SINGLE-SOURCE(G)
for each vertex u, taken in topologically sorted order
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

Shortest paths in DAGs: Illustration



Runtime

- line 1 takes $\Theta(V + E)$.
- line 2 takes $\Theta(V)$.
- **for** loop of lines 3-5 makes one iteration each vertex.
- Altogether, the for loop of lines 4–5 relaxes each edge exactly once.
- Because each iteration of the inner for loop takes $\Theta(1)$ time, the total running time is $\Theta(V + E)$, which is linear in the size of an adjacency-list representation of the graph.