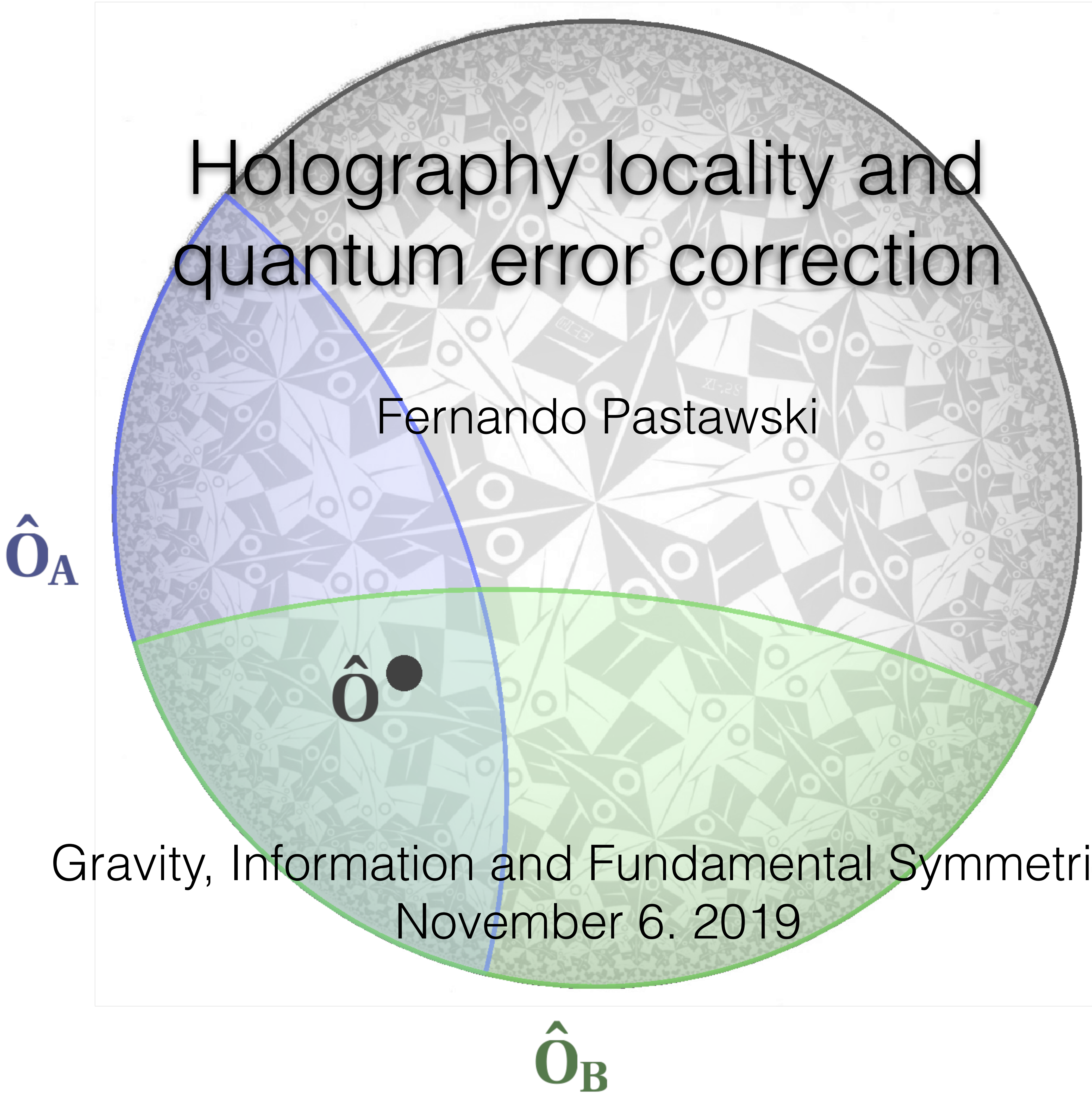
Holography locality and quantum error correction

Fernando Pastawski



Gravity, Information and Fundamental Symmetries
November 6. 2019

Quantum Mechanics -> Quantum Field Theory

Superposition

(Entanglement)

Unitarity

$$H(t)\,|\,\psi(t)\rangle = i\hbar\frac{d}{dt}\,|\,\psi(t)\rangle$$

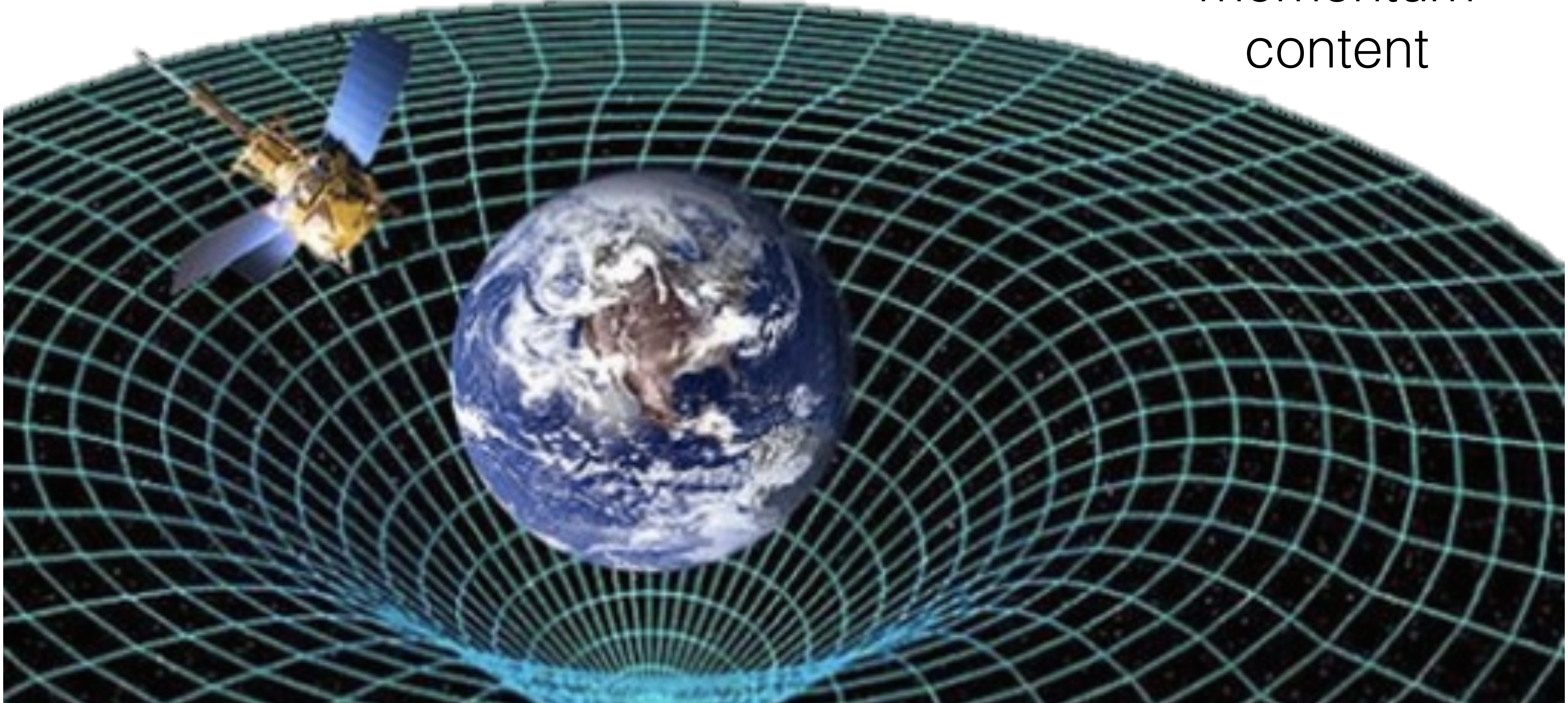
(Conservation of information)

General relativity

Intrinsic curvature

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

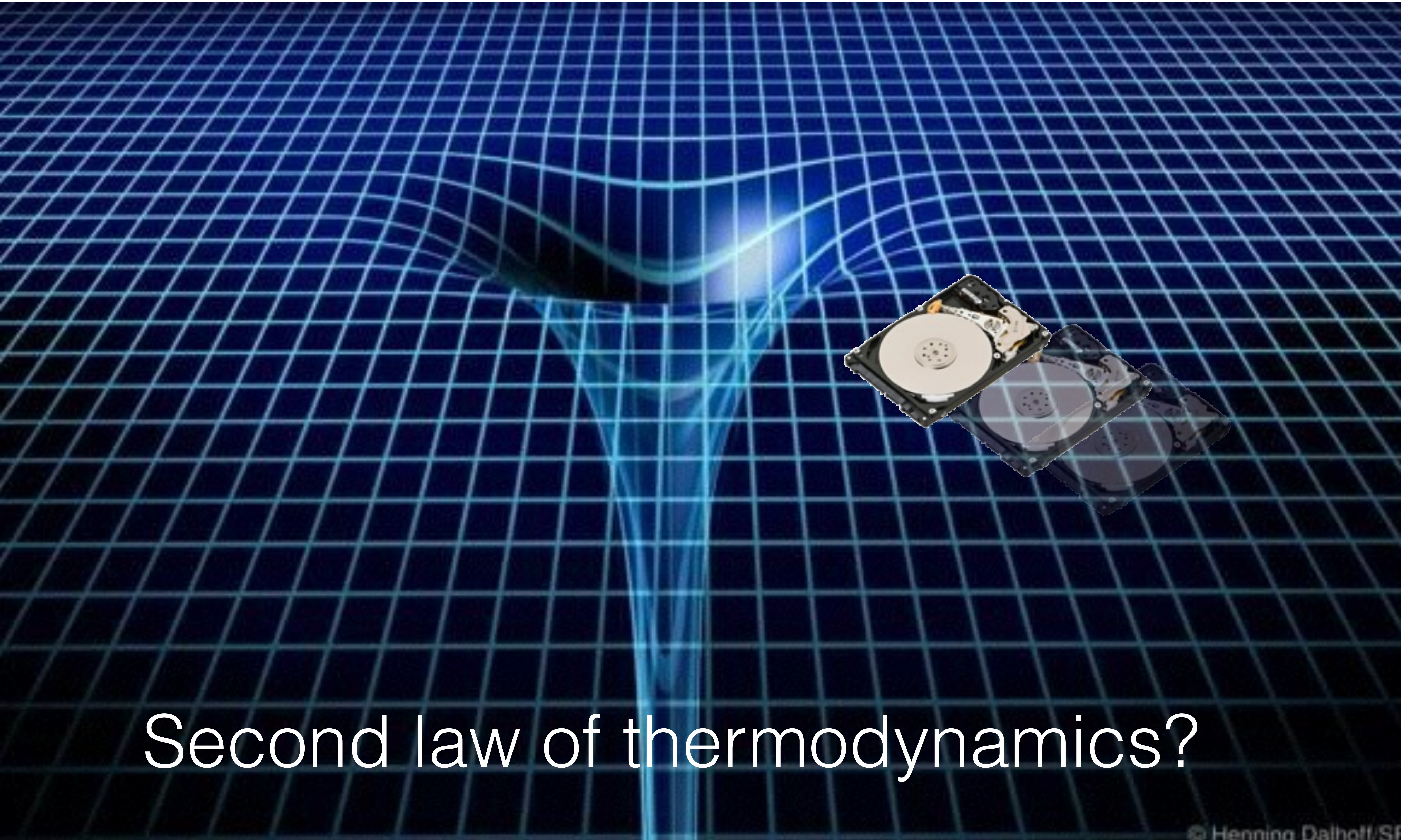
Energy, momentum content



Einstein, A. "Näherungsweise Integration der Feldgleichungen der Gravitation". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin. 1916

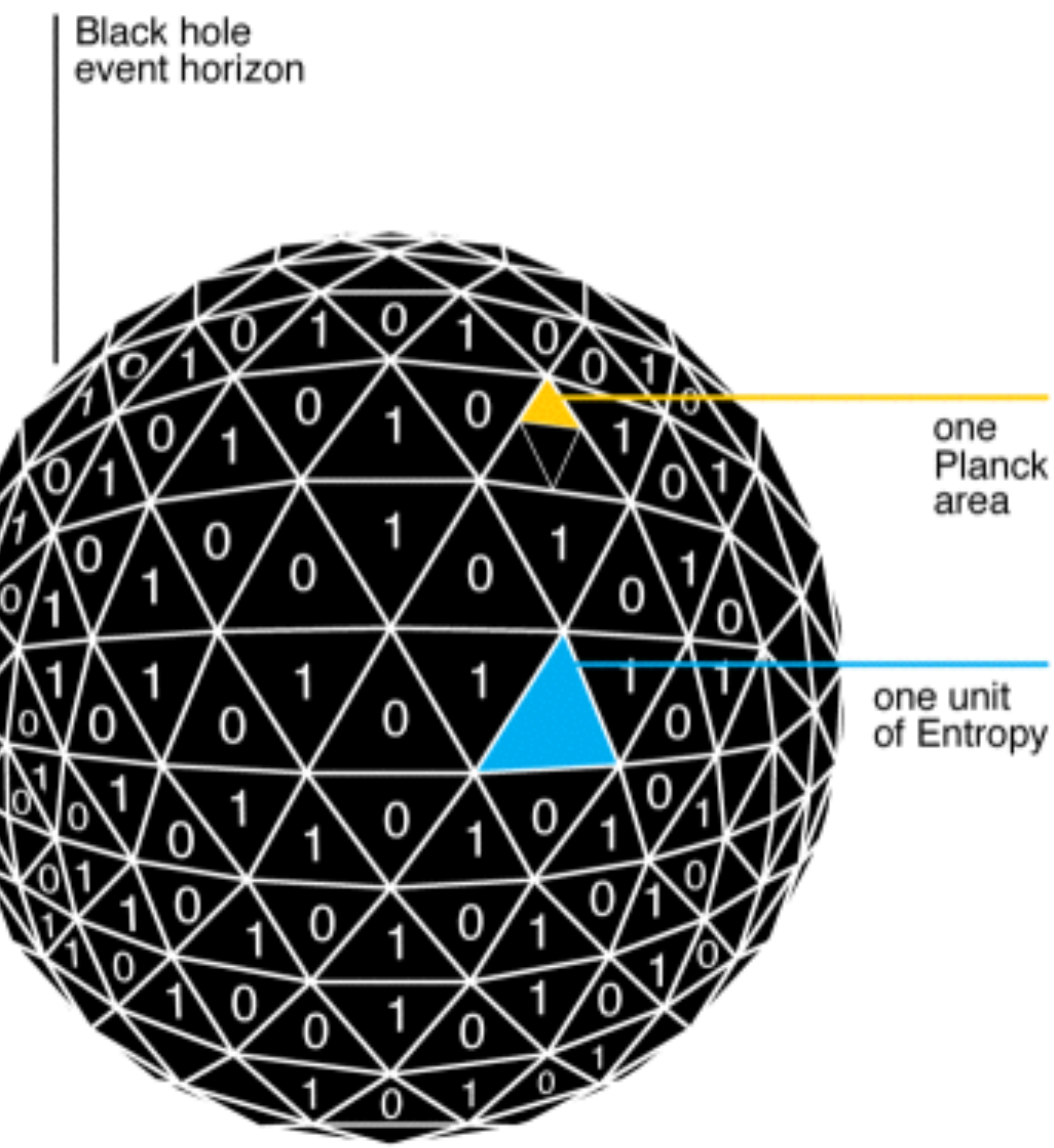
The puzzle 1.0 leading to holography

Black holes



$$A = 16\pi(GM/c^2)^2$$

Bekenstein/Hawking Black Hole entropy



$$A = 16\pi(GM/c^2)^2$$

Area

=

$$S_{BH} = \frac{A}{4L_p^2} = \frac{c^3 A}{4G\hbar}$$

Entropy

=

$$S = -\sum_j p_j \log(p_j)$$

Information

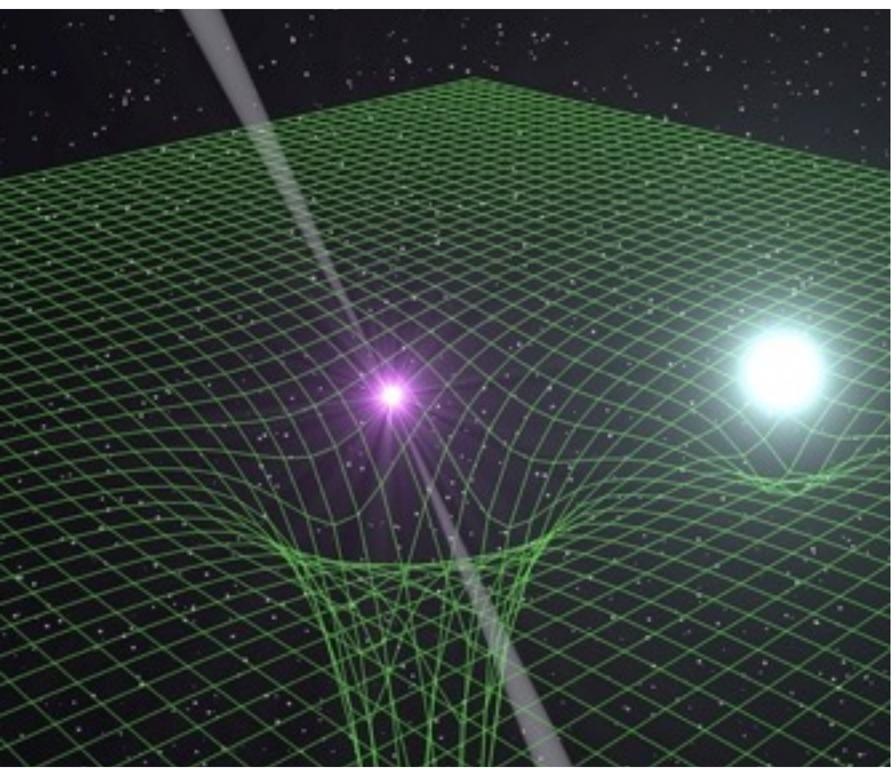
Holography anyone?

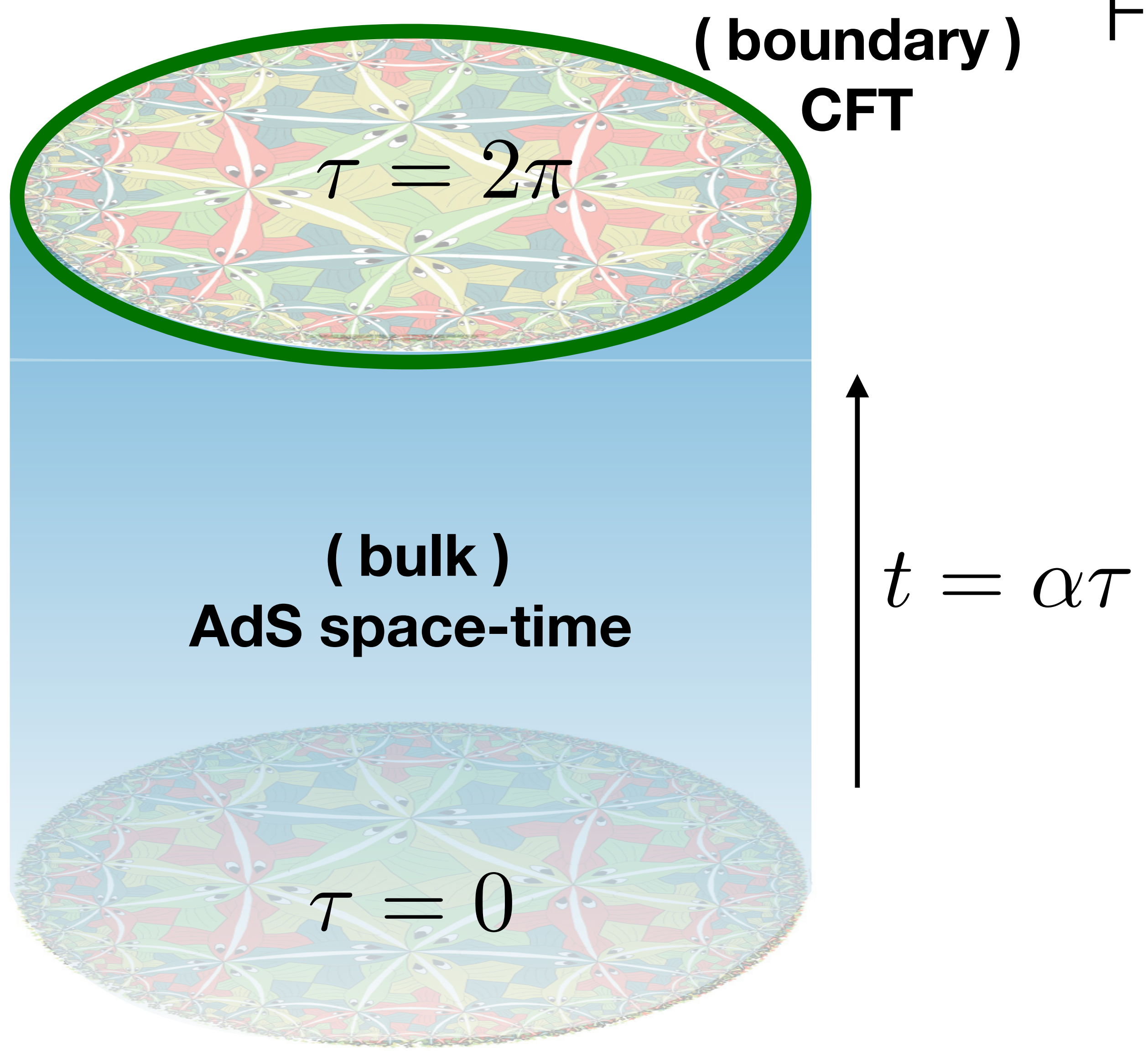
$$1[m^2] = 1.38 \times 10^{69} L_p^2 = 3.45 \times 10^{68} bits$$

Bekenstein J.D.: Lettere al Nuovo Cimento, 4, 737, (1972)
Hawking S. W. Physical Review Letters, 26, 1344 (1971)

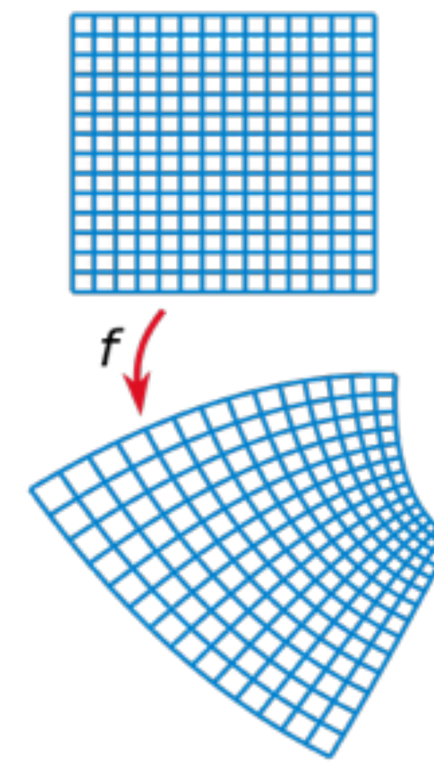
Holographic principle
and the AdS / CFT correspondence

GR on anti-de Sitter background.





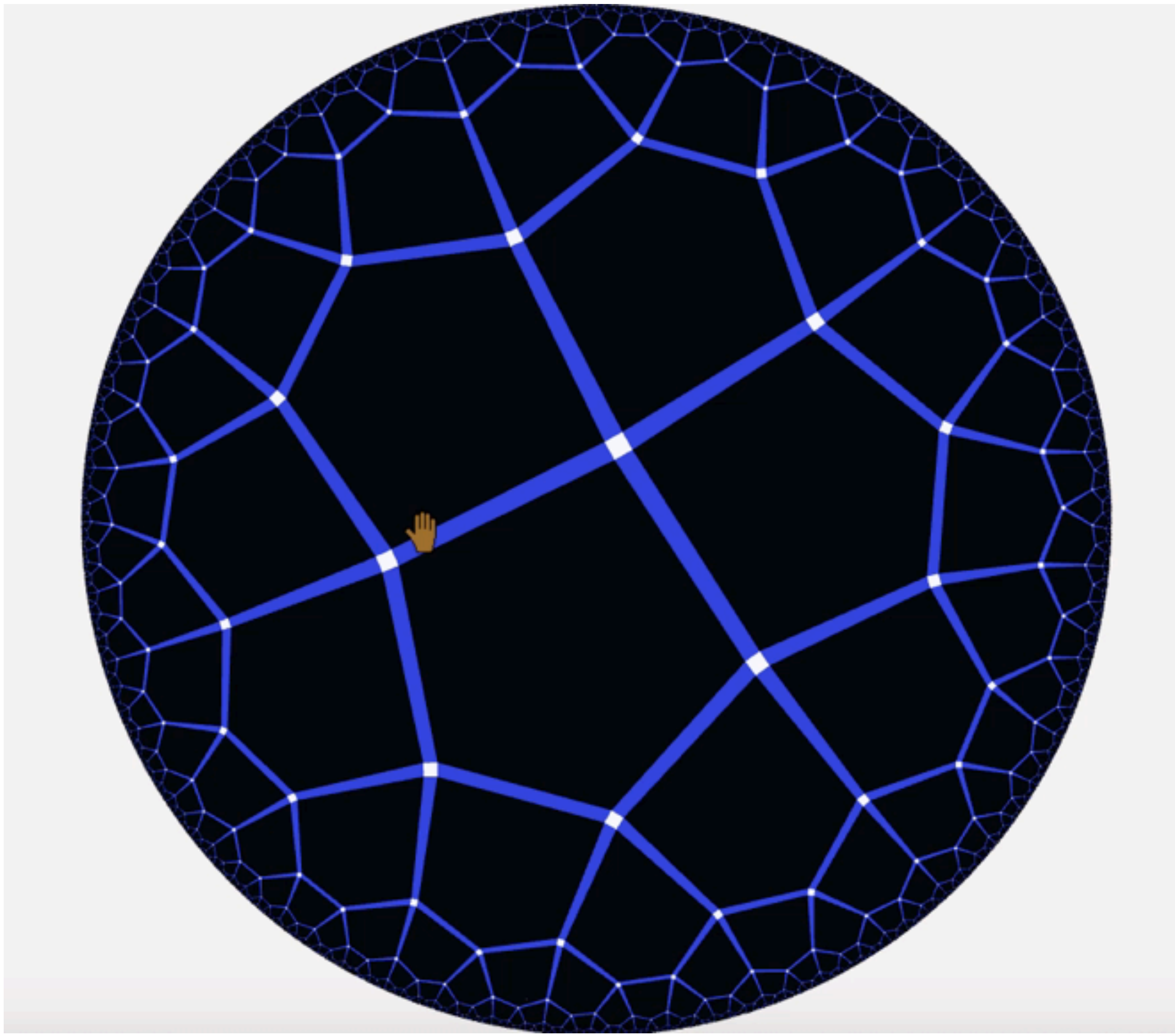
Field theory w. Conformal symmetry



$$ds^2 = \alpha^2(-\cosh^2\rho \; dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{n-1}^2)$$

Maldacena, J. (1998) The Large-N Limit of Superconformal Field Theories and Supergravity. IJTP, 38(4), 1113–1133.

Poincare disc representation of Hyperbolic space (AdS time-slice)



Entanglement entropy in the AdS/CFT correspondence

Entropy as entanglement

Shannon entropy

$$S = -\sum_{j=1}^{D} p_j \log(p_j)$$

Information

$$\log D \geq S \geq 0$$

Von Neumann entropy

$$S_A = -\mathrm{tr}[\rho_A \log(\rho_A)]$$

Schmidt-decomposition

$$|\psi\rangle = \sum_j \sqrt{p_j} |j_A\rangle |j_B\rangle$$

Reduced density matrix

$$\rho_A = \mathrm{tr}_B[|\psi\rangle\langle\psi|] = \sum_j p_j |j_A\rangle\langle j_A|$$

Entanglement entropy

$$S_A = S_B \leq \log \min[D_A, D_B]$$

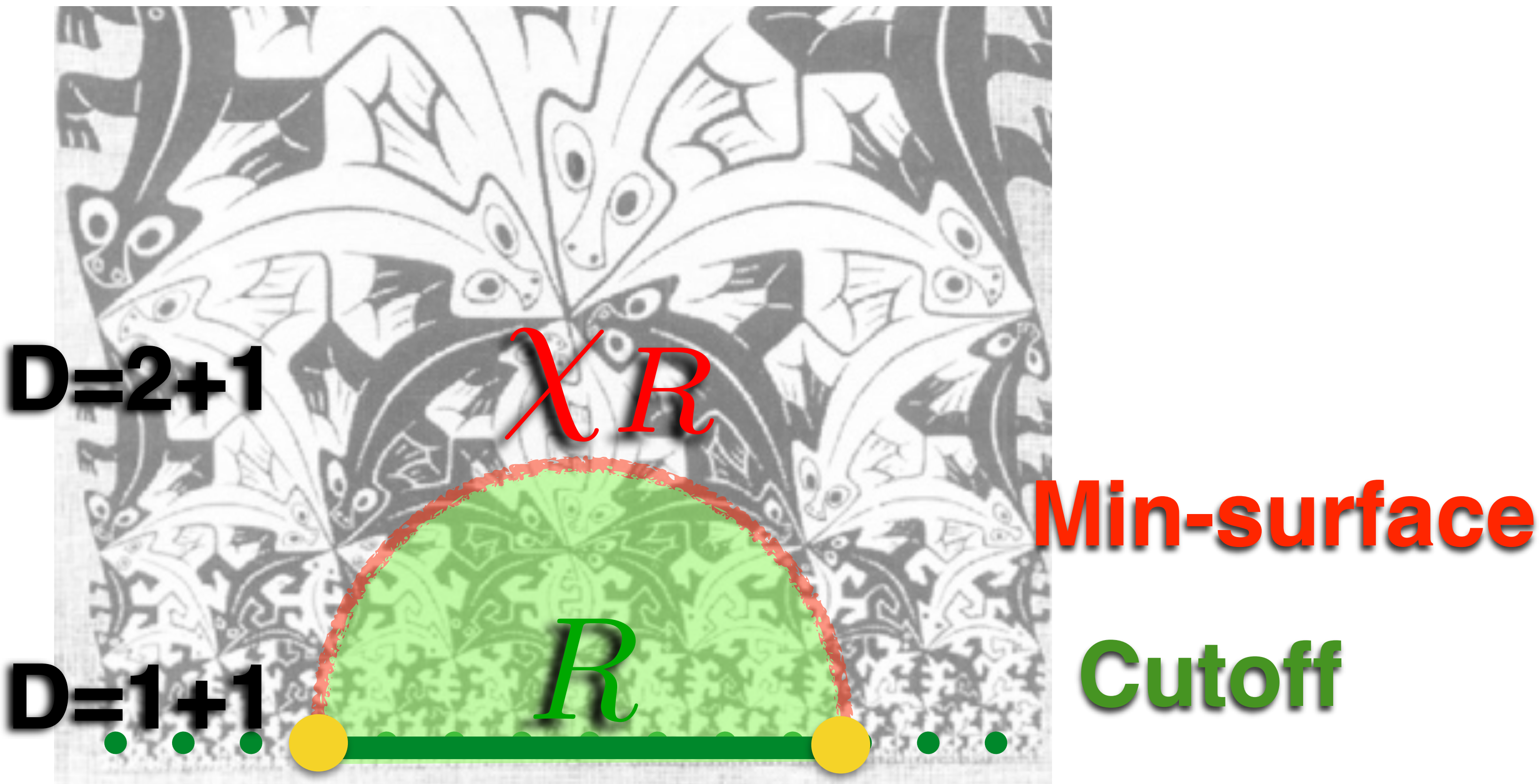
Ryu-Takayanagi

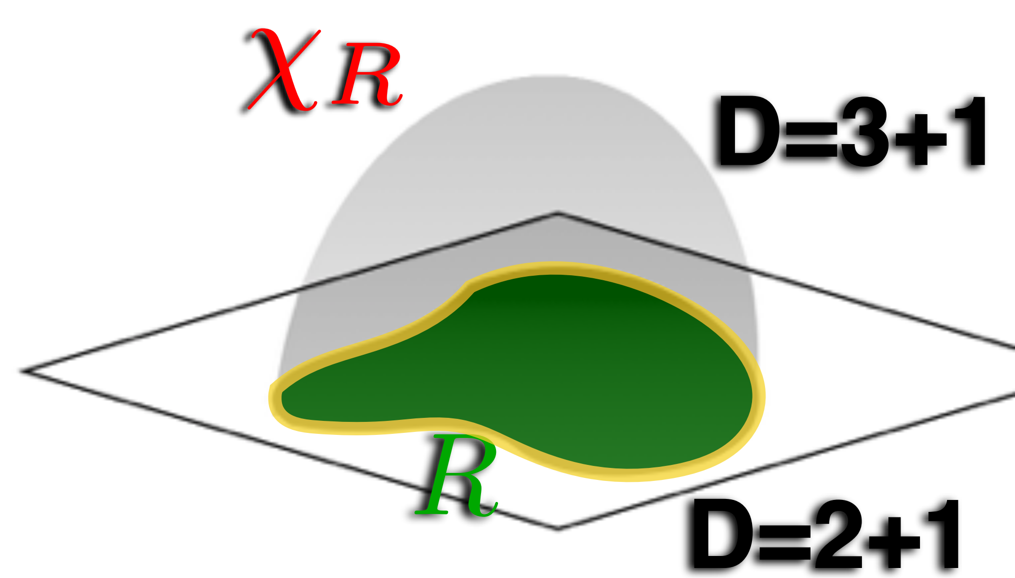
Bulk/Boundary duality to Geometry/Entanglement duality

Minimal area = Entropy = **Entanglement**

$$|\chi_R| \propto \log(|R|/\varepsilon)$$

$$S(R) = \min_{\partial\chi_R = \partial R} \frac{area(\chi_R)}{4L_p^2} + \dots$$

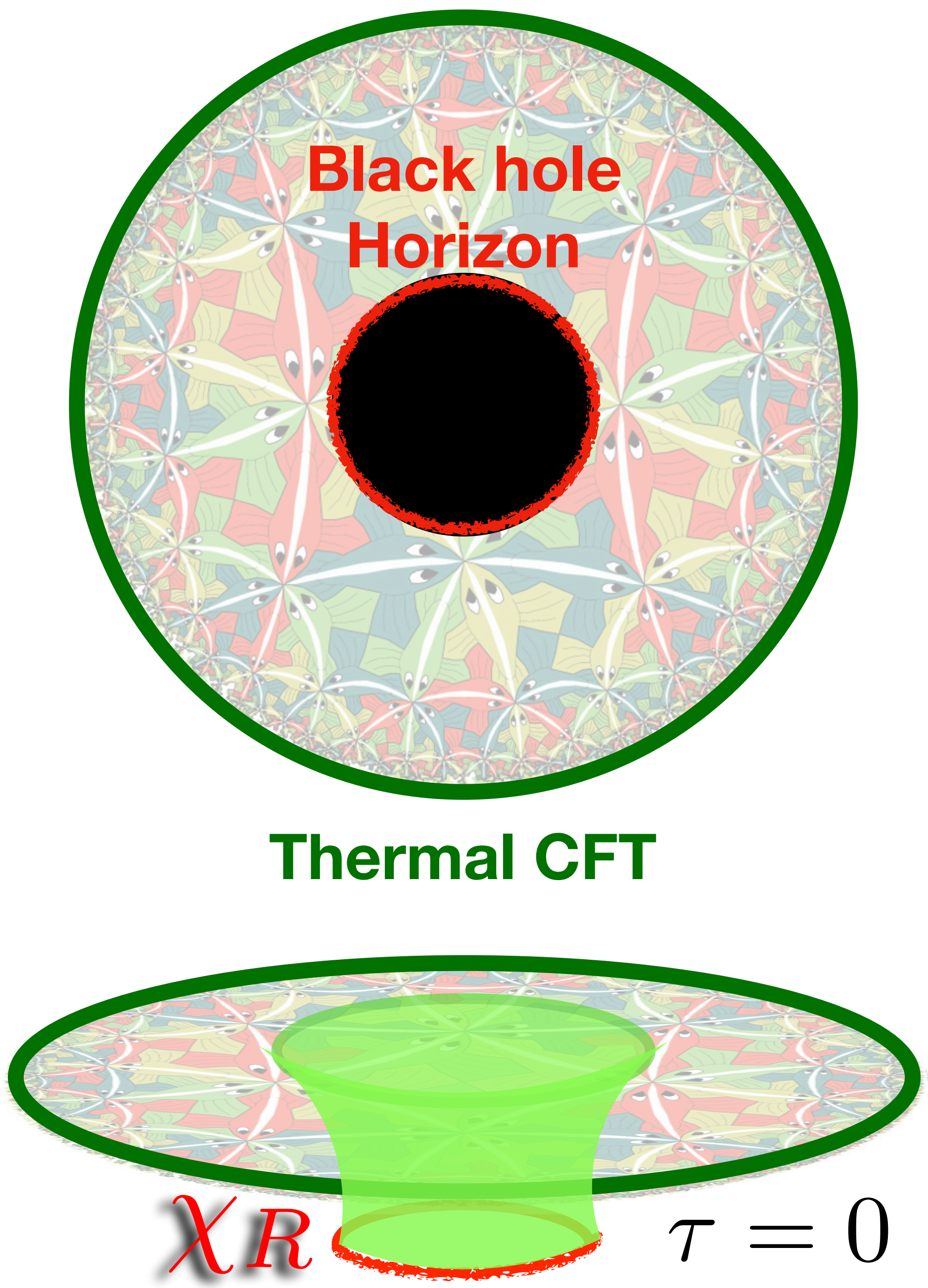


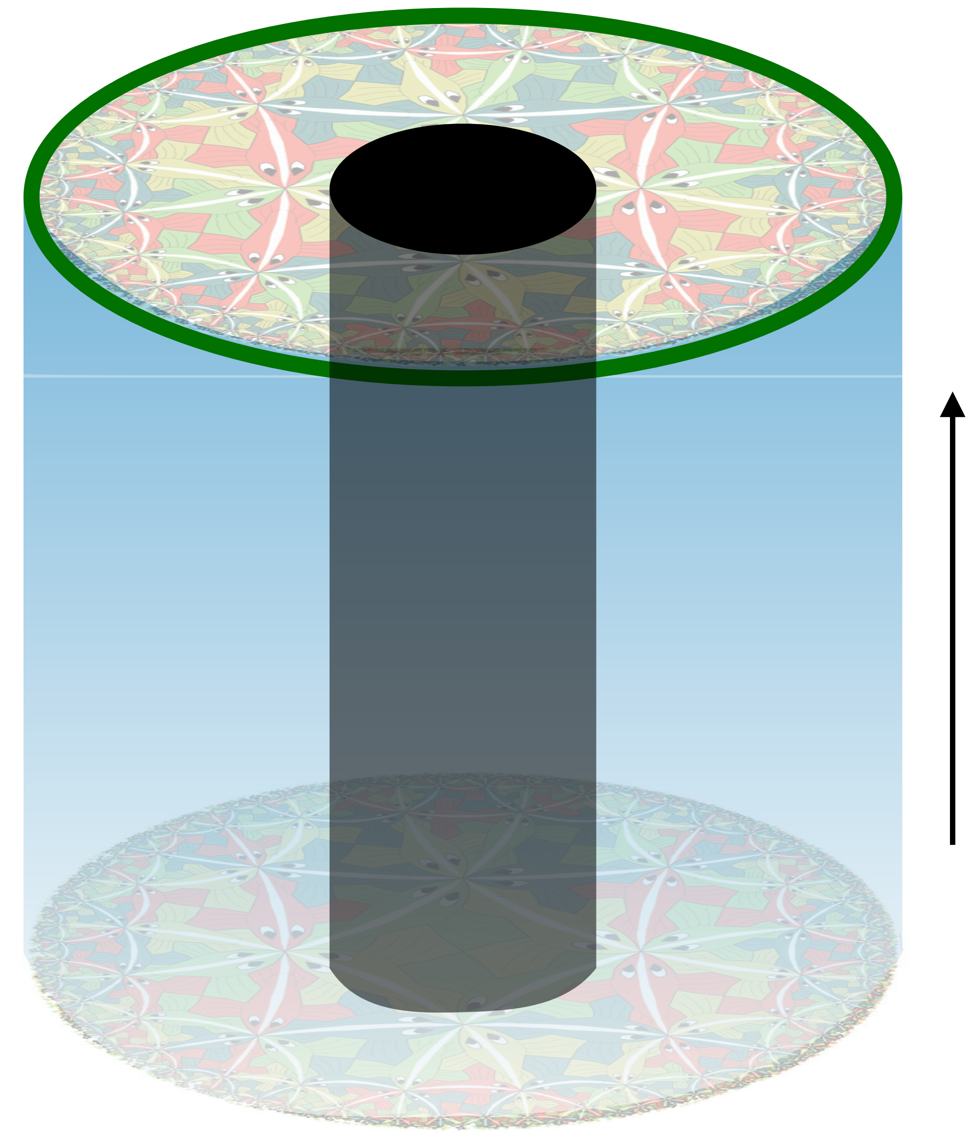


Generalization of Bekenstein-Hawking black hole entropy

Ryu, S., & Takayanagi, T. (2006). Holographic Derivation of Entanglement Entropy from the anti–de Sitter Space/Conformal Field Theory Correspondence. PRL, 96(18), 181602.

AdS Schwarzschild black hole

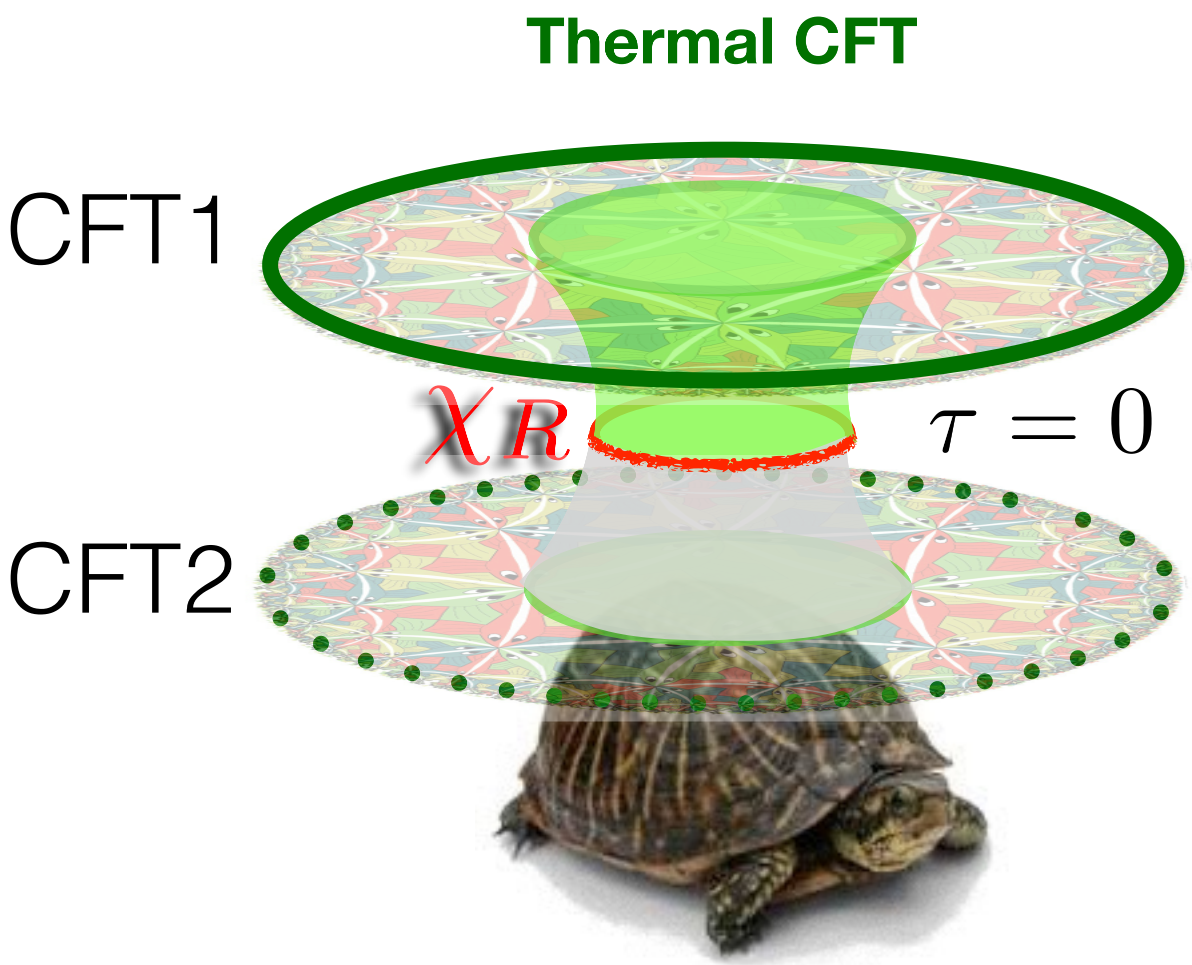




Thermal Gibbs state

$$\rho_{\mathrm{CFT}_{1/2}} = \sum_j \frac{e^{-\beta \varepsilon_j}}{\mathcal{Z}} |j\rangle\langle j|$$

ER = EPR



Einstein-**R**osen bridge
"Wormhole"

Thermal Gibbs state

$$\rho_{\mathrm{CFT}_{1/2}} = \sum_j \frac{e^{-\beta\varepsilon_j}}{\mathcal{Z}} |j\rangle\langle j|$$

Einstein-**P**odolski-**R**osen
Entangled pair

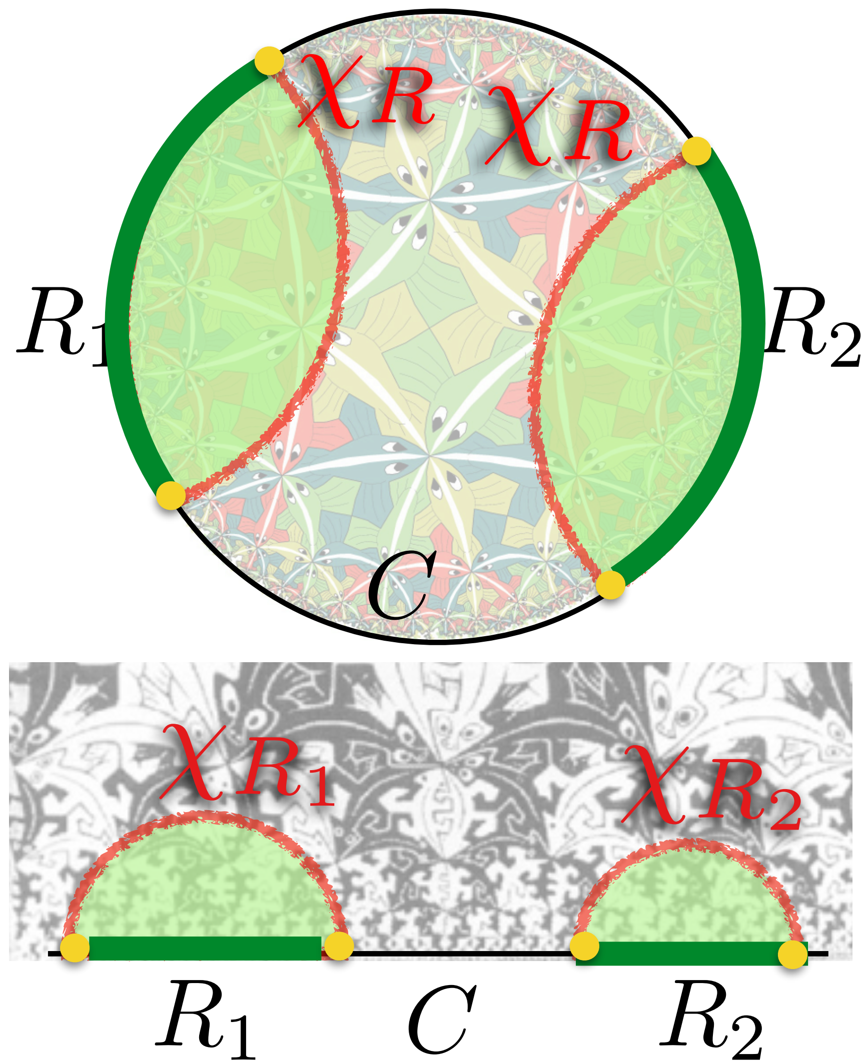
$$|\Psi_{\mathrm{TFD}}\rangle = \sum_j \frac{e^{-\beta\varepsilon_j/2}}{\sqrt{\mathcal{Z}}} \underbrace{|j\rangle}_{\text{CFT1}} \underbrace{|j\rangle}_{\text{CFT2}}$$

Extend black hole to a worm hole = Purifying a thermal state

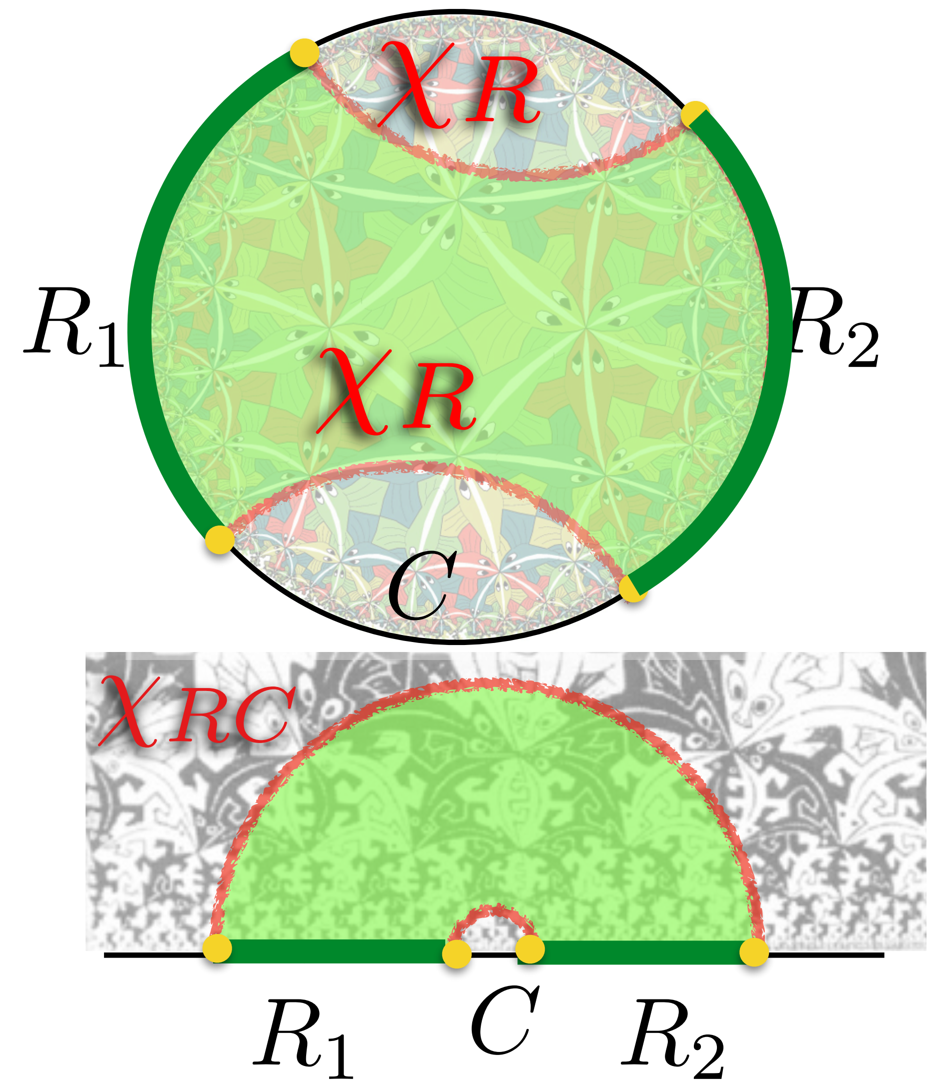
M. van Raamsdonk (2010). Building up spacetime with quantum entanglement. *Gen. Rel. Grav.* 42: 2323–2329.
J. Maldacena & L. Susskind (2013). Cool horizons for entangled black holes. Fortschritte der Physik, 61(9), 781-811.

Entanglement wedge

Disconnected | Transition | Connected



AdS slice



$$|C||RC| \geq |R_1||R_2|$$

$$S_R \approx S_{R_1} + S_{R_2} < S_{RC} + S_C$$

$$|C||RC| < |R_1||R_2|$$

$$S_R \approx S_{RC} + S_C \leq S_{R_1} + S_{R_2}$$

V. E. Hubeny et al (2007). A covariant holographic entanglement entropy proposal JHEP 07(2007)062

Entanglement structure

More structure | Less structure

Less entanglement | More entanglement

Product

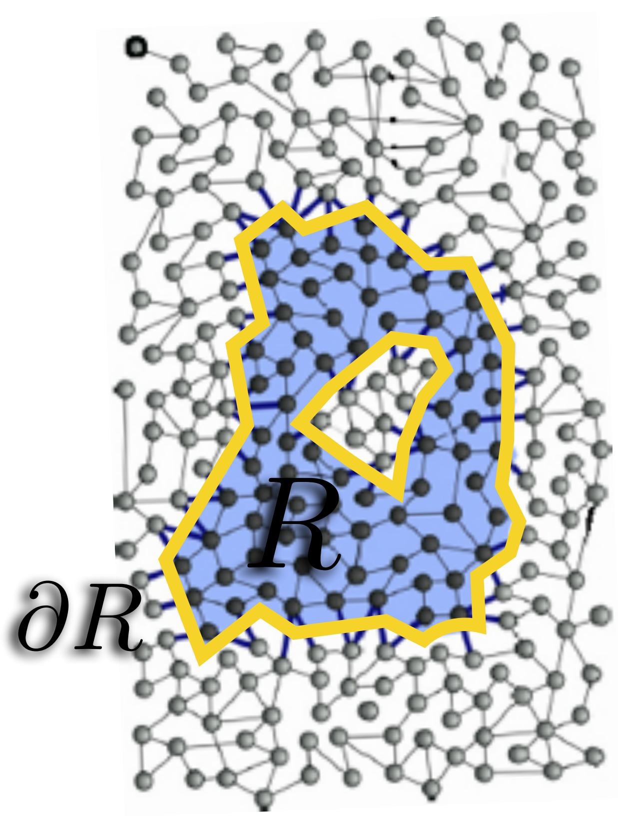
$$S(R) = 0$$

$$\Pi$$

Product state
no entanglement
or correlations

Area law

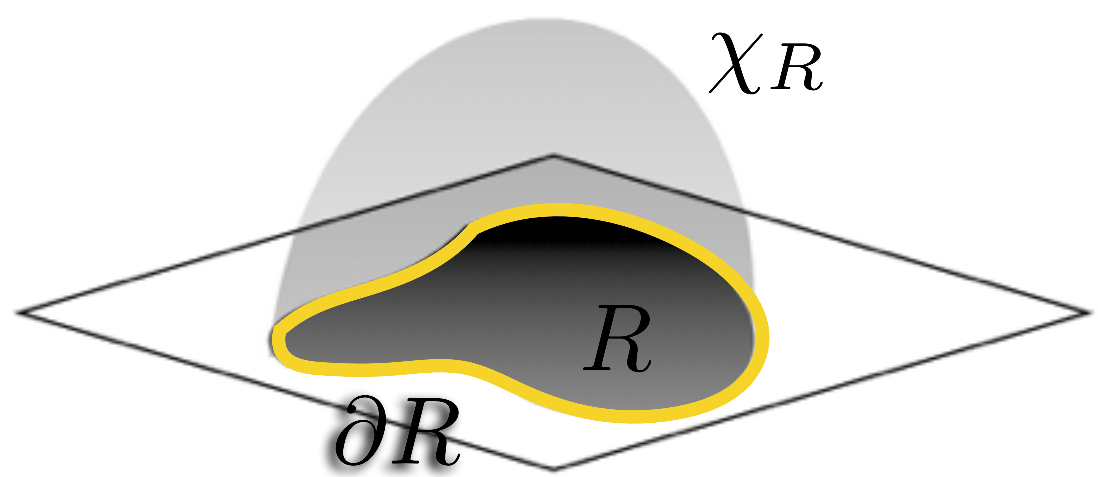
$$S(R) \propto |\partial R|$$



Area law state;
entanglement grows with
subsys boundary area in
"actual" geomety.

Holographic area law

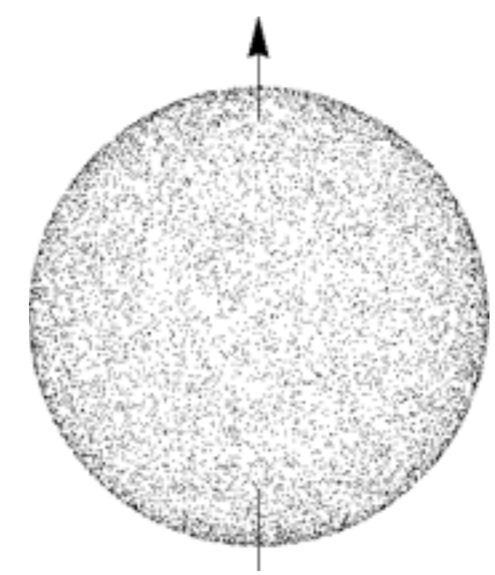
$$S(R) \propto |\chi_R|$$



The Ryu-Takayanagi
prescription or
Holographic area law
entanglement grows with
boundary area in emergent
holographic geometry.

Haar random

$$S(R) \propto |R|$$



"**Generic**" pure states
entanglement grow with
volume law.
As subsystem dimension.

Holography

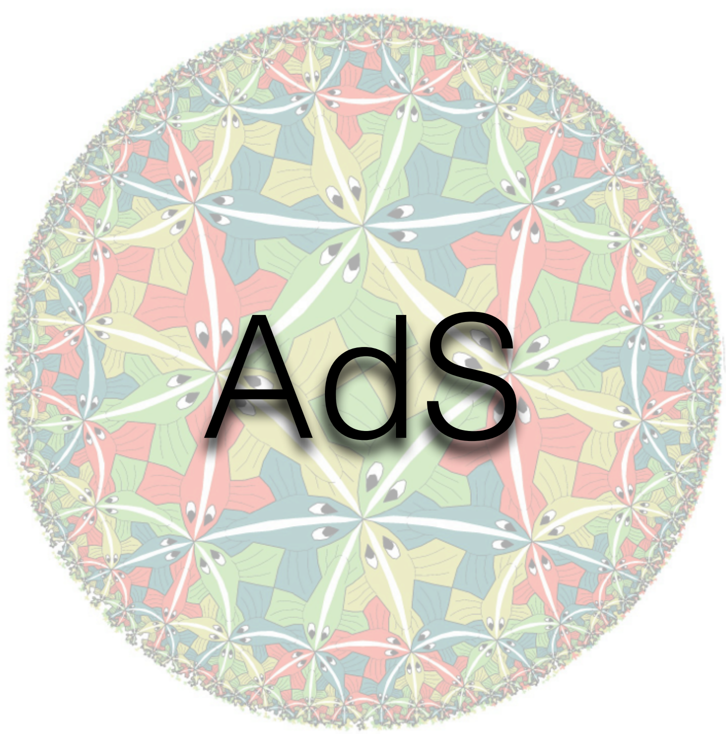
Entangelement structure

Holographic
duality

Holographic geometry

based on example:

AdS/CFT



Entanglement:
emergent space-time glue.

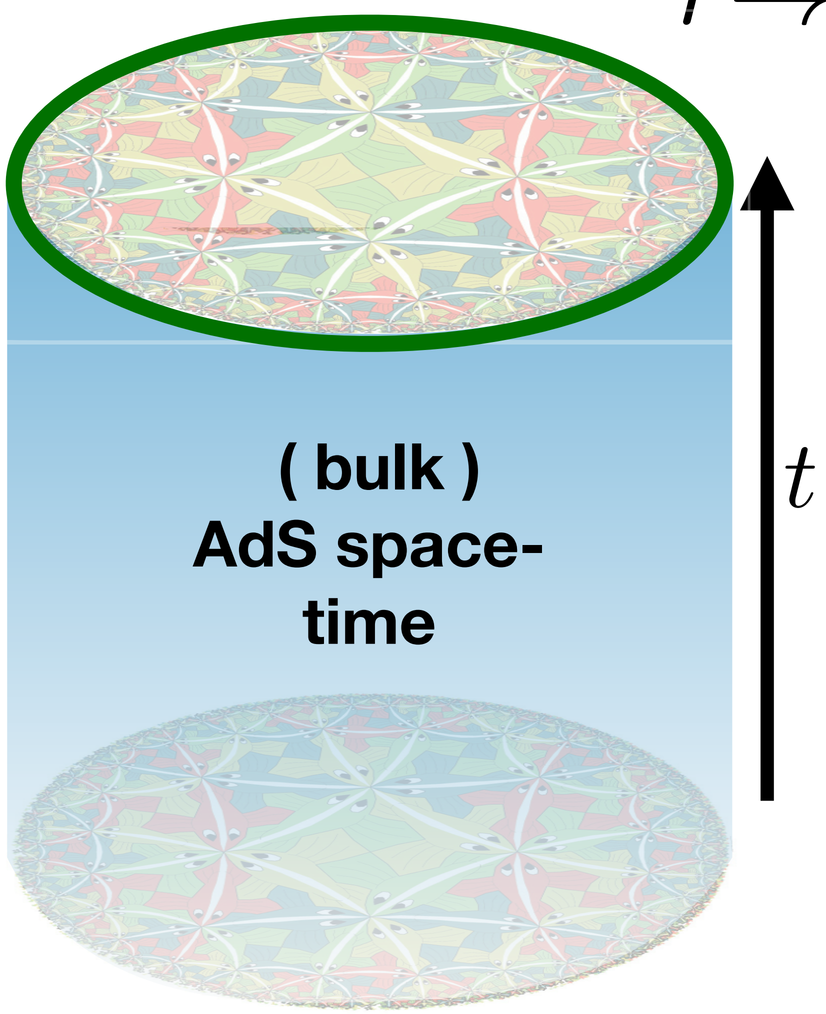
Holography as Quantum Error-Correction

Local bulk operators as boundary operators

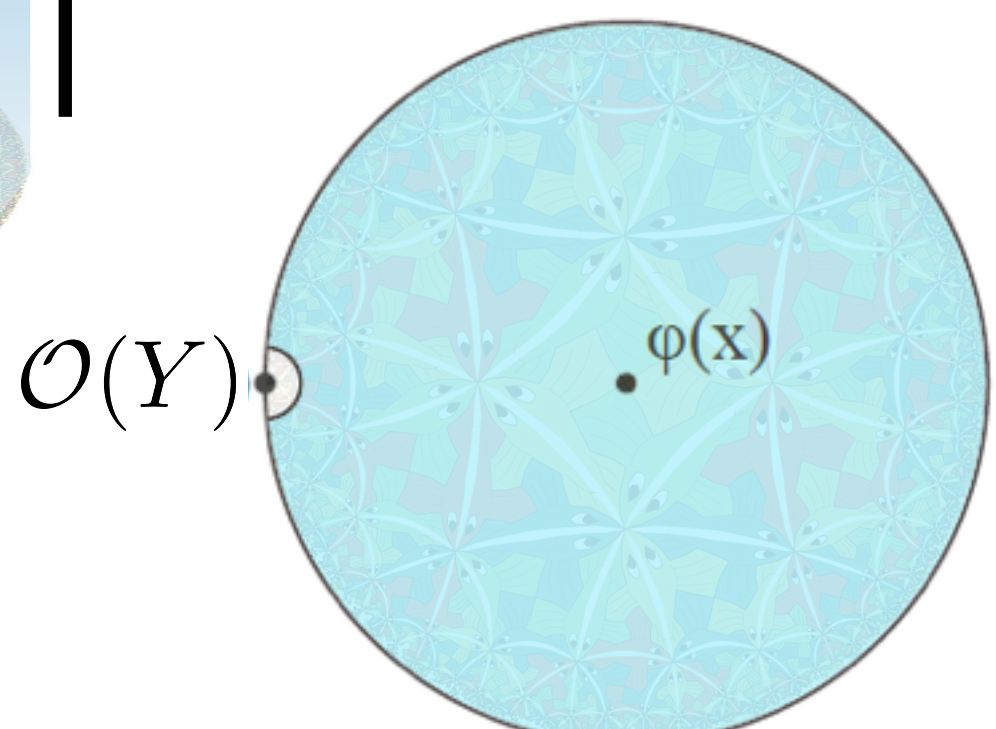
Almheiri, A., Dong, X., & Harlow, D. (2015).
Bulk locality and quantum error correction in AdS/CFT. JHEP, 2015(4), 163.

Global reconstruction

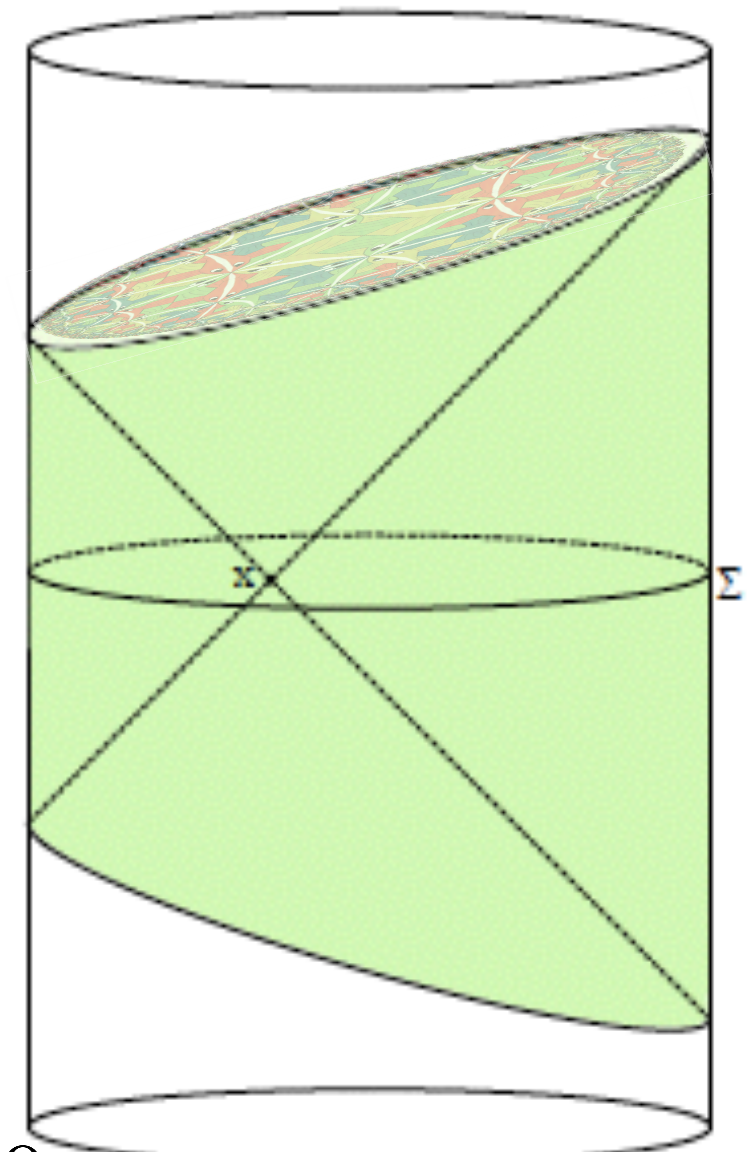
$$\lim_{r\to\infty} r^{\Delta}\phi(r,x) = \mathcal{O}(x)$$



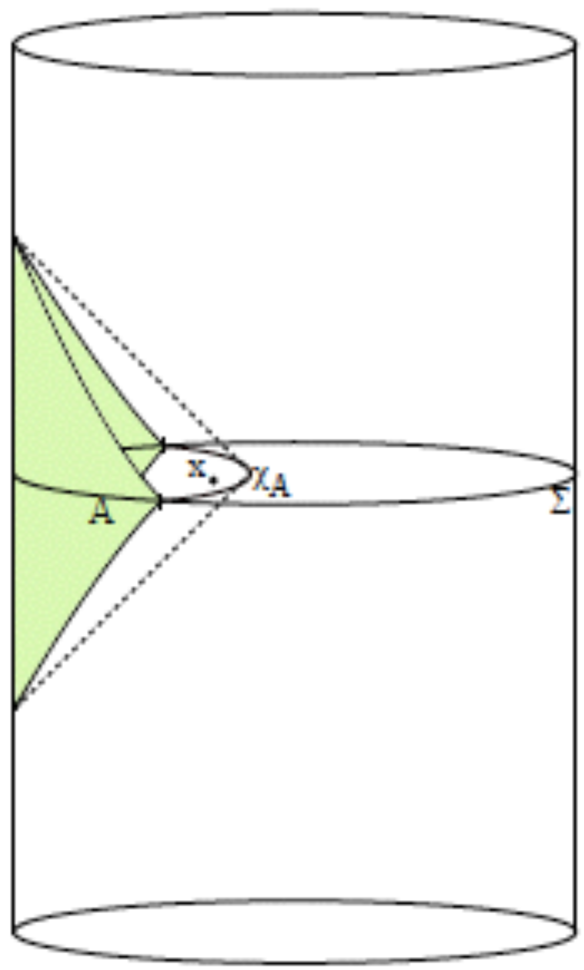
Radial commutativity.



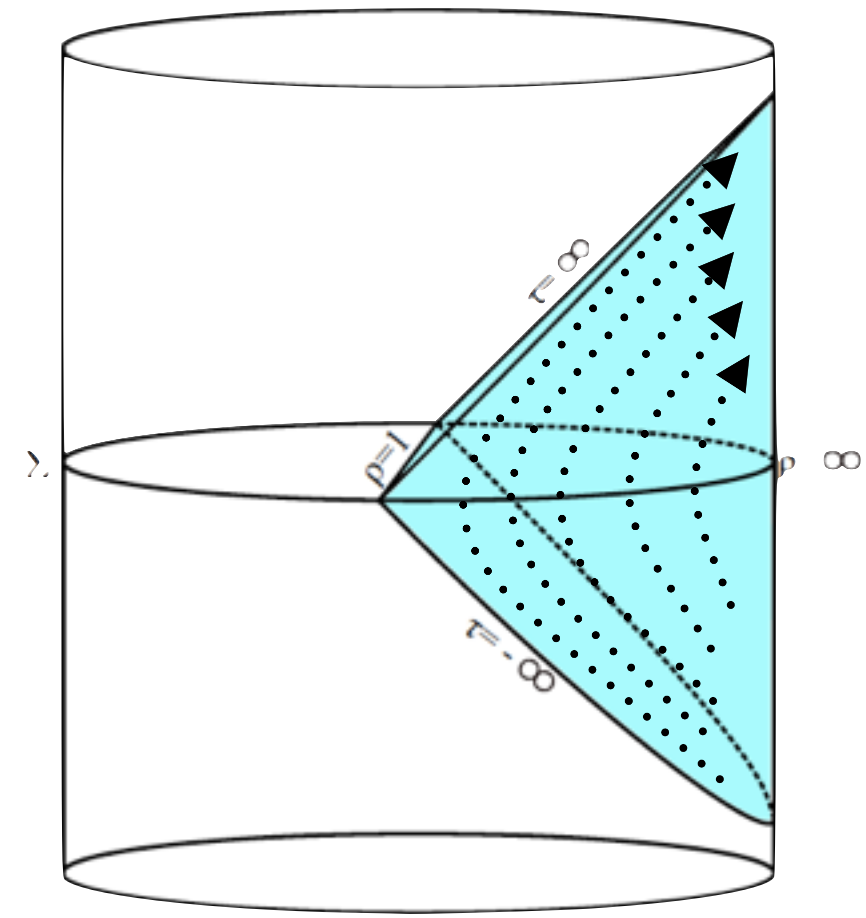
$t = 0$



AdS-Rindler wedge reconstruction



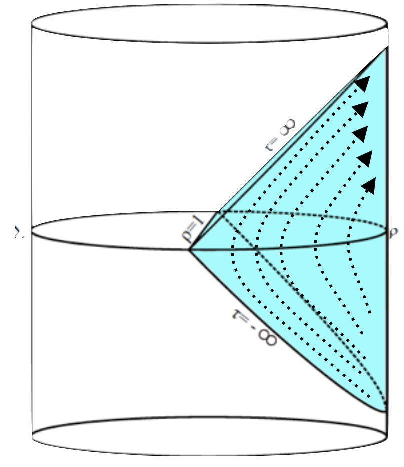
Causal
Wedge

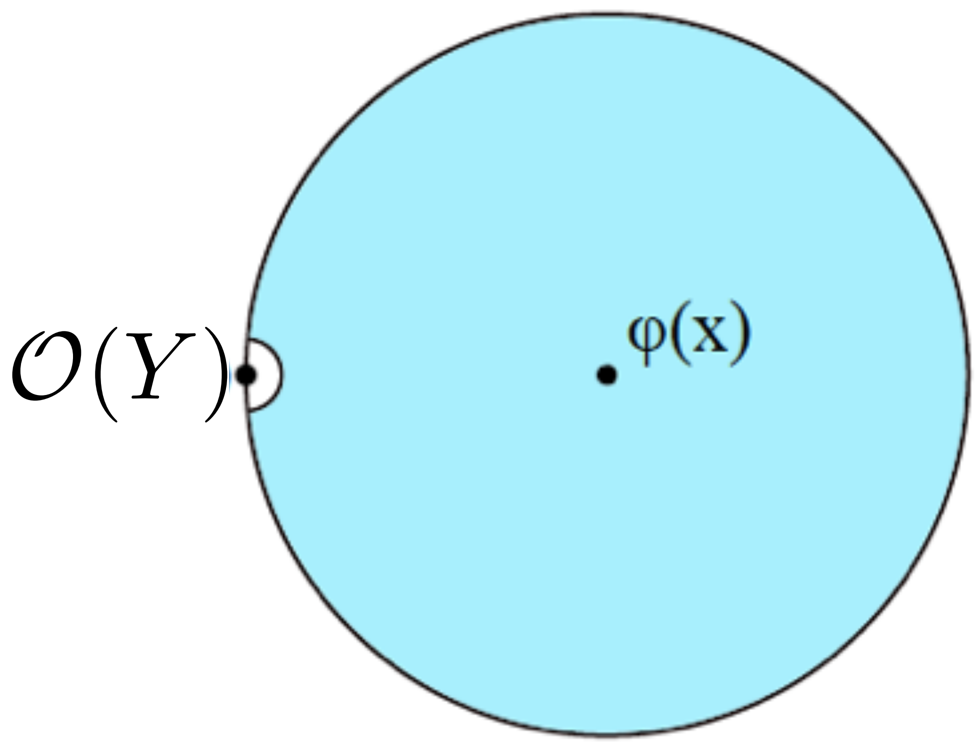


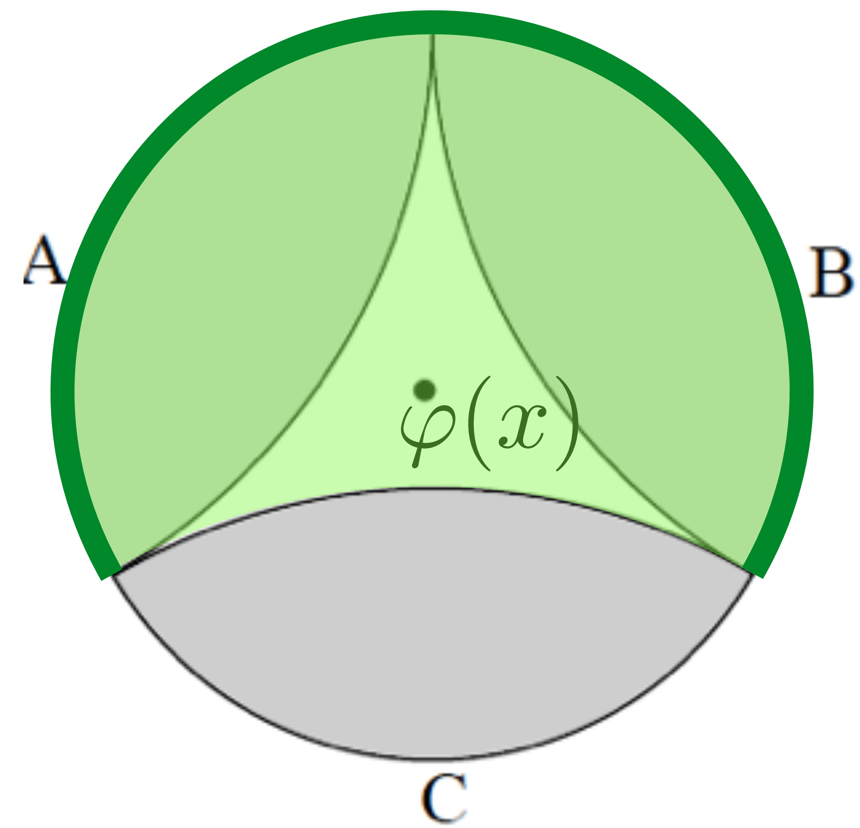
Uniformly accelerated coordinate system.

Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. (2006).
Holographic representation of local bulk operators. PRD, 74(6), 066009.

Reduction to spacelike slice







$$\varphi(x) \to \Phi_{AB}(x) \qquad \varphi(x) \to \Phi_{BC}(x) \qquad \varphi(x) \to \Phi_{CA}(x)$$

Almheiri, A., Dong, X., & Harlow, D. (2015).
Bulk locality and quantum error correction in AdS/CFT. JHEP, 2015(4), 163.

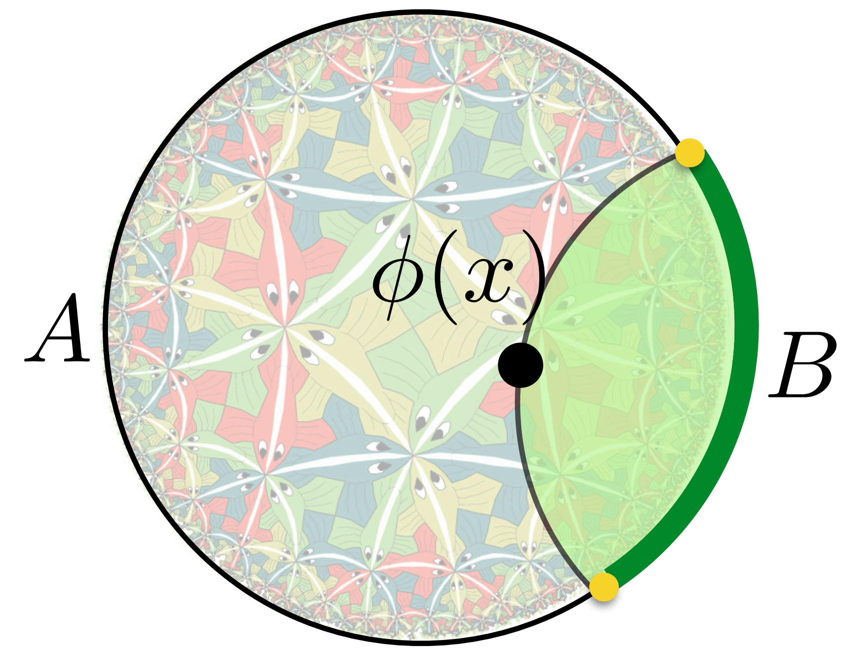
Different operators - same effect

Singlet $$|\Psi^-\rangle := \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

$$X \otimes I|\Psi^-\rangle = -I \otimes X|\Psi^-\rangle$$

$$Y \otimes I|\Psi^-\rangle = -I \otimes Y|\Psi^-\rangle$$

$$Z \otimes I|\Psi^-\rangle = -I \otimes Z|\Psi^-\rangle$$



Operator "teleportation"

$$O_A|\Psi^-\rangle = O_B|\Psi^-\rangle$$

Resolution: Low energy sector made of entangled states.

Example:$[[3,1,2]]_3$ quantum code

[[n,k,d]]   Protect non-commuting observables

$$\mathcal{H}_C = \text{span}\{|\tilde{0}\rangle, |\tilde{1}\rangle, |\tilde{2}\rangle\}$$

$$|0\rangle \to |\tilde{0}\rangle = |000\rangle + |111\rangle + |222\rangle$$
$$|1\rangle \to |\tilde{1}\rangle = |012\rangle + |120\rangle + |201\rangle$$
$$|2\rangle \to |\tilde{2}\rangle = |021\rangle + |102\rangle + |210\rangle$$

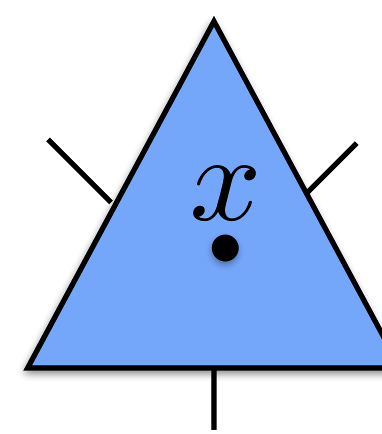
$$Z|j\rangle = \omega^j |j\rangle \qquad \omega = e^{\frac{2i\pi}{3}}$$
$$X|j\rangle = |j+1 \bmod (3)\rangle$$

$$E = \sum_j |\tilde{j}\rangle\langle j| \qquad EE^\dagger = P_C \qquad \mathcal{E}nc(\rho) = E\rho E^\dagger$$

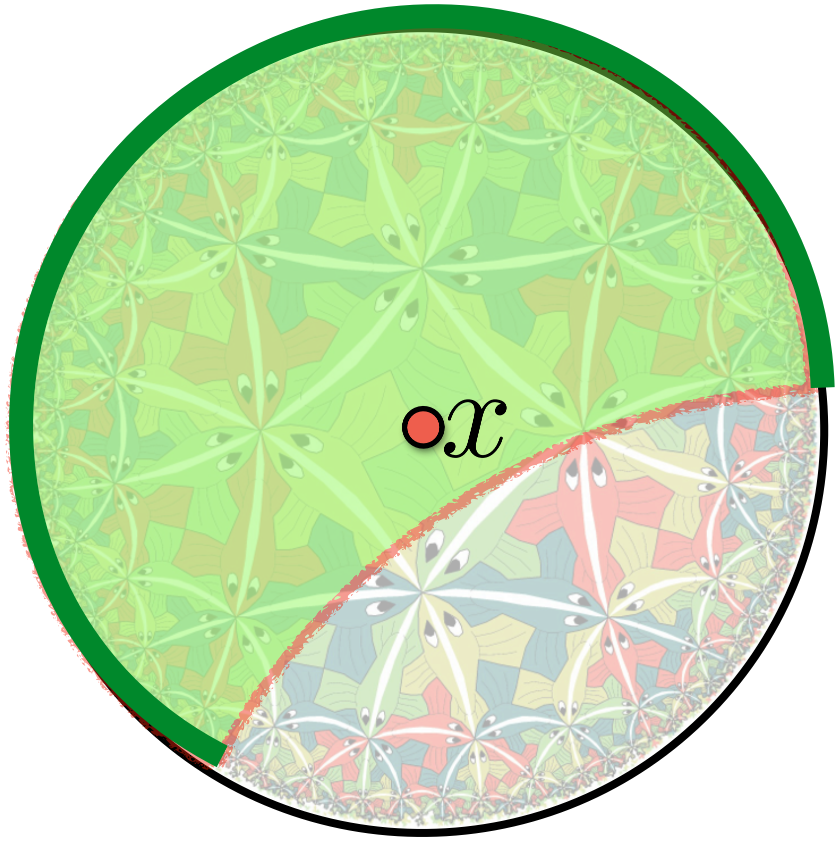
$$\bar{Z} \sim_C Z \otimes Z^\dagger \otimes 1 \sim_C 1 \otimes Z \otimes Z^\dagger \sim_C Z^\dagger \otimes 1 \otimes Z$$

$$\bar{X} \sim_C X \otimes X^\dagger \otimes 1 \sim_C 1 \otimes X \otimes X^\dagger \sim_C X^\dagger \otimes 1 \otimes X$$

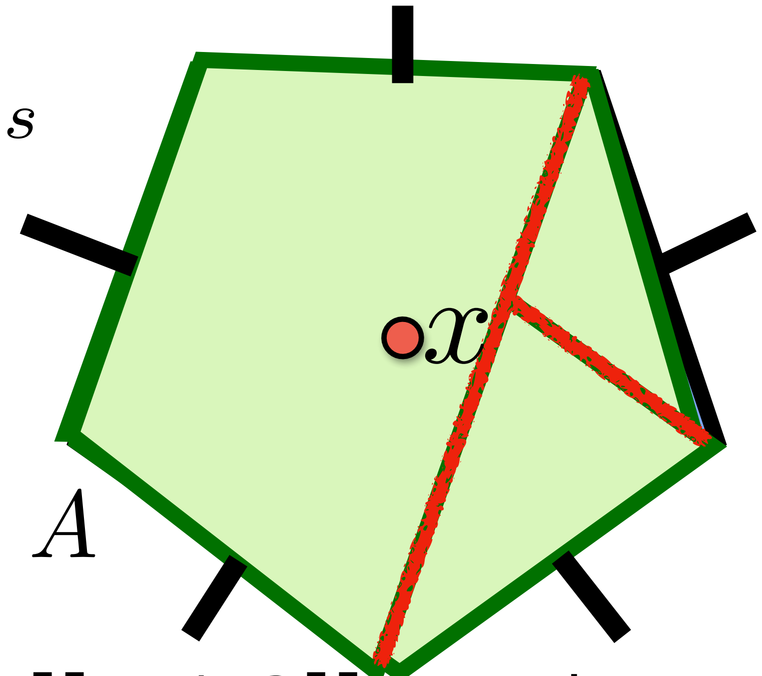
$$d(\bar{X}) = d(\bar{Z}) = d = 2$$



Holography-QEC dictionary



$$\mathcal{H}_{Code} \subset \mathcal{H}_{Phys}$$

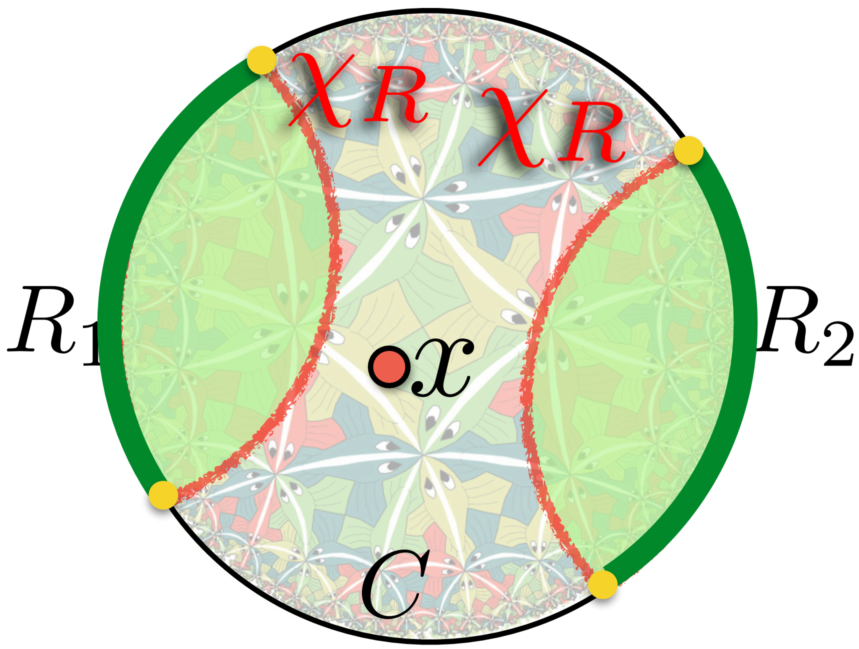


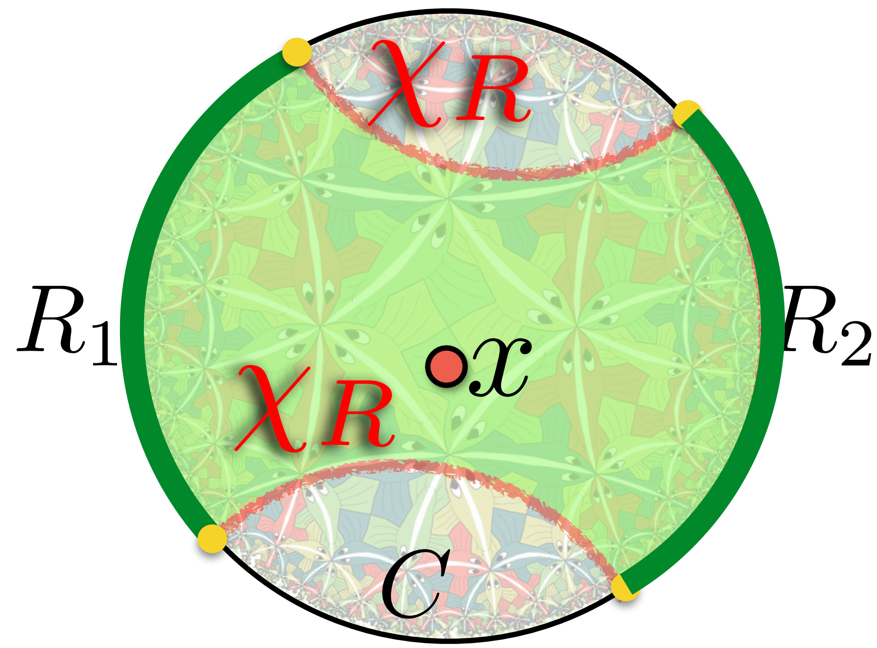
[[5,1,3]]$_2$ code

Single bulk location

Holography language	QEC language
Bulk operators	Logical “message" operators
Boundary operators	Physical “carrier” operators
AdS-Rindler reconstruction	Systematic physical representation
AdS bulk low energy subspace	Code subspace

Entanglement wedge reconstruction

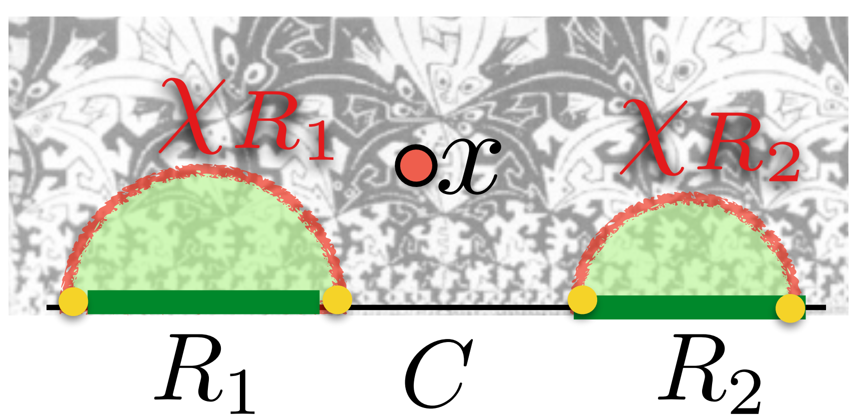


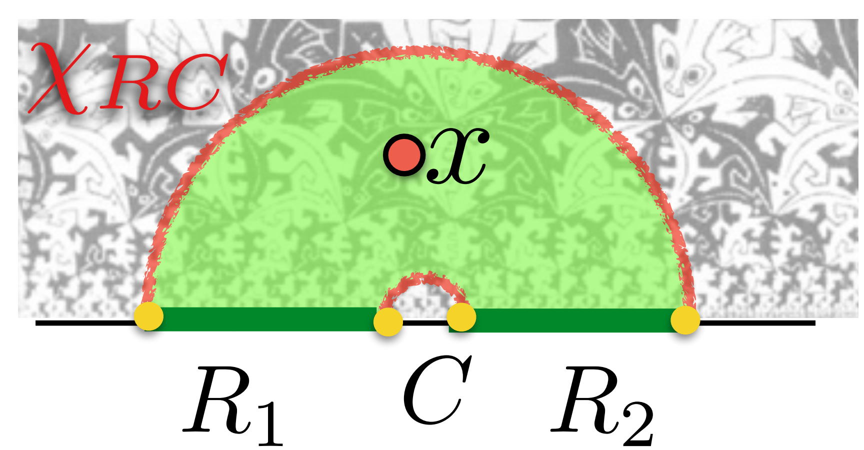


AdS
slice

x not reconstructible on R

x reconstructible on R





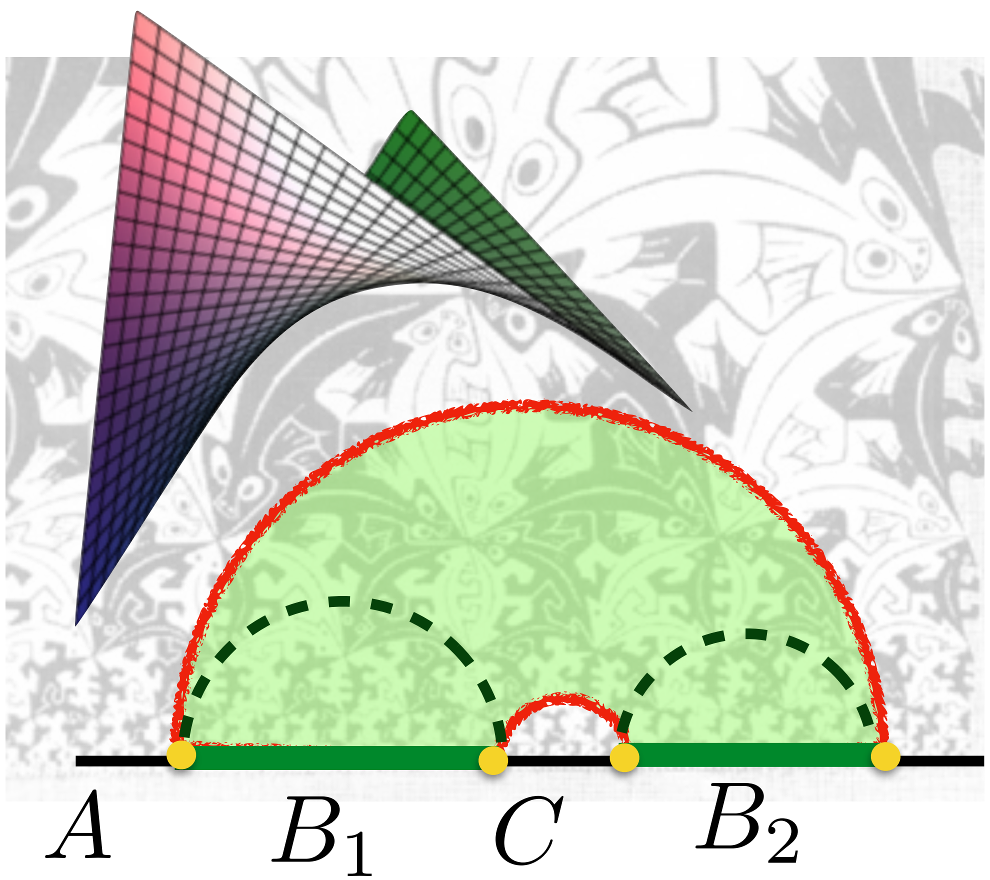
$$S_R \approx S_{R_1} + S_{R_2} < S_{RC} + S_C$$

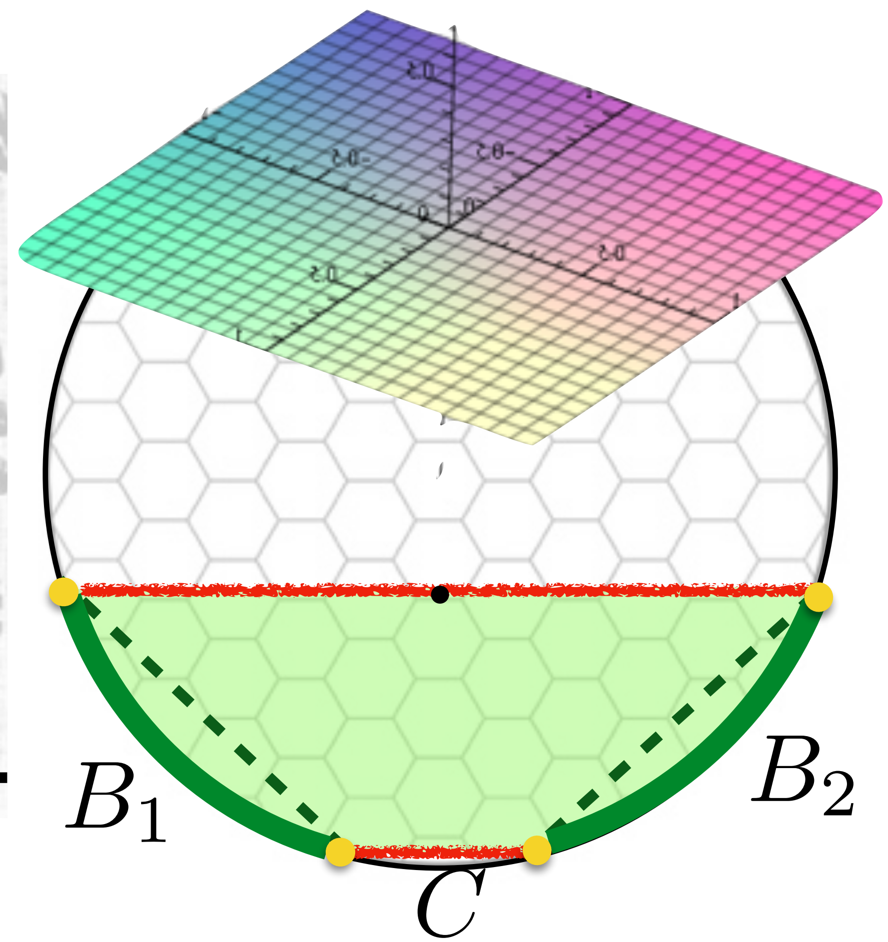
$$S_R \approx S_{RC} + S_C \leq S_{R_1} + S_{R_2}$$

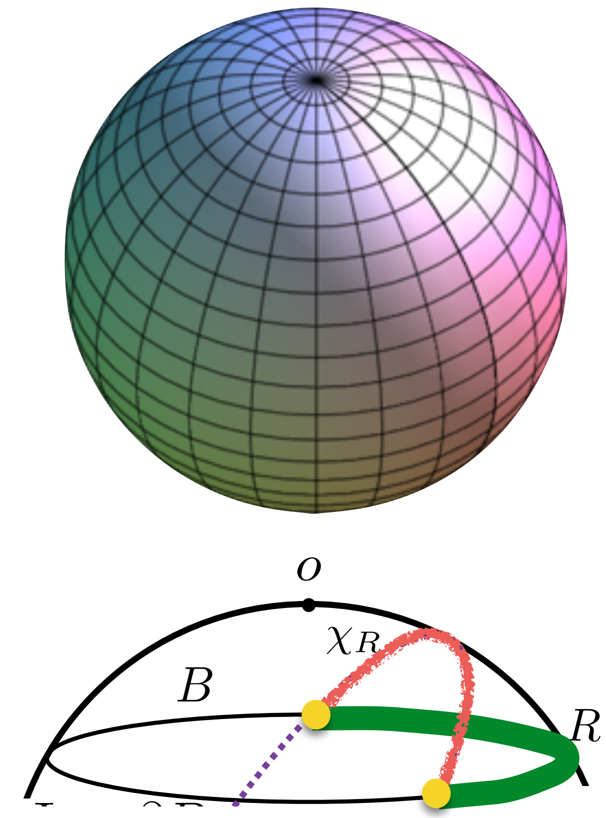
Xi Dong, Daniel Harlow, and Aron C. Wall (2016)
Reconstruction of Bulk Operators within the Entanglement Wedge in Gauge-Gravity Duality
Phys. Rev. Lett. 117, 021601

QEC directly from RT - holographic entanglement & Entanglement wedge hypothesis

Local recovery in other holographic geometries ?







Ryu-Takayanagi (connected wedge): $S(B) = S(BC) + S(C)$

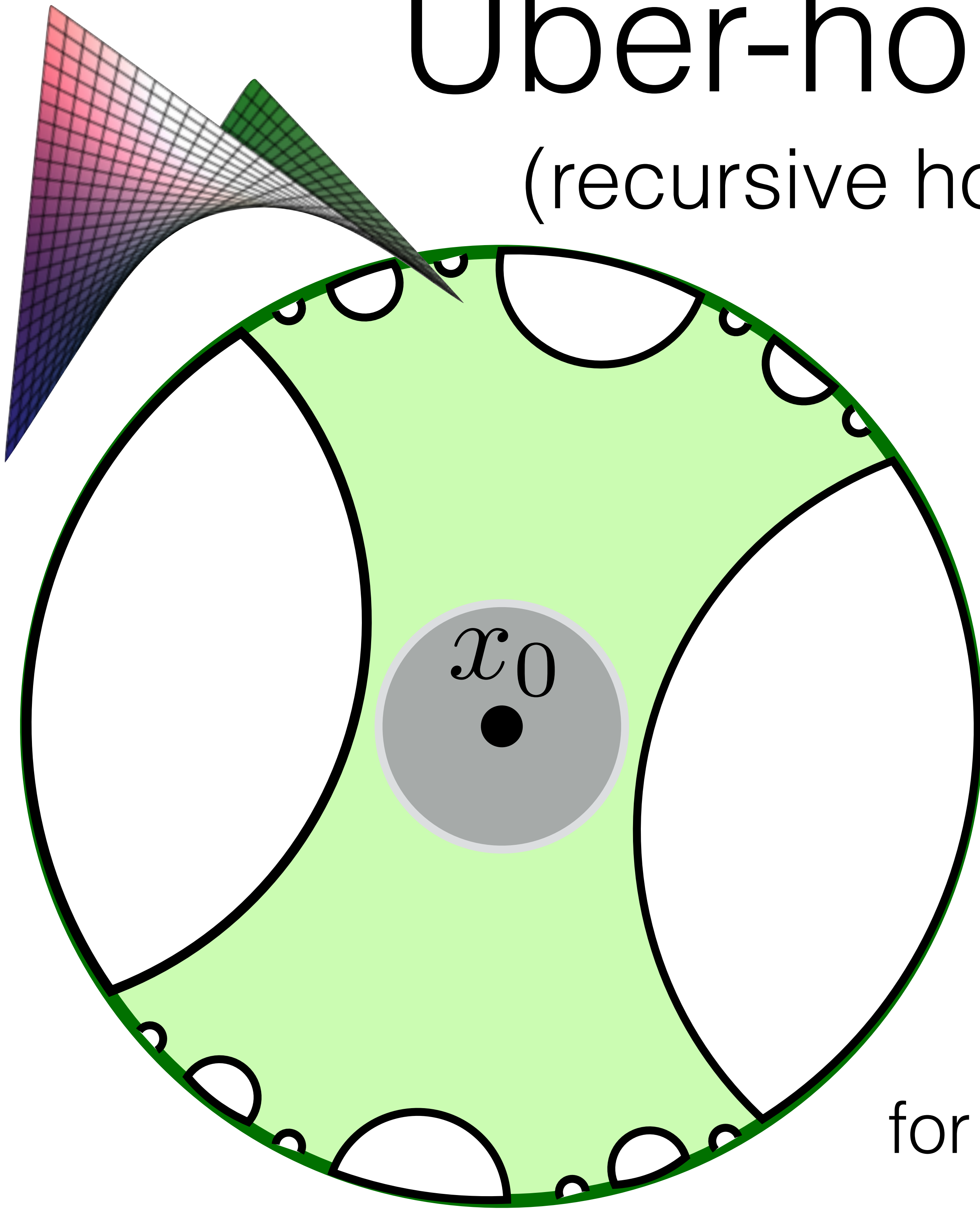
Implies existence of **local** recovery map.

$$\mathcal{R}^{B \to BC} : \rho_{AB} \mapsto \rho_{ABC},$$

FP & J. Preskill (2016). Code properties from holographic geometries.
Phys. Rev. X 7, 021022. arXiv:1612.00017

Über-holography

(recursive hole punching)



Calculating necessary fraction from Cantor type boundary

$$d(\mathcal{A}_{x_0}) = O(n^{0.786})$$

fractal dimension

$$\alpha = \frac{\log(2)}{\log(\sqrt{2}+1)} \approx 0.786$$

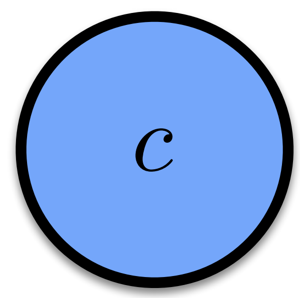
for uniform negatively curved bulk

FP & J. Preskill (2016). Code properties from holographic geometries. Phys. Rev. X 7, 021022. arXiv:1612.00017

Entanglement structures tensor networks and QECCs

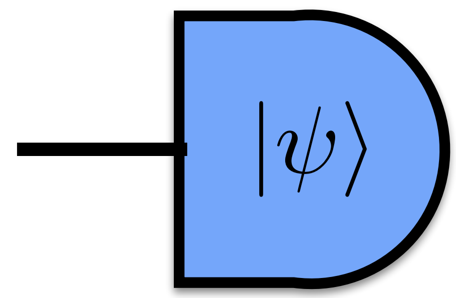
Tensors

Scalars
c-numbers



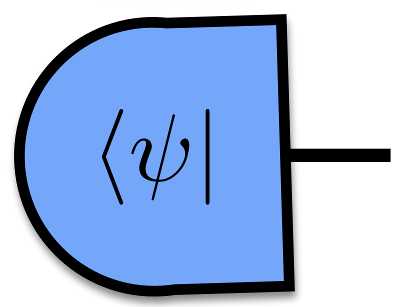
c

Ket
pure state



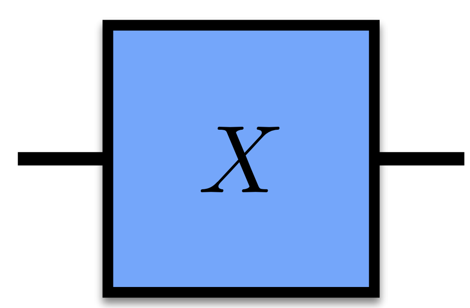
$$\sum_j \psi_j |j\rangle$$

Bra
pure state

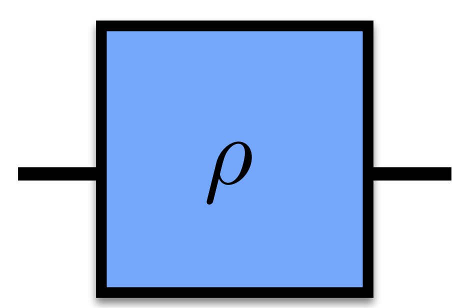


$$\sum_j \bar{\psi}_j \langle j|$$

Operator:
i.e. observable,
unitary,
density matrix



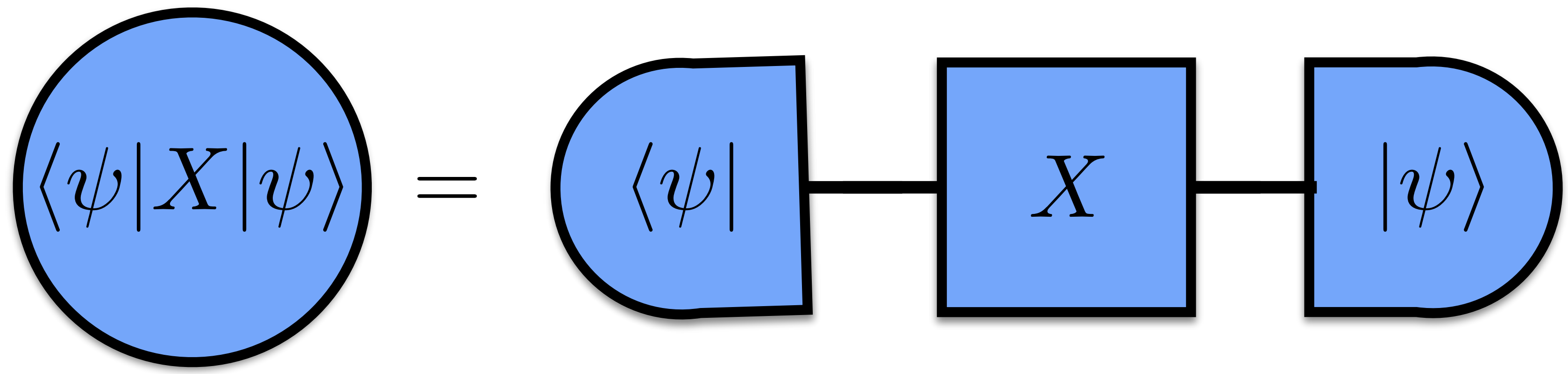
$$\sum_{i,j} X_{i,j} |i\rangle\langle j|$$



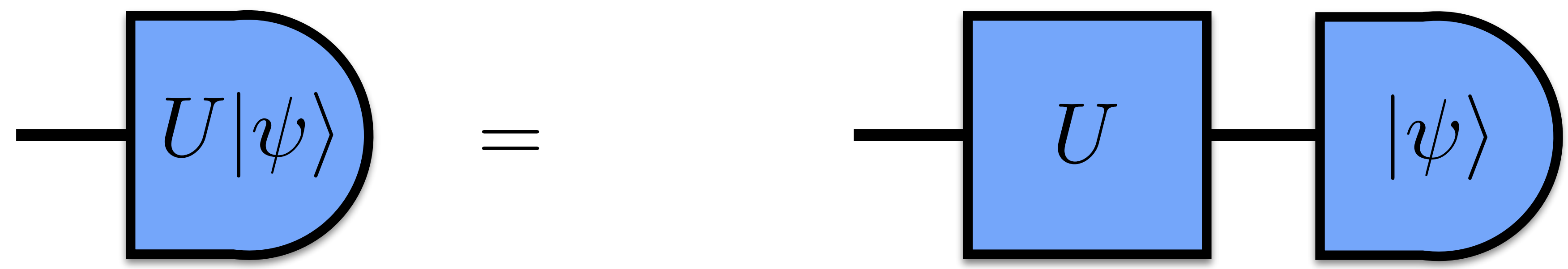
$$\sum_{i,j} \rho_{i,j} |i\rangle\langle j|$$

Contraction

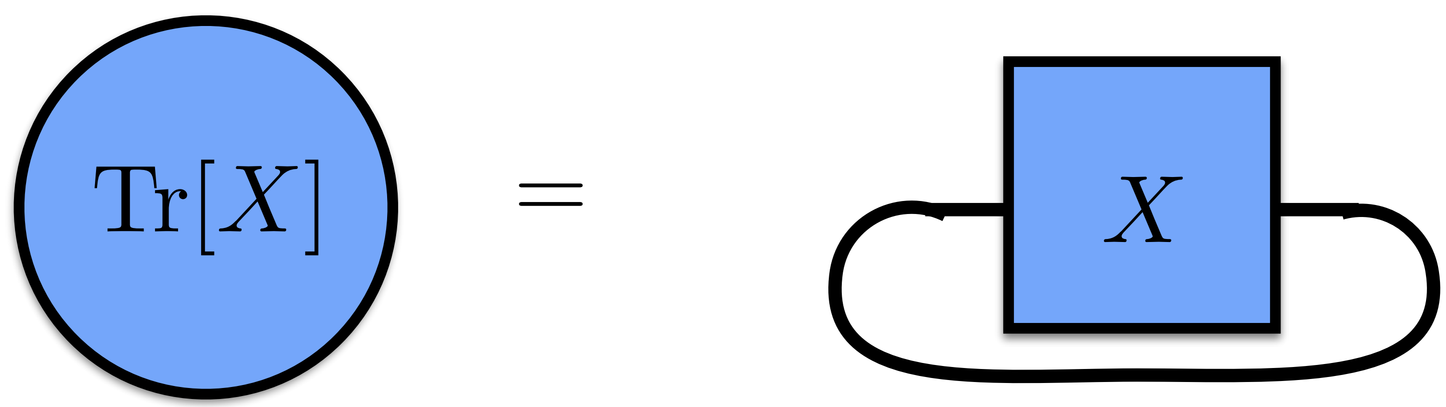
Expectation value $X_i^j \bar{\psi}^i \psi_j$



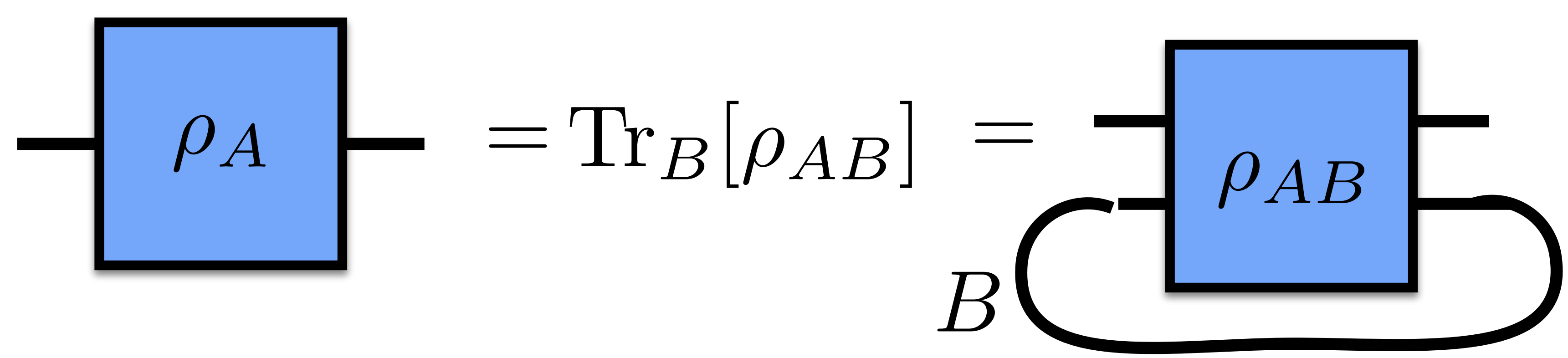
Unitary evolution $U_i^j \psi_j$



Trace $X_i^j \delta_j^i$



Partial trace $\rho_{i_a,i_b}^{j_a,j_b} \delta_{j_b}^{i_b}$

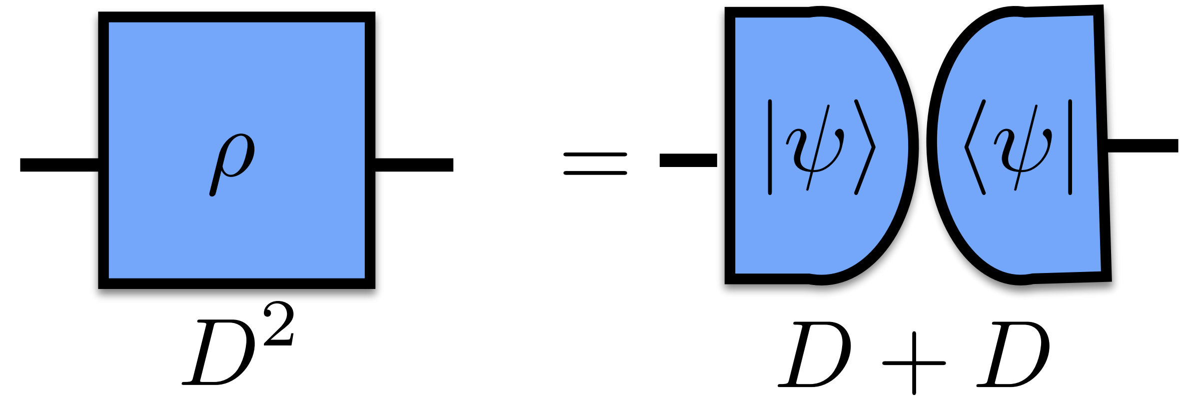


Products / Factorization(s)

Pure state density matrix

$$\rho = |\psi\rangle\langle\psi|$$

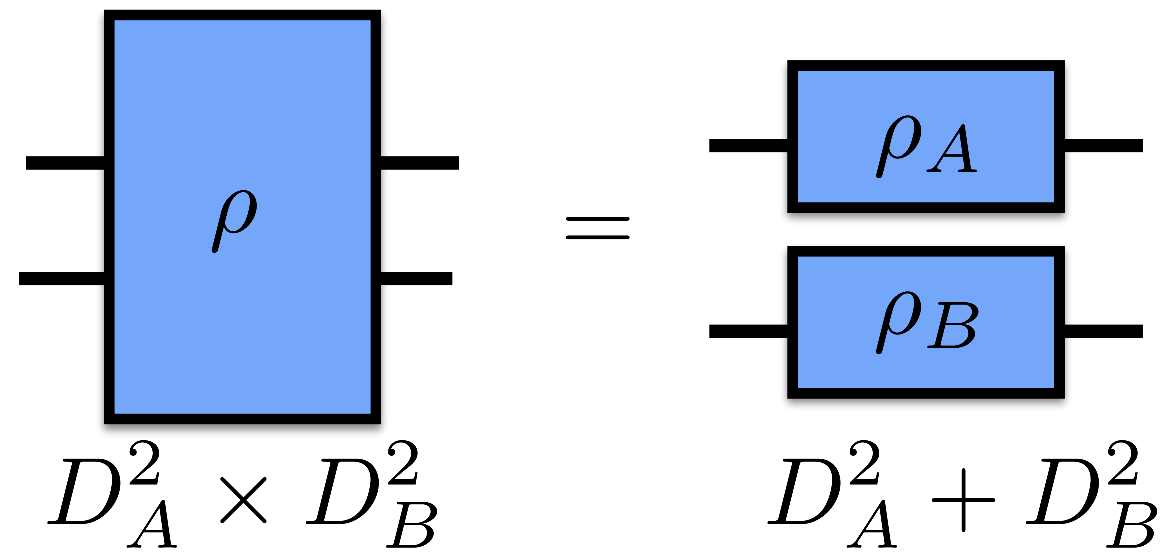
$$\rho_i^j = \psi_i \bar{\psi}^j$$



Product state (similarly op.)

$$\rho = \rho_A \otimes \rho_B$$

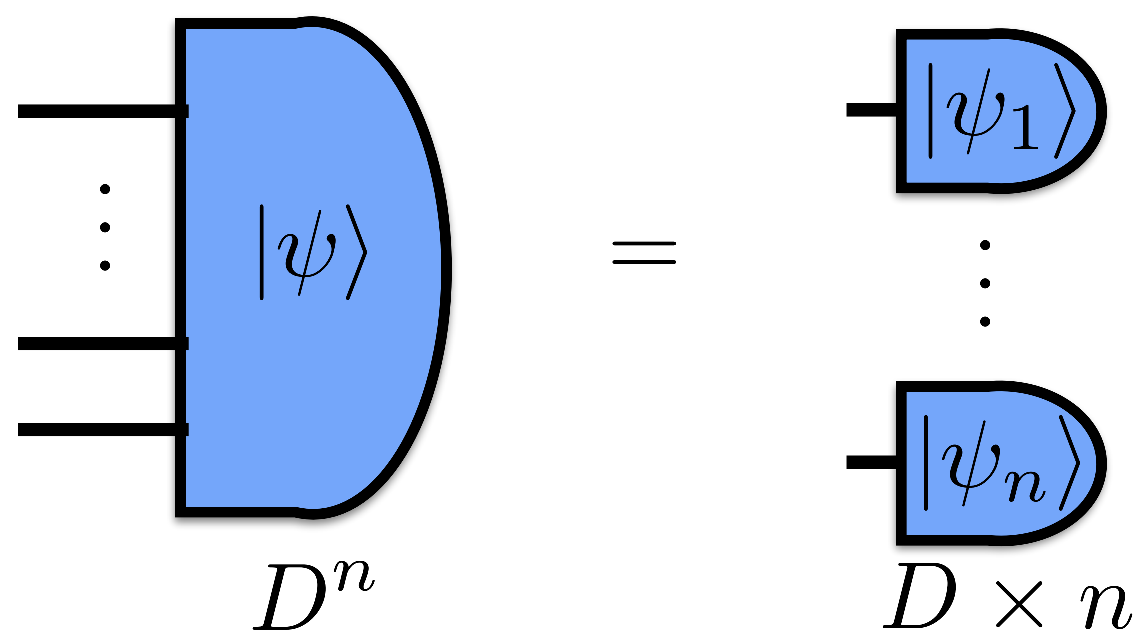
$$\rho_{i_a, i_b}^{j_a, j_b} = \rho_{i_a}^{j_a} \rho_{i_b}^{j_b}$$



Pure product state (ket)

$$|\psi\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$$

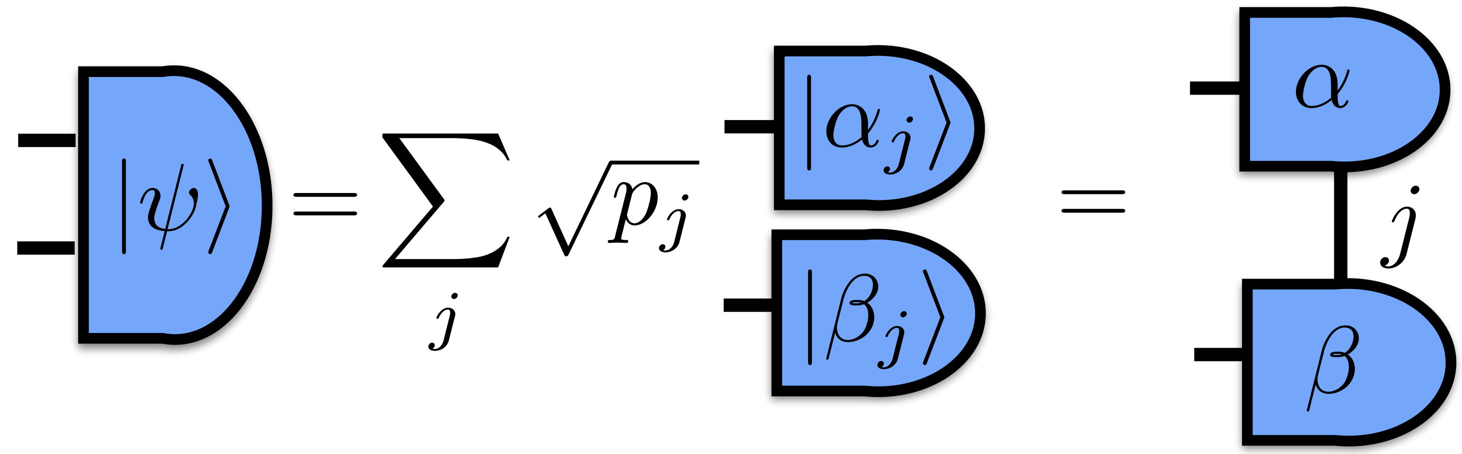
$$\psi_{\vec{i}} = \psi_{i_1} \ldots \psi_{i_n}$$



Schmidt decomposition

Entangled state

$$|\psi\rangle = \sum_j \sqrt{p_j} |\alpha_j\rangle \otimes |\beta_j\rangle$$



$D_A \times D_B$ $\qquad$ $s \times (D_A + D_B)$

$$\psi_{i_a, i_b} = \alpha_{i_a, s_a} \beta_{i_b, s_b} \delta^{s_a, s_b}$$

low entanglement states well approximated by **low Schmidt** rank *s*.

$+$

Entanglement area law (low)

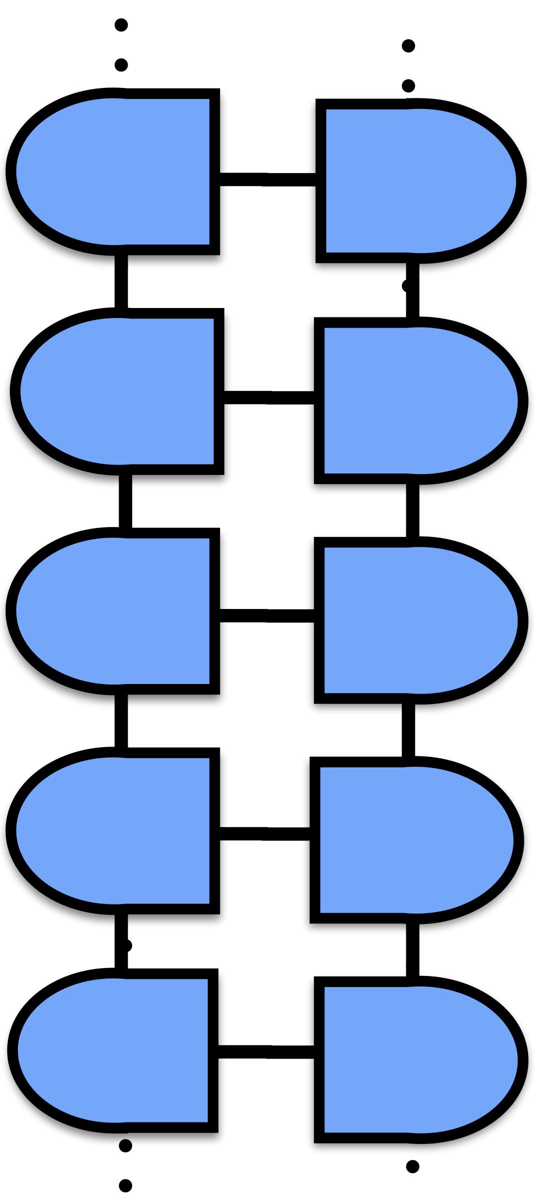
Basis for tensor network ansatz

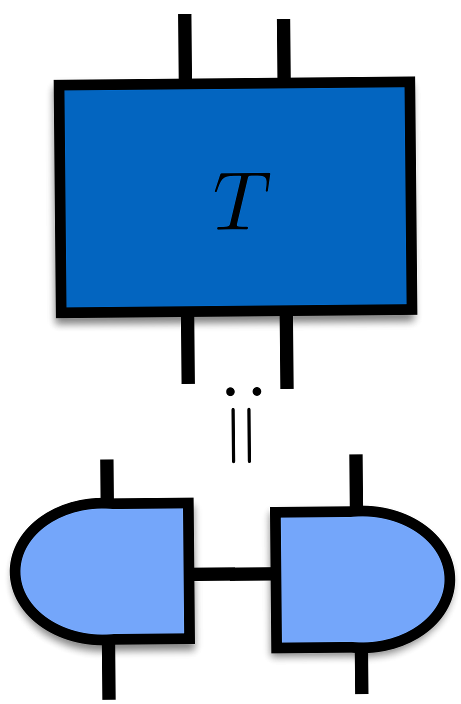
Example: Matrix Product States **(MPS)**

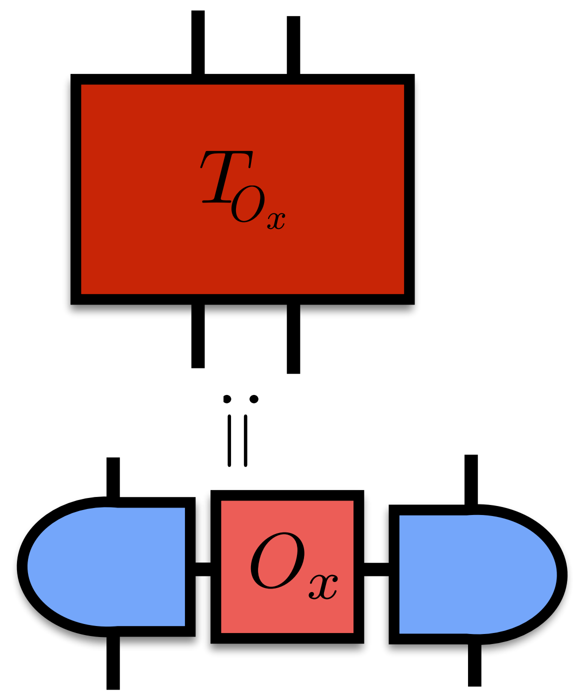
Normalization and expectation values

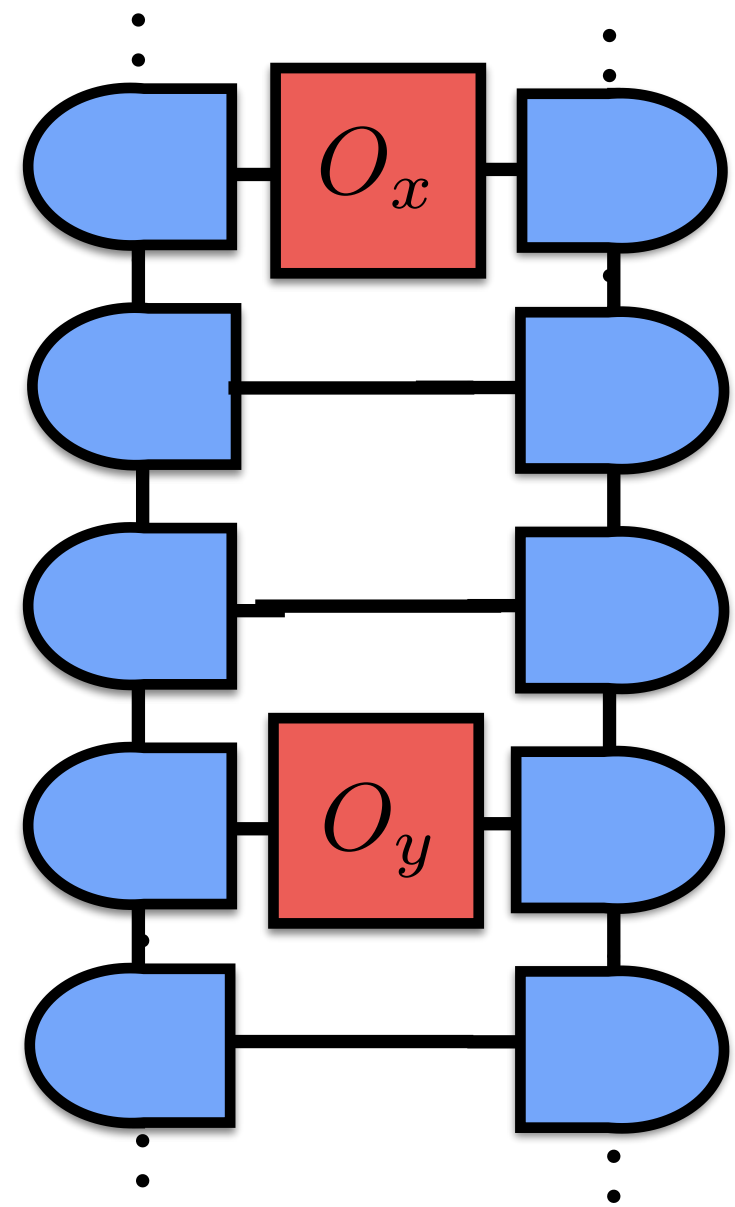
$$1 = \langle\psi|\psi\rangle = Tr[T^n]$$

$$\langle\psi|O_x O_y|\psi\rangle = Tr[T_{O_x} T^{d_1} T_{O_y} T^{d_2}]$$









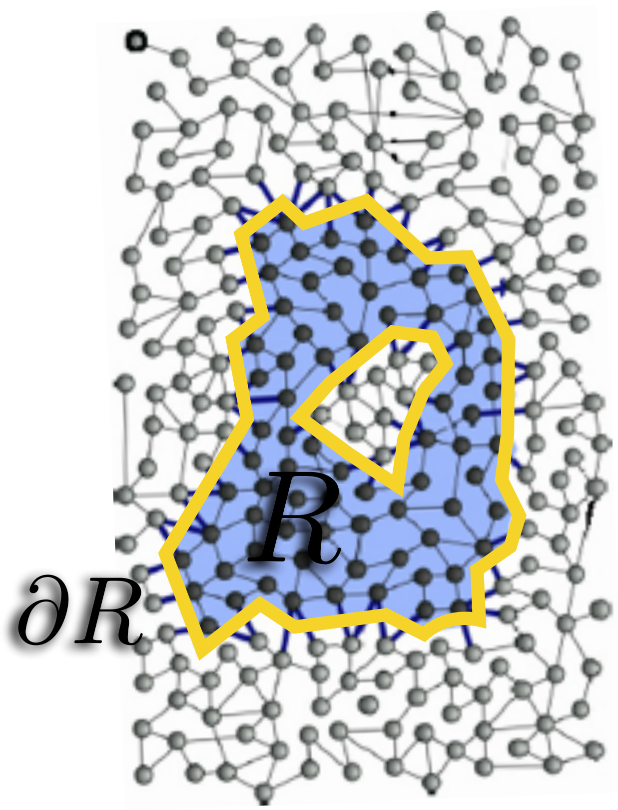
Entanglement structure

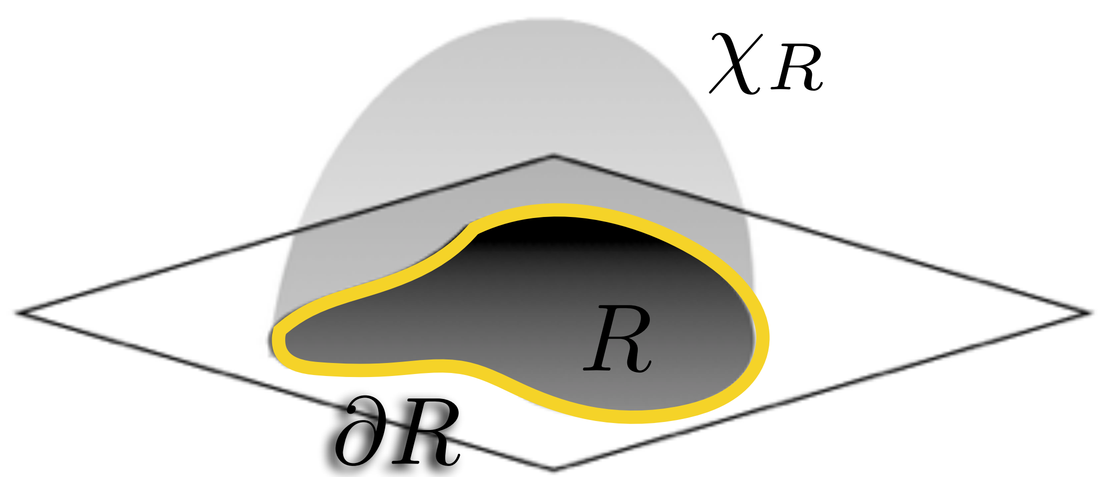
More structure ⟵ ⟶ Less structure

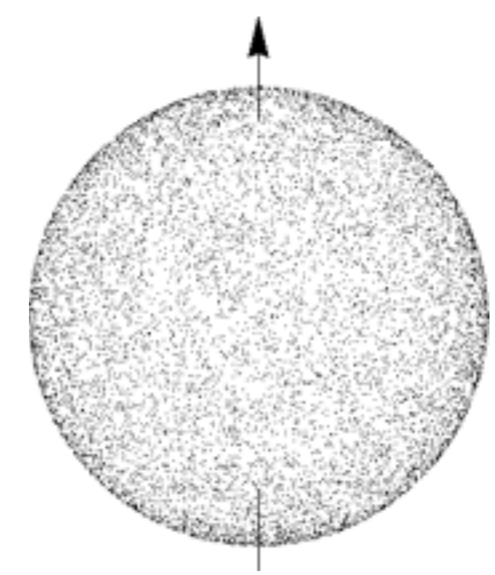
Less entanglement | More entanglement

Product	**Area law**	**Holographic area law**	**Haar random**
$S(R) = 0$	$S(R) \propto \lvert \partial R \rvert$	$S(R) \propto \lvert \chi_R \rvert$	$S(R) \propto \lvert R \rvert$

$$\Pi$$







Product state
no entanglement
or correlations

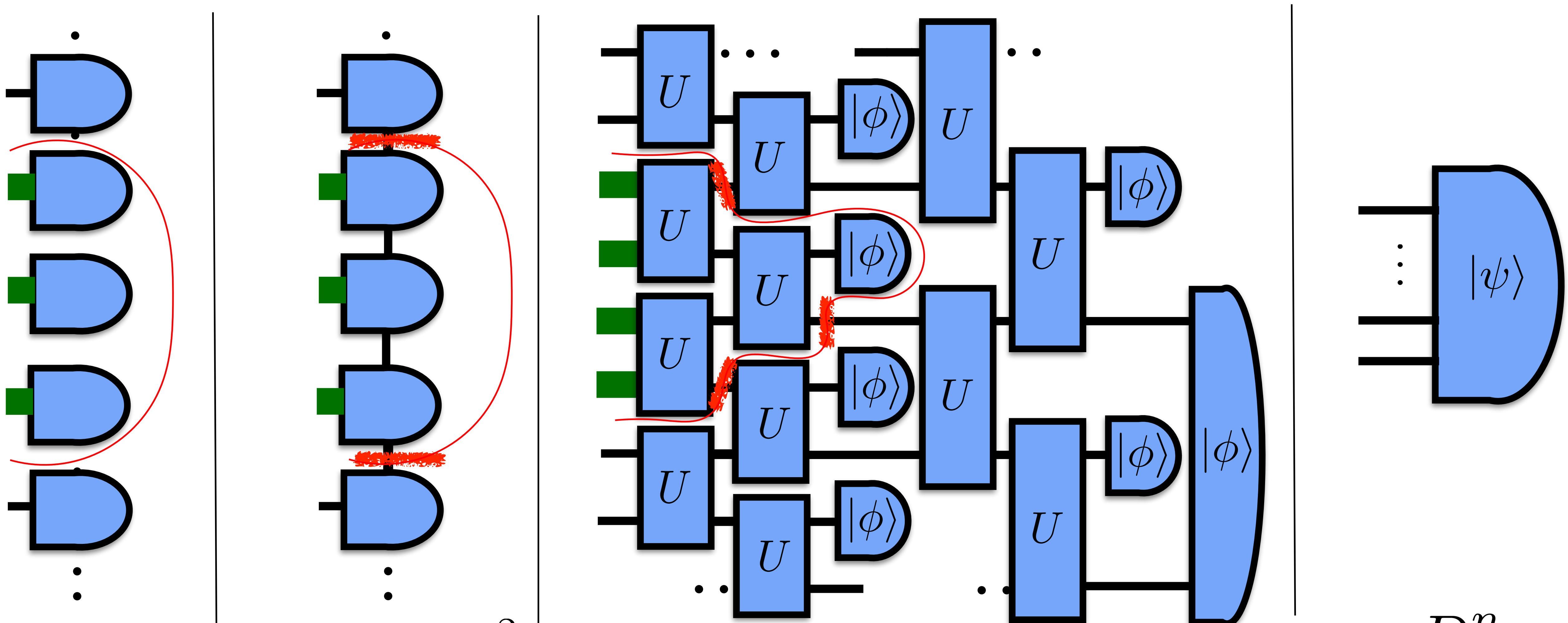
Area law state;
entanglement grows with
subsys boundary area in
“actual” geometry.

The Ryu-Takayanagi
prescription or
Holographic area law
entanglement grows with
boundary area in emergent
holographic geometry.

"**Generic**" pure states
entanglement grow with
volume law.
As subsystem dimension.

Tensor network ansatz(e)

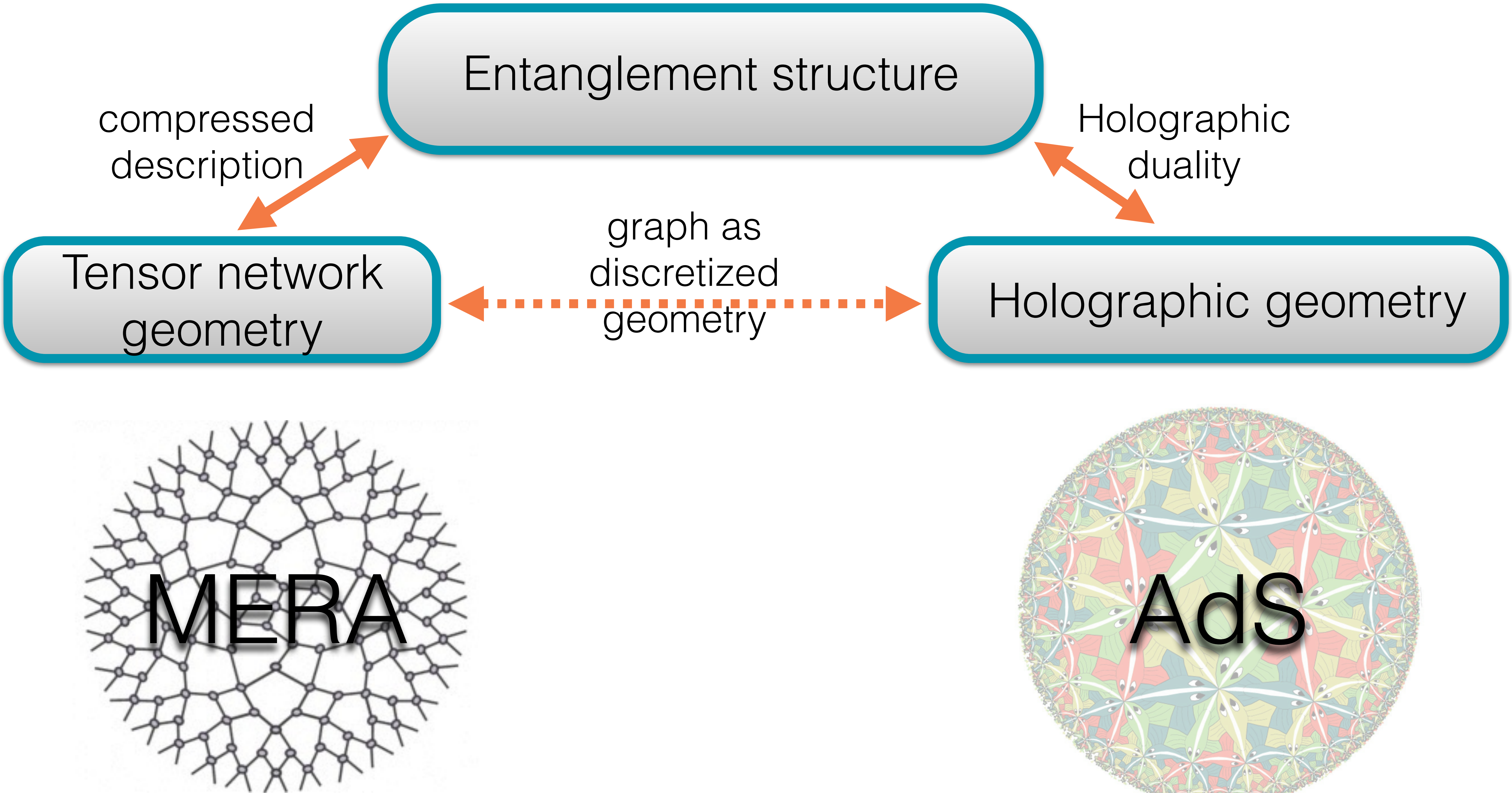
Example: product state	Example: **(MPS)** matrix product states	Example: **(MERA)** Multiscale entanglement renormalisation ansatz	No compressed representation possible



$n \times D$	$n \times (D \times s^2)$	$2n(2D^4 + D)$	D^n

Graph Min-Cut ≳ Entanglement ~ Holographic minimal surface

Holography & TN



Swingle, B. (2012). Entanglement renormalization and holography. PRD, 86(6), 065007.

Holographic QECCs

Tensor network construction.

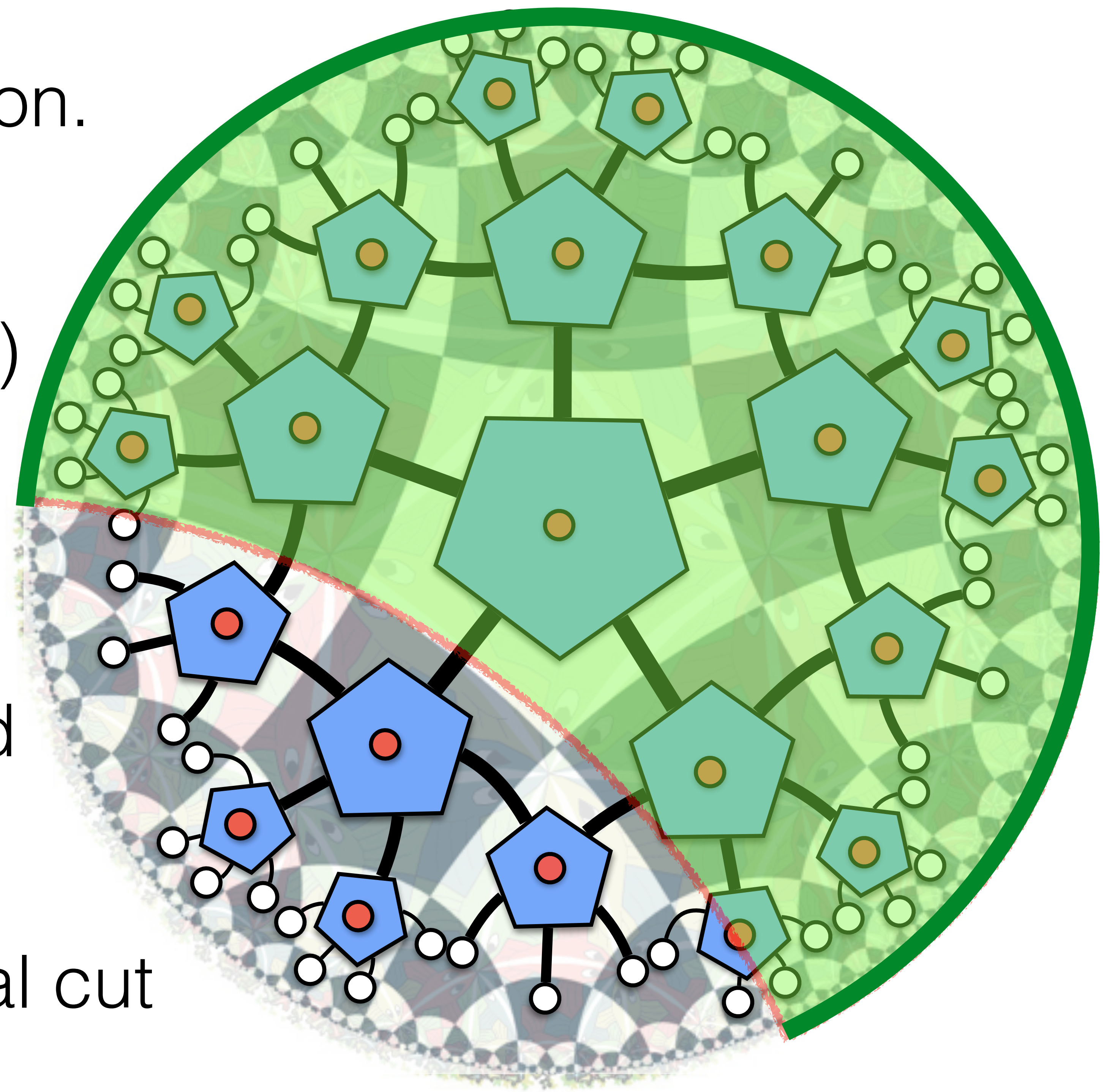
Isometry interpretation.

Boundary: physical (white)

Bulk: logical inputs (red)

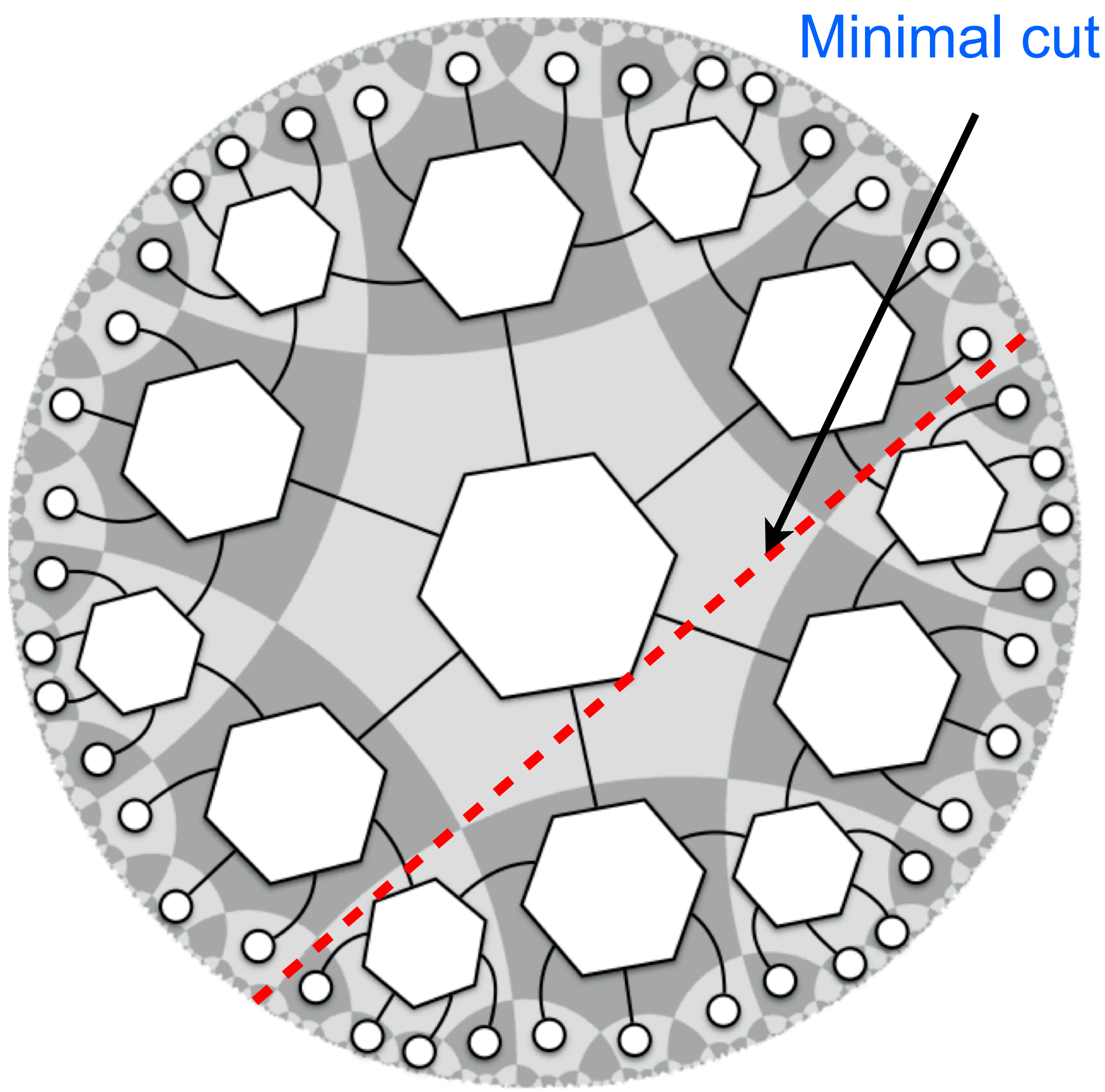
Code family interpolating uniform concatenated and convolutional codes

Minimal surface -> minimal cut



PF, Yoshida, B., Harlow, D., & Preskill, J. (2015). Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. JHEP, 2015(6), 149.

Holographic states



Entanglement = (Minimal cut) x log(d_{bond})

FP, Yoshida, B., Harlow, D., & Preskill, J. (2015). *Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence.* JHEP, 2015(6), 149.

Hayden, P., Nezami, S., Qi, X.-L., Thomas, N., Walter, M., & Yang, Z. (2016). *Holographic duality from random tensor networks*. JHEP, 2016(11), 009.

Conclusion

- Entanglement and QEC are key holography structures
  - RT: Bulk minimal surface area -> Boundary entanglement !
  - QEC: Bulk to boundary mapping “locality”
- Tensor networks
  - Synthetic entanglement structures
  - Toy QEC models for bulk -> boundary locality
  - Many isomorphisms become manifest