Electron stopping power via phonon production in the ion lattice

Paul Riggins

July 3, 2017

1 Goal and Setup

Our ultimate goal is to compute the stopping power

$$\frac{dE}{dx} = \int dE' \frac{d\sigma}{dE'} E' \tag{1}$$

for an electron scattering off an ion coulomb lattice. Here E' is the energy transfer. For energy transfers below the lattice binding energy, we expect phonon production to become relevant, and the cross-section must take this into account. To capture this behavior we will use the structure factor formalism developed, for instance, in [1] Chapter 19.

The double differential cross-section for a particle to scatter into solid angle Ω while transferring energy ω and momentum $\mathbf{q}$ is given by ([1] 19.13)

$$\frac{d\sigma}{d\Omega d\omega} = A_{\mathbf{q}} S(\omega, \mathbf{q}) \tag{2}$$

where ([1] 19.14)

$$A_{\mathbf{q}} = \frac{k'}{k} \left(\frac{m}{2\pi}\right)^2 |V_{\mathbf{q}}|^2 \tag{3}$$

Here m is the incident particle mass, and k and k' are the magnitudes of the incident particle's initial and final momentum, respectively. These momenta are related to the momentum transfer by $\mathbf{k}' = \mathbf{k} - \mathbf{q}$ ([1] 19.3). The fourier transform of the interaction potential is ([1] 19.6)

$$V_{\mathbf{q}} = \int d^3x \, e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \tag{4}$$

For momentum and energy transfers much smaller than the Fermi momentum and energy, we should be able to use a simple coulomb potential for $V(\mathbf{x})$. For small transfers, however, we must take into account screening by the Fermi sea, for instance via a dielectric constant (see [2] eq. 9 and [3] 5.67). We will return to this later after we have developed the formalism further.

The structure factor $S(\omega, q)$ is the space-time fourier transform of the density correlation function ([1] 19.15-17, 19.40)

$$S(\omega, \mathbf{q}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} F(\mathbf{q}, t) \tag{5}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \sum_{jl} \left\langle e^{-i\mathbf{q}\cdot\mathbf{x}_j(0)} e^{i\mathbf{q}\cdot\mathbf{x}_l(t)} \right\rangle_T \tag{6}$$

$$= \frac{1}{2\pi} \int d^3x\, d^3x'\, e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x'})} \int dt\, e^{-i\omega t} \langle \rho(\mathbf{x}', 0) \rho(\mathbf{x}, t) \rangle_T \tag{7}$$

where the intermediate scattering function $F(\mathbf{q}, t)$ is given by

$$F(\mathbf{q}, t) = \sum_{jl} \left\langle e^{-i\mathbf{q}\cdot\mathbf{x}_j(0)} e^{i\mathbf{q}\cdot\mathbf{x}_l(t)} \right\rangle_T \tag{8}$$

and the density operator is

$$\rho(\mathbf{x}, t) = \sum_j \delta[\mathbf{x} - \mathbf{x}_j(t)] \tag{9}$$

The expectation values are taken over the thermal phonon distribution at temperature T. In the one-phonon approximation, the elastic and inelastic parts of $F(\mathbf{q}, t)$ are given by ([1] 19.40, 19.51)

$$F_{\text{el}}(\mathbf{q}, t) = (2\pi)^3 n_i e^{-2W} \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q} - \mathbf{G}) \tag{10}$$

$$F_{\text{in}}(\mathbf{q}, t) = (2\pi)^3 n_i e^{-2W} \sum_{\nu} \frac{q^2}{2NM\omega_\nu} \left(\langle n_\nu + 1 \rangle e^{i\omega_\nu t} \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q} - \mathbf{k}_\nu - \mathbf{G}) \right. \\ \left. + \langle n_\nu \rangle e^{-i\omega_\nu t} \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q} + \mathbf{k}_\nu - \mathbf{G}) \right) \tag{11}$$

where N is the number of ions in the lattice, M is the ion mass, n_i is the ion number density, ν indexes the phonon modes of energy ω_ν and quasimomentum $\mathbf{k}_\nu$, $\mathbf{G}$ is a reciprocal lattice factor, and $\langle n_\nu \rangle$ is the expected number of phonons of index ν at temperature T. This last quantity is given by the usual boson statistics:

$$\langle n_\nu \rangle = \frac{e^{-\omega_\nu/T}}{1 - e^{-\omega_\nu/T}} = \frac{1}{e^{\omega_\nu/T} - 1} \tag{12}$$

The quantity $W = \frac{1}{6} q^2 \langle u_j^2 \rangle$ is the "Debye-Waller factor", reflecting decoherence that arises from zero-point and finite temperature effects.

1.1 Debye approximation

For simplicity of calculation, we will work in the Debye approximation. In the Debye approximation the Debye-Waller factor is given by ([1] 19.56)

$$W = \frac{1}{6} q^2 \langle u_j^2 \rangle = \frac{3q^2}{8M\Theta} \left\{ 1 + \frac{2\pi^2}{3} \left(\frac{T}{\Theta} \right)^2 + \cdots \right\} \tag{13}$$

Here Θ there is the Debye temperature. The Debye approximation describes a phonon distribution of N phonon modes with energies between $\sim \omega_D/N$ and ω_D, where the maximum photon energy is given by ([4] 5.21-24)

$$\omega_D = c_s (6\pi^2 n_i)^{1/3} \tag{14}$$

and c_s is the sound speed. The Debye temperature is given by $\Theta = \omega_D$ ([4] 5.28). The dispersion relation for phonons in this approximation is given by

$$\omega_\nu = c_s k_\nu \tag{15}$$

and the density of states is given by

$$D(\omega_\nu) = \frac{V \omega_\nu^2}{2\pi^2 c_s^3} \tag{16}$$

where V is the lattice volume.

2 Putting it all together

Let us now put all this together to calculate the double differential cross-section for inelastic scattering leading to phonon production in the medium. We neglect elastic scattering, since it does not contribute to the stopping power, and we will ignore the possibility of phonon absorption on the assumption that it will be negligible. (We can check this later.)

The phonon production term in $F(\mathbf{q}, t)$ is $\propto \langle n_\nu + 1 \rangle$. Selecting this term, we find the structure factor to be

$$S(\omega, \mathbf{q}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} (2\pi)^3 n_i e^{-2W} \sum_\nu \frac{q^2}{2NM\omega_\nu} \langle n_\nu + 1 \rangle e^{i\omega_\nu t} \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q} - \mathbf{k}_\nu - \mathbf{G}) \tag{17}$$

Evaluating the time integral yields a delta function in energy

$$S(\omega, \mathbf{q}) = \frac{1}{2\pi} (2\pi)^3 n_i e^{-2W} \sum_\nu \frac{q^2}{2NM\omega_\nu} \langle n_\nu + 1 \rangle \delta(\omega - \omega_\nu) \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q} - \mathbf{k}_\nu - \mathbf{G}) \tag{18}$$

We know now that we have two delta functions that are not independent of one another. The momentum transfer $\mathbf{q}$ and energy transfer ω are related to each other by the kinematics of the incident particle, and the phonon energy ω_ν and crystal momentum $\mathbf{k}_\nu$ related to each other by the dispersion relation. Specifically, we have the relations

$$\omega = \sqrt{m^2 + k^2} - \sqrt{m^2 + |\mathbf{k} - \mathbf{q}|^2} \tag{19}$$

$$\omega_\nu = c_s k_\nu \tag{20}$$

where $\mathbf{k}$ is the initial momentum of the incident particle of mass m. Let's begin by considering a highly relativistic particle with small momentum transfer, so that

$$\begin{aligned} \omega &\approx k - |\mathbf{k} - \mathbf{q}| && (21)\\ &= k - \sqrt{k^2 + q^2 - 2kq\cos\theta_q} && (22)\\ &\approx k - k\left(1 + \frac{1}{2}\frac{q^2}{k^2} - \frac{q}{k}\cos\theta_q\right) && (23)\\ &\approx q\cos\theta_q && (24) \end{aligned}$$

where θ_q is the angle of the momentum transfer vector relative to the incident momentum $\mathbf{k}$. To simplify, let us also suppose that we are only interested in energy transfers within first Brillioun zone, corresponding to momentum transfers $q \lesssim k_D$ and energy transfers $\omega \lesssim \omega_D$. Then we may take only the $\mathbf{G} = 0$ term in the reciprocal lattice sum. We may then rewrite the structure factor in terms of momenta as

$$\begin{aligned} S(\omega, \mathbf{q}) = {} & \frac{1}{2\pi}(2\pi)^3 n_i e^{-2W} \\ & \sum_\nu \frac{q^2}{2NM(c_s k_\nu)} \langle n_\nu + 1\rangle \delta(q\cos\theta_q - c_s k_\nu)\delta^{(3)}(\mathbf{q} - \mathbf{k}_\nu) \end{aligned} \tag{25}$$

We can approximate the sum over phonons as an integral over the first Brillioun zone, including a factor of 3 for the various polarizations:

$$\sum_\nu \to 3\int^{k_D} d^3k_\nu \frac{V}{(2\pi)^3} \tag{26}$$

We use this to collapse the three-dimensional delta function (remember we are assuming q lies within the first Brillioun zone):

$$\begin{aligned} S(\omega, \mathbf{q}) = {} & \frac{1}{2\pi}(2\pi)^3 n_i e^{-2W} \\ & \left(\frac{3V}{(2\pi)^3}\right) \frac{q^2}{2NM(c_s q)} \left[\frac{1}{e^{c_s q/T} - 1} + 1\right] \delta(q\cos\theta_q - qc_s) \end{aligned} \tag{27}$$

We note that $V/N = n_i^{-1}$. Simplifying this expression, we find

$$S(\omega, \mathbf{q}) = \frac{3}{4\pi M c_s}\left[\frac{1}{e^{c_s q/T} - 1} + 1\right]\delta(\cos\theta_q - c_s) \tag{28}$$

Note that we have extracted the momenta factor q from the remaining delta function. We may do this because we are only considering processes with non-zero momentum transfer. We have also neglected the Debye-Waller factor, since it is small and the exponential is $\mathcal{O}(1)$.

3 Calculating the stopping power

In order to find the singly-differential cross-section $d\sigma/d\omega$ and the stopping power, we are about to integrate over $\cos\theta_q$, ϕ_q, and ω. It will be convenient therefore change variables so our structure factor is in terms of these. Recall that $\omega \approx q\cos\theta_q$ (24). Then the structure factor becomes

$$S(\omega, \cos\theta_q, \phi_q) = \frac{3}{4\pi M c_s}\left[\frac{1}{e^{c_s\omega/T\cos\theta_q} - 1} + 1\right]\delta(\cos\theta_q - c_s) \tag{29}$$

The singly-differential cross-section is given by (see (2) and following)

$$\frac{d\sigma}{d\omega} = \int d\Omega\, \frac{k'}{k}\left(\frac{m}{2\pi}\right)^2 |V_{\mathbf{q}}|^2 S(\omega, \cos\theta_q, \phi_q) \tag{30}$$

We must now calculate the interaction potential and the differential $d\Omega$.

3.1 Interaction potential

We will use Heaviside-Lorentz units for electromagnetism, so that $\epsilon_0 = 1$. Then the coulomb force is given by $V(\mathbf{r}) = e^2/4\pi r = \alpha/r$. Accounting for screening, the fourier transform of this interaction potential is given by [3]

$$V_{\mathbf{q}} = \frac{4\pi Z\alpha}{q^2\epsilon^l(q,0)} \tag{31}$$

where the electron longitudinal static dielectric function relative to the vacuum $\epsilon^l(q,0)$ is given by

$$\epsilon^l(q,0) = 1 + \frac{4e^2 E_F\sqrt{E_F^2 + m^2}}{\pi q^2} \tag{32}$$

so we can write

$$V_{\mathbf{q}} = \frac{4\pi Z\alpha}{q^2 + \lambda^2} = \frac{4\pi Z\alpha}{(\omega/\cos\theta_q)^2 + \lambda^2} \tag{33}$$

with the screening length

$$\lambda = \left(\frac{\pi}{4\alpha E_F \sqrt{E_F^2 + m^2}} \right)^{-1/2} \approx \left(\frac{\pi}{4\alpha E_F m} \right)^{-1/2} \tag{34}$$

We cannot ignore the dielectric function because our momentum transfers are coincidentally smaller than the Fermi energy by virtue of being smaller than the lattice binding energy.

3.2 Solid angle $d\Omega$

We wish to rewrite the solid angle differential $d\Omega = d(\cos\theta_{k'})d\phi_{k'}$ in terms of $d(\cos\theta_q)$ and $d\phi_q$. To begin, we recall that $\mathbf{k}' = \mathbf{k} - \mathbf{q}$. The momentum transfer vector $\mathbf{q}$ entirely determines the transverse component of $\mathbf{k}'$, so we may relate

$$q \sin\theta_q = k' \sin\theta_{k'} \approx k \sin\theta_{k'} \tag{35}$$

where in the last step we used the fact that $q \ll k$. Manipulating and using the trig identity $\sin\theta = \sqrt{1 - \cos^2\theta}$, we find

$$\begin{aligned}
\cos\theta_{k'} &\approx \sqrt{1 - \left(\frac{q}{k}\right)^2 \sin^2\theta_q} && (36)\\
&\approx 1 - \frac{1}{2}\left(\frac{q}{k}\right)^2 \sin^2\theta_q && (37)\\
&\approx 1 - \frac{1}{2}\left(\frac{q}{k}\right)^2 (1 - \cos^2\theta_q) && (38)\\
&\approx 1 - \frac{\omega^2}{2k^2}\left(\frac{1}{\cos^2\theta_q} - 1\right) && (39)\\
& && (40)
\end{aligned}$$

Thus we have the differential

$$d(\cos\theta_{k'}) \approx \frac{\omega^2}{k^2} \frac{d(\cos\theta_q)}{\cos^3\theta_q} \tag{41}$$

The azimuthal angles are simply the same, so the solid angle differential becomes

$$\begin{aligned}
d\Omega &= d(\cos\theta_{k'})\, d\phi_{k'} && (42)\\
&\approx d(\cos\theta_q)\, d\phi_q \, \frac{\omega^2}{k^2} \frac{1}{\cos^3\theta_q} && (43)
\end{aligned}$$

3.3 Singly-differential cross-section

We are now ready to calculate the singly-differential cross-section (30). Approximating $k' \approx k$ and inserting the results (29), (33), and (43), we can easily evaluate the integrals (with the help of the delta function) to find

$$\frac{d\sigma}{d\omega} = \int d\Omega \left(\frac{m}{2\pi}\right)^2 |V_{\mathbf{q}}|^2 S(\omega, \cos\theta_q, \phi_q) \tag{44}$$

$$\approx \int \left[d(\cos\theta_q)\, d\phi_q \frac{\omega^2}{k^2} \frac{1}{\cos^3\theta_q}\right] \left(\frac{m}{2\pi}\right)^2 \left[\frac{4\pi Z\alpha}{(\omega/\cos\theta_q)^2 + \lambda^2}\right]^2 \tag{45}$$

$$\times \left\{\frac{3}{4\pi M c_s}\left[\frac{1}{e^{c_s\omega/T\cos\theta_q} - 1} + 1\right]\delta(\cos\theta_q - c_s)\right\}$$

$$= 2\pi \left[\frac{\omega^2}{k^2}\frac{1}{c_s^3}\right]\left(\frac{m}{2\pi}\right)^2 \left[\frac{4\pi Z\alpha}{(\omega/c_s)^2 + \lambda^2}\right]^2 \frac{3}{4\pi M c_s}\left[\frac{1}{e^{\omega/T} - 1} + 1\right] \tag{46}$$

We recall that the speed of sound $c_s \sim 10^{-2}$.

3.4 Stopping power

All that remains is to integrate this overall possible energy transfers within the first Brillioun zone to find the stopping power. The result is

$$\frac{dE}{dx} = \int d\omega \frac{d\sigma}{d\omega} n_i \omega \tag{47}$$

$$\approx \frac{3\alpha^2 m^2 n_i Z^2}{k^2 M}\left(\frac{\omega_D^2}{c_s^2\lambda^2 + \omega_D^2} + \ln\left[\frac{c_s^2\lambda^2}{c_s^2\lambda^2 + \omega_D^2}\right]\right) \tag{48}$$

$$\sim \frac{3\alpha^2 m^2 n_i Z^2}{k^2 M}(10) \tag{49}$$

4 Beyond the one-phonon approximation

In this section we will attempt to go beyond the one-phonon approximation to allow for larger transfers of energy and momentum from the electron to the lattice. If we do not employ the one-phonon approximation, the inelastic intermediate scattering function (11) becomes

$$F_{\text{in}}(\mathbf{q}, t) = \sum_{ij} e^{i\mathbf{q}\cdot(\mathbf{R}_i - \mathbf{R}_j)} \sum_{n=1}^{\infty} \frac{1}{n!}\left\{\frac{1}{2MN}\sum_{\nu}\frac{q^2}{\omega_\nu}\langle n_\nu + 1\rangle e^{-i\mathbf{k}_\nu\cdot(\mathbf{R}_i - \mathbf{R}_j) + i\omega_\nu t}\right\}^n \tag{50}$$

Note we have dropped the Debye-Waller factor e^{-2W} and the phonon-absorption terms because of the low temperature of the star relative to the lattice binding energy. The nth term in this expansion corresponds to the creation of n phonons.

The reciprocal lattice vectors in (11) result from the sum

$$\sum_{ij} e^{i(\mathbf{q} - \mathbf{k}_\nu)\cdot(\mathbf{R}_i - \mathbf{R}_j)} = (2\pi)^3 n_i \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q} - \mathbf{k}_\nu - \mathbf{G}) \tag{51}$$

The sum over ij here will similarly serve to relate the momentum transfer $\mathbf{q}$ to the total momentum of the created phonons in each of the higher phonon terms. For simplicity (and because it is a good approximation), we will take the star to be at zero temperature so that $\langle n_\nu + 1\rangle = 1$ for all phonon modes. Let's see how this plays out in the 1- and 2-phonon terms $F_{\text{in}}^{(1)}$ and $F_{\text{in}}^{(2)}$:

$$\begin{aligned}
F_{\text{in}}^{(1)} &= \sum_{ij} e^{i\mathbf{q}\cdot(\mathbf{R}_i-\mathbf{R}_j)} \frac{1}{1!}\frac{1}{(2MN)} \sum_{\nu_1} \frac{q^2}{\omega_{\nu_1}} e^{-i\mathbf{k}_{\nu_1}\cdot(\mathbf{R}_i-\mathbf{R}_j)+i\omega_{\nu_1}t} \\
&= \frac{1}{1!(2MN)} \sum_{\nu_1} \frac{q^2}{\omega_{\nu_1}} (2\pi)^3 n_i \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q}-\mathbf{k}_{\nu_1}-\mathbf{G}) e^{i\omega_{\nu_1}t} \\
&= \frac{3}{1!(2Mn_i)} \int \frac{d^3k_{\nu_1}}{(2\pi)^3} \frac{q^2}{\omega_{\nu_1}} (2\pi)^3 n_i \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q}-\mathbf{k}_{\nu_1}-\mathbf{G}) e^{i\omega_{\nu_1}t}
\end{aligned} \tag{52}$$

$$\begin{aligned}
F_{\text{in}}^{(2)} &= \sum_{ij} e^{i\mathbf{q}\cdot(\mathbf{R}_i-\mathbf{R}_j)} \frac{1}{2!}\frac{1}{(2MN)^2} \sum_{\nu_1}\sum_{\nu_2} \frac{q^4}{\omega_{\nu_1}\omega_{\nu_2}} \\
&\qquad \times e^{-i(\mathbf{k}_{\nu_1}+\mathbf{k}_{\nu_2})\cdot(\mathbf{R}_i-\mathbf{R}_j)+i(\omega_{\nu_1}+\omega_{\nu_2})t} \\
&= \frac{1}{2!(2MN)^2} \sum_{\nu_1}\sum_{\nu_2} \frac{q^4}{\omega_{\nu_1}\omega_{\nu_2}} (2\pi)^3 n_i \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q}-\mathbf{k}_{\nu_1}-\mathbf{k}_{\nu_2}-\mathbf{G}) e^{i(\omega_{\nu_1}+\omega_{\nu_2})t} \\
&= \frac{3^2}{2!(2Mn_i)^2} \int \frac{d^3k_{\nu_1}}{(2\pi)^3} \int \frac{d^3k_{\nu_2}}{(2\pi)^3} \frac{q^4}{\omega_{\nu_1}\omega_{\nu_2}} \\
&\qquad \times (2\pi)^3 n_i \sum_{\mathbf{G}} \delta^{(3)}(\mathbf{q}-\mathbf{k}_{\nu_1}-\mathbf{k}_{\nu_2}-\mathbf{G}) e^{i(\omega_{\nu_1}+\omega_{\nu_2})t}
\end{aligned} \tag{53}$$

At this point we can infer the pattern and write down the result for the n-phonon process:

$$F_{\text{in}}^{(n)} = \frac{3^n(2\pi)^3 n_i}{n!(2Mn_i)^n} \sum_{\mathbf{G}} q^{2n} \prod_{j=1}^{n} \left(\int \frac{d^3k_{\nu_i}}{(2\pi)^3} \frac{e^{i\omega_{\nu_i}t}}{\omega_{\nu_i}} \right) \delta^{(3)}\left(\mathbf{q}-\mathbf{G}-\sum_{j=1}^{n}\mathbf{k}_{\nu_j}\right) \tag{54}$$

We will begin by considering only phonons created within the first Brillioun zone, so that we may take $\mathbf{G} = 0$. Once we have explored this we may try a non-zero $\mathbf{G}$ to see what the effects might be. With this simplification, the intermediate scattering function becomes

$$\begin{aligned}
F_{\text{in}}^{(n)} &= \frac{3^n(2\pi)^3 n_i}{n!(2Mn_i)^n} q^{2n} \prod_{j=1}^{n} \left(\int \frac{d^3k_{\nu_j}}{(2\pi)^3} \frac{e^{i\omega_{\nu_j}t}}{\omega_{\nu_j}} \right) \delta^{(3)}\left(\mathbf{q}-\sum_{j=1}^{n}\mathbf{k}_{\nu_j}\right) \\
&= \frac{3^n n_i}{n!(2Mn_i)^n} q^{2n} \prod_{j=2}^{n} \left(\int \frac{d^3k_{\nu_j}}{(2\pi)^3} \frac{e^{i\omega_{\nu_j}t}}{\omega_{\nu_j}} \right) \frac{e^{i\omega_{\nu_1}t}}{\omega_{\nu_1}} \bigg|_{\omega_{\nu_1}=c_s|\mathbf{q}-\sum_{j=2}^{n}\mathbf{k}_{\nu_j}|}
\end{aligned} \tag{55}$$

The assumption that $\mathbf{G} = 0$ is probably a bad and overcomplicating one. Various papers (like this one) argue that umklapp processes dominate in a coulomb lattice, such that it is only the distant Brillioun zones that contribute. This seems to be because the typical (maximum?) momentum transfer $\sim k_F$ is large compared to the size of the Brillioun zone. (This is probably the same for us? Review the arguments that lead to their conclusion.) In these distant zones, one can approximate $\mathbf{q} \sim \mathbf{G}$, and then we simply need $\cos\theta_{kq} \ll 1$ so that $\omega \sim G\cos\theta_{kq}$ is small enough to match a phonon energy. (Recall θ_{kq} is the angle between the momentum transfer and the incident momentum.)

So perhaps I can integrate over $\mathbf{G}$, simply using the delta function to set $\mathbf{q} \approx \mathbf{G}$. Then the integration over time will match the energy to a phonon energy, and the integral over phonon momentum will ... restrict the allowed angles of $\mathbf{q}$?

I expect the end result of this to be a larger cross-section and stopping power than the one I first calculated. The energy transfers aren't any larger, but there is a much larger phase space that the incident particle could be scattered into.

These same references say that multi-phonon excitations are only relevant near the melting point. I should review the argument behind this and see whether it still holds for high-energy incident particles.

References

[1] Kittel, *Quantum Theory of Solids*

[2] Baiko, *Ion Structure Factors and Electron Transport in Dense Coulomb Plasmas*

[3] Jancovici, *On the Relativistic Degenerate Electron Gas*

[4] Kittel, *Introduction to Solid State Physics*