CSCI 470: Single-Source Shortest Paths, Dijkstra's algorithm

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Overview

1. Shortest-paths problems
2. Dijkstra's algorithm

Shortest-paths problems

Shortest-paths problems

- In a ***shortest-paths problem***, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights.
- The ***weight*** $w(p)$ of path $p = \langle v_0, v_1, ..., v_k \rangle$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i).$$

Shortest-paths problems

- We define the ***shortest-path weights*** $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} min\{w(p) : u \rightsquigarrow v\} & \exists \text{ a path } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

- A ***shortest path*** from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.
- Edge weights can represent metrics other than distances, such as time, cost, penalties, loss, or any other quantity than accumulates linearly along a path and that we would want to minimize.

Shortest Paths: Figure

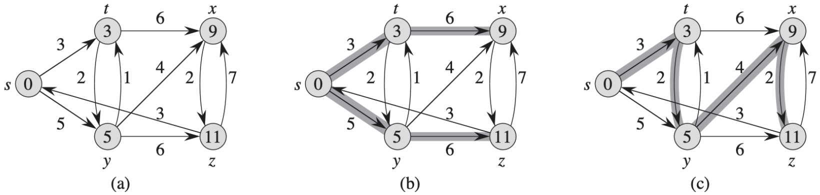

Figure 24.2 **(a)** A weighted, directed graph with shortest-path weights from source s. **(b)** The shaded edges form a shortest-paths tree rooted at the source s. **(c)** Another shortest-paths tree with the same root.

Relaxation

INITIALIZE-SINGLE-SOURCE(G, s)

for each vertex $v \in G.V$
    $v.d = \infty$
    $v.\pi = \text{NIL}$
$s.d = 0$

- After initialization, we have $v.\pi = \text{NIL}$ for all $v \in V$, $s.d = 0$, and $v.d = \infty$ for $v \in V - \{s\}$.

Relax

$\textsc{Relax}(u, v, w)$

if $v.d > u.d + w(u, v)$
  $v.d = u.d + w(u, v)$
  $v.\pi = u$

Relaxation: Figure

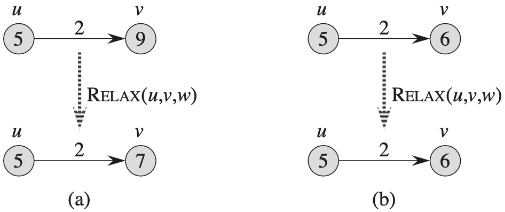

Figure 24.3 Relaxing an edge (u, v) with weight $w(u, v) = 2$. The shortest-path estimate of each vertex appears within the vertex. **(a)** Because $v.d > u.d + w(u, v)$ prior to relaxation, the value of $v.d$ decreases. **(b)** Here, $v.d \le u.d + w(u, v)$ before relaxing the edge, and so the relaxation step leaves $v.d$ unchanged.

Dijkstra’s algorithm

Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

Dijkstra: Figure