1. Getting Started with Probability

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Outline

1. Experiments, Sample Spaces, and Events
2. What is Probability?
3. Basic Probability Results
4. Finite Sample Spaces
5. Counting Techniques: Baby Examples
6. Counting Techniques: Permutations
7. Counting Techniques: Combinations
8. Hypergeometric, Binomial, and Multinomial Problems
9. Permutations vs. Combinations
10. The Birthday Problem
11. The Envelope Problem
12. Poker Problems
13. Conditional Probability
14. Independence Day
15. Partitions and the Law of Total Probability
16. Bayes Theorem

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- Intro to Experiments, Sample Spaces, and Events
- Definition of Probability
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Examples:

- Toss a coin.
- Toss a coin 3 times.
- Ask 10 people if they prefer Coke or Pepsi.
- See how long a light bulb lasts.

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- Ask 10 people if they prefer Coke or Pepsi: $S = \{0, 1, \ldots, 10\}$.
- Light bulb life: $S = \{t | t \geq 0\}$.

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B &= \text{“no } T\text{'s observed”} \;=\; \{HHH\} \\
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Then

$$\begin{aligned} A \cup B &= \text{"at most one } T \text{ observed"} \\ &= \{HHT, HTH, THH, HHH\} \\ A \cap C &= \{HHT, HTH\}. \quad \square \end{aligned}$$

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Example: Toss a fair die. $S = \{1, 2, 3, 4, 5, 6\}$, where each individual outcome has probability 1/6. Then $P(1 \text{ or } 2) = 1/3$.

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(4) Suppose $A_1, A_2, \ldots$ is a sequence of *disjoint* events (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$). Then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

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(We'll eventually see why that last equality holds, though it may already be intuitively obvious.)

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Example: The probability that it'll rain tomorrow is 1 minus the probability that it won't rain.

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Remark: The converse is *false*: $P(A) = 0$ does *not* imply $A = \emptyset$.

Example: Pick a random number between 0 and 1. Later on, we'll show why any particular outcome actually has probability 0!

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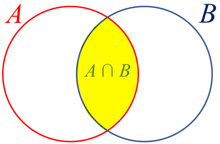

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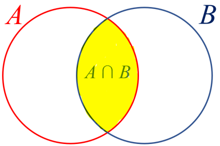

Remark: Axiom (3) is a "special case" of this theorem with $A \cap B = \emptyset$.

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$$\begin{aligned} P(R) &= P(R \cup C) - P(C) + P(R \cap C) \\ &= 0.8 - 0.4 + 0.1 \;=\; 0.5. \quad \Box \end{aligned}$$

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The formal proof is a bit tedious. You can try an informal proof via Venn diagrams, but you'll need to be careful about double and triple counting events.

Example: 75% of Atlantans jog (J), 20% like ice cream (I), and 40% enjoy music (M). Also, 15% J *and* I, 30% J *and* M, 10% I *and* M, and 5% do all three. Find the probability that a random resident will engage in at least one of the three activities.

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$$\begin{aligned} &P(J \cup I \cup M) \\ &\quad = \quad P(J) + P(I) + P(M) \\ &\qquad\qquad -P(J \cap I) - P(J \cap M) - P(I \cap M) \\ &\qquad\qquad +P(J \cap I \cap M) \end{aligned}$$

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&\qquad\qquad -P(J \cap I) - P(J \cap M) - P(I \cap M) \\
&\qquad\qquad +P(J \cap I \cap M) \\
&\quad = \quad 0.75 + 0.20 + 0.40 - 0.15 - 0.30 - 0.10 + 0.05 \\
&\quad = \quad 0.85. \quad \Box
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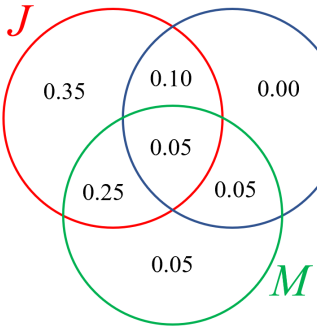

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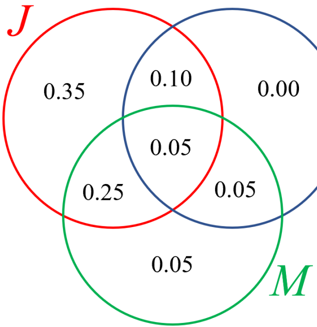

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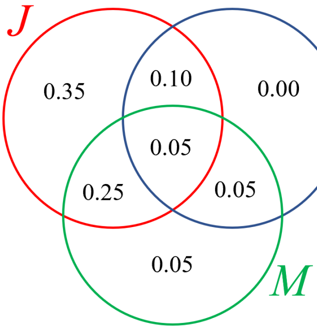
Now find the probability of precisely one activity. We can use a Venn diagram, starting from the center (since $P(J \cap I \cap M) = 0.05$) and building out.



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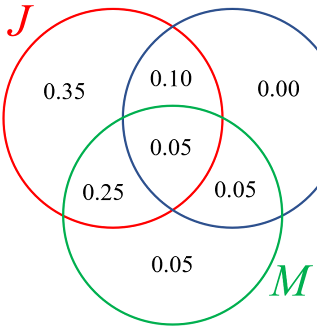

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The proof of this thing is quite tedious. In any case, the previous two theorems are special cases.

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Lesson 1.4 — Finite Sample Spaces

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The sample space $S = \{\text{red, blue, yellow}\} = \{s_1, s_2, s_3\}$.

$P(s_1) = 1/2, P(s_2) = 1/4, P(s_3) = 1/4.$

$P(\text{red or yellow}) = P(s_1) + P(s_3) = 3/4. \quad \Box$

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Example: Toss a die. Let $A = \{1, 2, 4, 6\}$. Each outcome has probability 1/6, so $P(A) = 4/6$. $\Box$

Example: Roll a pair of dice. Possible results (each w.p. 1/36):

$$\begin{array}{cccc} 1,1 & 1,2 & \cdots & 1,6 \\ 2,1 & 2,2 & \cdots & 2,6 \\ & \vdots & & \\ 6,1 & 6,2 & \cdots & 6,6 \end{array}$$

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Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

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With this material in mind, we can now move on to more-complicated counting problems. . . .

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Baby Example: $n_{\text{AB}} = 3$ ways to go from City A to B (walk, car, bus), and $n_{\text{BC}} = 4$ ways to go from B to C (car, bus, train, plane). Then you can go from A to C (via B) using $n_{\text{AB}}\, n_{\text{BC}} = 12$ itineraries. $\Box$

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Georgia Tech

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Example: A baseball manager has 9 players on his team. Find the number of possible batting orders. Answer: $9! = 362880$. $\Box$

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Answer: $P_{4,2} = 4!/(4-2)! = 12$. Let's list them:

$$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc. \quad \Box$$

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P_{n,r} &= \text{(choose first)(choose second)} \cdots \text{(choose } r\text{th)} \\
&= n(n-1)(n-2)\cdots(n-r+1) \\
&= \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1}{(n-r)\cdots 2\cdot 1} \\
&= \frac{n!}{(n-r)!}. \quad \Box
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$P_{9,4} = 9!/(9-4)! = 3024$ ways. $\square$

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Method 2: It's clear that each of the 9 players is equally likely to bat first. Thus, $3024/9 = 336$. $\square$

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(c) containing repetitions? $531441 - 60480 = 470961$. $\square$

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Lesson 1.7 — Counting Techniques: Combinations

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Definition: The number of subsets with r elements of a set with n elements is called the **number of combinations of n things taken r-at-a-time**.

Notation: $\binom{n}{r}$ or $C_{n,r}$ (read as "n choose r"). These are also called **binomial coefficients**. It turns out (see below) that $C_{n,r} = \frac{n!}{r!(n-r)!}$.

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Binomial Theorem: We won't prove it here, but you may know this famous result.

$$(x+y)^n \;=\; \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

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Example: Smith is one of the players on the team. How many of the 792 starting line-ups include him?

$$\binom{11}{4} = \frac{11!}{4!7!} = 330.$$

(Smith gets one of the five positions for free; there are now 4 left to be filled by the remaining 11 players.) $\square$

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Answer: $\binom{12}{7}$. $\square$

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Answer: $\binom{12}{7}$. $\Box$

How many ways to put 5 blues in 12 slots? Same answer. $\Box$

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Lesson 1.8 — Hypergeometric, Binomial, and Multinomial Problems

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Next Few Lessons — all involve interesting applications:

- Hypergeometric Distribution (sampling without replacement)
- Binomial Distribution (sampling with replacement)
- Multinomial Coefficients (generalizes binomial)
- Permutations vs. Combinations
- The Birthday Problem
- The Envelope Problem
- Poker Probabilities

Hypergeometric Distribution

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The number of type 1's chosen is said to have the **hypergeometric distribution**. We'll have a very thorough discussion on "distributions" later.

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Make sure to compare these answers with the answers to the analogous hypergeometric examples.

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Lesson 1.9 — Permutations vs. Combinations

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Example: 4 red marbles, 2 whites. Put them in a row in random order. Find…

(a) $P(2 \text{ end marbles are W})$.

(b) $P(2 \text{ end marbles aren't both W})$.

(c) $P(2 \text{ W's are side by side})$.

Method 1 (using permutations): Let the sample space

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$$\begin{aligned} |A| &= (\text{\# ways to permute the 2 W's in the end slots}) \\ &\quad \times (\text{\# ways to permute the 4 R's in the middle slots}) \end{aligned}$$

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$$S = \{\text{every random ordering of the 6 marbles}\}.$$

(a) A: 2 end marbles are W — WRRRRW.

$$\begin{aligned} |A| &= (\text{\# ways to permute the 2 W's in the end slots}) \\ &\quad \times (\text{\# ways to permute the 4 R's in the middle slots}) \\ &= 2! \times 4! = 48. \end{aligned}$$

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But — The above method took too much time! Here's an easier way….

Method 2 (using combinations): Which 2 positions do the W's occupy?

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(c) $|B| = 5 \Rightarrow P(B) = 5/15 = 1/3$. $\square$

(That was much nicer!)

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Lesson 1.10 — The Birthday Problem

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There are n people in a room. Find the probability that at least two have the same birthday. (Ignore Feb. 29, and assume that all 365 days have equal probability.)

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The (simple) sample space is $S = \big\{(x_1, \ldots, x_n) : x_i \in \{1, 2, \ldots, 365\},\ \forall i\big\}$ (x_i is person i's birthday), and note that $|S| = (365)^n$.

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Let A: All birthdays are different. Then

$$|A| \ = \ P_{365,n} \ = \ (365)(364)\cdots(365 - n + 1).$$

Thus, we have

$$
\begin{aligned}
P(A) &= \frac{(365)(364)\cdots(365-n+1)}{(365)^n} \\
&= 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365}.
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When $n = 50$, $P(\bar{A}) = 0.97$. $\Box$

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Let A_i: Person i receives the correct envelope.

We obviously want $P(A_1 \cup A_2 \cup \cdots \cup A_n)$.

By the general principle of inclusion-exclusion, we have....

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\begin{aligned}
&P(A_1 \cup A_2 \cup \cdots \cup A_n) \\
&\quad = \sum_{i=1}^{n} P(A_i) - \sum\sum\nolimits_{i<j} P(A_i \cap A_j) \\
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Example: If there are just $n = 4$ envelopes, then

Finally, $P(A_1) = 1/n$, $P(A_1 \cap A_2) = 1/(n(n-1))$, etc. (why?) imply that

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$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = 0.625,$$

which is right on the asymptotic money! $\square$

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Lesson 1.12 — Poker Problems

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The number of possible hands is $|S| = \binom{52}{5} = 2{,}598{,}960$.

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Terminology (not that I'm advocating gambling, but if you aren't familiar with poker, take a few minutes to learn the basics):

rank $= 2, 3, \ldots, Q, K, A$,
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We will calculate the probabilities of obtaining various special "hands"....

(a) **2 pairs** — e.g., $A\heartsuit$, $A\clubsuit$, $3\heartsuit$, $3\diamondsuit$, $10\spadesuit$. (This hand does *not* include a full house, which will be explained below.)

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Select 2 suits for second pair (e.g., $\heartsuit, \diamondsuit$). $\binom{4}{2}$ ways.

Select remaining card to complete the hand. 44 ways (because we are not allowing a full house).

$$|\text{2 pairs}| \;=\; \binom{13}{2}\binom{4}{2}\binom{4}{2}44 \;=\; 123{,}552$$

$$P(\text{2 pairs}) \;=\; \frac{123{,}552}{2{,}598{,}960} \;\doteq\; 0.0475. \quad \square$$

(b) **Full house** (1 pair, 3-of-a-kind) — e.g., $A\heartsuit, A\clubsuit, 3\heartsuit, 3\diamondsuit, 3\spadesuit$

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$$P(\text{full house}) \;=\; \frac{3744}{2{,}598{,}960} \;\doteq\; 0.00144. \quad \Box$$

(c) **Flush** (all 5 cards from same suit) (This includes all kinds of flushes.)

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(d) **Straight** (5 ranks in a row) (This includes all straights.)

Select a starting point for the straight $(A, 2, 3, \ldots, 10)$. $\binom{10}{1}$ ways.

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$$P(\text{straight}) = \frac{10 \cdot 4^5}{2{,}598{,}960} \doteq 0.00394. \quad \Box$$

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Remark: Can you do bridge problems? Yahtzee?

Outline

1. Experiments, Sample Spaces, and Events
2. What is Probability?
3. Basic Probability Results
4. Finite Sample Spaces
5. Counting Techniques: Baby Examples
6. Counting Techniques: Permutations
7. Counting Techniques: Combinations
8. Hypergeometric, Binomial, and Multinomial Problems
9. Permutations vs. Combinations
10. The Birthday Problem
11. The Envelope Problem
12. Poker Problems
13. Conditional Probability
14. Independence Day
15. Partitions and the Law of Total Probability
16. Bayes Theorem

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Lesson 1.13 — Conditional Probability

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Next Few Lessons:

- Conditional Probability
- Independent Events
- Partition of a Sample Space and the Law of Total Probability
- Bayes Theorem (updating probabilities in a clever way)

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Example: If A is the event that a person weighs at least 150 pounds, then $P(A)$ certainly depends on the person's height, e.g., if B is the event that the person is at least 6 feet tall vs. B being the event that the person is < 5 feet tall.

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Remarks: If A and B are disjoint, then $P(A|B) = 0$. (If B occurs, there's no chance that A can also occur.)

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Remarks: If A and B are disjoint, then $P(A|B) = 0$. (If B occurs, there's no chance that A can also occur.)

What happens if $P(B) = 0$? Don't worry! In this case, makes no sense to consider $P(A|B)$.

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Thus, in light of the information provided by B, we see that $P(A) = 1/2$ increases to $P(A|B) = 2/3$. $\square$

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Could you have gotten this result without thinking?

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As you get more information, you can make some surprising findings. . . .

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Outline

1. Experiments, Sample Spaces, and Events
2. What is Probability?
3. Basic Probability Results
4. Finite Sample Spaces
5. Counting Techniques: Baby Examples
6. Counting Techniques: Permutations
7. Counting Techniques: Combinations
8. Hypergeometric, Binomial, and Multinomial Problems
9. Permutations vs. Combinations
10. The Birthday Problem
11. The Envelope Problem
12. Poker Problems
13. Conditional Probability
14. Independence Day
15. Partitions and the Law of Total Probability
16. Bayes Theorem

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Remark: So if A and B are independent, the probability of A doesn't depend on whether or not B occurs.

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Remark: In fact, independence and disjointness are almost opposites. If A and B are disjoint and A occurs, then you have *information* that B cannot occur — so A and B can't be independent!

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Remark: For independent trials, you just multiply the individual probabilities.

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Lesson 1.15 — Partitions and the Law of Total Probability

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Example: "Vowels" and "consonants" form a partition of the letters (if you pretend that only a,e,i,o,u are vowels).

Remark: It's often convenient to choose all of the A_i's such that $P(A_i) > 0$, but this is not actually a requirement.

Suppose $A_1, A_2, \ldots, A_n$ form a partition of S, and B is some arbitrary event.

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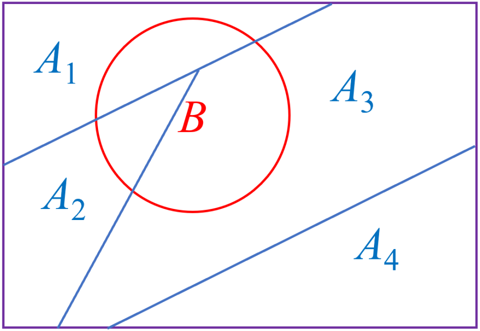

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This is the **Law of Total Probability**.

Example: $P(B) = P(A)P(B|A) + P(\bar{A})P(B|\bar{A})$, which we saw in the previous lesson.

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$$\begin{aligned} P(\text{passes}) &= P(\text{GT})P(\text{passes}\,|\,\text{GT}) + P(\text{UGA})P(\text{passes}\,|\,\text{UGA}) \\ &= (1/3)(0.95) + (2/3)(0.5) \;=\; 0.65. \quad \Box \end{aligned}$$

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The $P(A_j|B)$'s add up to 1. That's why the funny-looking denominator.

Example: Two political candidates are at a debate. Candidate A is asked 60% of the questions, and candidate B is asked just 40% (for some reason). Candidate A is likely to make a stupid answer 20% of the time, and B makes a dumb answer a whopping 50% of the time. One of the candidates is asked a question and makes a dumb answer. What's the probability that it was A?

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Notice how the posterior probabilities depend strongly on the priors and the information we receive. $\square$

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$$= \frac{(0.01)(0.95)}{(0.01)(0.95) + (0.99)(0.05)} \;=\; 0.161. \quad \square$$

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\begin{aligned}
& P(\text{Car behind } 3 \,|\, \text{Monty shows you door } 2) \\
&\quad = \frac{P(\text{Monty shows you door } 2 \,|\, \text{Car behind } 3) P(\text{Car behind } 3)}{\sum_{i=1}^{3} P(\text{Monty shows you door } 2 \,|\, \text{Car behind } i) P(\text{Car behind } i)} \\
&\quad = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} \;=\; 2/3.
\end{aligned}
$$

On the other hand,

$$\begin{aligned} & P(\text{Car behind } 3 \,|\, \text{Monty shows you door } 2) \\ &= \frac{P(\text{Monty shows you door } 2 \,|\, \text{Car behind } 3) P(\text{Car behind } 3)}{\sum_{i=1}^{3} P(\text{Monty shows you door } 2 \,|\, \text{Car behind } i) P(\text{Car behind } i)} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} \;=\; 2/3. \end{aligned}$$

Thus, the prudent action is to switch to door 3! $\Box$

On the other hand,

$$\begin{aligned}
&P(\text{Car behind 3}\,|\,\text{Monty shows you door 2}) \\
&\quad= \frac{P(\text{Monty shows you door 2}\,|\,\text{Car behind 3})P(\text{Car behind 3})}{\sum_{i=1}^{3} P(\text{Monty shows you door 2}\,|\,\text{Car behind } i)P(\text{Car behind } i)} \\
&\quad= \frac{1\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}} \;=\; 2/3.
\end{aligned}$$

Thus, the prudent action is to switch to door 3! $\square$

If you don't quite believe this, you aren't alone. But think what you would do if there were 1000 doors and Monty revealed 998 of them — of course you would switch from your door to the remaining one!