

White Dwarfs as Dark Matter Detectors

Peter W. Graham,¹ Ryan Janish,² Vijay Narayan,² Surjeet Rajendran,² and Paul Riggins²

¹*Stanford Institute for Theoretical Physics, Department of Physics, Stanford University, Stanford, CA, 94305*

²*Berkeley Center for Theoretical Physics, Department of Physics,
University of California, Berkeley, CA 94720, USA*

If dark matter (DM) is capable of sufficiently heating a local region in a white dwarf, it will trigger runaway fusion and ignite a type 1a supernova. This was originally proposed in [4] and used to constrain primordial black holes which transit and heat a white dwarf via dynamical friction. In this paper, we extend the analysis of white dwarf DM detection to candidates with non-gravitational interactions that heat through the production of standard model (SM) particles. We consider a general class of models in which DM-DM collisions, DM decays, or DM transits including a SM scattering interaction produce particles that subsequently deposit energy inside the star. The existence of long-lived white dwarfs and the observed supernovae rate provide constraints for such models. As a concrete example, we rule out supersymmetric Q-ball DM in a large region of parameter space fundamentally inaccessible to terrestrial detection. It is also intriguing that the DM-induced ignition discussed in this work provide an alternative mechanism of triggering supernovae from sub-Chandrasekhar Mass progenitors.

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I. INTRODUCTION

Identifying the nature of dark matter (DM) remains one of the clearest paths beyond the Standard Model (SM) and it is thus fruitful to study the observable signatures of any yet-allowed candidate. Many terrestrial direct detection experiments are designed to search for DM, e.g. [2, 3], yet these lose sensitivity to heavier DM due to its diminished number density. Even for a strongly-interacting candidate, if the DM mass is above $\sim 10^{22}$ GeV a detector of size $\sim (100 \text{ m})^2$ will register fewer than one event per year. While these masses are large compared to those of fundamental particles, it is reasonable to suppose that DM may exist as composite states just as the SM produces complex structures with mass much larger than fundamental scales (e.g. you, dear reader). Currently there is a wide range of unexplored parameter space for DM candidates less than $\sim 10^{48}$ GeV, above which the DM will have observable gravitational microlensing effects [1]. For such ultra-heavy DM, indirect signatures in astrophysical systems are a natural way forward. One possibility proposed by [4] is that DM can trigger runaway fusion and ignite type 1a supernovae (SN) in sub-Chandrasekhar white dwarf (WD) stars. This is even more interesting in light of recent observations that an $\mathcal{O}(1)$ fraction of type 1a SN are due to sub-Chandrasekhar progenitors [23, 24]. While astrophysical mechanisms for these events have been proposed [25, 26], the situation is yet unclear and it is an exciting possibility that these SN arise from DM interactions.

Runaway thermonuclear fusion requires both a heating event and the lack of significant cooling which might quench the process. The WD medium is particularly suited to this as it is dominated by degeneracy pressure and undergoes minimal thermal expansion, which is the mechanism that regulates fusion in main sequence stars. Thermal diffusion is the primary cooling process in a WD and it can be thwarted by heating a large enough region. The properties of a localized heating necessary to trigger runaway fusion were computed in [5]. Consequently, it was realized [4] that if DM is capable of sufficiently heating a WD in this manner, it will result in a SN with sub-Chandrasekhar Mass progenitor. This was used to constrain primordial black holes which transit a WD and cause heating by dynamical friction, although the authors of [4] identify several other heating mechanisms which may be similarly constrained.

In this paper, we examine DM candidates with non-gravitational interactions that cause heating through the production of SM particles. An essential ingredient in this analysis is understanding the length scales over which SM particles deposit energy in a WD medium. We find that most high energy particles thermalize efficiently with ions in the star, nearly independent of species or initial energy. Particle production is thus an effective means of heating WDs. Constraints on these DM candidates come from either observing specific, long-lived WDs or by comparing the measured rate of type 1a SN with

that expected due to DM. It is important to note that these constraints are complementary to direct searches—it is more massive DM that is likely to trigger SN, but also more massive DM that has low terrestrial flux. The WD detector excels in this regime due to its large surface area $\sim (10^4 \text{ km})^2$, long lifetime $\sim \text{Gyr}$, and galactic abundance. We demonstrate these constraints for generic classes of DM models that produce SM particles via DM-DM collisions, DM decays, or DM transits including a SM scattering interaction. As a concrete example we consider ultra-heavy Q-ball DM as found in supersymmetric extensions of the SM.

The rest of the paper is organized as follows. We begin in Section ?? by reviewing the mechanism of runaway fusion in a WD. In Section ?? we study the non-gravitational heating of a WD due to the production of high-energy SM particles. Detailed calculations of the stopping of such particles are provided in Appendix A. In Section ?? we parameterize the explosiveness and event rate for generic classes of DM-WD encounters, and in Section V we derive schematic constraints on such models. Finally we specialize to the case of Q-balls in Section ??, and conclude in Section ??.

II. WHITE DWARF RUNAWAY FUSION

We first review the conditions for which a local energy deposition in a WD results in runaway fusion. Any energy deposit will eventually heat ions within some localized region—parameterize this region by its linear size L_0 , total kinetic energy \mathcal{E}_0 and typical temperature T_0 . These scales evolve in time, but it will be useful to describe a given heating event by their initial values.

The fate of a heated region is either a nonviolent diffusion of the excess energy across the star, or a runaway fusion chain-reaction that destroys the star. The precise outcome depends on L_0 , \mathcal{E}_0 and T_0 . There is a critical temperature T_f , set by the energy required for ions to overcome their mutual Coulomb barrier, above which fusion occurs. For carbon burning, $T_f \sim \text{MeV}$ [6]. Any heated region $T_0 > T_f$ will initially support fusion, although this is not sufficient for runaway as cooling processes may rapidly lower the temperature below T_f . This cooling will not occur if the corresponding timescale is larger than the timescale at which fusion releases energy. Cooling in a WD is dominated by thermal diffusion, and the diffusion time increases as the size of the heated region. However, the timescale for heating due to fusion is independent of region size. Thus, for a region at temperature $\gtrsim T_f$, there is a critical size above which the heated region does not cool but instead initiates runaway. For a region at the critical fusion temperature T_f , we call this critical size the *trigger size* λ_T . The value of λ_T is highly dependent on density, and in a WD is set by the thermal diffusivity of either photons or degenerate electrons. This critical length scale has been computed numerically in [5] for a narrow range of WD densities and analyti-

cally scaled for other WD masses in [4]. As in [4], we will restrict our attention to carbon-oxygen WDs in the upper mass range $\sim 0.85 - 1.4 M_\odot$ (these will yield the most stringent constraints on DM). This corresponds to a central number density of ions $n_{\text{ion}} \sim 10^{30} - 10^{32} \text{ cm}^{-3}$ and a trigger size of $\lambda_T \sim 10^{-3} - 10^{-5} \text{ cm}$.

If a heated region is smaller than the trigger size, its thermal evolution is initially dominated by diffusion. However, this will still result in runaway fusion if the temperature is of order T_f by the time the region diffuses out to the trigger size. For our purposes it is more natural to phrase this in terms of the total energy \mathcal{E}_0 deposited during a heating event. Of course, the relation between energy \mathcal{E}_0 and temperature T_0 depends on the rate at which WD constituents—ions, electrons, and photons—thermalize with each other within the region size L_0 . In Section ?? we explicitly calculate the length scales over which a hot bath of ions thermalizes electrons and photons, focusing our attention on ions because they are what ultimately must be heated in order for fusion to take place. Here we simply state the results: if ions are heated to temperatures $T_0 \sim 1 - 10 \text{ MeV}$, typical for a wide variety of heating processes, then electrons and photons are also heated to T_0 within the trigger size. Therefore, any heating event which results in runaway fusion necessarily has ions, electrons, and photons in thermal equilibrium once a region of size λ_T or greater is at the critical temperature T_f . The excess energy in a volume V required to heat all these species to T_f is given by a sum of their heat capacities

$$\frac{\mathcal{E}_0}{V} \gtrsim \int_0^{T_f} dT (n_{\text{ion}} + n_e^{2/3} T + T^3), \quad (1)$$

where n_e is the number density of electrons. Note that we use the heat capacity of a degenerate gas of electrons, since the Fermi energy $E_F \gtrsim \text{MeV}$ for the densities we consider. The minimum energy deposit necessary to trigger runaway fusion is simply

$$\begin{aligned} \mathcal{E}_{\text{boom}} &\sim \lambda_T^3 (n_{\text{ion}} T_f + n_e^{2/3} T_f^2 + T_f^4) \\ &\approx 10^{15} - 10^{23} \text{ GeV}. \end{aligned} \quad (2)$$

$\mathcal{E}_{\text{boom}}$ varies with λ_T over the range of WD densities and is plotted in Figure 1. Thus for a heating event characterized by its L_0 , \mathcal{E}_0 , and $T_0 \gtrsim T_f$, there is a *boom condition*:

$$\mathcal{E}_0 \gtrsim \mathcal{E}_{\text{boom}} \cdot \max \left\{ 1, \frac{L_0}{\lambda_T} \right\}^3. \quad (3)$$

Any \mathcal{E}_0 satisfying this condition is minimized for L_0 less than the trigger size, where it is also independent of the precise value of L_0 . For broader deposits, the necessary energy is parametrically larger than $\mathcal{E}_{\text{boom}}$ by a volume ratio $(L_0/\lambda_T)^3$. As a result, understanding the L_0 for different kinds of heating events in a WD is critical to determining whether or not they are capable of destroying the star.



FIG. 1: The minimum energy deposit (2) necessary to trigger runaway fusion, based on numerical results for λ_T [5] and the WD mass-density relation [7]

III. NON-GRAVITATIONAL HEATING OF WHITE DWARFS

We address now the possibility of DM heating the WD medium via the production of SM particles. The critical quantity is the length scale over which such SM particles heat the medium—this scale determines their efficiency in triggering runaway fusion as described by condition (3). Note that this is a question of purely SM physics. The unknown physics of DM will serve only to set the initial properties of the SM particles.

One may have expected that efficient heating occurs only for a limited range of SM species and energies, thus restricting the set of DM candidates capable of producing SN. However, we find that SM particles tend to efficiently heat the WD regardless of species or energy—the length scale of heating is typically less than or of order the trigger size λ_T , and is never parametrically larger. This is accomplished primarily through hadronic showers initiated by collisions with carbon ions. In some cases electromagnetic shower processes are important, however at high energies these are suppressed by density effects and photons/electrons are dominated by hadronic interactions. These interactions rapidly stop high-energy particles due to the logarithmic nature of showers, converting them into a cloud of low-energy particles which efficiently heat the WD medium through elastic scatters. In this light, the WD operates analogously to a particle detector, including hadronic and electromagnetic “calorimeter” components. Runaway fusion provides the necessary amplification to convert a detected event into a recordable signal, in this case a violent SN.

In the remainder of this section we present the above energy deposition process in detail—a schematic for this heating is summarized in Figure ?. We review the dominant source of energy loss and the resulting stopping lengths λ for SM particles of incident kinetic energy ϵ ,

approximated by

$$\lambda \sim \frac{\epsilon}{dE/dx}, \quad (4)$$

where dE/dx is the stopping power in the WD medium. These are plotted in Figures 2, 3, 4, and 5. A detailed treatment of the stopping powers is reserved for Appendix A. We will consider incident light hadrons, photons, electrons, and neutrinos—as we are concerned with triggering runaway fusion, we restrict our attention to energies $\epsilon \gg T_f \sim \text{MeV}$.

A. High-Energy Showers

a. Hadronic Showers. Incident hadrons with kinetic energy larger than the nuclear binding scale $E_{\text{nuc}} \sim 10 \text{ MeV}$ will undergo violent inelastic collisions with carbon ions resulting in an $\mathcal{O}(1)$ number of secondary hadrons. This results in a roughly collinear shower of hadrons which ends when the constituents reach an energy $\sim E_{\text{nuc}}$. This occurs over a shower length

$$X_{\text{had}} \sim l_{\text{inel}} \log \left(\frac{\epsilon}{E_{\text{nuc}}} \right) \approx 10^{-6} \text{ cm} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right) \quad (5)$$

where l_{inel} is the mean free path for nuclear scatters, set by $\sigma_{\text{inel}} \approx 100 \text{ mb}$, and we have taken the logarithm to be ~ 10 . The shower terminates into an exponential number of $\sim 10 \text{ MeV}$ hadrons with roughly equal fractions of pions, protons, and neutrons. For a more detailed discussion of hadronic showers, see Appendix A B. Note that neutral pions of energy $10 - 100 \text{ MeV}$ have a decay length to photons of order $\delta_{\pi^0} \sim 10^{-6} \text{ cm}$. Hadronic showers will therefore generate an electromagnetic component carrying an $\mathcal{O}(1)$ fraction of the energy.

b. Electromagnetic Showers. Electrons and photons also shower through successive bremsstrahlung and pair-production interactions. An EM shower proceeds until a critical energy $\sim 100 \text{ MeV}$ set by the scale at which these radiative processes become subdominant to Coulomb and Compton scattering. Note that bremsstrahlung and pair-production are strictly forbidden for incident energies below the electron Fermi energy.

At sufficiently high electron/photon energies and nuclear target densities, EM showers are elongated due to the “Landau-Pomeranchuk-Migdal” (LPM) effect. High-energy radiative processes necessarily involve small momentum transfers to nuclei. These soft virtual photons cannot be exchanged with only a single ion, but rather interact simultaneously with multiple ions. This generates a decoherence, suppressing bremsstrahlung/pair-production above an energy E_{LPM} which scales inversely with density:

$$E_{\text{LPM}} \approx 1 \text{ MeV} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right) \quad (6)$$

The corresponding shower lengths are

$$X_{\text{EM}} \approx X_0 \cdot \begin{cases} \left(\frac{\epsilon}{E_{\text{LPM}}} \right)^{1/2} & \epsilon > E_{\text{LPM}} \\ 1 & \epsilon < E_{\text{LPM}} \end{cases} \quad (7)$$

where

$$X_0 \approx 10^{-7} \text{ cm} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right). \quad (8)$$

See Appendix A C for further details. At the highest WD densities radiative processes are always LPM-suppressed, while at lower densities we observe both regimes. In high-density WDs, EM showers completely carry the energy of $\sim 10^2 - 10^4 \text{ MeV}$ electrons and photons. The lower end of this range corresponds to the scale at which elastic scatters dominate, and the upper end corresponds to the scale at which photonuclear interactions dominate. In low-density WDs, the latter will occur at an energy $\sim 10^6 \text{ GeV}$.

c. Photonuclear and Electronuclear Showers. The LPM effect ensures that hadronic interactions become important at sufficiently high energies even for electrons and photons. Photons can interact hadronically via quark-antiquark pairs and directly induce hadronic showers off ions. The only quantitative difference between these showers and purely hadronic ones is that they require a slightly longer distance to initiate. Roughly, the photonuclear cross section is suppressed relative to the hadronic inelastic cross section by a factor of α required to produce a quark-antiquark pair. This gives a photon range

$$\lambda_{\gamma A} \sim \frac{l_{\text{inel}}}{\alpha} \approx 10^{-5} \text{ cm} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right). \quad (9)$$

Note that $\lambda_{\gamma A}$ is the distance to begin a hadronic shower, whereas the shower itself extends a distance X_{had} .

In the regime of strong LPM suppression, high-energy electrons will also initiate hadronic showers through virtual photons. This is suppressed by an additional factor of α relative to the photonuclear interaction, however a full calculation also yields an $\mathcal{O}(10)$ logarithmic phase-space enhancement. The electronuclear stopping length is

$$\lambda_{eA} \approx 10^{-4} \text{ cm} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right). \quad (10)$$

For electron energies greater than $\sim 10^6 - 10^8 \text{ GeV}$ (depending on the WD density), electronuclear stopping even dominates over the bremsstrahlung of real photons that then interact hadronically. This process is best described as a continuous energy loss of the electron into minimal $\sim 10 \text{ MeV}$ hadronic showers.

d. Neutrino-induced Showers. Neutrinos scatter off ions with a cross section that increases with energy. In these interaction, an $\mathcal{O}(1)$ fraction of the neutrino energy

is transferred to the nucleus with the rest going to produced electrons [8]—this is sufficient to start a hadronic shower. At an energy of $\sim 10^{11}$ GeV, [8] calculates the neutrino-nuclear cross section to be $\sim 10^{-32}$ cm², which we will conservatively take as an estimate for even higher energies. This gives a distance of order $\lambda_{\nu A} \sim$ meter to initiate a shower. While it is much too large to efficiently heat a WD via the release of multiple neutrinos, $\lambda_{\nu A}$ is simply the displacement for a single high-energy neutrino to begin a shower of size X_{had} . As such, $\lambda_{\nu A}$ is qualitatively similar to the photonuclear length (9).

B. Low-Energy Elastic Heating

The showers of high-energy particles described above terminate in a cloud of low-energy $\epsilon \sim 10$ MeV neutrons, protons, and charged pions, and $\epsilon \sim 100$ MeV electrons and photons. Of course, particles at these energies may also be directly produced by the DM. At these energies, Coulomb, Compton, and elastic nuclear scatters dominate and eventually lead to the thermalization of ions. The stopping powers for these processes are calculated in Appendix AD0a, AD0c, and AD0b.

a. Nucleons and Pions. Neutrons and neutral pions are the simplest species we consider, interacting at low-energies only through elastic nuclear scatters characterized by a cross section $\sigma_{\text{el}} \approx 1$ b. Note that the mass hierarchy between these particles and the ions requires 10 – 100 scatters to transfer the hadron’s energy in the form of a random-walk. This elastic heating range is approximately

$$\lambda_{\text{el}} \approx 10^{-7} \text{ cm} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right), \quad (11)$$

and is always less than the trigger size. Note that this may or may not be shorter than the neutral pion decay length δ_{π^0} , depending on the WD density. Low-energy neutrons thus always provide efficient heating, low-energy neutral pions provide efficient heating at high densities, and the effectiveness of neutral pions at low densities depends on the stopping of 70 MeV photons.

We now turn to charged hadrons, which are subject to Coulomb interactions with ions and electrons as well as elastic nuclear scatters similar to their neutral brethren. These ranges are plotted in Figures 5 and 4. It is curious that at energies $\epsilon \lesssim 10$ MeV, scattering off electrons is the least dominant of these processes, contrary to the behavior of terrestrial detectors. This is due to the significant Pauli-blocking of electron interactions near the Fermi energy $E_F \sim 1 - 10$ MeV. Low-energy charged hadrons will thus predominantly transfer energy to ions, which occurs below the trigger size for both nuclear-dominated and EM-dominated stopping.

In the context of a hadronic shower, the final-state hadrons are ~ 10 MeV nucleons and pions, with each species carrying an $\mathcal{O}(1)$ fraction of the initial energy. These products will thermalize within a trigger size and

thus hadronic showers are also an efficient heating mechanism.

b. Electrons and Photons. As shown in Figures 2 and 3, electron stopping at low energy is dominated by bremsstrahlung and photon stopping by Compton scatters. Thus, at these energies electrons and photons first thermalize into an EM gas with size $\lambda_{\text{brem}} < \lambda_T$. This gas cools and diffuses to larger length scales, eventually allowing subdominant processes to thermalize carbon ions. The details of this evolution depend on the initial EM gas temperature, which is set by the total SM energy released by the DM.

If the gas remains above $T \sim 10$ MeV by the time it has diffused to a scale $\lambda_{\gamma-\text{nuc}}$, it will begin a phase of photonuclear showers. These will transfer an $\mathcal{O}(1)$ fraction of energy into neutrons and other hadrons, which as discussed above will efficiently heat ions. This heating will extend over the scale $\lambda_{\gamma-\text{nuc}}$ needed to begin photonuclear showers, which is indeed below the trigger size.

If the temperature falls below 10 MeV before the photonuclear scale, the dominant process will be Coulomb scattering of hot electrons off WD ions. Note that for the most dense WDs, the electron Fermi energy is $E_F \sim 10$ MeV. At temperatures $T \lesssim E_F$, the electrons are partially degenerate and heating proceeds via the thermal tail with kinetic energies $\epsilon \sim E_F + T$. (Note that the stopping length plots are plotted according to kinetic energy ϵ .) This range is plotted in Figures 2 and 3, where it is seen that for EM gas temperatures 1 – 10 MeV the electrons thermalize over a distance of order the trigger size. The fact that this length scale is slightly larger than the trigger size simply means that constraints relying on this heating mechanism will have some volume dilution, increasing the necessary energy deposit according to (3).

In the context of EM showers terminating at $E_{\text{crit}} \sim 100$ MeV, the final-state electrons and photons will establish an EM gas of initial temperature $T \lesssim E_{\text{crit}}$ with length scale λ_{brem} . At these temperatures the heat capacity is dominated by photons, so diffusion out to $\lambda_{\gamma-\text{nuc}}$ dilutes the temperature to $T \lesssim (\lambda_{\text{brem}}/\lambda_{\gamma-\text{nuc}})^{3/4} E_{\text{crit}}$. Inspecting Figures 2 and 3 near E_{crit} , we find at high and low densities respectively $\lambda_{\text{brem}}/\lambda_{\gamma-\text{nuc}} \sim 10^{-1}$ and 10^{-2} . The final temperatures at the photonuclear scale are thus $T \lesssim 20$ MeV and $T \lesssim 3$ MeV respectively. This barely allows photonuclear showers at high density, and so we conservatively take EM shower products to thermalize ions through Coulomb interactions.

c. Ions are a Leaky Bucket. Finally, we revisit a claim of Section II that heating ions to a temperature $\gtrsim T_f$ over a size $\gtrsim \lambda_T$ also necessarily heats the WD electrons and photons in that region. Carbon ions themselves rapidly lose energy to cold electrons via Coulomb interactions. This range is also given in Figures 5 and 4. These heated electrons in turn radiate photons that will continually to thermalize with electrons via Compton scatters. The stopping lengths of bremsstrahlung and Compton scattering are shown in Figures 3 and 2. As



FIG. 2: Stopping lengths of incident photons and electrons as a function of kinetic energy in a WD of density $n_{\text{ion}} \sim 10^{32} \text{ cm}^{-3}$ ($\approx 1.37 M_{\odot}$). Orange lines are incident photons, purple lines are incident electrons. The dashed line is the EM shower length X_{EM} .



FIG. 3: Stopping lengths of incident photons and electrons as a function of kinetic energy in a WD of density $n_{\text{ion}} \sim 10^{30} \text{ cm}^{-3}$ ($\approx 0.85 M_{\odot}$). Orange lines are incident photons, purple lines are incident electrons. The dashed line is the EM shower length X_{EM} .

claimed, these lengths are all well below the trigger size λ_T at energies MeV. Note one subtlety here: electron bremsstrahlung has a hard cutoff at the electron Fermi energy, which is possibly greater than the heated ion temperature $T \gtrsim \text{MeV}$. However, after the ions warm the degenerate electron gas to temperature T , there is thermal population of electrons carrying energy $E_F + T$ that is able to radiate photons up to an energy T , and thus still thermalize photons. We therefore see that any processes which heats ions to an MeV also establishes a hot EM bath within the trigger size.

IV. DARK MATTER-INDUCED IGNITION

The unknown physics of DM sets the rate of SM particle production within the star as well as the initial distribution in space, momentum, and species of the products. This information is needed to determine if a given DM encounter with a WD results in runaway fusion and with



FIG. 4: Stopping lengths for incident hadrons as a function of kinetic energy in a WD of density $n_{\text{ion}} \sim 10^{32} \text{ cm}^{-3}$ ($\approx 1.37 M_{\odot}$). The magenta line is the hadronic shower length X_{had} .



FIG. 5: Stopping lengths for incident hadrons as a function of kinetic energy in a WD of density $n_{\text{ion}} \sim 10^{30} \text{ cm}^{-3}$ ($\approx 0.85 M_{\odot}$). The magenta line is the hadronic shower length X_{had} .

what frequency. Of course, this can be done precisely for a specific DM model. In this Section, we describe several general, illustrative classes of DM-WD encounters which demonstrate the explosiveness of ultra-heavy DM interactions. We also calculate the typical rates at which these events take place in a WD.

A. Classifying DM-WD Encounters

DM can generically heat the WD medium through the three schematic interactions depicted in Figure 6: DM-DM collisions, DM decays, and DM-SM scattering. Note that for ultra-heavy DM these can be complicated events involving many (possibly dark) final states, analogous to the interactions of heavy nuclei. We classify DM candidates into three types according to the interaction that provides the dominant source of heating, and refer to these as collision, decay, and transit candidates. We additionally make simplifying assumptions about the spatial extent of these interactions. For collisions and decays, we consider only “point-like” heating events with all SM

products produced in a localized region (smaller than the trigger size). For transits, we consider only DM-SM scatters that result in a continuous release of particles along the DM trajectory.

We can also classify candidates according to the evolution of the DM itself inside the star. Generally there will be some loss of DM kinetic energy due to DM-SM scatters—this is either incidental to the eventual heating of the star or represents the dominant heating mechanism. We consider two simple, limiting cases depending on the magnitude of this energy loss relative to the DM kinetic energy: “DM wind” and “DM capture”. In the DM wind scenario, there is negligible energy loss and the DM simply passes through the star. In the DM capture scenario, the energy loss due to DM-SM scatters is not capable of igniting runaway but is sufficient to stop the DM and cause it to accumulate inside the star. We consider both scenarios for collision and decay candidates, for which the capture results in a significantly enhanced rate of events. For simplicity we consider only the wind scenario for transit candidates, though an enhanced explosiveness from slowed DM continuously scattering off stellar constituents is certainly possible in some models.

B. Transits

a. Boom Condition. The energy deposited during a continuous heating event such as a DM transit is best described in terms of a linear energy transfer $(dE/dx)_{\text{LET}}$, the kinetic energy of SM particles produced per distance traveled by the DM. If these products have a heating length L_0 then the relevant energy deposit must at minimum be taken as the energy transferred over the transit distance L_0 . Of course, we can always choose to consider energy deposits over a longer segment of the DM trajectory. Importantly, as per the general condition (3) such a deposition is *less* explosive unless L_0 is smaller than the trigger size λ_T . Thus, we consider the energy deposited in a transit over the larger of these two length scales. Assuming the energy of the DM is roughly constant over this heating event, the boom condition for transit heating is:

$$\left(\frac{dE}{dx}\right)_{\text{LET}} \gtrsim \frac{\mathcal{E}_{\text{boom}}}{\lambda_T} \cdot \text{Max} \left\{ \frac{L_0}{\lambda_T}, 1 \right\}^2. \quad (12)$$

The above argument sums the individual energy deposits along the DM trajectory as though they are all deposited simultaneously. This is possible if the DM moves sufficiently quickly so that this energy does not diffuse out of the region of interest before the DM has traversed the region. We therefore require that the diffusion time $\tau_{\text{diff}} \approx 10^{-12}$ s across a heated region at temperature T_f be larger than the DM crossing-time:

$$\tau_{\text{diff}} \sim \frac{L^2}{\alpha(T_f)} \gg \frac{L}{v_{\text{esc}}}, \quad (13)$$

where $\alpha(T)$ is the temperature-dependent diffusivity, and the DM transits at the stellar escape velocity $v_{\text{esc}} \approx 10^{-2}$. This condition is more stringent for smaller regions, so we focus on the smallest region of interest, $L = \lambda_T$. (13) is then equivalent to demanding that the escape speed is greater than the conductive speed of the fusion wave front, $v_{\text{cond}} \sim \alpha(T_f)/\lambda_T$. Numerical calculations of v_{cond} are tabulated in [5], and indeed condition (13) is satisfied for all WD densities.

b. Event Rate: Wind Scenario. The rate of transit events is given by the flux of DM passing through a WD

$$\Gamma_{\text{transit}} \sim \frac{\rho_\chi}{m_\chi} R_{\text{WD}}^2 \left(\frac{v_{\text{esc}}}{v} \right) v_{\text{esc}}, \quad (14)$$

where m_χ is the DM mass, ρ_χ is the local DM density near the WD, and $R_{\text{WD}} \approx 4000$ km is the WD radius. Here $v \sim 10^{-3}$ is galactic virial velocity, and the transit rate contains an $\mathcal{O}(100)$ enhancement due to gravitational focusing.

c. WD Shielding. Runaway fusion only occurs in the degenerate WD interior where thermal expansion is suppressed as a cooling mechanism. The outer layers of the WD, however, are composed of a non-degenerate gas and it is therefore essential that a DM candidate penetrate this layer in order to ignite a SN. We parameterize this by a DM stopping power $(dE/dx)_{\text{SP}}$, the kinetic energy lost by the DM per distance traveled in the non-degenerate layer, and demand that

$$\left(\frac{dE}{dx}\right)_{\text{SP}} \ll \frac{m_\chi v_{\text{esc}}^2}{R_{\text{envelope}}}, \quad (15)$$

where $R_{\text{envelope}} \approx 50$ km is the width of a WD envelope [35]. Note that the DM stopping power in the non-degenerate layer $(dE/dx)_{\text{SP}}$ and the linear energy transfer in the degenerate interior $(dE/dx)_{\text{LET}}$ are possibly controlled by different physics and may have very different numerical values. In addition, a transit heating event satisfying condition (15) will have negligible energy loss over the parametrically smaller trigger size or heating length L_0 , validating the boom condition (12).

C. Collisions and Decays

a. Boom Condition. For a point-like DM-DM collision or DM decay event releasing particles of heating length L_0 , ignition will occur if the total energy in SM products satisfies condition (3). Such an event will likely result in both SM and dark sector products, so we parameterize the resulting energy in SM particles as a fraction f_{SM} of the DM mass. For non-relativistic DM, the DM mass is the dominant source of energy and therefore $f_{\text{SM}} \lesssim 1$ regardless of the interaction details, although we may well suspect that $f_{\text{SM}} \ll 1$ for realistic models. With this parameterization, a single DM-DM collision or DM decay has a boom condition:

$$m_\chi f_{\text{SM}} \gtrsim \mathcal{E}_{\text{boom}} \cdot \max \left\{ \frac{L_0}{\lambda_T}, 1 \right\}^3. \quad (16)$$

We are thus sensitive to DM masses $m_\chi \gtrsim 10^{16}$ GeV.

b. Event Rate: DM Wind. DM with negligible energy loss in the WD medium will traverse the star in $\sim R_{\text{WD}}/v_{\text{esc}} \approx 0.1$ s and have a number density within the WD enhanced relative to the average galactic density by a factor $v_{\text{esc}}/v \sim \mathcal{O}(10)$. In the wind scenario, the DM-DM collision rate inside the WD parameterized by a cross-section $\sigma_{\chi\chi}$ is:

$$\Gamma_{\text{collision}} \sim \left(\frac{\rho_\chi}{m_\chi}\right)^2 \sigma_{\chi\chi} \left(\frac{v_{\text{esc}}}{v}\right)^2 v_{\text{esc}} R_{\text{WD}}^3. \quad (17)$$

Similarly the net DM decay rate inside the WD parameterized by a lifetime τ_χ is:

$$\Gamma_{\text{decay}} \sim \frac{1}{\tau_\chi} \frac{\rho_\chi}{m_\chi} \left(\frac{v_{\text{esc}}}{v}\right) R_{\text{WD}}^3. \quad (18)$$

c. Event Rate: DM Capture. For the DM to be able to stop in a WD, it must have sufficiently strong interactions with stellar constituents:

$$\left(\frac{dE}{dx}\right)_{\text{SP}} \gtrsim \frac{m_\chi v^2}{R_{\text{WD}}}. \quad (19)$$

Note that the DM need only lose velocity of order its galactic virial velocity to get captured. Implicitly, we are making the assumption that the scatters responsible for slowing the DM are not sufficient to blow up the WD, i.e. (??) is never satisfied. This is reasonable: if $m_\chi \gg m_{\text{ion}}$,

First we review the evolution of DM within the star once it has been captured. The DM eventually thermalizes at a velocity

$$v_{\text{th}} \sim \sqrt{\frac{T}{m_\chi}} \approx 10^{-12} \left(\frac{10^{16} \text{ GeV}}{m_\chi}\right)^{1/2}, \quad (20)$$

where $T \sim \text{keV}$ is the WD temperature. It will then accumulate at the thermal radius set by v_{th}

$$R_{\text{th}} \sim \left(\frac{T}{G m_\chi \rho_{\text{WD}}}\right)^{1/2} \quad (21)$$

$$\approx 0.1 \text{ cm} \left(\frac{10^{16} \text{ GeV}}{m_\chi}\right)^{1/2} \left(\frac{10^{31} \text{ cm}^{-3}}{n_{\text{ion}}}\right)^{1/2},$$

where we have assumed a constant WD density $\rho_{\text{WD}} \sim n_{\text{ion}} m_{\text{ion}}$ within R_{th} . DM will collect at this radius until its total mass exceeds the WD mass within R_{th} ,

$$M_{\text{crit}} \sim \rho_{\text{WD}} R_{\text{th}}^3 \quad (22)$$

$$\approx 10^{29} \text{ GeV} \left(\frac{10^{16} \text{ GeV}}{m_\chi}\right)^{3/2} \left(\frac{10^{31} \text{ cm}^{-3}}{n_{\text{ion}}}\right)^{1/2}.$$

The DM cloud will then begin self-gravitational collapse. The exact nature of this collapse is model-dependent, eventually being arrested by DM-DM interactions or the formation of a black hole. For composite DM, it is reasonable to suspect that the collapse stabilizes into a core

of radius R_{sta} larger than the Schwarzschild radius due to some unknown physics.

There are several timescales relevant to the DM capture scenario. Initially there is a timescale for DM to slow down to the thermal velocity v_{th} and “drift” down to the thermal radius. This is roughly given by

$$t_{\text{drift}} \sim \frac{R_{\text{WD}}}{v_{\text{th}}} \approx 50 \text{ yr} \left(\frac{m_\chi}{10^{16} \text{ GeV}}\right)^{1/2}. \quad (23)$$

Evidently for $m_\chi \gtrsim 10^{30}$ GeV, a DM cloud at R_{th} does not even form within the lifetime of the star. Once formed, the DM cloud will continue to collect DM until it reaches a critical core mass (22) in a time

$$t_{\text{collect}} \sim \left(\frac{M_{\text{crit}}}{m_\chi}\right) \frac{1}{\Gamma_{\text{transit}}} \quad (24)$$

$$\approx 10 \text{ s} \left(\frac{10^{16} \text{ GeV}}{m_\chi}\right)^{3/2} \left(\frac{0.4 \text{ GeV/cm}^3}{\rho_\chi}\right),$$

evaluated at $n_{\text{ion}} \sim 10^{31} \text{ cm}^{-3}$. Note that t_{collect} has a non-trivial dependence on WD density: this is manifest in the values for v_{esc} and R_{WD} . Subsequently, the actual timescale for the collapse is independent of DM mass:

$$t_{\text{collapse}} \sim \frac{R_{\text{th}}}{v_{\text{th}}} \approx 3 \text{ s} \left(\frac{10^{31} \text{ cm}^{-3}}{n_{\text{ion}}}\right)^{1/2}. \quad (25)$$

This expression is valid for the two cases that the DM core either initially remains thermalized while self-gravitating or collapses at free-fall velocity.

We now turn towards the rate of explosive collisions and decays for captured DM. Of course, if the slowing of DM itself is sufficient to ignite a SN, then the boom condition is instead given by the transit condition (12). Assuming this is not the case, we calculate the rate of DM events during the various stages of the capture described above.

For decay heating, capture gives an enhancement due to the increased number of DM particles within the WD. This can be very large if the DM core admits decays, however it is still significantly enhanced over the wind scenario even for inert cores (as in the case that the DM forms a black hole). We have an enhancement of the net decay rate (18) by a factor

$$\frac{v_{\text{esc}}}{v_{\text{th}}} \approx 10^{10} \left(\frac{m_\chi}{10^{16} \text{ GeV}}\right)^{1/2} \quad (26)$$

due to the increased time spent by the DM in the WD medium before joining the inert core.

In the case of DM-DM collision heating, it is possible that the collapse of the core will induce an ignition event due to the enhancement of DM number density during the collapse. This would set the lifetime of WDs to t_{collect} . During the collapse, the rate of collisions taking place at a radius r within the enclosed volume is given by

$$\Gamma_{\text{collision}}(r) \sim \left(\frac{M_{\text{core}}}{m_\chi}\right)^2 \frac{1}{r^3} \sigma_{\chi\chi} v(r), \quad (27)$$



FIG. 6: Schematic of possible non-gravitational DM interactions in a WD which release SM (and possibly dark sector) particles.

where $v(r)$ is the velocity of DM - this could be either free-fall velocity or v_{th} if the DM remains thermalized. Integrating to the stable radius R_{sta} , we find the total number of collisions during the collapse is

$$N_{\text{col}} \sim \left(\frac{M_{\text{core}}}{m_{\chi}} \right)^2 \frac{\sigma_{\chi\chi}}{R_{\text{sta}}^2}. \quad (28)$$

Assuming the collapse proceeds until the DM core becomes a black hole, the number of collisions is

$$N_{\text{col}} \sim \frac{\sigma_{\chi\chi}}{G^2 m_{\chi}^2}. \quad (29)$$

If the collapse itself is not explosive, there is still an enhanced collision rate relative to the wind scenario due to DM colliding while in-falling to the core. Again we look at the conservative situation of an inert core - the rate is obviously much greater if the core is stabilized in a fluid state which admits DM-DM collisions. The rate of in-falling collisions is enhanced over the wind collision rate (17) by a factor

$$\frac{R_{\text{WD}}}{R_{\text{sta}}} \times \frac{v_{\text{esc}}}{v_{\text{th}}}, \quad (30)$$

which again depends on the physics of R_{sta} .

d. Event Rate: DM Capture and Multiple Collisions

Up until now, we only considered the case that a single DM-DM collision releases sufficient energy (16) in order to trigger runaway fusion. However, the possibility of a DM core collapse in the star provides up an interesting alternative:

V. DARK MATTER CONSTRAINTS

We now constrain some simplified models of DM which will ignite a WD via one of the processes parameterized in Section IV. First, however, we review how WD observables constrain DM candidates capable of triggering SN.

A. Review of WD Observables

Following the discussion of [4], our constraints come from (1) the existence of heavy, long-lived white dwarfs, or (2) the measured type 1a SN rate. The typical age of a WD is of order the age of the universe $\sim \text{Gyr}$. RX J0648.04418 is a nearby star and one of the heavier known WDs, with a mass $\sim 1.25 M_{\odot}$ [10] and local dark matter density which we will take to be $\rho_{\chi} \sim 0.4 \text{ GeV/cm}^3$. Of course, this is not the only known heavy WD—the Sloan Digital Sky Survey [11] has found 20+ others. The NuStar collaboration has also recently uncovered evidence for the likely existence of $\sim 1.25 M_{\odot}$ WDs in the galactic center [12], where it is estimated that $\rho_{\chi} \sim 10^3 \text{ GeV/cm}^3$ [13]. Such heavy candidates are particularly suited for our constraints as the energy deposit necessary to trigger SN $\mathcal{E}_{\text{boom}}$ is a decreasing function of WD mass. However, less dense white dwarfs are significantly more abundant in the galaxy. Thus, even if a sufficiently massive DM is unable to trigger a violent heating event within the lifetime of a WD, it could still ignite enough lighter WDs to affect the measured SN rate of ~ 0.3 per century. The DM-induced SN rate is estimated using the expected number of white dwarfs per galaxy $\sim 10^{10}$ and their mass distribution [11]. Simulations indicate that only WD masses heavier than $\sim 0.85 M_{\odot}$ will result in optically visible SN [4]. Therefore, most of the stars exploded in this manner will be in the mass range $\sim 0.85 - 1 M_{\odot}$, resulting in weaker SN than expected of typical Chandrasekhar mass WDs.

To summarize, a bound on DM parameters can be placed if either a single explosive event occurs during the lifetime of an observed star such as RX J0648.04418, or the SN rate due to such DM events throughout the galaxy exceeds the measured value. Note that for low-mass WDs dominated by photon diffusion, $\mathcal{E}_{\text{boom}}$ is a strong function of WD density. In [4] the central WD density is used to constrain black hole transits with the justification that the density is nearly constant for much of the star. The average density for WDs is typically a factor $\sim 10^{-2} - 10^{-1}$ less than the central density, although it is found that the WD density only changes by an $\mathcal{O}(1)$ fraction from the central value up to a distance $\sim R_{\text{WD}}/2$ [9]. Therefore the central density is a valid approximation as long as we consider heating events within this “modified” WD volume. For simplicity, we employ this approach.

B. Transit Constraints

In order to constrain a DM model through its transit interaction with a WD, we require that it satisfy the boom condition (12). This is given in terms of an LET, which parameterizes the ability for DM to release sufficient energy to the star in the form of SM particles. $(dE/dx)_{\text{LET}}$ for any realistic DM model would necessarily involve a sum over stellar targets along with species

that could be produced, as well as an integral over the produced particle spectrum. However, we will consider a simplified interaction in which $\sigma_{Ni\epsilon}$ denotes the cross-section for DM to scatter off a stellar constituent (e.g. ions), producing N particles of SM species i and individual energy ϵ . If this were the only available channel for the DM to deposit energy, then the LET could be written as

$$\left(\frac{dE}{dx}\right)_{\text{LET}} = n_{\text{ion}} \sigma_{Ni\epsilon} N \epsilon. \quad (31)$$

The heating length for such a DM-SM scattering interaction is computed in Section III.

Additionally, consider the case that the LET $(dE/dx)_{\text{LET}}$ and DM stopping power $(dE/dx)_{\text{SP}}$ are equal—that is, the DM loses kinetic energy at the same rate as energy is deposited to the WD. While such a statement is certainly not true for all DM models (such as the Q-ball, which liberates binding energy rather than transferring kinetic energy), it provides a useful benchmark to express constraints. It is interesting to note that in this case combining the transit explosion condition (12) with (31) yields a lower bound on DM mass such that the DM is able to both penetrate the envelope *and* trigger an explosion:

$$m_\chi > \mathcal{E}_{\text{boom}} \left(\frac{R_{\text{envelope}}}{\lambda_T} \right) \left(\frac{\rho_{\text{envelope}}}{\rho_{\text{central}}} \right) \frac{1}{v_{\text{esc}}^2}. \quad (32)$$

For the typical parameters of a $1.25 M_\odot$ WD we find that the DM mass must be greater than $\sim 10^{28}$ GeV to ensure a penetrating and explosive transit, taking the density of the WD envelope ρ_{envelope} to be a nominal $\mathcal{O}(10^{-3})$ fraction of the central density ρ_{central} [35]. In other words, if (32) were violated then the DM interaction is either not strong enough to ignite the WD or is so strong that the DM cannot penetrate the envelope without losing appreciable kinetic energy. We reiterate, however, that this bound is only applicable when the energy input to the WD is chiefly coming from the DM kinetic energy, rather than binding energy or other sources.

With the above schematic for a DM transit, we use the rates and heating lengths computed in previous sections to constrain the parameter $\sigma_{Ni\epsilon}$ as a function of DM mass m_χ . This is done in Figure 7 using the different classes of observation available and for representative choices of ϵ and SM species i released.

C. Collision and Decay Constraints

In order to constrain a DM model through its annihilations or decays within a WD, we require that it satisfy the boom condition (16). Consider a simplified interaction where an annihilation or decay releases N particles of SM species i and individual energy ϵ . If we assume a fractional parameter $f_{\text{SM}} = 1$, this corresponds to the entire mass of DM being converted into SM products i ,

each with energy m_χ/N . These will deposit their energy and thermalize ions within a distance described in Section III.

With this schematic for DM-DM collisions, we use the rates and heating lengths computed in previous sections to constrain the cross section $\sigma_{\chi\chi}$ as a function of m_χ using the different classes of observation available and for representative choices of f_{SM} and SM species i released. This is done in Figure 8 for the wind scenario and Figure ?? for the capture scenario. The constraints obtained by considering multiple collisions are shown in Figure ?. In a similar manner, we constrain the lifetime τ_χ as a function of m_χ in Figures 9 and ?.

Complementary Limits It is important to note that there are additional limits on DM interactions of this kind, complementary to the limits placed from WDs. For instance, DM can annihilate or decay into ultra-high energy particles within our galactic halo and therefore contribute to the cosmic ray flux seen in terrestrial air shower detectors. As cosmic rays of energy greater than $\sim 10^{12}$ GeV have not yet been observed [14, 15], this places a concrete limit on DM interaction parameters $\sigma_{\chi\chi}$ and τ_χ which involve the release of such ultra-high energy particles. In theory, a constraint may also be placed on lower-energy SM products from DM annihilations or decays, which would provide an additional source for the measured cosmic ray flux, although such a detailed analysis is beyond the scope of this work.

The cosmic ray constraint on DM is derived by requiring that the expected time for an event to strike earth is less than the lifetime of the detector ~ 10 yr. For a cosmic ray detector with area $\sim (50 \text{ km})^2$ [14], we find that the cosmic ray bounds are weaker than the WD bounds except in the DM decay “wind scenario”, where the cosmic rays bounds are comparable within a couple orders of magnitude to those due to the observation of a local WD. This coincidence is simply a consequence of the similar “space-time volumes” of the two systems. A cosmic ray detector sees events within a space-time volume $\sim (R_{\text{det}}^2 R_{\text{halo}} \times 10 \text{ yr})$ which is comparable to the WD space-time volume for decay events $\sim (R_{\text{WD}}^3 \times 10^9 \text{ yr} \times 10)$, including the factor of 10 gravitational enhancement.

In addition, there are various cosmological bounds on DM interactions. By requiring that the galactic halo has not diminished by more than an $\mathcal{O}(1)$ factor during its lifetime, we constrain $\sigma_{\chi\chi}/m_\chi \lesssim \text{barn/GeV}$, regardless of the precise details of the collision. This is similar in magnitude to the DM self-interaction bounds from colliding galaxy clusters [17]. The cosmological bound on DM lifetime $\tau_\chi \gtrsim 100 \text{ Gyr}$ is also independent of the nature of the decay products (see [16] for details). Since the limits imposed by the WD scale as $\sigma_{\chi\chi} \propto m_\chi^2$ and $\tau_\chi \propto m_\chi^{-1}$, there will necessarily be a sufficiently large DM mass for which the above cosmological considerations are the more stringent constraints on its interactions. This occurs for DM masses $m_\chi \sim 10^{25} - 10^{30} \text{ GeV}$.

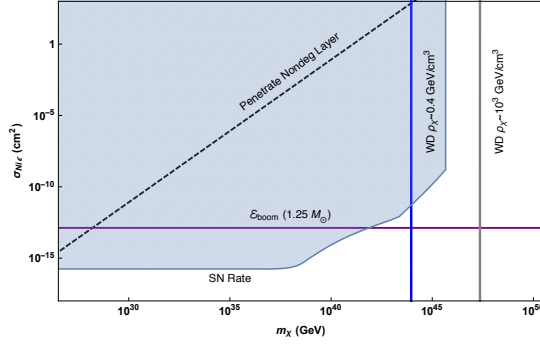


FIG. 7: Constraints on a DM-nuclei scattering cross-section to produce a single TeV photon. Bounds come from demanding that heating events satisfy (12) and occur at a rate (14) rapid enough to either ignite a single observed $1.25 M_\odot$ WD in its lifetime (local and galactic center) or exceed the measured SN rate in our galaxy.

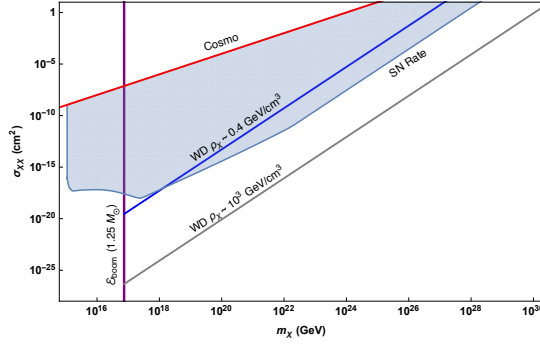


FIG. 8: Constraints on DM-DM collision cross-section into photons with individual energy $\epsilon > 10$ MeV and $f_{\text{SM}} = 1$. Bounds come from demanding that heating events satisfy (16) and occur at a rate (17) (“wind scenario”) rapid enough to either ignite a single observed $1.25 M_\odot$ WD in its lifetime (local and galactic center) or exceed the measured SN rate in our galaxy.

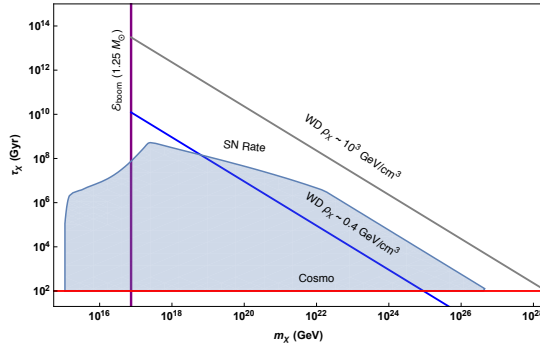


FIG. 9: Constraints on DM decay lifetime into photons with individual energy $\epsilon > 10$ MeV and $f_{\text{SM}} = 1$. Bounds come from demanding that heating events satisfy (16) and occur at a rate (18) (“wind scenario”) rapid enough to either ignite a single observed $1.25 M_\odot$ WD in its lifetime (local and galactic center) or exceed the measured SN rate in our galaxy.

VI. Q-BALLS

Having derived constraints on generic models of ultra-heavy DM, we turn towards a concrete example. In various supersymmetric extensions of the SM, non-topological solitons called Q-balls can be produced in the early universe [19, 20]. If these Q-balls were stable, they would comprise a component of the DM today. For gauge-mediated models with flat scalar potentials, the Q-ball mass and radius are given by

$$M_Q \sim m_S Q^{3/4}, \quad R_Q \sim m_S^{-1} Q^{1/4}, \quad (33)$$

where m_S is related to the scale of supersymmetry breaking, and Q is the global charge of the Q-ball—in our case, baryon number. The condition $M_Q/Q < m_p$ ensures that the Q-ball is stable against decay to nucleons. When an electrically neutral baryonic Q-ball interacts with a nucleon, it absorbs its baryonic charge and induces the dissociation of the nucleon into free quarks. During this proton decay-like process, $\sim \text{GeV}$ of energy is released through the emission of 2–3 pions [21]. We assume that for each Q-ball collision, there is equal probability to produce π^0 and π^\pm under the constraint of charge conservation. Note that a sufficiently massive Q-ball will become a black hole if $R_Q \lesssim GM_Q$. In the model described above, this translates into a condition $(M_{\text{pl}}/m_S)^4 \lesssim Q$.

We now determine the explosiveness of a Q-ball transit. As in Section V, this process is described by the parameter

$$\left(\frac{dE}{dx}\right)_{\text{LET}} \sim n_{\text{ion}} \sigma_Q N \epsilon, \quad (34)$$

where the nuclear collision results in $N \sim 30$ pions released, each with kinetic energy $\epsilon \sim 500$ MeV. These pions induce hadronic showers which terminate in low-energy hadrons that rapidly transfer their energy to ions via elastic scatters, as discussed in Section III. Thus the Q-ball transit has a heating length within the trigger size, and the Q-ball cross-section necessary to trigger runaway fusion is given by equations (12) and (34):

$$\sigma_Q \gtrsim \frac{1}{n_{\text{ion}}} \frac{\mathcal{E}_{\text{boom}}}{\lambda_T} \left(\frac{1}{10 \text{ GeV}} \right). \quad (35)$$

We see $\sigma_Q \approx 10^{-12} \text{ cm}^2$ is sufficient to blow up a $\sim 1.25 M_\odot$ WD. The cross-section for this interaction is approximately geometric

$$\sigma_Q \sim \pi R_Q^2, \quad (36)$$

and so $Q \gtrsim 10^{42} (m_S/\text{TeV})^4$ can be adequately constrained from the observation of a single, heavy WD. Note that the Q-ball interaction described above results in minimal slowing or transfer of kinetic energy for Q-balls this massive, so transits will easily penetrate the non-degenerate WD layer (15).

The strongest previous constraints on Q-balls come from Super-Kamiokande as well as air fluorescence detectors of cosmic rays [22]. However, the constraints due to

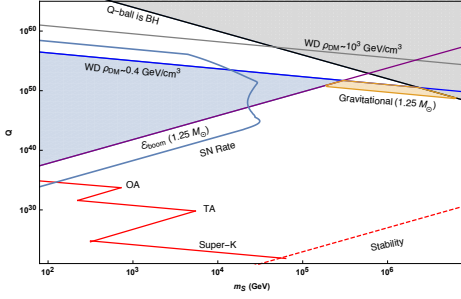


FIG. 10: Constraints on baryonic Q-balls from transits of a $\sim 1.25 M_\odot$ WD in the galactic center, $\rho_\chi \sim 10^3 \text{ g/cm}^3$. We include constraints from heating by strong interactions with nuclei and gravitational interactions (dynamical friction). Also shown are the limits from Super-K and the OA, TA cosmic ray detectors, extracted from [22].

white dwarfs are in a fundamentally complementary region of parameter space. These are plotted in Figure 10. As a comparison, the combined limits from Super-K and the OA, TA cosmic ray detectors are shown in red. We have also included the constraints from a WD which result from considering gravitational heating during a Q-ball transit, as in [4].

VII. DISCUSSION

The detection of ultra-heavy DM is an open problem which will likely require a confluence of astrophysical probes. Here we present a comprehensive guide to containing these DM candidates through annihilates, decays, and transits within a WD that release sufficient SM energy to trigger a type 1a supernova. In particular, we calculate the energy loss of high-energy particles due to SM interactions within the WD medium and determine the conditions for which a general energy deposition will heat a WD and ignite thermonuclear runaway. The formalism provided will enable WDs to be applied as detectors for any DM models capable of heating the star through non-gravitational interactions, and as a concrete example we are able to place bounds on supersymmetric Q-ball DM over a wide region of parameter space.

In general, the phenomenology of such a DM-induced event will be the ignition of sub-Chandrasekhar mass progenitors. This raises the tantalizing possibility that DM encounters with a WD can act as an alternative explosion mechanism and progenitor system for type 1a SN. For decades, standard lore has been that type 1a SN are caused by accretion onto carbon-oxygen white dwarfs in binary systems that reach the critical $\sim 1.4 M_\odot$ Chandrasekhar mass limit. Nevertheless, it is well-known that such a mechanism cannot account for all observed type 1a SN. Recent observations [23, 24] suggest that an $\mathcal{O}(1)$ fraction of the observed type 1a SN appear to have sub-Chandrasekhar progenitors. A leading explanation for this phenomenon is the detonation of a surface layer of

helium which drives a shock into the interior of a sub-Chandrasekhar-mass WD [25, 26]. However, in light of the lack of understanding of DM and its interactions, it is worthwhile to consider whether a DM-WD encounter may also give rise to type 1a SN progenitor.

A. PARTICLE STOPPING IN A WHITE DWARF

Here we provide a more detailed analysis of the stopping power (energy loss per distance traveled) of high-energy SM particles in a carbon-oxygen WD due to electromagnetic and strong interactions. We consider incident electrons, photons, pions, and nucleons with kinetic energy greater than an MeV.

A. WD Medium

For the WD masses that we consider, the stellar medium consists of electrons and fully-ionized carbon nuclei with densities in the range $n_e = Zn_{\text{ion}} \sim 10^{31} - 10^{33} \text{ cm}^{-3}$ where $Z = 6$. The internal temperature is $T \sim \text{keV}$ [35]. The electrons are degenerate and predominantly relativistic free gas, with Fermi energy

$$E_F \sim (3\pi^2 n_e)^{1/3} \sim 1 - 10 \text{ MeV}. \quad (37)$$

The carbon ions, however, are non-degenerate and do not form a free gas. The plasma frequency due to ion-ion Coulomb interactions is given by

$$\Omega_p = \left(\frac{4\pi n_{\text{ion}} Z^2 \alpha}{m_{\text{ion}}} \right)^{1/2} \sim 1 - 10 \text{ keV}, \quad (38)$$

where m_{ion} is the ion mass. Finally, the medium also contains thermal photons, though these are never significant for stopping particles as the photon number density $n_\gamma \sim T^3$ is much smaller than that of electrons or ions.

B. Nuclear Interactions

a. Elastic Scattering of Hadrons. Hadrons with energy less than 10 MeV will predominantly stop due to elastic nuclear scatters with ions. These are hard scatters, resulting in a stopping power

$$\frac{dE}{dx} \sim \frac{m}{m_{\text{ion}}} E n_{\text{ion}} \sigma_{\text{el}} \quad (39)$$

for a hadron of mass $m \ll m_{\text{ion}}$ and kinetic energy E . σ_{el} is the elastic nuclear scattering cross-section, which is of order $\sim b$ at these energies and drops to $\sim 0.1b$ above 10 MeV [33], ignoring the presence of various nuclear resonances in the range 1 MeV to 10 MeV.

b. Inelastic Scattering of Hadrons. For energies above 10 MeV, the stopping of hadrons is dominated by inelastic nuclear scatters. In such a collision, an incoming hadron interacts with one or more nucleons to produce a $\mathcal{O}(1)$ number of additional hadrons which approximately split the initial energy. For incident energy greater than \sim GeV, the majority of secondary hadrons are pions with transverse momenta \sim 100 MeV [33]. Below \sim GeV, it is found that roughly equal fractions of protons, neutrons, and pions are produced in each collision [34]. We will thus have a roughly collinear shower terminating at an energy \sim 10 MeV which consists of pions for most of the shower's development and converts to a mix of pions and nucleons in the final decade of energy. This cascade is described by a radiative stopping power

$$\frac{dE}{dx} \sim n_{\text{ion}} \sigma_{\text{inel}} E, \quad (40)$$

where the inelastic nuclear cross-section σ_{inel} is given by $\sigma_{\text{inel}} \approx 100$ mb and is roughly constant in energy [33]. The total length of the shower is only logarithmically dependent on the initial hadron energy E ,

$$X_{\text{had}} \sim \frac{1}{n_{\text{ion}} \sigma_{\text{inel}}} \log \left(\frac{E}{10 \text{ MeV}} \right). \quad (41)$$

c. Photonuclear Interactions. Photons of energy $k \gtrsim 10$ MeV can also strongly interact with nuclei through the production of virtual quark-antiquark pairs. This is the dominant mode of photon energy loss at high energy. The photonuclear scatter destroys the photon and fragments the nucleus, producing secondary hadrons in a shower analogous to that described above. The photonuclear cross-section $\sigma_{\gamma A}$ is roughly given by $\sigma_{\gamma A} \approx \alpha \sigma_{\text{inel}}$, again ignoring the nuclear resonances that occur for $E \lesssim$ GeV. [33] For $E \gtrsim$ GeV, $\sigma_{\gamma A}$ is likely a slowly increasing function of energy due to the coherent interaction of the photon over multiple nucleons [31], however, instead of extrapolating this behavior we conservatively take a constant photonuclear cross-section $\sigma_{\gamma A} \approx 1$ mb.

d. Electronuclear Interactions. Electrons can similarly lose energy to nuclei by radiating a virtual photon that undergoes a photonuclear scatter, which indeed provides the dominant energy loss for high energy electrons. The cross-section for this process is roughly given by the photonuclear cross-section and scaled by a factor representing the probability to radiate such a photon. This can be estimated with the Weizsacker-Williams approximation, which gives a stopping power that is scaled from the photonuclear result by α as well as an $\mathcal{O}(10)$ kinematic factor [31]:

$$\frac{dE}{dx} \sim 10^{-3} \cdot \sigma_{\text{inel}} n_{\text{ion}} E. \quad (42)$$

Unlike the photonuclear interaction, the electronuclear event is a radiative process that preserves the original electron while leaving hadronic showers in its wake.

C. Radiative Processes

EM showers due to successive bremsstrahlung and pair production events off carbon ions are the dominant stopping mechanisms for intermediate-energy electrons and photons. Both of these processes result in radiative stopping powers, expressed semi-classically as [36]

$$\frac{dE}{dx} \sim n_{\text{ion}} \frac{Z^2 \alpha^3}{m_e^2} E \log \Lambda \quad (43)$$

where $\log \Lambda$ is a Coulomb form factor given by the range of effective impact parameters b :

$$\Lambda = \frac{b_{\text{max}}}{b_{\text{min}}} \sim \lambda_{\text{TF}} m_e. \quad (44)$$

The maximal impact parameter is set by the plasma screening length and the minimum by the electron mass, below which the semi-classical description breaks down. For the highest WD densities, we may indeed have that $\Lambda \lesssim 1$, in which case equation (43) ought to be replaced by a quantum mechanical result, such as the Bethe-Heitler formula [37]. This still results in a stopping power $\sim n_{\text{ion}} Z^2 \alpha^3 E / m_e^2$, and so for simplicity we employ equation (43) and take $\log \Lambda \sim \mathcal{O}(1)$ for all WD densities.

a. LPM Suppression A radiative event involving momentum transfer q to an ion must, quantum mechanically, occur over a length $\sim q^{-1}$. All ions within this region contribute to the scattering of the incident particle, and for sufficiently small q this results in a decoherence that suppresses the formation of photons or electron-position pairs. This is the ‘‘Landau-Pomeranchuk-Midgal’’ (LPM) effect. The momentum transfer q in a given event decreases with increasing incident particle energy, and so the LPM effect will suppress radiative processes for energies greater than some scale E_{LPM} . This can be calculated semi-classically [36],

$$E_{\text{LPM}} = \frac{m_e^4}{4\pi Z^2 \alpha^2 n_{\text{ion}}} \approx 1 \text{ MeV} \left(\frac{10^{32} \text{ cm}^{-3}}{n_{\text{ion}}} \right). \quad (45)$$

which is quite small due to the high ion density in the WD. The stopping power for $E > E_{\text{LPM}}$ is

$$\left(\frac{dE}{dx} \right) \sim n_{\text{ion}} \frac{Z^2 \alpha^3}{m_e^2} E \cdot \left(\frac{E_{\text{LPM}}}{E} \right)^{1/2} \quad (46)$$

In addition to the LPM effect, soft bremsstrahlung may be suppressed in a medium as the emitted photon acquires an effective mass of order the plasma frequency Ω_p . However, for high-energy electrons this dielectric suppression only introduces a minor correction to (46), in which soft radiation is already suppressed [36].

D. Elastic EM Scattering

a. Coulomb Scattering off Carbon Ions. Coulomb collisions with ions provide the dominant mechanism by

which electrons with energy $1 \text{ MeV} \lesssim E \lesssim 10 \text{ MeV}$ thermalize ions. In this scenario we may treat the ions as stationary and ignore their recoil during collisions. The ionic charge will be screened by the mobile electrons of the medium, so incident particles will scatter via a potential

$$V(\mathbf{r}) = \frac{Z\alpha}{r} e^{-\lambda_{\text{TF}} r}. \quad (47)$$

The screening length λ_{TF} is given in the Thomas-Fermi approximation by [29]

$$\lambda_{\text{TF}}^2 = \frac{E_F}{6\pi\alpha n_e} \sim \frac{1}{\alpha n_e^{2/3}} \quad (48)$$

where E_F is the electron Fermi energy. This plasma screening suppresses scatters with momentum transfers below $\sim \lambda_{\text{TF}}^{-1}$, corresponding to a minimal energy transfer of $\omega_{\text{min}} = \lambda_{\text{TF}}^{-2}/2m_{\text{ion}}$. Ions may in principle also cause screening through lattice distortion, however this may be ignored as the sound speed of the lattice $c_s \sim 10^{-2}$ is much smaller than the speed of an incident relativistic electron. Using the Born approximation, we have a cross-section for energy transfer ω

$$\frac{d\sigma}{d\omega} = \frac{2\pi Z^2 \alpha^2}{m_{\text{ion}} \beta^2} \frac{1}{(\omega + \omega_{\text{min}})^2} \quad (49)$$

and a stopping power

$$\begin{aligned} \frac{dE}{dx} &= \int_0^{\omega_{\text{max}}} d\omega n_{\text{ion}} \frac{d\sigma}{d\omega} \omega \\ &\approx \frac{2\pi n_{\text{ion}} Z^2 \alpha^2}{m_{\text{ion}} \beta^2} \log \left(\frac{\omega_{\text{max}}}{\omega_{\text{min}}} \right) \end{aligned} \quad (50)$$

where the second line is valid if $\omega_{\text{max}} \gg \omega_{\text{min}}$. ω_{max} is the maximum possible energy transfer. This may be due to 4-momentum conservation, or in the case of incident electrons, the impossibility of scattering to a final energy less than E_F . 4-momentum conservation sets an upper bound ω_{kin} , which for a stationary target is

$$\omega_{\text{kin}} = \frac{2m_{\text{ion}} p^2}{m_{\text{ion}}^2 + m^2 + 2Em_{\text{ion}}} \quad (51)$$

with p , E the incoming momentum and energy. The Fermi upper bound is simply $\omega_F = E - E_F$ and for incident electrons $\omega_{\text{max}} = \min(\omega_{\text{kin}}, \omega_F)$.

For scatters that transfer energy less than the plasma frequency Ω_p , one may be concerned about phonon excitations. We estimate this by treating each ion as an independent oscillator with frequency Ω_p (an Einstein solid) and compute the stopping power due to scatters which excite a single oscillator quanta. There are two key differences between this and the free ion case: incident particles must transfer an energy Ω_p and the cross-section to transfer momentum q is suppressed by a factor $q^2/2m_{\text{ion}}\Omega_p = \omega_{\text{free}}/\Omega_p$. ω_{free} is the energy transfer that would accompany a free ion scatter with momentum

transfer q . The resulting stopping power is unchanged from the free case (50), as the increased energy transfer compensates for the suppressed cross-section:

$$d\sigma \cdot \omega \sim d\sigma_{\text{free}} \frac{\omega_{\text{free}}}{\Omega_p} \cdot \Omega_p \sim d\sigma_{\text{free}} \cdot \omega_{\text{free}}. \quad (52)$$

Finally, we note that for more massive or more energetic incident particles the cross-section (49) must be replaced with a more complicated expression to account for the recoil of the ion during collisions. In these scenarios, however, this stopping power is far subdominant to hadronic or electromagnetic showers.

b. Coulomb Scattering off Electrons. The scattering of incident electrons with the degenerate electron sea determines the termination energy of electromagnetic showers, and so we focus here on that scenario. This calculation demands two considerations not present when scattering off ions: the targets are not stationary and they require a threshold energy transfer in order to be scattered out of the Fermi sea. In the scenario of interest, however, these do not result in a parametrically different stopping power than was found for ions in equation (50).

For incident momenta much greater than the Fermi momentum, the relative velocity is of order the incident velocity and the deflection of the incident particle will generally be small. It is reasonable then that scattering proceeds, up to $\mathcal{O}(1)$ factors, as though a heavy incident particle is striking a light, stationary target. The cross-section is then given by the usual result,

$$\frac{d\sigma}{d\omega} \approx \frac{2\pi\alpha^2}{E_F} \frac{1}{\omega^2}, \quad (53)$$

where we have accounted for the target's motion by replacing its mass with the relativistic inertia $\sqrt{m_e^2 + p^2} \approx E_F$. We have ignored plasma screening, as Pauli-blocking will provide a more stringent cutoff on soft scatters in this case. Scatters which transfer an energy $\omega \leq E_F$ will have a suppressed contribution to the stopping power as they can only access a fraction of the Fermi sea. For incident energies $E \gg E_F$ it is sufficient to ignore these suppressed scatters, i.e.,

$$\begin{aligned} \frac{dE}{dx} &= \int_{E_F}^{\omega_{\text{max}}} d\omega n_e \frac{d\sigma}{d\omega} \omega \\ &\approx \frac{2\pi n_e \alpha^2}{E_F} \log \left(\frac{\omega_{\text{max}}}{E_F} \right) \end{aligned} \quad (54)$$

where, as above, $\omega_{\text{max}} = \min(\omega_{\text{kin}}, \omega_F)$. This derivation is admittedly quite heuristic, and so it has been checked with a detailed numerical calculation accounting fully for the target's motion and degeneracy. Equation (54) is indeed a good approximation to the stopping power for incident energies larger than the Fermi energy.

c. Compton Scattering off Electrons. For incident photon energies less than the Fermi energy, the dominant stopping is provided by Compton scatters with degenerate electrons. This stopping power differs parametrically

from its usual form due to the target electron motion and degeneracy.

Consider an incident photon with energy $k \leq E_F$. Compton scatters will only occur off electrons moving roughly collinear with the photon momentum - a head-on collision would result in an energy loss for the electron, which is forbidden by Pauli exclusion. In the electron rest frame these collinear scatters are Thompson-like, and so the photon energy loss is dominated by backward scatters. For relativistic electrons near the Fermi surface, these scatters transfer an energy

$$\omega \approx k \left(1 - \frac{m_e^2}{4E_F^2} \right) \sim k. \quad (55)$$

The cross-section can be taken in the electron rest frame, $\sigma_T \sim \alpha^2/m_e^2$, along with an ‘aiming’ factor $1/4\pi$ to account for the restriction to initially parallel trajectories. Finally, for $k \ll E_F$ only those electrons near the top of the Fermi sea are available to scatter, so the photon

interacts with an effective electron density of

$$n_{\text{eff}} = \int_{E_F - k}^{E_F} g(E) dE \approx 3n_e \frac{k}{E_F} \quad (56)$$

where $g(E)$ is the Fermi density of states. This gives a stopping power

$$\frac{dk}{dx} \approx \frac{\alpha^2 n_e k^2}{m_e^2 E_F}. \quad (57)$$

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