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DIGITAL SIGNAL PROCESSING

Covers Signals and Systems

Includes MATLAB Programs

S Salivahanan
A Vallavaraj
C Gnanapriya

McGraw Hill Education

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Chapter 1

Classification of Signals and Systems

1.1 INTRODUCTION

Signals play a major role in our life. In general, a signal can be a function of time, distance, position, temperature, pressure, etc., and it represents some variable of interest associated with a system. For example, in an electrical system the associated signals are electric current and voltage. In a mechanical system, the associated signals may be force, speed, torque, etc. In addition to these, some examples of signals that we encounter in our daily life are speech, music, picture and video signals. A signal can be represented in a number of ways. Most of the signals that we come across are generated naturally. However, there are some signals that are generated synthetically. In general, a signal carries information, and the objective of signal processing is to extract this information.

Signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the nature of information it carries. Thus signal processing is concerned with representing signals in mathematical terms and extracting the information by carrying out algorithmic operations on the signal. Mathematically, a signal can be represented in terms of basic functions in the domain of the original independent variable or it can be represented in terms of basic functions in a transformed domain. Similarly, the information contained in the signal can also be extracted either in the original domain or in the transformed domain.

A system may be defined as an integrated unit composed of diverse, interacting structures to perform a desired task. The task may vary such as filtering of noise in a communication receiver, detection of range of a target in a radar system, or monitoring steam pressure in a boiler. The function of a system is to process a given input sequence to generate an output sequence.

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It is said that digital signal processing techniques origin in the seventeenth century when finite difference methods, numerical integration methods, and numerical interpolation methods were developed to solve physical problems involving continuous variables and functions. There has been a tremendous growth since then and today digital signal processing techniques are applied in almost every field. The main reasons for such wide applications are due to the numerous advantages of digital signal processing techniques. Some of these advantages are discussed subsequently.

Digital circuits do not depend on precise values of digital signals for their operation. Digital circuits are less sensitive to changes in component values. They are also less sensitive to variations in temperature, ageing and other external parameters.

In a digital processor, the signals and system coefficients are represented as binary words. This enables one to choose any accuracy by increasing or decreasing the number of bits in the binary word.

Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time-sharing. This reduces the processing cost per signal.

Digital implementation of a system allows easy adjustment of the processor characteristics during processing. Adjustments in the processor characteristics can be easily done by periodically changing the coefficients of the algorithm representing the processor characteristics. Such adjustments are often needed in adaptive filters.

Digital processing of signals also has a major advantage which is not possible with the analog techniques. With digital filters, linear phase characteristics can be achieved. Also multirate processing is possible only in the digital domain. Digital circuits can be connected in cascade without any loading problems, whereas this cannot be easily done with analog circuits.

Storage of digital data is very easy. Signals can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

For processing very low frequency signals like seismic signals, analog circuits require inductors and capacitors of a very large size whereas, digital processing is more suited for such applications.

Though the advantages are many, there are some drawbacks associated with processing a signal in the digital domain. Digital processing needs 'pre' and 'post' processing devices like analog-to-digital and digital-to-analog converters and associated reconstruction filters. This increases the complexity of the digital system. Also, digital techniques suffer from frequency limitations. For reconstructing a signal from its sample, the sampling frequency must be atleast twice the highest frequency component present in that signal. The available frequency range of operation of a digital signal processor is primarily

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determined by the sample-and-hold circuit and the analog-to-digital converter, and as a result is limited by the technology available at that time. The highest sampling frequency is presently around 1GHz reported by K.Poulton, etal., in 1987. However, such high sampling frequencies are not used since the resolution of the A/D converter decreases with an increase in the speed of the converter. But the advantages of digital processing techniques outweigh the disadvantages in many applications. Also, the cost of DSP hardware is decreasing continuously. Consequently, the applications of digital signal processing are increasing rapidly.

1.2 CLASSIFICATION OF SIGNALS

Signals can be classified based on their nature and characteristics in the time domain. They are broadly classified as (i) *continuous-time signals* and (ii) *discrete-time signals*. A continuous-time signal is a mathematically continuous function and the function is defined continuously in the time domain. On the other hand, a discrete-time signal is specified only at certain time instants. The amplitude of the discrete-time signal between two time instants is just not defined. Figure 1.1 shows typical continuous-time and discrete-time signals.

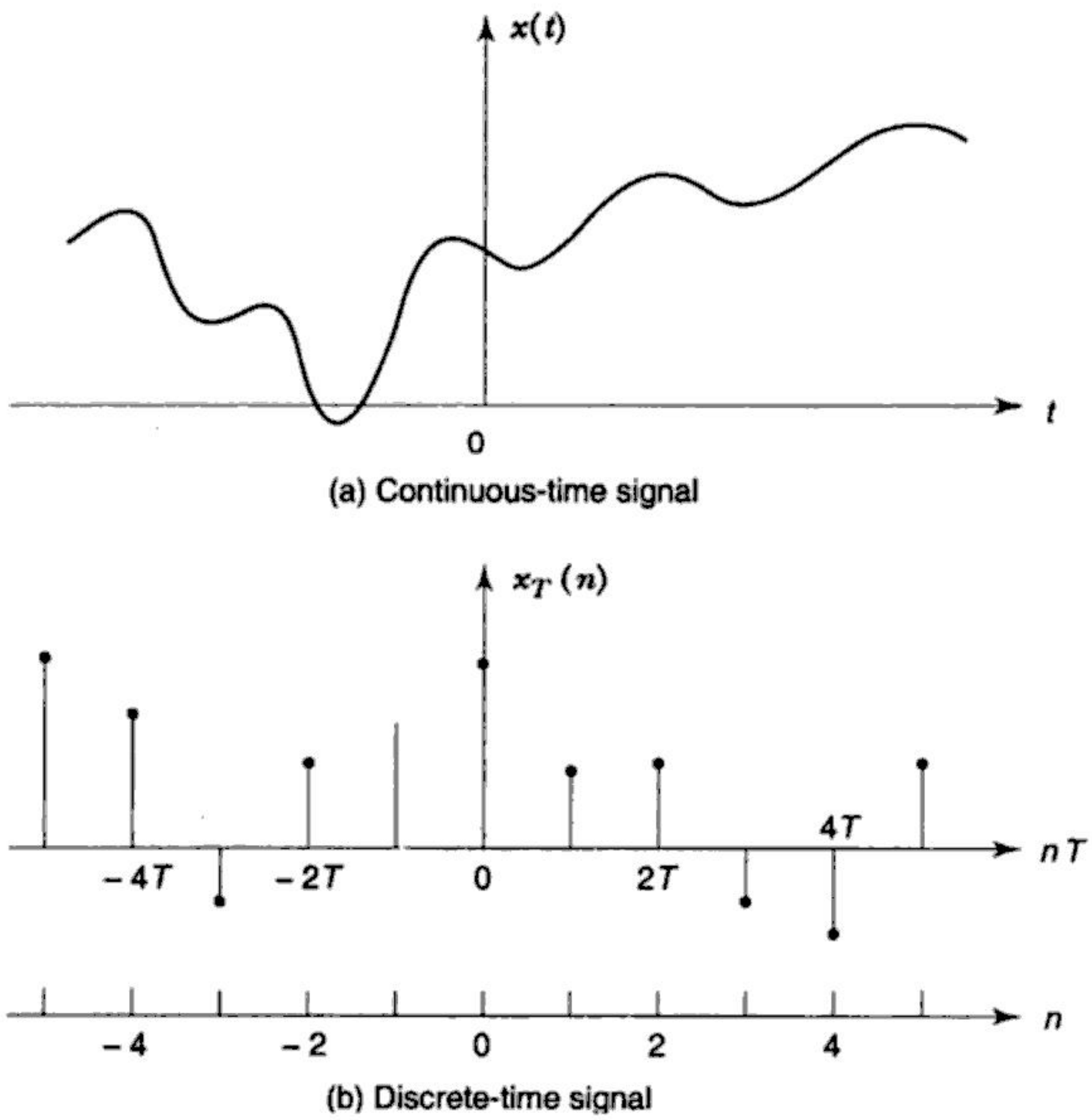

Fig. 1.1 *Continuous-Time and Discrete-Time Signals*

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Both continuous-time and discrete-time signals are further classified as

(i) Deterministic and non-deterministic signals
(ii) Periodic and aperiodic signals
(iii) Even and odd signals, and
(iv) Energy and power signals.

1.2.1 Deterministic and Non-deterministic Signals

Deterministic signals are functions that are completely specified in time. The nature and amplitude of such a signal at any time can be predicted. The pattern of the signal is regular and can be characterised mathematically. Examples of deterministic signals are

(i) $x(t) = \alpha\, t$ This is a ramp whose amplitude increases linearly with time and slope is α.

(ii) $x(t) = A \sin \omega t$. The amplitude of this signal varies sinusoidally with time and its maximum amplitude is A.

(iii) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$ This is a discrete-time signal whose amplitude is 1 for the sampling instants $n \geq 0$ and for all other samples, the amplitude is zero.

For all the signals given above, the amplitude at any time instant can be predicted in advance. Contrary to this, a non-deterministic signal is one whose occurrence is random in nature and its pattern is quite irregular. A typical example of a non-deterministic signal is **thermal noise** in an electrical circuit. The behaviour of such a signal is probabilistic in nature and can be analysed only stochastically. Another example which can be easily understood is the number of accidents in an year. One cannot exactly predict what would be the figure in a particular year and this varies randomly. Non-deterministic signals are also called **random signals**.

1.2.2 Periodic and Aperiodic Signals

A continuous-time signal is said to be periodic if it exhibits periodicity, i.e.

$$x(t + T) = x(t), \quad -\infty < t < \infty \tag{1.1}$$

where T is the period of the signal. The smallest value of T that satisfies Eq. 1.1 is called the fundamental period, T_o, of the signal. A periodic signal has a definite pattern that repeats over and over, with a repetition period of T_o. For a discrete-time signal, the condition for periodicity can be written as,

$$x(n + N_o) = x(n), \quad -\infty < n < \infty \tag{1.2}$$

where N_o is the sampling period measured in units of number of sample spacings. Periodic signals can be in general, expressed as

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(i) Continuous-Time Periodic Signals

$$x_p(t) = \sum_{i=-\infty}^{\infty} X(t - iT_o) \tag{1.3}$$

where

$$X(t) = \begin{Bmatrix} x(t), & t_1 \le t < t_1 + T_o \\ 0, & \text{elsewhere} \end{Bmatrix} \tag{1.4}$$

(ii) Discrete-Time Periodic Signals

$$x_p(n) = \left[\sum_{i=-\infty}^{\infty} X(n - iN_o), T \right] \tag{1.5}$$

where

$$X(n) = \begin{Bmatrix} x(n), & n_1 \le n < (n_1 + N_o) \\ 0, & \text{elsewhere} \end{Bmatrix} \tag{1.6}$$

and T is the sampling period in seconds.

A signal which does not satisfy either Eq. 1.1 or 1.2 is called an **aperiodic signal**. Some examples of periodic signals are shown in Fig. 1.2. Some periodic signals can be simply modelled using a single equation. For example,

$$x(t) = A \sin\left(\frac{2\pi t}{T_o} - \varphi\right) \tag{1.7}$$

is a continuous-time sinusoidal signal which is valid for all t. The constant A represents the maximum amplitude and φ represents the phase shift of the sinusoidal signal. Similarly, a periodic discrete-time sinusoidal signal is represented as

$$x(n) = \left[A \sin\left(\frac{2\pi n}{N_0} - \beta\right), T \right] \tag{1.8}$$

The term β represents the delay and T represents the sampling period.

The sum of two or more periodic continuous-time signals need not be periodic. They will be periodic if and only if the ratio of their fundamental periods is rational. In order to determine whether the sum of two or more periodic signals is periodic or not, the following steps may be used.

(i) Determine the fundamental period of the individual signals in the sum signal.
(ii) Find the ratio of the fundamental period of the first signal with the fundamental periods of every other signal.
(iii) If all these ratios are rational, then the sum signal is also periodic.

In the case of discrete-time signals, the sum of a number of periodic signals is always periodic because the ratio of individual periods is always the ratio of integers, which is rational.

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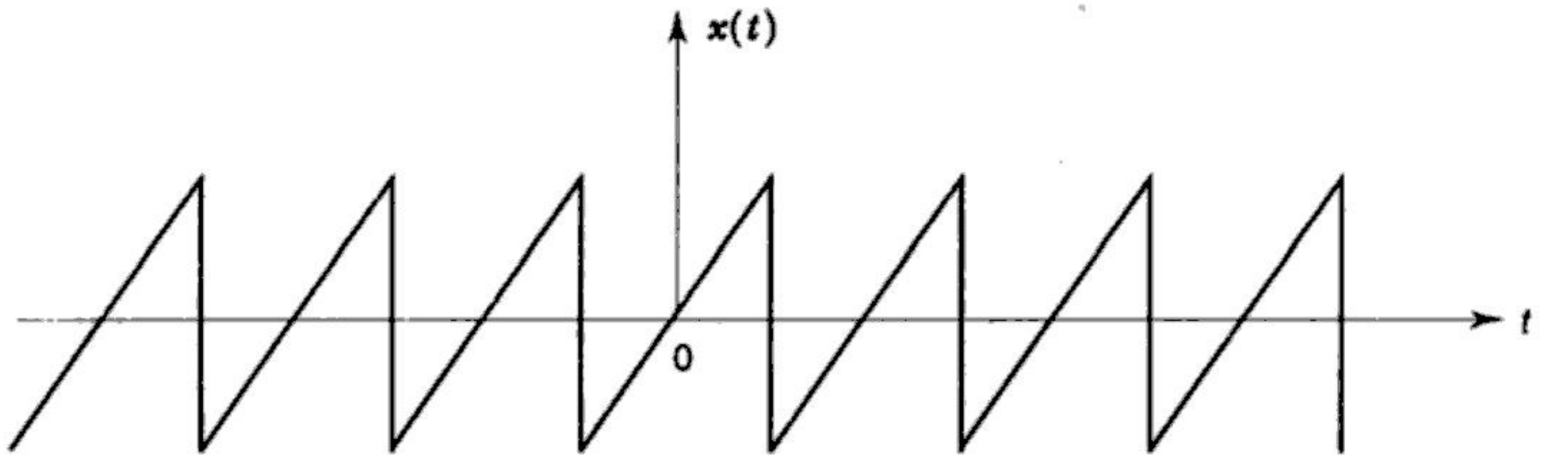
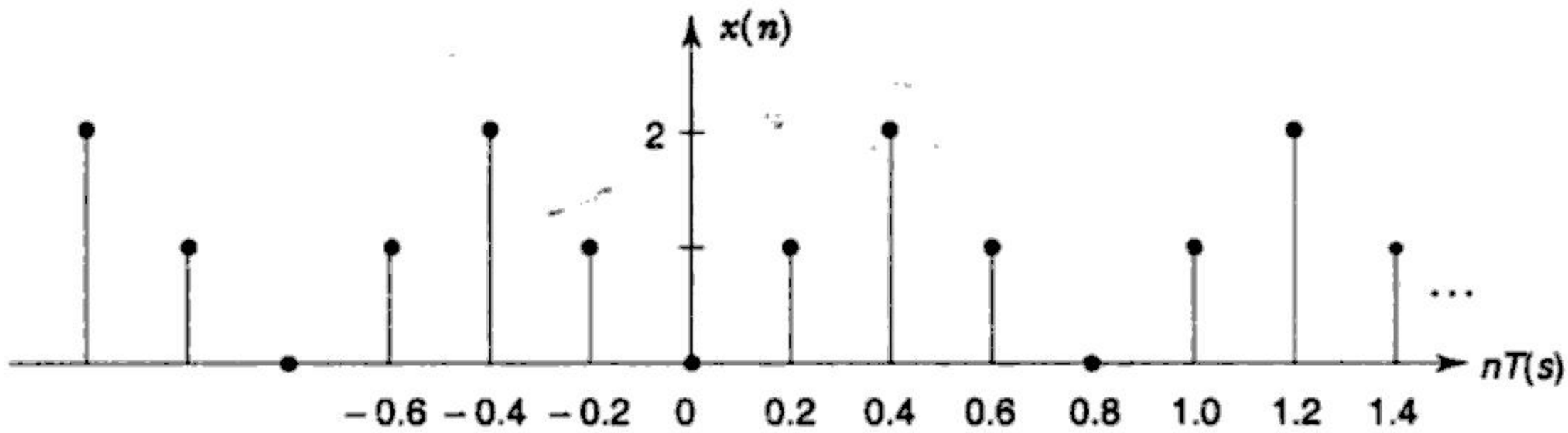


(a) Continuous-time



$$x_p = \left[\sum_{i=-\infty}^{\infty} X[n - iN_0], T \right]$$

Where

$$x[n] = \begin{cases} -n, & -2 \le n < 0 \\ n, & 0 \le n < 2 \\ 0, & \text{otherwise} \end{cases}$$

$N_0 = 4$ and $T = 0.25$

(b) *Discrete-time*

Fig. 1.2 *Some Examples of Periodic Signals*

Example 1.1 Determine which of the following signals are periodic.

(a) $x_1(t) = \sin 15\,\pi t$ (b) $x_2(t) = \sin 20\,\pi t$ (c) $x_3(t) = \sin \sqrt{2}\,\pi t$
(d) $x_4(t) = \sin 5\pi t$ (e) $x_5(t) = x_1(t) + x_2(t)$
(f) $x_6(t) = x_2(t) + x_4(t)$

Solution

(a) $x_1(t) = \sin 15\,\pi t$ is periodic.

The fundamental period is $T_o = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{15\pi}$

$= 0.1333333333...$ seconds

(b) $x_2(t) = \sin 20\,\pi t$ is periodic.

The fundamental period is $T_o = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{20\pi} = 0.1$ seconds

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(c) $x_3(t) = \sin \sqrt{2}\,\pi t$ is periodic.

The fundamental period is $T_o = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{2}\,\pi}$

$= 1.41421356...$ seconds

(d) $x_4(t) = \sin 5\pi t$ is periodic.

The fundamental period is $T_o = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5\pi} = 0.4$ seconds

(e) $x_5(t) = x_1(t) + x_2(t)$

The fundamental period of $x_1(t) = T_{o1} = 0.13333333...$ seconds and the fundamental period of $x_2(t) = T_{o2} = 0.1$ seconds. The ratio of fundamental frequencies, $\dfrac{T_{o1}}{T_{o2}} = \dfrac{0.13333333333...}{0.1}$, cannot be expressed as a ratio of integers. Hence, $x_5(t)$ is not periodic.

(f) $x_6(t) = x_2(t) + x_4(t)$

The fundamental period of $x_2(t) = T_{o2} = 0.1$ seconds and the fundamental period of $x_4(t) = T_{o4} = 0.4$ seconds. The ratio of fundamental frequencies, $\dfrac{T_{o2}}{T_{o4}} = \dfrac{0.1}{0.4} = \dfrac{1}{4}$, can be expressed as a ratio of integers. Hence, $x_5(t)$ is periodic.

1.2.3 Even and Odd Signals

If a signal exhibits symmetry in the time domain, it is called an *even signal*. The signal must be identical to its reflection about the origin. Mathematically, an even signal satisfies the following relation.

For a continuous-time signal, $x(t) = x(-t)$ (1.9a)

For a discrete-time signal, $x(n) = x(-n)$ (1.9b)

An *odd signal* exhibits anti-symmetry. The signal is not identical to its reflection about the origin, but to its negative. An odd signal satisfies the following relation.

For a continuous-time signal, $x(t) = -x(-t)$ (1.10a)

For a discrete-time signal, $x(n) = -x(-n)$ (1.10b)

$x_1(t) = \sin \omega t$ and $x_2(t) = \cos \omega t$ are good examples of odd and even signals, respectively. Figure 1.3 shows the typical odd and even signals. An even signal which often occurs in the analysis of signals is the sinc function. The sinc function may be expressed in the following two ways according to our convenience: (i) $\text{sinc}(x) = \dfrac{\sin x}{x}$ and (ii) $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$. In chapter 2, the first expression is used.

The area under the sinc function is unity. The sinc function is shown in Fig. 1.3c. The positive portions of the sinc function have angles of

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$\pm n\pi$ where n is an even integer, and the negative portions of the sinc function have angles of $\pm m\pi$ where m is odd. It can be seen from Fig. 1.3c that the sinc function exhibits symmetry about $x = 0$.

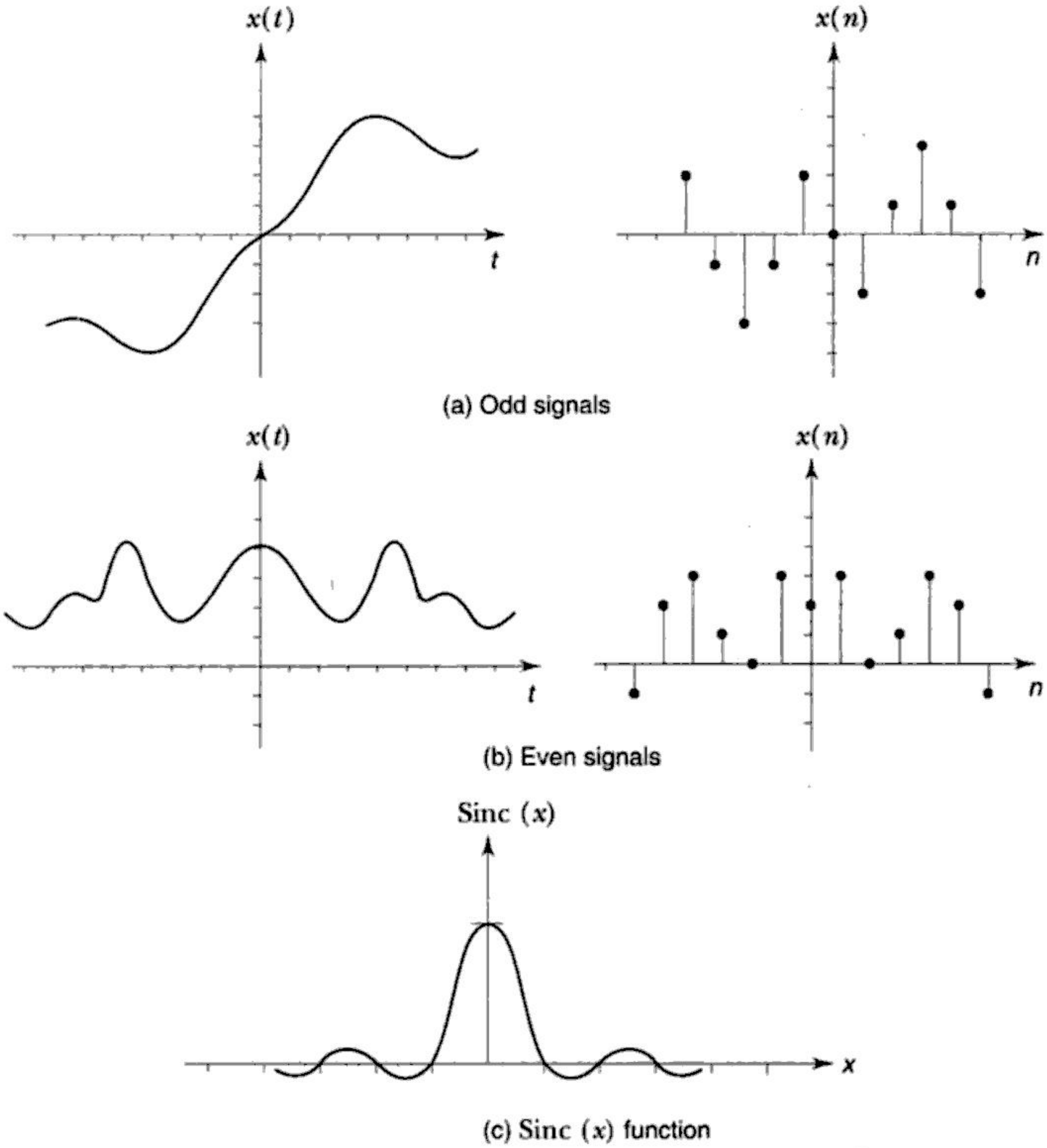

Fig. 1.3 *Typical Examples for (a) Odd Signal and (b) Even Signal (c) the Sinc (x) Function*

A signal can be expressed as a sum of two components, namely, the even component of the signal and the odd component of the signal. The even and odd components can be obtained from the signal itself, as given below.

$$x(t) = x_{\text{even}}(t) + x_{\text{odd}}(t) \tag{1.11}$$

where

$$x_{\text{even}}(t) = \frac{1}{2}[x(t) + x(-t)] \text{ and } x_{\text{odd}}(t) = \frac{1}{2}[x(t) - x(-t)]$$

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1.2.4 Energy and Power Signals

Signals can also be classified as those having finite energy or finite average power. However, there are some signals which can neither be classified as energy signals nor power signals. Consider a voltage source $v(t)$, across a unit resistance R, conducting a current $i(t)$. The instantaneous power dissipated by the resistor is

$$p(t) = v(t)\, i(t) = \frac{v^2(t)}{R} = i^2(t)R$$

Since $R = 1$ ohm, we have

$$p(t) = v^2(t) = i^2(t) \tag{1.12}$$

The total energy and the average power are defined as the limits

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)\, \mathrm{d}t, \quad \text{joules} \tag{1.13}$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, \mathrm{d}t\,, \text{watts} \tag{1.14}$$

The total energy and the average power normalised to unit resistance of any arbitrary signal $x(t)$ can be defined as

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2\, \mathrm{d}t\,, \text{Joules} \tag{1.15}$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, \mathrm{d}t\,, \text{watts} \tag{1.16}$$

The ***energy signal*** is one which has finite energy and zero average power, i.e. $x(t)$ is an energy signal if $0 < E < \infty$, and $P = 0$. The ***power signal*** is one which has finite average power and infinite energy, i.e. $0 < P < \infty$, and $E = \infty$. If the signal does not satisfy any of these two conditions, then it is neither an energy nor a power signal.

1.3 SINGULARITY FUNCTIONS

Singularity functions are an important classification of non-periodic signals. They can be used to represent more complicated signals. The unit-impulse function, sometimes referred to as delta function, is the basic singularity function and all other singularity functions can be derived by repeated integration or differentiation of the delta function. The other commonly used singularity functions are the unit-step and unit-ramp functions.

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1.3.1 Unit-Impulse Function

The unit-impulse function is defined as

$$\delta(t) = 0,\ t \neq 0 \tag{1.17}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 \tag{1.18}$$

The Equations 1.17 and 1.18 indicate that the area of the impulse function is unity and this area is confined to an infinitesimal interval on the t-axis and concentrated at $t = 0$. The unit impulse function is very useful in continuous-time system analysis. It is used to generate the system response providing fundamental information about the system characteristics. In discrete-time domain, the unit-impulse signal is called a unit-sample signal. It is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.19}$$

1.3.2 Unit-step Function

The integral of the impulse function $\delta(t)$ gives,

$$\int_{-\infty}^{t} \delta(t)\,\mathrm{d}t = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.20}$$

Since, the area of the impulse function is all concentrated at $t = 0$, for any value of $t < 0$ the integral becomes zero and for $t > 0$, from Eq.1.18, the value of the integral is unity. The integral of the impulse function is also a singularity function and called the unit-step function and is represented as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{1.21}$$

The value at $t = 0$ is taken to be finite and in most cases it is unspecified. The discrete-time unit-step signal is defined as

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{1.22}$$

1.3.3 Unit-ramp Function

The unit-ramp function, $r(t)$ can be obtained by integrating the unit-impulse function twice or integrating the unit-step function once, i.e.

$$r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\alpha} \delta(\tau)\,\mathrm{d}\tau\,\mathrm{d}\alpha \tag{1.23}$$

$$= \int_{-\infty}^{t} u(\alpha)\,\mathrm{d}\alpha$$

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$$= \int_{0^+}^{t} 1 \cdot d\alpha$$

That is,

$$r(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases} \tag{1.24}$$

A ramp signal starts at $t = 0$ and increases linearly with time, t. In discrete-time domain, the unit-ramp signal is defined as

$$r(n) = \begin{cases} 0, & n < 0 \\ n, & n \geq 0 \end{cases} \tag{1.25}$$

1.3.4 Unit-pulse Function

An unit-pulse function, $\Pi(t)$, is obtained from unit-step signals as shown below.

$$\Pi(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right) \tag{1.26}$$

The signals $u\left(t + \frac{1}{2}\right)$ and $u\left(t - \frac{1}{2}\right)$ are the unit-step signals shifted by $\frac{1}{2}$ units in the time axis towards the left and right, respectively.

Figure 1.4 shows some of the singularity functions. The advantage of the singularity function is that any arbitrary signal that is made up of straight line segments can be represented in terms of step and ramp functions.

1.3.5 Properties of δ (t)

1. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$
2. $\int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt = x(0)$
3. $\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$
4. $\int_{-\infty}^{\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$
5. $\delta(at) = \frac{1}{|a|}\, \delta(t)$
6. $x(t)\, \delta(t - t_0) = x(t_0)$
7. $x(t_0)\, \delta(t - t_0) = x(t_0)$
8. $\int_{t_1}^{t_2} x(t)\, \delta^n(t - t_0)\, dt = (-1)^n x^n(t_0)$

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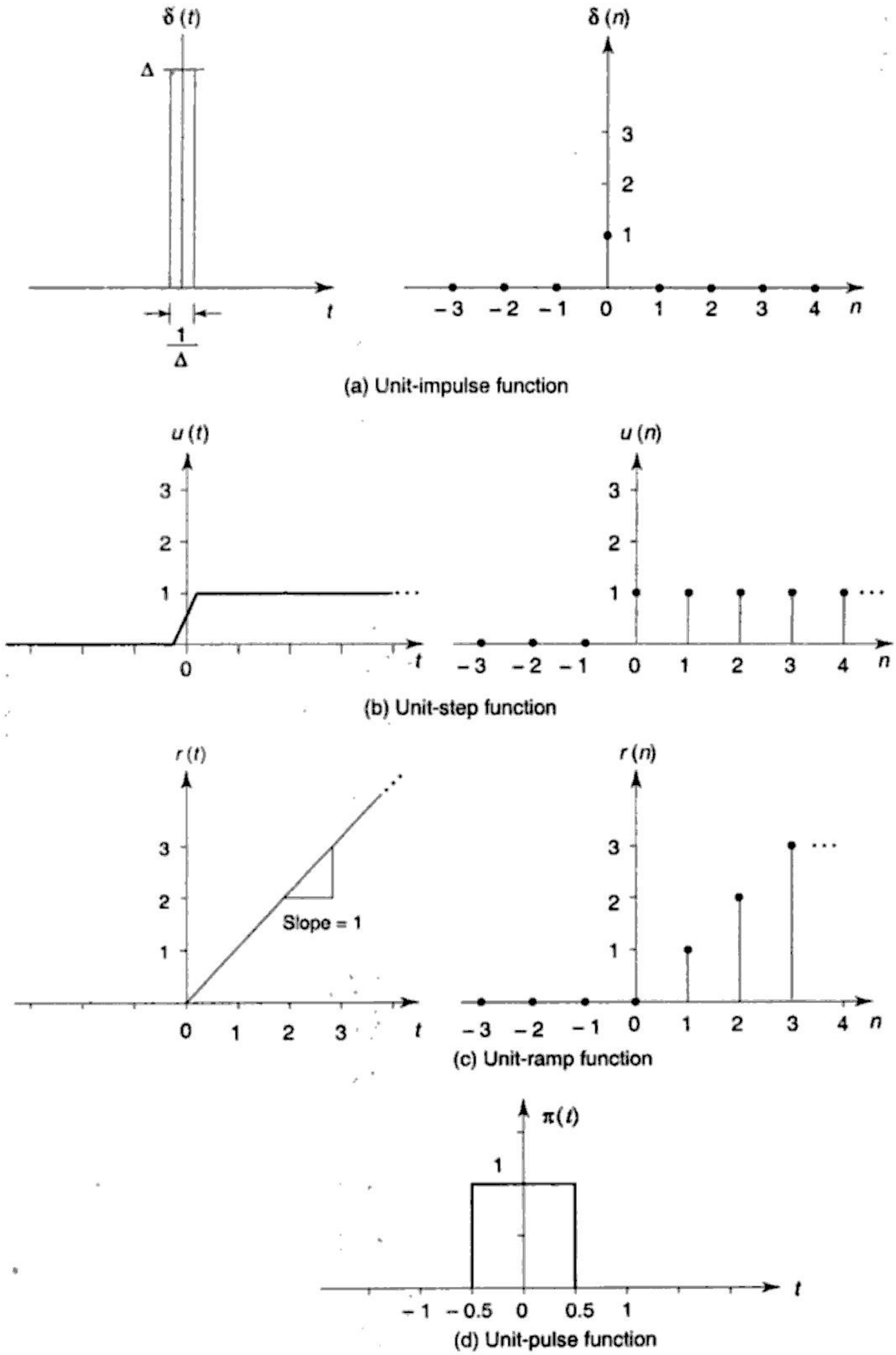

Fig. 1.4 *Singularity Functions (a) Unit-Impulse Function (b) Unit-Step Function (c) Unit-Ramp Function (d) Unit-Pulse Function*

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Proof

$$\frac{d}{dt}[x(t)\,\delta(t-t_0)] = x(t)\,\dot{\delta}(t-t_0) + \dot{x}(t)\,\delta(t-t_0)$$

$$= x(t)\,\dot{\delta}(t-t_0) + \dot{x}\,(t_0)\,\delta(t-t_0),\ t_1 < t_0 < t_2$$

Integrating, we get

$$\int_{t_1}^{t_2} \frac{d}{dt}[x(t)\,\delta(t-t_0)]\,dt = \int_{t_1}^{t_2} [x(t)\,\dot{\delta}(t-t_0)]\,dt + \int_{t_1}^{t_2} [\dot{x}(t_0)\,\delta(t-t_0)]\,dt$$

$$[x(t)\,\delta(t-t_0)]_{t_1}^{t_2} = \int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt + \dot{x}(t_0)$$

LHS = 0.

Therefore, $\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt + \dot{x}(t_0) = 0$

i.e. $\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt = -\dot{x}(t_0)$

Similarly,

$$\int_{t_1}^{t_2} x(t)\,\ddot{\delta}(t-t_0)\,dt = \ddot{x}(t_0)$$

Hence, $\int_{t_1}^{t_2} x(t)\,\delta^n(t-t_0)\,dt = (-1)^n\,x^n(t_0)$

1.3.6 Representation of Signals

In the signal given by $x(at+b)$, i.e., $x(a(t+b/a))$, a is a scaling factor and b/a is a pure shift version in the time domain.

If b/a is positive, then the signal $x(t)$ is shifted to left.

If b/a is negative, then the signal $x(t)$ is shifted to right.

If a is positive, then the signal $x(t)$ will have positive slope.

If a is negative, then the signal $x(t)$ will have negative slope.

If a is less than 0, then the signal $x(t)$ is reflected or reversed through the origin.

If $|a| < 1$, $x(t)$ is expanded, and if $|a| > 1$, $x(t)$ is compressed.

Example 1.2 Sketch the following signals

(a) $x(t) = \Pi(2t+3)$ (b) $x(t) = 2\Pi(t-1/4)$

(c) $x(t) = \cos(20\,\pi t - 5\pi)$ and (d) $x(t) = r\,(-0.5t+2)$

Solution

(a) $\Pi(2t+3) = \Pi(2(t+3/2))$

Here the signal shown in Fig. E1.2(a) is shifted to left, with centre at $-3/2$. Since $a = 2$, i.e. $|a| > 1$, the signal is compressed. The signal width becomes 1/2 with unity amplitude.

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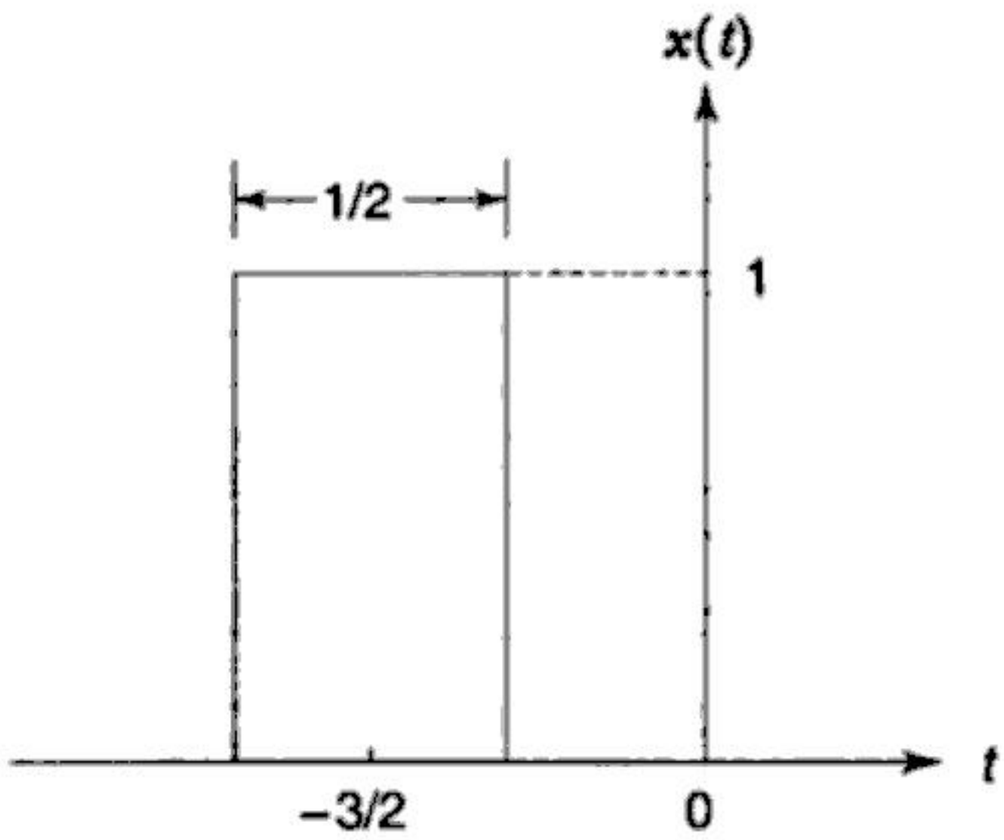

Fig. E1.2 (a)

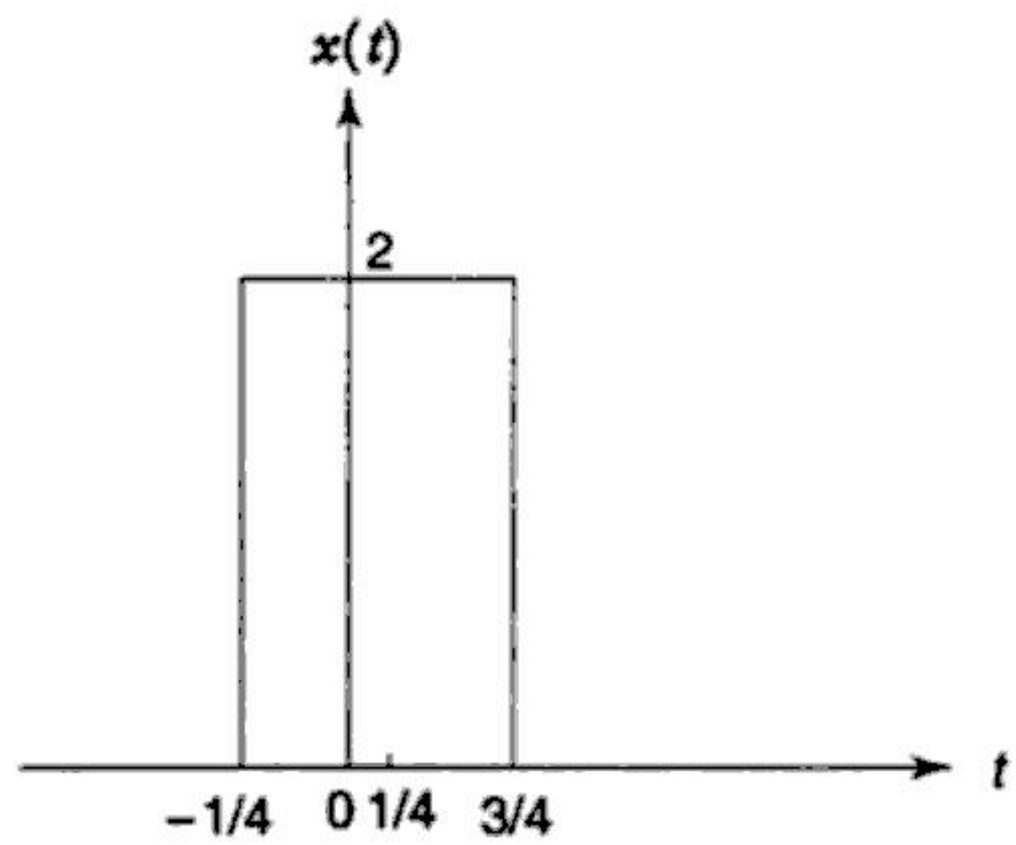

Fig. E1.2 (b)

(b) $x(t) = 2\Pi(t - 1/4)$

Here the signal shown in Fig. E1.2(b) is shifted to the right, with centre at 1 /4. Since $a = 1$, the signal width is 1 and amplitude is 2.

(c) $x(t) = \cos(20\,\pi t - 5\pi)$

$$= \cos\left(20\pi\left(t - \frac{5\pi}{20\pi}\right)\right)$$

$$= \cos\left(20\pi\left(t - \frac{1}{4}\right)\right)$$

Here the signal $x(t)$ shown in Fig. E1.2(c) is shifted by quarter cycle to the right.

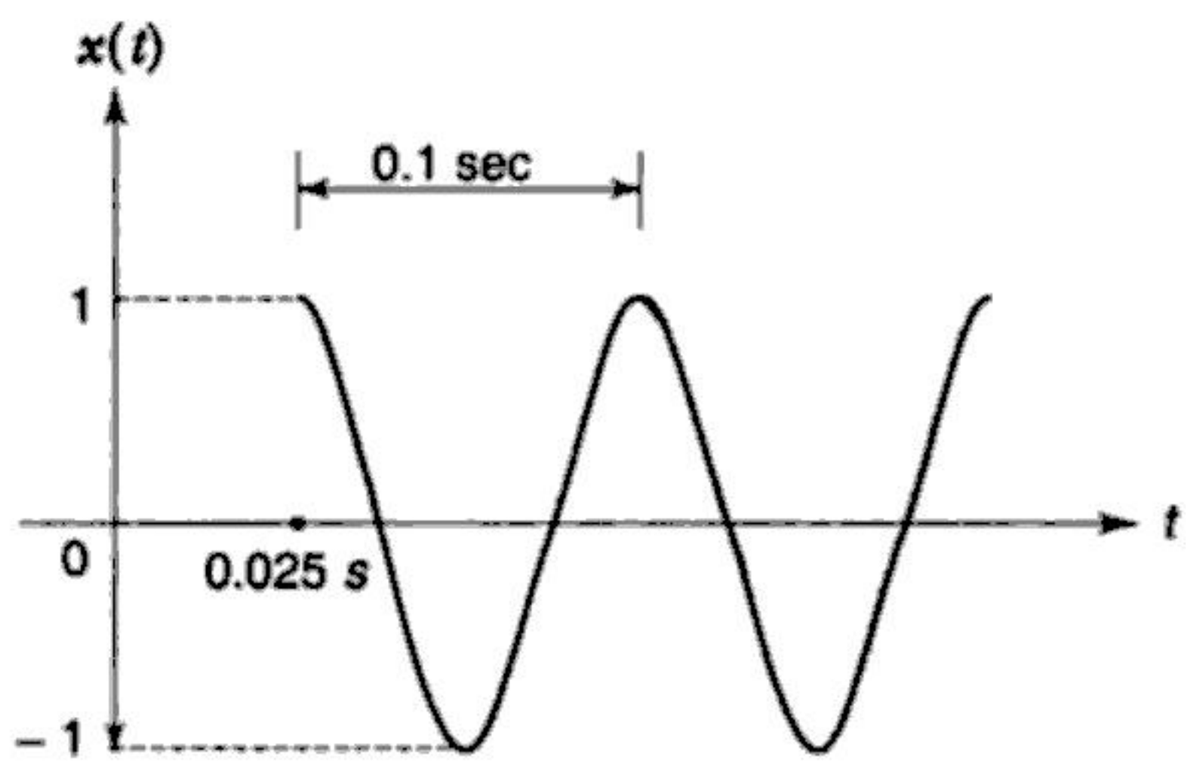

Fig. E1.2(c)

(d) $x(t) = r(-0.5t + 2)$

$$= r\left(-0.5\left(t - \frac{2}{0.5}\right)\right)$$

$$= r(-0.5\,(t - 4))$$

The given ramp signal is reflected through the origin and shifted to right at $t = 4$.

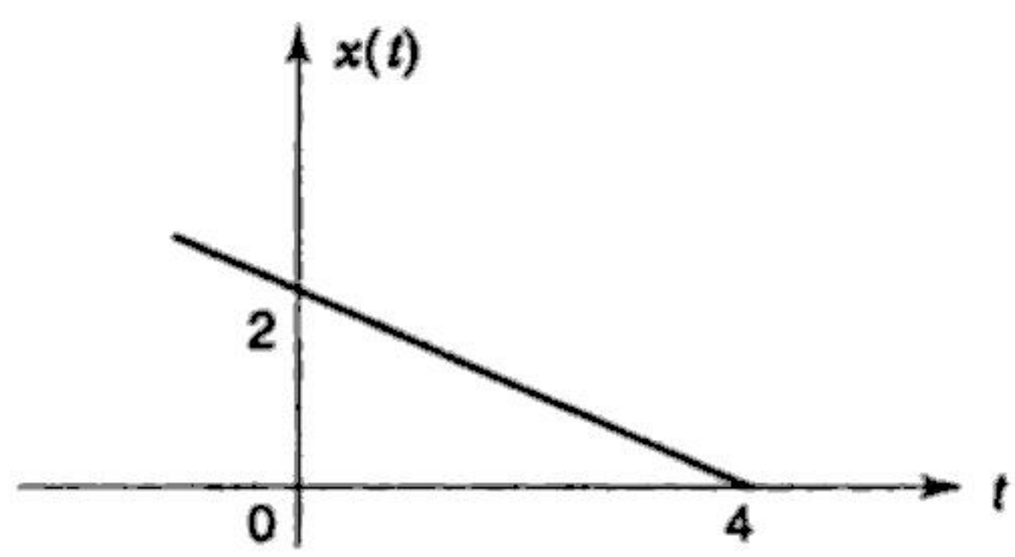

Fig. E1.2 (d)

The signal is expanded by $\frac{1}{0.5} = 2$. When $t = 0$, the magnitude of the signal $x(t) = 2$, shown in Fig. E1.2(d).

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Example 1.3 **Write down the corresponding equation for the given signal.**

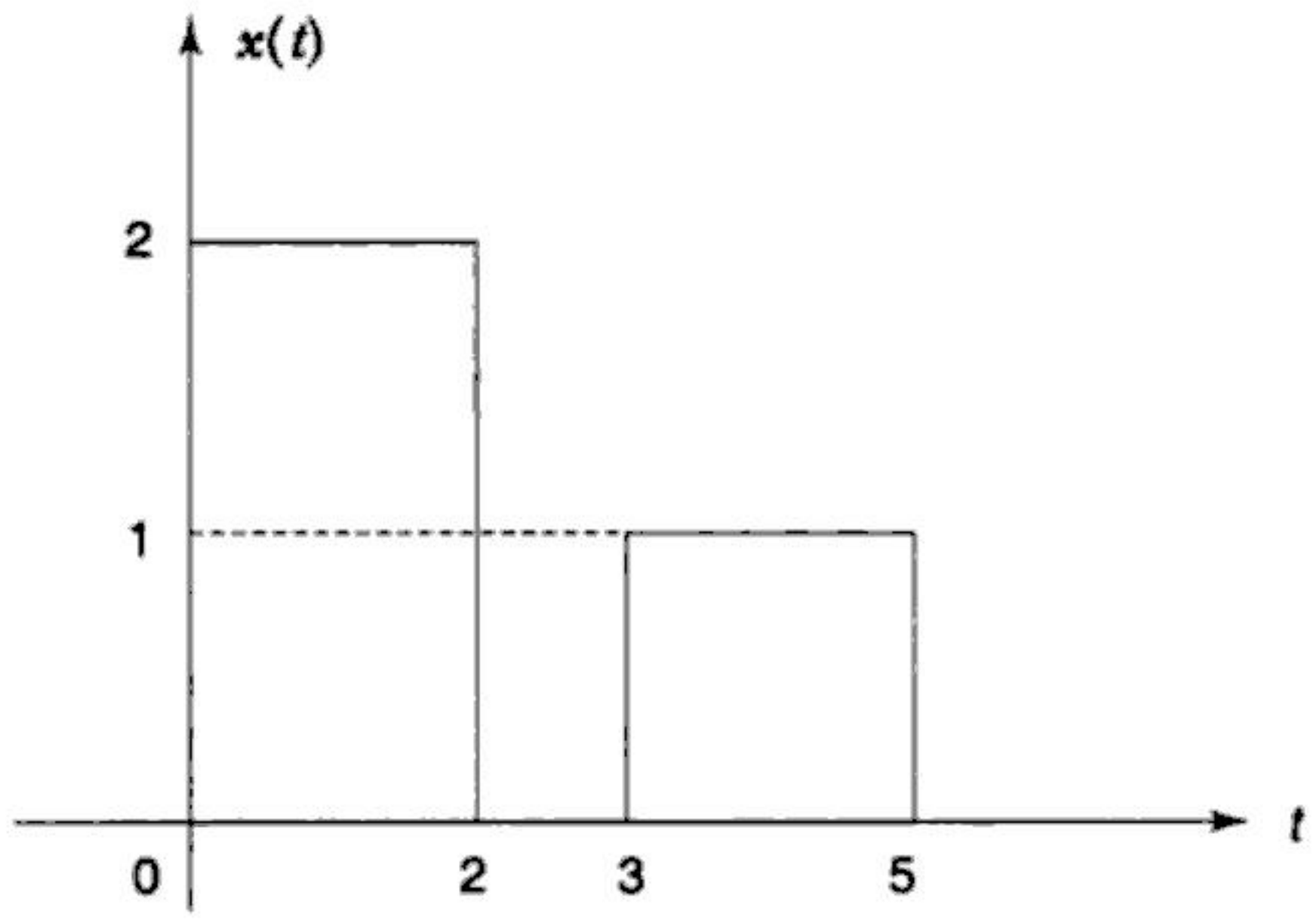

Fig. E1.3

Solution

Representation through addition of two unit step functions

The signal $x(t)$ can be obtained by adding both the pulses, i.e.

$$x(t) = 2[u(t) - u(t-2)] + [u(t-3) - u(t-5)]$$

Representation through multiplication of two unit step functions

$$x(t) = 2[u(t)\,u(-t+2)] + [u(t-3)\,u(-t+5)]$$
$$= 2(u(t)\,u(2-t) + u(t-3)\,u(5-t))$$

1.4 AMPLITUDE AND PHASE SPECTRA

Let us consider a cosine signal of peak amplitude A, frequency f and phase shift Φ, in order to introduce the concept of amplitude and phase spectra, i.e.,

$$x(t) = A\cos(2\pi ft + \Phi) \tag{1.27}$$

The amplitude and phase of this signal can be plotted as a function of frequency. The amplitude of the signal as a function of frequency is referred to as *amplitude spectrum* and the phase of the signal as a function of frequency is referred to as *phase spectrum* of the signal. The amplitude and phase spectra together is called the *frequency spectrum* of the signal. The units of the amplitude spectrum depends on the signal. For example, the unit of the amplitude spectrum of a voltage signal is measured in volts, and the unit of the amplitude spectrum of a current signal is measured in amperes. The unit of the phase spectrum is usually radians. The frequency spectrum drawn only for positive values of frequencies alone is called a *single-sided spectrum*.

The cosine signal can also be expressed in phasor form as the sum of the two counter rotating phasors with complex-conjugate magnitudes, i.e.

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$$x(t) = A\left[\frac{e^{j(2\pi f t + \Phi)} + e^{-j(2\pi f t + \Phi)}}{2}\right]$$

From this the amplitude spectrum for the signal $x(t)$ consists of two components of amplitude, viz. $A/2$ at frequency 'f' and $A/2$ at frequency '$-f$'. Similarly, the phase spectrum also consists of two phase components one at 'f' and the other at '$-f$'. The frequency spectrum of the signal, in this case, is called a *double-sided spectrum*. The following example illustrates the single-sided and double-sided frequency spectra of a signal.

Example 1.4 Sketch the single-sided and double-sided amplitude and phase spectra of the signal

$$x(t) = 8 \sin\left(20\pi t - \frac{\pi}{6}\right), -\infty < t < \infty$$

Solution: The single-sided spectra is plotted by expressing $x(t)$ as the real part of the rotating phasor. Using the trigonometric identity, $\cos\left(u - \frac{\pi}{2}\right) = \sin u$, the given signal is converted into a form as in Eq.1.27, i.e.

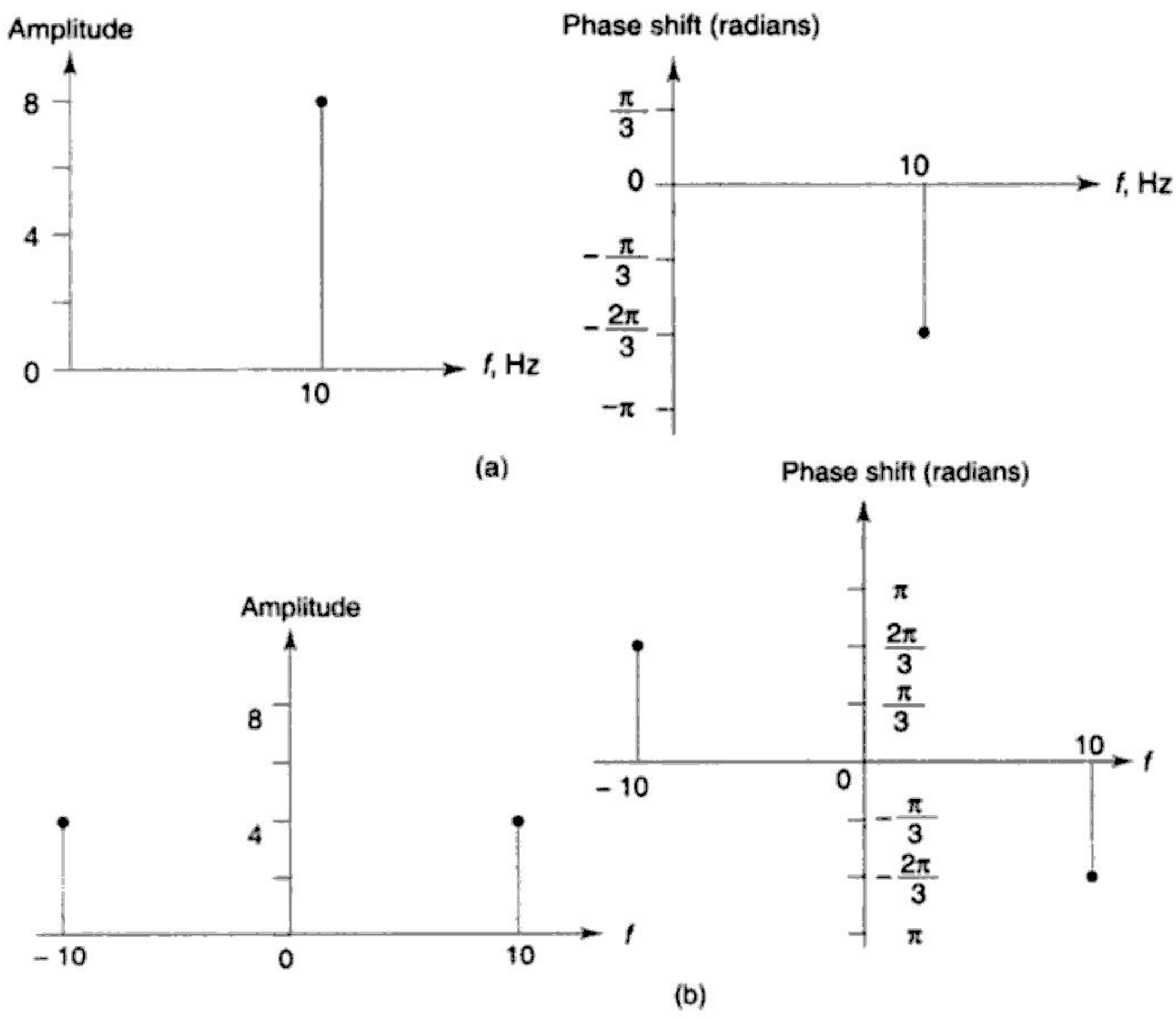

Fig. E.1.4 *Amplitude and Phase Spectra (a) Single-Sided and (b) Double-Sided*

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$$x(t) = 8 \sin\left(20\pi t - \frac{\pi}{6}\right) = 8\cos\left(20\pi t - \frac{\pi}{6} - \frac{\pi}{2}\right)$$
$$= 8 \cos\left(20\pi t - \frac{2\pi}{3}\right)$$

The single-sided amplitude and phase spectra are shown in Fig. E.1.4a. The signal has an amplitude of 8 units at $f = 10$ Hz and a phase angle of $-\frac{2\pi}{3}$ radians at $f = 10$ Hz. To plot the double-sided spectrum, the signal is converted into the form as in Eq.1.28. Therefore,

$$x(t) = 4e^{j\left(20\pi t - \frac{2\pi}{3}\right)} + 4e^{-j\left(20\pi t - \frac{2\pi}{3}\right)}$$

The double-sided amplitude and phase spectra are shown in Fig. E.1.4b. The signal has two components at $f = 10$ Hz and $f = -10$ Hz. The amplitude of these components are 4 units each and the phase of these components are $-\frac{2\pi}{3}$ and $\frac{2\pi}{3}$ radians, respectively.

1.5 CLASSIFICATION OF SYSTEMS

As with signals, systems are also broadly classified into continuous-time and discrete-time systems. In a continuous-time system, the associated signals are also continuous, i.e. the input and output of the system are both continuous-time signals. On the other hand, a discrete-time system handles discrete-time signals. Here, both the input and output signals are discrete-time signals.

Both continuous and discrete-time systems are further classified into the following types.

(i) Static and dynamic systems
(ii) Linear and non-linear systems
(iii) Time-variant and time-invariant systems
(iv) Causal and non-causal systems, and
(v) Stable and unstable systems.

1.5.1 Static and Dynamic Systems

The output of a static system at any specific time depends on the input at that particular time. It does not depend on past or future values of the input. Hence, a static system can be considered as a system with no memory or energy storage elements. A simple resistive network is an example of a static system. The input/output relation of such systems does not involve integrals or derivatives.

The output of a dynamic system, on the other hand at any specified time depends on the inputs at that specific time and at other times. Such systems have memory or energy storage elements. The equation characterising a dynamic system will always be a differential equation

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for continuous-time system or a difference equation for a discrete-time system. Any electrical circuit consisting of a capacitor or an inductor is an example of a dynamic system. The following equations characterise the dynamic systems.

(i) $\dfrac{d^2y(t)}{dt^2} + \dfrac{dy(t)}{dt} + 3y(t) = \dfrac{dx(t)}{dt} + x(t)$

(ii) $y(n-1) + 2y(n) = 4x(n) - x(n-1)$

1.5.2 Linear and Non-linear Systems

A linear system is one in which the principle of superposition holds. See Fig. 1.5. For a system with two inputs $x_1(t)$ and $x_2(t)$, the superposition is defined as follows.

$$H[a_1x_1(t) + a_2x_2(t)] = a_1H[x_1(t)] + a_2H[x_2(t)] \tag{1.28}$$

where, a_1 and a_2 are the weights added to the inputs, and $H[x(t)] = y(t)$ is the response of the continuous-time system to the input $x(t)$. Thus, a *linear system* is defined as one whose response to the sum of the weighted inputs is same as the sum of the weighted responses. If a system does not satisfy Eq.1.28, then the system is *non-linear*. For a discrete-time system, the condition for linearity is given by Eq.1.29.

$$H[a_1x_1(n) + a_2x_2(n)] = a_1H[x_1(n)] + a_2H[x_2(n)] \tag{1.29}$$

where $H[x(n)] = y(n)$ is the response of the discrete-time system to the input $x(n)$.

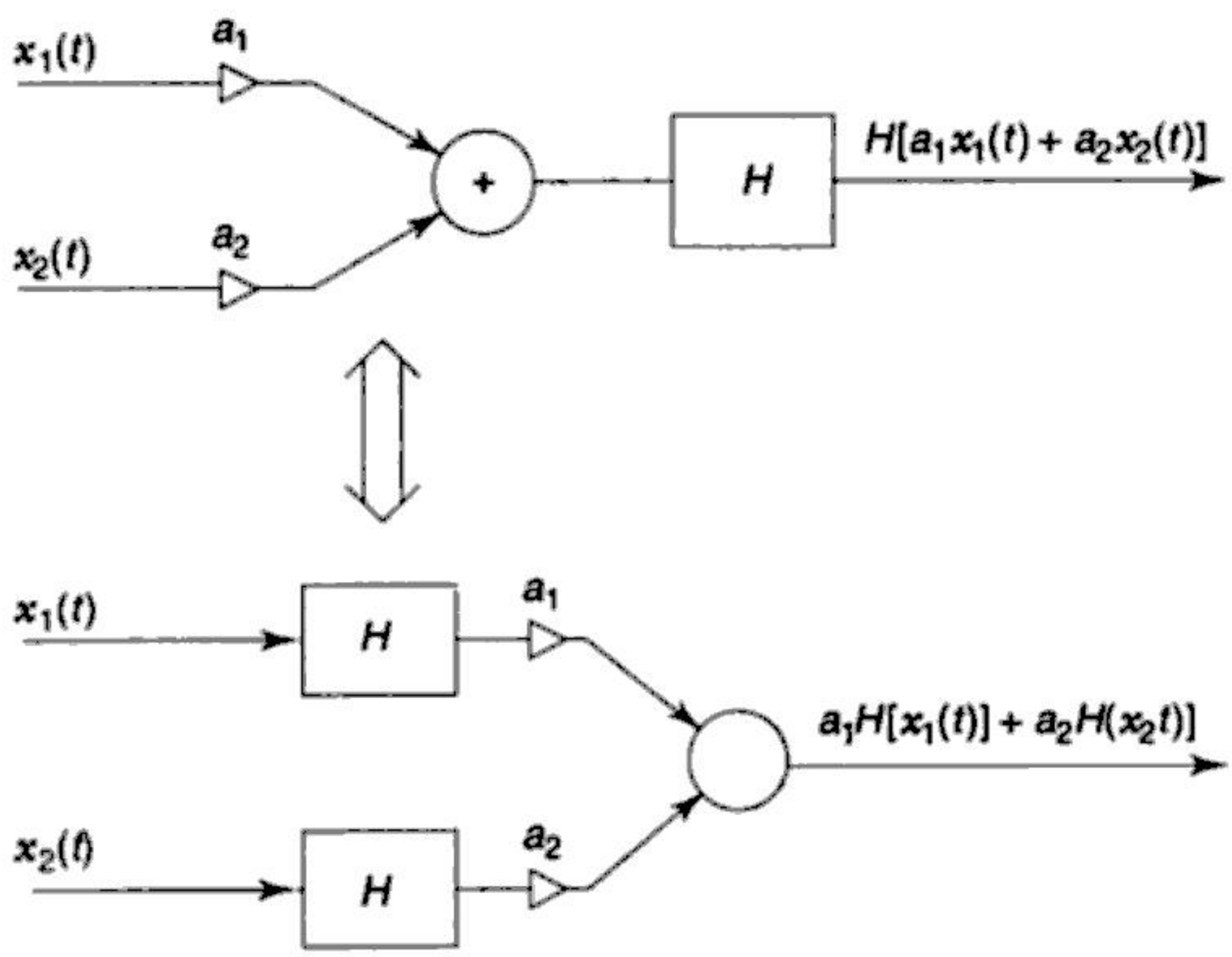

Fig. 1.5 *Illustration of the Superposition Principle*

Example 1.5 Determine whether the system described by the differential equation $\dfrac{dy(t)}{dt} + 2y(t) = x(t)$ is linear.

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Solution Let the response of the system to $x_1(t)$ be $y_1(t)$ and the response of the system to $x_2(t)$ be $y_2(t)$. Thus, for the input $x_1(t)$, the describing equation is

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t)$$

and for the input $x_2(t)$,

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t)$$

Multiplying these equations by a_1 and a_2, respectively, and adding yields,

$$a_1 \frac{dy_1(t)}{dt} + a_2 \frac{dy_2(t)}{dt} + 2a_1 y_1(t) + 2a_2 y_2(t) = a_1 x_1(t) + a_2 x_2(t)$$

i.e.

$$\frac{d}{dt}(a_1 y_1(t) + a_2 y_2(t)) + 2(a_1 y_1(t) + a_2 y_2(t)) = a_1 x_1(t) + a_2 x_2(t)$$

The response of the system to the input $a_1 x_1(t) + a_2 x_2(t)$ is $a_1 y_1(t) + a_2 y_2(t)$. Thus, the superposition condition is satisfied and hence the system is linear.

Example 1.6 Determine whether the system described by the differential equation $\frac{dy(t)}{dt} + y(t) + 4 = x(t)$ is linear.

Solution Let the response of the system to $x_1(t)$ be $y_1(t)$ and the response of the system to $x_2(t)$ be $y_2(t)$. Thus, for input $x_1(t)$, the describing equation is

$$\frac{dy_1(t)}{dt} + y_1(t) + 4 = x_1(t)$$

and for input $x_2(t)$,

$$\frac{dy_2(t)}{dt} + y_2(t) + 4 = x_2(t)$$

Multiplying these equations by a_1 and a_2, respectively, and adding yields,

$$\frac{d}{dt}(a_1 y_1(t) + a_2 y_2(t)) + (a_1 y_1(t) + a_2 y_2(t)) + 4(a_1 + a_2)$$
$$= a_1 x_1(t) + a_2 x_2(t)$$

This equation cannot be put into the same form as the original differential equation describing the system. Hence, the system is non-linear.

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1.5.3 Time-variant and Time-invariant Systems

A time-invariant system is one whose input-output relationship does not vary with time. A time-invariant system is also called a *fixed system*. The condition for a system to be fixed is

$$H\,[x(t-\tau)] = y\,(t-\tau) \tag{1.30}$$

A time-invariant system satisfies Eq.1.30 for any $x\,(t)$ and any value of τ. Equation 1.30 states that **if $y\,(t)$ is the response of the system to any input $x(t)$, then the response of the system to the time-shifted input is the response of the system to $x\,(t)$ time shifted by the same amount.** In discrete time, this property is also referred to as shift-invariance. For a discrete-time system, the condition for shift-invariance is given by

$$H\,[x\,(n-k)] = y\,(n-k) \tag{1.31}$$

where k is an integer. A system not satisfying either Eq.1.30 or 1.31 is said to be time-variant. The systems satisfying both linearity and time-invariant conditions are called ***linear, time-invariant*** systems, or simply ***LTI systems***.

1.5.4 Causal and Non-causal Systems

A causal system is non-anticipatory. The response of the causal system to an input does not depend on future values of that input, but depends only on the present and/or past values of the input. If the response of the system to an input depends on the future values of that input, then the system is non-causal or anticipatory. Non-causal systems are unrealisable. The following difference equations describe causal systems.

(i) $y(n) = 0.5\,x\,(n) - x\,(n-2)$

(ii) $y(n) = x\,(n)$

(iii) $y(n-2) + y\,(n) = x(n) + 0.98\,x\,(n-1)$

The following equations describe non-causal systems.

(i) $y(n-1) = x(n)$

(ii) $y(n) = 0.11x(n-1) + x(n) - 0.8\,x(n+1)$

1.5.5 Stable and Unstable Systems

A system is said to be bounded-input, bounded-output (BIBO) stable, if every bounded input produces a bounded output. A bounded signal has an amplitude that remains finite. Thus, a BIBO stable system will have a bounded output for any bounded input so that its output does not grow unreasonably large. The conditions for a system to be BIBO stable are given below.

(i) If the system transfer function is a rational function, the degree of the numerator must be no larger than the degree of the denominator.

(ii) The poles of the system must lie in the left half of the s-plane or within the unit circle in the z-plane.

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(iii) If a pole lies on the imaginary axis, it must be a single-order one, i.e. no repeated poles must lie on the imaginary axis.

The systems not satisfying the above conditions are unstable.

1.6 SIMPLE MANIPULATIONS OF DISCRETE-TIME SIGNALS

When a signal is processed, the signal undergoes many manipulations involving the independent variable and the dependent variable. Some of these manipulations include (i) shifting the signal in the time domain, (ii) folding the signal and (iii) scaling in the time-domain. A brief introduction of these manipulations here, will help the reader in the following chapters.

1.6.1 Transformation of the Independent Variable

Shifting

In the case of discrete-time signals, the independent variable is the time, n. A signal $x(n)$ may be shifted in time, i.e. the signal can be either advanced in the time axis or delayed in the time axis. The shifted signal is represented by $x(n-k)$, where k is an integer. If 'k' is positive, the signal is delayed by k units of time and if k is negative, the time shift results in an advance of signal by k units of time. However, advancing the signal in the time axis is not possible always. If the signal is available in a magnetic disk or other storage units, then the signal can be delayed or advanced as one wishes. But in real time, advancing a signal is not possible since such an operation involves samples that have not been generated. As a result, in real-time signal processing applications, the operation of advancing the time base of the signal is physically unrealizable.

Folding

This operation is done by replacing the independent variable n by $-n$. This results in folding of the signal about the origin, i.e. $n = 0$. Folding is also known as the reflection of the signal about the time origin $n = 0$. Folding of a signal is done while convoluting the signal with another.

Time scaling

This involves replacing the independent variable n by kn, where k is an integer. This process is also called as *down sampling*. If $x(n)$ is the discrete-time signal obtained by sampling the analog signal, $x(t)$, then $x(n) = x(nT)$, where T is the sampling period. If time-scaling is done, then the time-scaled signal, $y[n] = x(kn) = x(knT)$. This implies that the sampling rate is changed from $1/T$ to $1/kT$. This decreases the sampling rate by a factor of k. Down-sampling operations are discussed in detail in Chapter 11 of this book. The folding and time scaling operations are shown in Fig. 1.7(a) and (b).

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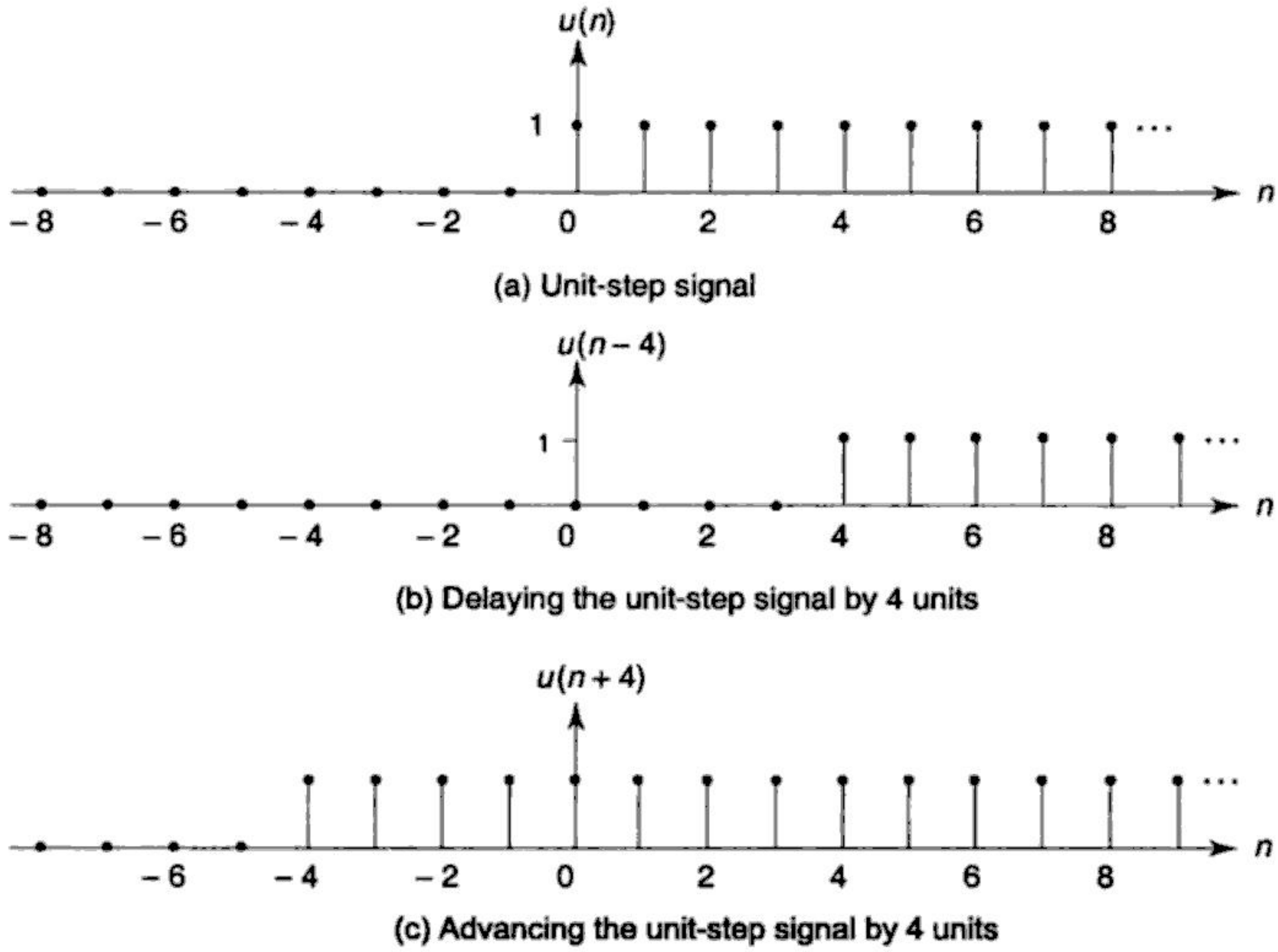

Fig. 1.6 *Graphical Representation of a Signal, and Its Delayed and Advanced Versions*

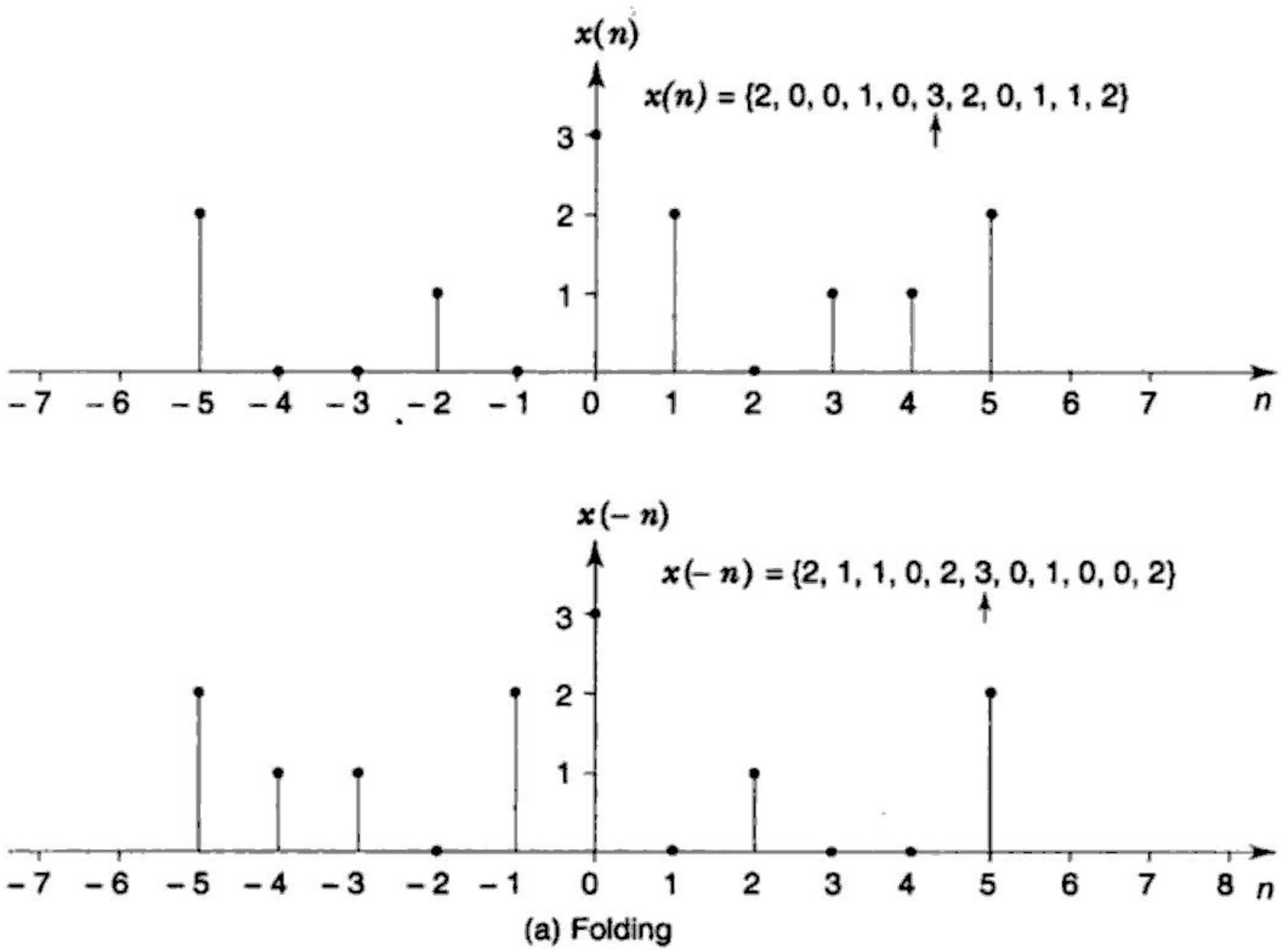

Fig. 1.7 *(Contd.)*

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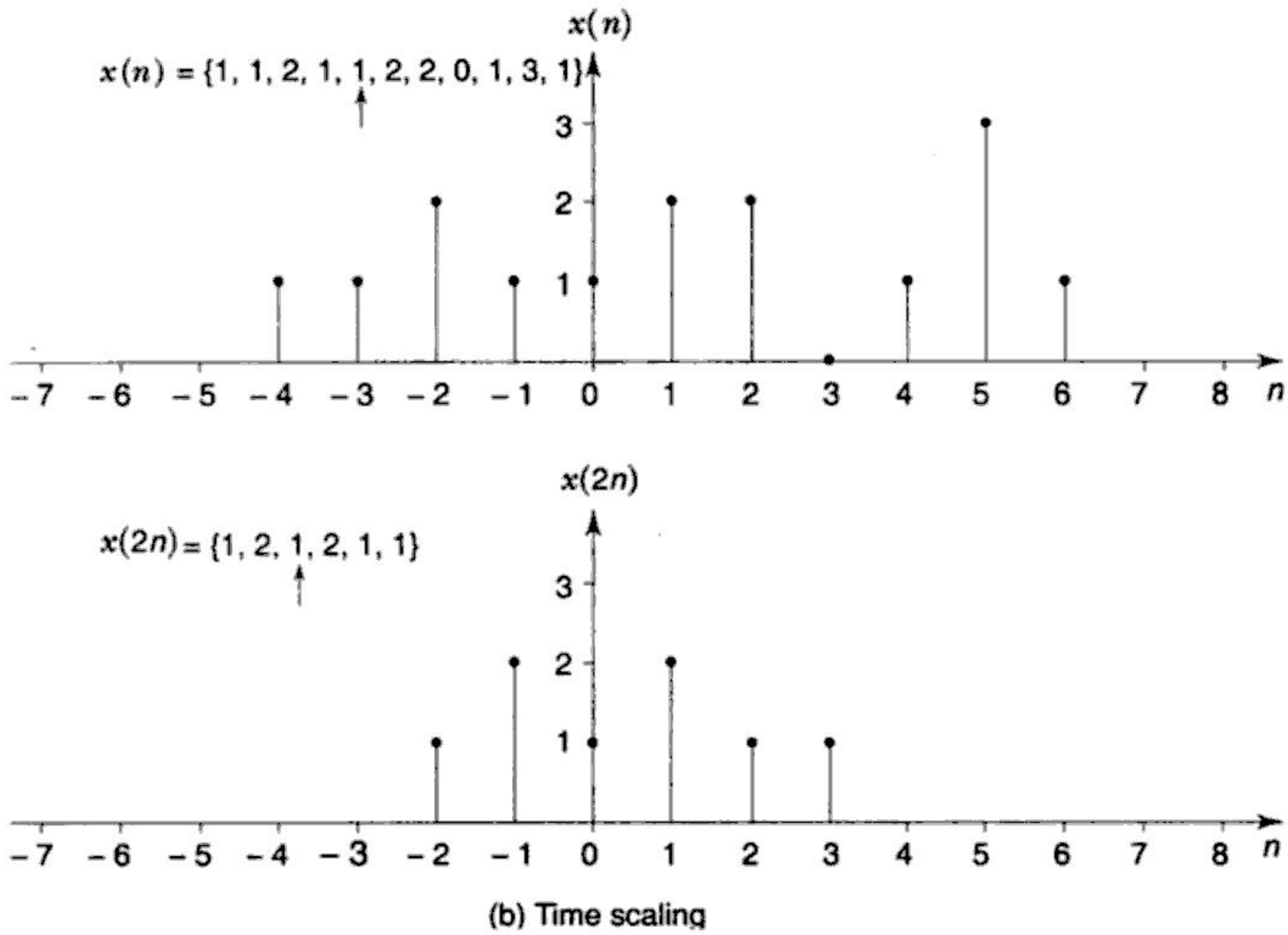

(b) Time scaling

Fig. 1.7 *Illustration of Folding and Time Scaling Operations*

1.7 REPRESENTATIONS OF SYSTEMS

The representation of a system helps in visualising the system with its components and their interconnections. A system can be represented using a diagram featuring various components of the system. These components are represented by symbols. An electrical system is thus represented by a diagram consisting of different symbols representing resistors, capacitors or any other device. A mechanical system can be represented using symbols for different elements like damper, acceleration, zero friction, etc. Thus, a system can be visualised when represented in the form of a diagram.

A more convenient form of representing a system is the block-diagram representation. In this form of representation, each box is an operator on the input signal and the operation is shown on the box itself. Some typical operations include integration, differentiation, scalar multiplication, delay, etc. The lines connecting the individual boxes are directional and these lines show the direction of the signal flow. A number of lines may terminate at a node. The node may be either an accumulator or a multiplier. These nodes are represented by circles with the symbols '+' or '×' marked on them. Figure 1.8 shows the block diagram representation of continuous-time and discrete-time systems.

A system can also be represented using mathematical models. The analysis of system characteristics and performance can be carried out using the mathematical model of a system. The mathematical model of a

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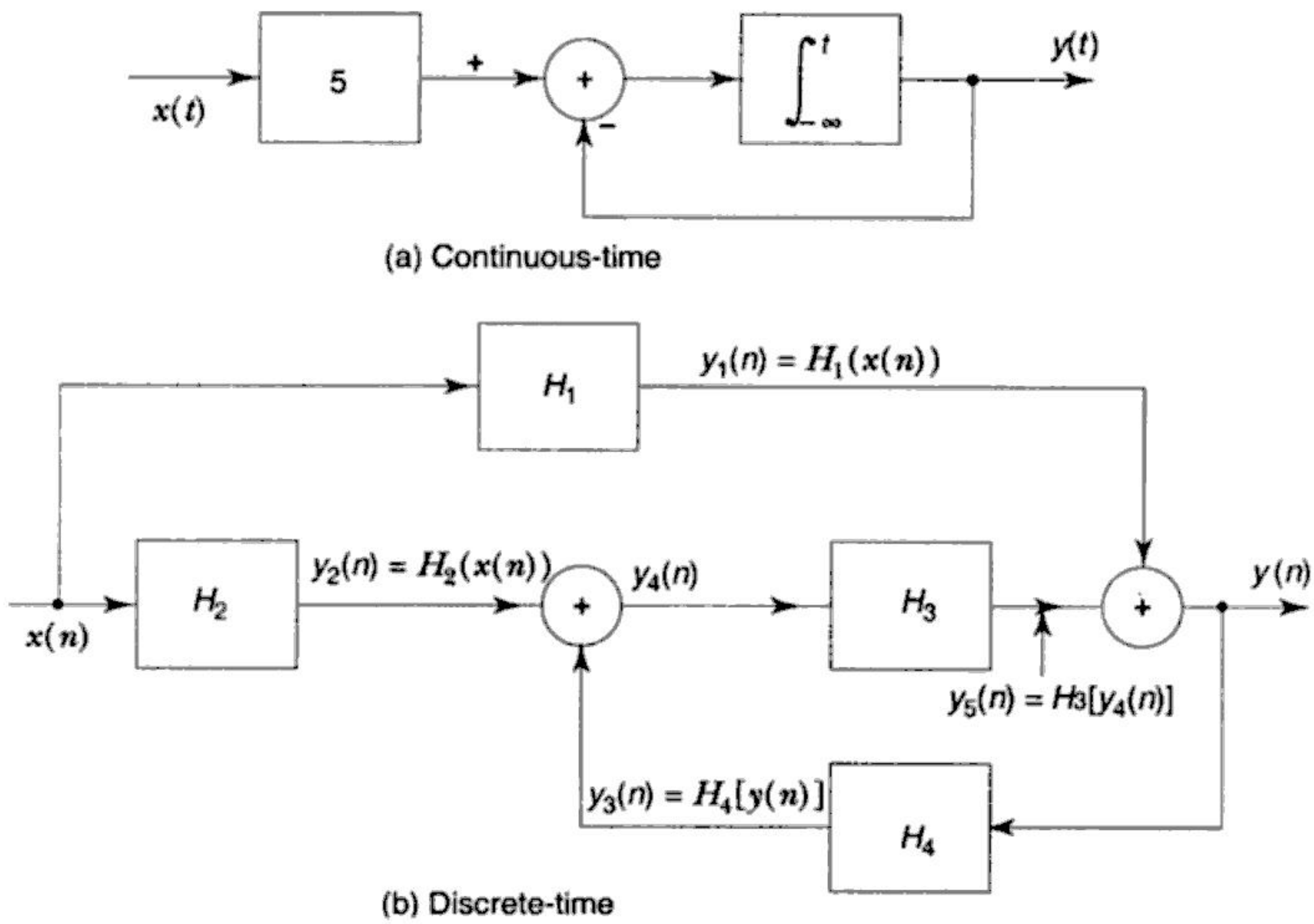

Fig. 1.8 *Block Diagram Representation of (a) Continuous-Time System (b) Discrete-Time System*

system consists of equations relating to the signals of interest. Both discrete-time and continuous-time signals can be modelled using the following three methods.

(i) A linear difference/differential equation
(ii) The impulse-response sequence
(iii) A state-variable or matrix description.

All the above three methods help in determining the output of the system from the knowledge of the input to the system. A system can be interpreted in different ways depending on the model since each model emphasise certain aspects of the system. Together, these models provide a very good understanding of the system and how the system works. In the following sections, these models are discussed further.

1.7.1 Linear Difference/Differential Equations

A discrete-time system is modelled by a difference equation, whereas a continuous-time system is modelled by a differential equation. In a linear discrete-time system, the input sequence $\{x_n\}$ is transformed into an output sequence $\{y_n\}$ according to some difference equation. For example,

$$y(n) = x(n) + 3x(n-1) + 2x(n-2) \tag{1.32}$$

is a linear difference equation which tells that the nth member of the output sequence $y(n)$ is obtained by accumulating (adding) the input at the present moment, $x(n)$, with thrice the previous input, $x(n-1)$ and

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twice the input delayed twice, $x(n-2)$. Let the input sequence be $x(n) = \{0,1, 1, 2, 0, 0, 0, \ldots\}$. The output sequence for the system as described by Eq. 1.32 is $y(n) = \{0, 1, 4, 7, 8, 4, 0, 0, \ldots\}$. The block diagram representation of the system described by Eq.1.32 is shown in Fig. 1.9.

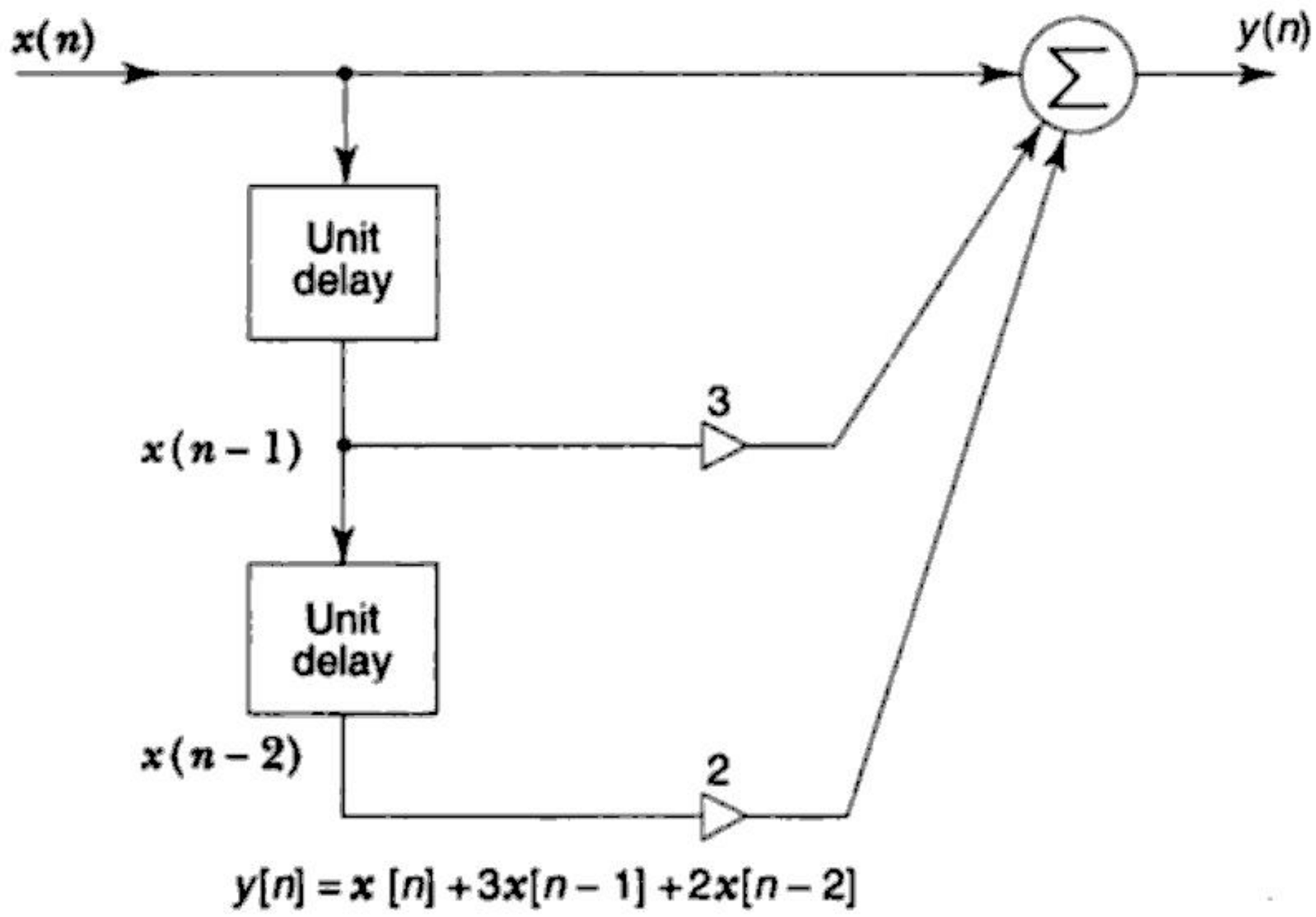

Fig. 1.9 *Discrete-Time System Corresponding to Eq. 1.32*

In digital signal processing applications, our prime concern is of linear, time-invariant discrete-time systems. Such systems are modelled using linear difference equations with constant coefficients. The block diagram representation of these systems contain only unit delays, constant multipliers and adders.

A continuous-time system is modelled by a linear differential equation. An ordinary linear differential equation with constant coefficients characterises linear, constant parameter systems. For example, an nth order system is represented by

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + y(t) = x(t) \qquad (1.33)$$

The general solution of the above equation consists of two components, namely, the homogeneous solution and the particular solution. The homogeneous solution is the sourcefree, natural solution of the system, whereas the particular solution is the component due to the source $x(t)$.

1.7.2 Impulse Response of a System

The impulse response of a system is another method for modelling a system. The impulse response of a linear, time-invariant system is the response of the system when the input signal is an unit-impulse function. The system is assumed to be initially relaxed, i.e. the system has zero initial conditions. The impulse response of a system is represented by the notation $h(t)$ (continuous-time) or $h(n)$ (discrete-

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time). If $y(t)$ is the system response for an input $x(t)$, then the response of the system when $x(t) = \delta(t)$ is $y(t) = h(t)$.

The impulse response of a system can be directly obtained from the solution of the differential or difference equation characterising the system. The impulse response is also determined by finding out the output of the system to the rectangular pulse input $x(t) = \frac{1}{\varepsilon}\Pi\left(\frac{t}{\varepsilon}\right)$ and then taking the limit of the resulting system response, $y(t)$ as $\varepsilon \to 0$. The unit-impulse function is nothing but the derivative of the unit-step signal. Therefore, the impulse response of the system can also be obtained by computing the derivative of the step response of the system.

1.7.3 State-Variable Technique

The state-variable technique provides a convenient formulation procedure for modelling a multi-input, multi-output system. This technique also facilitates the determination of the internal behaviour of the system very easily. The *state* of a system at time t_0 is the minimum information necessary to completely specify the condition of the system at time t_0 and it allows determination of the system outputs at any time $t > t_0$, when inputs upto time t are specified. The state of a system at time t_0 is a set of values, at time t_0, of a set of variables. These variables are called the *state variables*. The number of state variables is equal to the order of the system. The state variables are chosen such that they correspond to physically measurable quantities. It is also convenient to consider an n-dimensional space in which each coordinate is defined by one of the state variables $x_1, x_2, \ldots, x_n$, where n is the order of the system. This n-dimensional space is called the *state space*. The ***state vector*** is an n-vector x whose elements are the state variables. The state vector defines a point in the state space at any time t. As the time changes, the system state changes and a set of points, which is nothing but the locus of the tip of the state vector as time progresses, is called a *trajectory* of the system.

A linear system of order n with m inputs and k outputs can be represented by n first-order differential equations and k output equations as shown below.

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots + b_{1m}u_m \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots + b_{2m}u_m \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nm}u_m
\end{aligned} \tag{1.34}$$

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and

$$
\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots + d_{1m}u_m \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \ldots + d_{2m}u_m \\
&\vdots \\
y_k &= c_{k1}x_1 + c_{k2}x_2 + \ldots + c_{kn}x_n + d_{k1}u_1 + d_{k2}u_2 + \ldots + d_{km}u_m
\end{aligned}
\tag{1.35}
$$

where u_i, $i = 1, 2, \ldots, m$ are the system inputs, x_i, $i = 1, 2,3, \ldots, n$ are called the state variables and y_i, $i = 1, 2, 3, \ldots, k$ are the system outputs. Equations 1.34 are called the *state equations*, and Eqs 1.35 are the *output equations*. Equations 1.34 and 1.35 together constitute the state-equation model of the system. Generally, the a's, b's, c's and d's may be functions of time. The solution of such a set of time-varying state equations is very difficult. If the system is assumed to be time-invariant, then the solution of the state equations can be obtained without much difficulty.

The state variable representation of a system offers a number of advantages. The most obvious advantage of this representation is that multiple-input, multiple-output systems can be easily represented and analysed. The model is in the time-domain, and one can obtain the simulation diagram for the equations directly. This is of much use when computer simulation methods are used to analyse the system. Also, a compact matrix notation can be used for the state model and using the laws of linear algebra the state equations can be very easily manipulated. For example, Eqs 1.34 and 1.35 expressed in a compact matrix form is shown below. Let us define vectors

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} \tag{1.36}
$$

and matrices

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1m} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nm} \end{bmatrix} \tag{1.37}
$$

$$
C = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{k1} & c_{k2} & \cdot & \cdot & \cdot & c_{kn} \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdot & \cdot & \cdot & d_{1m} \\ d_{21} & d_{22} & \cdot & \cdot & \cdot & d_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{n1} & d_{k2} & \cdot & \cdot & \cdot & d_{km} \end{bmatrix}
$$

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Now, Eqs 1.34 and 1.35 can be compactly written as

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \tag{1.38a}$$

$$y = \boldsymbol{Cx} + \boldsymbol{Du} \tag{1.38b}$$

where $\dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$. Equations 1.38 may be illustrated schematically as shown in Fig.1.10. The double lines indicate a multiple-variable signal flow path. The blocks represent matrix multiplication of the vectors and matrices. The integrator block consists of n integrators with appropriate connections specified by the **A** and **B** matrices.

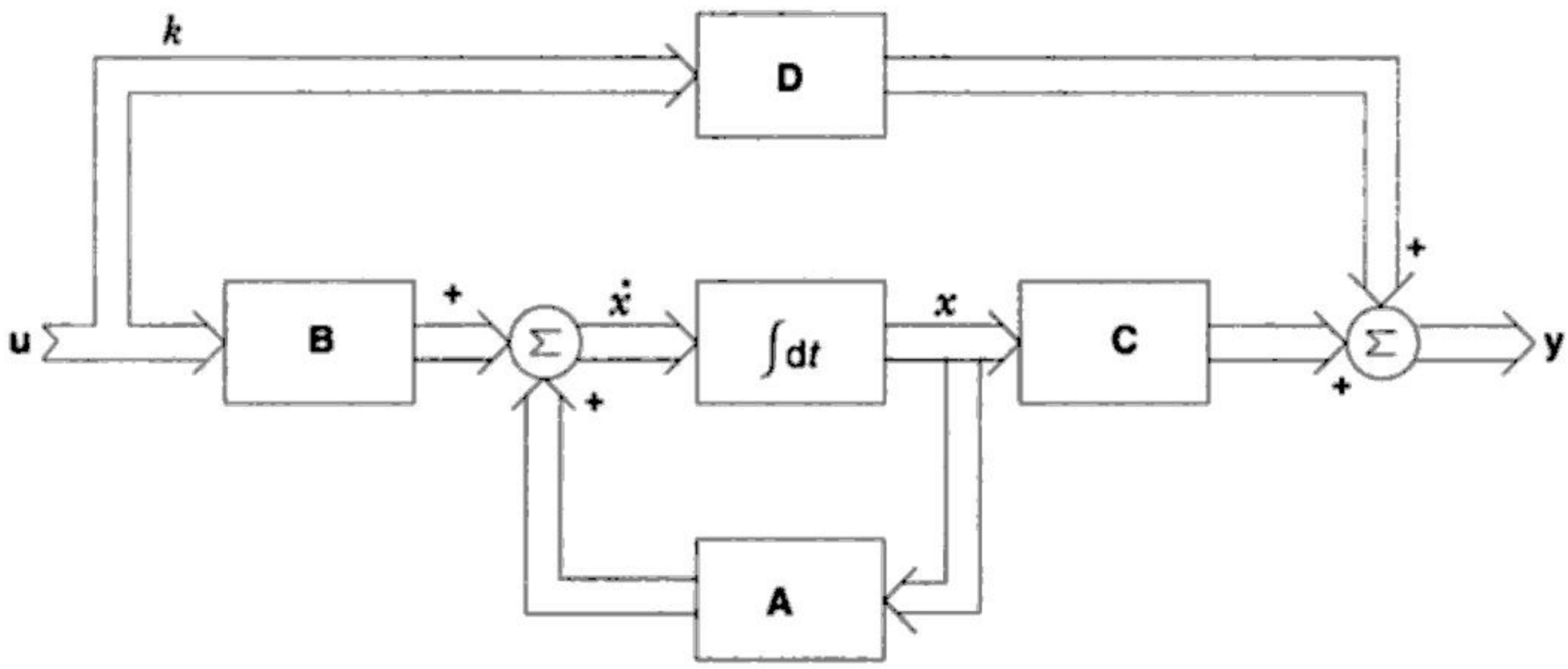

Fig. 1.10 *Block Diagram of the State-Variable Model of Eq. 1.38*

State Equations for Discrete-time Systems

For a discrete-time system, the state equations form a set of first-order difference equations constituting a recursion relation. This recursion relation allows determination of the state of a system at the sampling time kT from the state of the system and the input at the sampling time $(k-1)T$, where k is an integer. The state-equations for a discrete-time system can be modelled as shown below.

$$\boldsymbol{x}_{k+1} = \boldsymbol{Fx} + \boldsymbol{Gu}_k \tag{1.39a}$$

$$\boldsymbol{y}_k = \boldsymbol{Hx}_k + \boldsymbol{Ju}_k \tag{1.39b}$$

The dependence of these parameters on T is suppressed for simplicity. For a single input, single output system $\boldsymbol{u}_k$ and $\boldsymbol{y}_k$ are scalars and $\boldsymbol{G}$ and $\boldsymbol{H}$ become vectors $\boldsymbol{g}$ and $\boldsymbol{h}$, and $\boldsymbol{J}$ is a null in most cases. The state-variable modelling of a discrete-time system finds application in the digital simulation of a continuous time systems.

1.8 ANALOG-TO-DIGITAL CONVERSION OF SIGNALS

A discrete-time signal is defined by specifying its value only at discrete times, called *sampling instants*. When the sampled values are quantised and encoded, a digital signal is obtained. A digital signal can be obtained from the analog signal by using an analog-to-digital converter. In the following sections the process of analog-to-digital conversion is

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discussed in some detail and this enables one to understand the relationship between the digital signals and discrete-time signals.

Figure 1.11 shows the block diagram of an analog-to-digital converter. The *sampler* extracts the sample values of the input signal at the sampling instants. The output of the sampler is the discrete-time signal with continuous amplitude. This signal is applied to a *quantiser* which converts this continuous amplitude into a finite number of sample values. Each sample value can be represented by a digital word of finite word length. The final stage of analog-to-digital conversion is encoding. The *encoder* assigns a digital word to each quantised sample. Sampling, quantizing and encoding are discussed in the following sections.

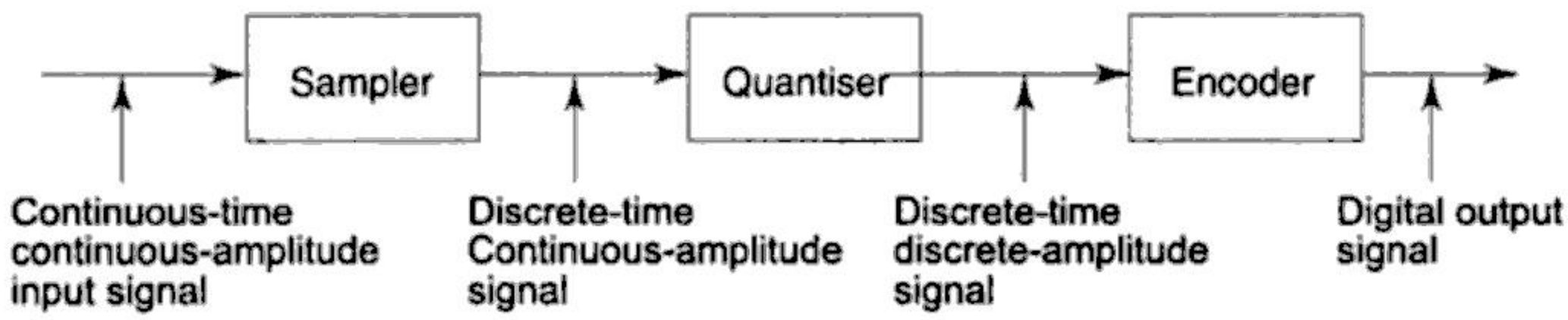

Fig. 1.11 *Analog-to-Digital Converter*

1.8.1 Sampling of Continuous-time Signals

Sampling is a process by which a continuous-time signal is converted into a discrete-time signal. This can be accomplished by representing the continuous-time signal $x(t)$, at a discrete number of points. These discrete number of points are determined by the sampling period, T, i.e. the samples of $x(t)$ can be obtained at discrete points $t = nT$, where n is an integer. The process of sampling is illustrated in Fig.1.12. The sampling unit can be thought of as a switch, where, to one of its inputs the continuous-time signal is applied. The signal is available at the output only during the instants the switch is closed. Thus, the signal at the output end is not a continuous function of time but only discrete samples. In order to extract samples of $x(t)$, the switch closes briefly every T seconds. Thus, the output signal has the same amplitude as $x(t)$ when the switch is closed and a value of zero when the switch is open. The switch can be any high speed switching device.

The continuous-time signal $x(t)$ must be sampled in such a way that the original signal can be reconstructed from these samples. Otherwise, the sampling process is useless. Let us obtain the condition necessary to faithfully reconstruct the original signal from the samples of that signal. The condition can be easily obtained if the signals are analysed in the frequency domain. Let the sampled signal be represented by $x_s(t)$. Then,

$$x_s(t) = x(t)\, g(t) \tag{1.40}$$

where $g(t)$ is the *sampling function*. The sampling function is a continuous train of pulses with a period of T seconds between the pulses, and it models the action of the sampling switch. The sampling function is shown in Fig. 1.12(c) and (d). The frequency spectrum of the sampled

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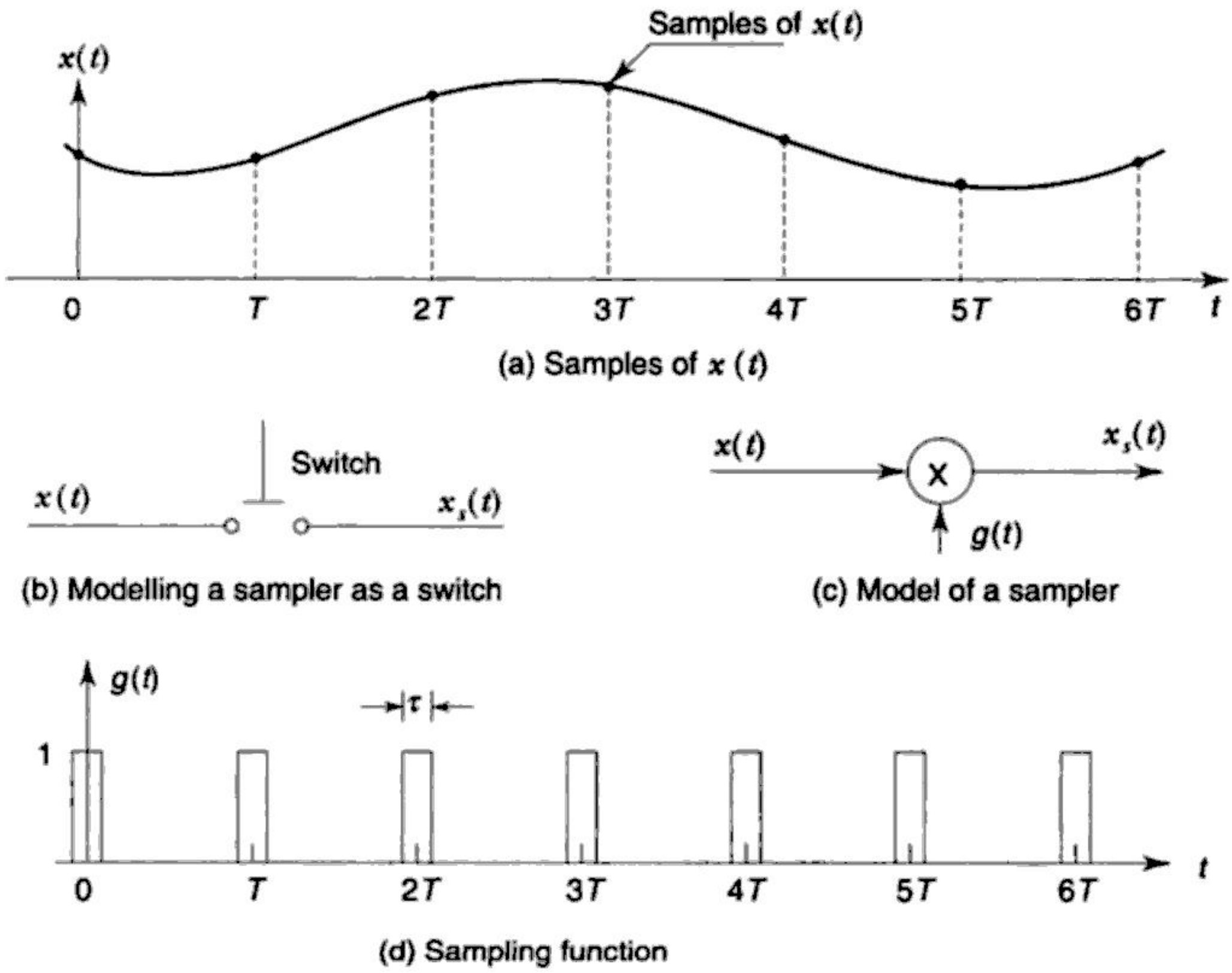

Fig. 1.12 *The Sampling Process*

signal $x_s(t)$ helps in determining the appropriate values of T for reconstructing the original signal. The sampling function $g(t)$ is periodic and can be represented by a Fourier series (Fourier Series and transforms are discussed in Chapter six), i.e.

$$g(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn2\pi f_s t} \tag{1.41}$$

where

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-jn2\pi f_s t} \, dt \tag{1.42}$$

is the nth Fourier coefficient of $g(t)$, and $f_s = \dfrac{1}{T}$ is the fundamental frequency of $g(t)$. The fundamental frequency, f_s is also called the *sampling frequency*. From Eqs.1.40, and 1.41, we have

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_s t} = \sum_{n=-\infty}^{\infty} C_n x(t) e^{jn2\pi f_s t} \tag{1.43}$$

The spectrum of $x_s(t)$, denoted by $X_s(f)$, can be determined by taking the Fourier transform of Eq. 1.43, i.e.

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$$X_s(f) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j2\pi f t}\, \mathrm{d}t \tag{1.44}$$

Using Eq.1.43 in the above equation,

$$X_s(f) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n x(t)\, e^{jn2\pi f_s t}\, e^{-j2\pi f t} \mathrm{d}t \tag{1.45}$$

Interchanging the order of integration and summation,

$$X_s(f) = \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{\infty} x(t) e^{-j2\pi(f - nf_s)t}\, \mathrm{d}t \tag{1.46}$$

But from the definition of the Fourier transform

$$\int_{-\infty}^{\infty} x(t) e^{-j2\pi(f - nf_s)t}\, \mathrm{d}t = X(f - nf_s)$$

Thus,

$$X_s(f) = \sum_{n=-\infty}^{\infty} C_n X(f - nf_s) \tag{1.47}$$

From Eq. 1.47, it is understood that the spectrum of the sampled continuous-time signal is composed of the spectrum of $x(t)$ plus the spectrum of $x(t)$ translated to each harmonic of the sampling frequency. The spectrum of the sampled signal is shown in Fig. 1.13. Each frequency translated spectrum is multiplied by a constant. To reconstruct the original signal, it is enough to just pass the spectrum of $x(t)$ and suppress the spectra of other translated frequencies. The amplitude response of such a filter is also shown in Fig. 1.13. As this filter is used to reconstruct the original signal, it is often referred to as a *reconstruction filter*. The output of the reconstruction filter will be $C_0X(f)$ in the frequency domain and $x(t)$ in the time-domain.

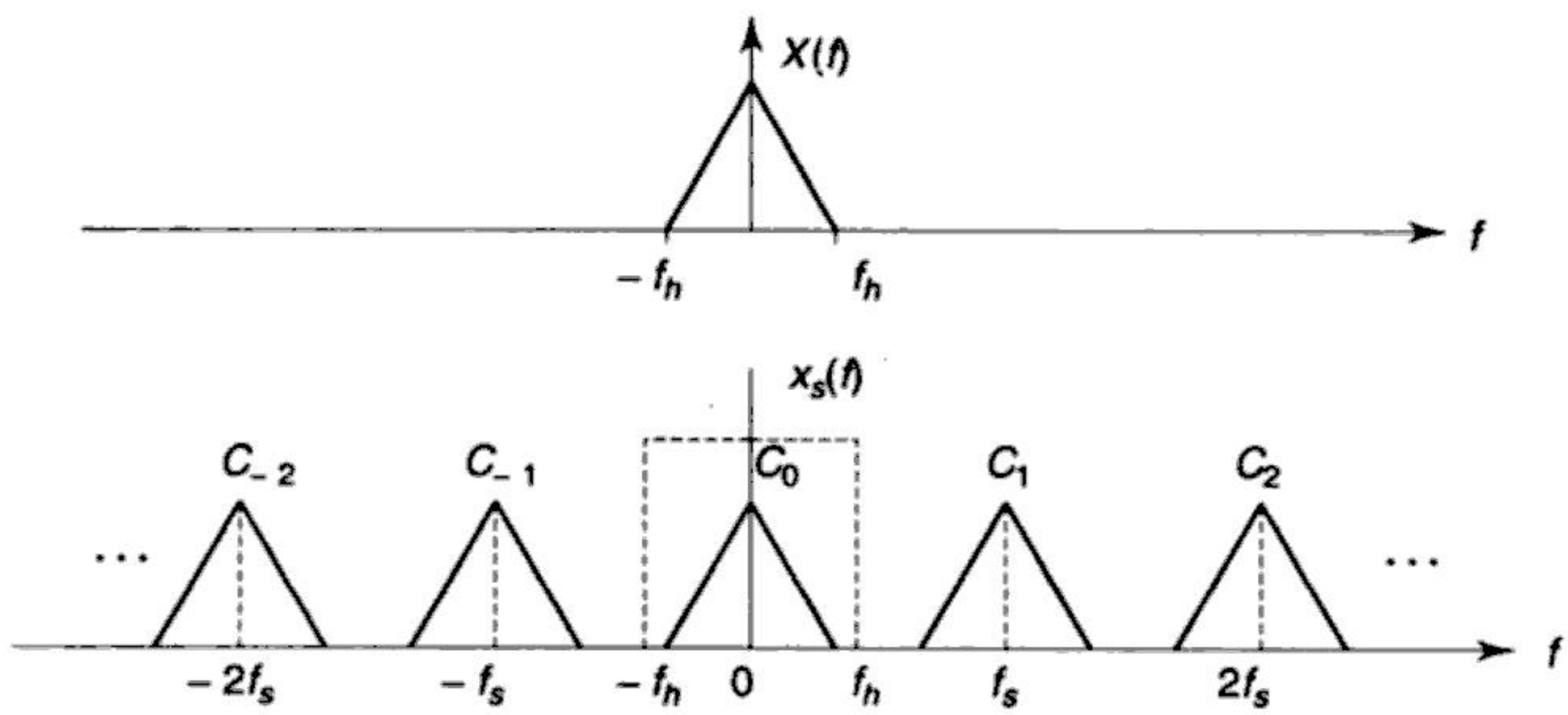

Fig. 1.13 *Spectrum of Sampled Signal*

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The signal $x(t)$, in this case, is assumed to have no frequency components above f_h, i.e. in the frequency domain, $X(f)$ is zero for $|f| \geq f_h$. Such a signal is said to be ***bandlimited***. From Fig.1.13, it is clear that in order to recover $X(f)$ from $X_s(f)$, we must have

$$f_s - f_h \geq f_h$$

or equivalently,

$$f_s \geq 2f_h, \text{ hertz} \tag{1.48}$$

That is, in order to recover the original signal from the samples, the sampling frequency must be greater than or equal to twice the maximum frequency in $x(t)$. *The sampling theorem is thus derived, which states that a bandlimited signal $x(t)$ having no frequency components above f_h hertz, is completely specified by samples that are taken at a uniform rate greater than $2f_h$ hertz.* The frequency equal to twice the highest frequency in $x(t)$, i.e. $2f_h$, is called the *Nyquist rate*.

Sampling by Impulse Function

The sampling function $g(t)$, discussed above, was periodic. The pulse width of the sampling function must be very small compared to the period, T. The samples in digital systems are in the form of a number, and the magnitude of these numbers represent the value of the signal $x(t)$ at the sampling instants. In this case, the pulse width of the sampling function is infinitely small and an infinite train of impulse functions of period T can be considered for the sampling function. That is,

$$g(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{1.49}$$

The sampling function as given in Eq.1.49 is shown in Fig.1.14. When this sampling function is used, the weight of the impulse carries the sample value.

The sampling function $g(t)$ is periodic and can be represented by a Fourier series as in Eq.1.41, which is repeated here.

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_s t}$$

where

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jn2\pi f_s t} dt \tag{1.50}$$

Since $\delta(t)$ has its maximum energy concentrated at $t = 0$, a more formal mathematical definition of the unit-impulse function may be defined as a functional

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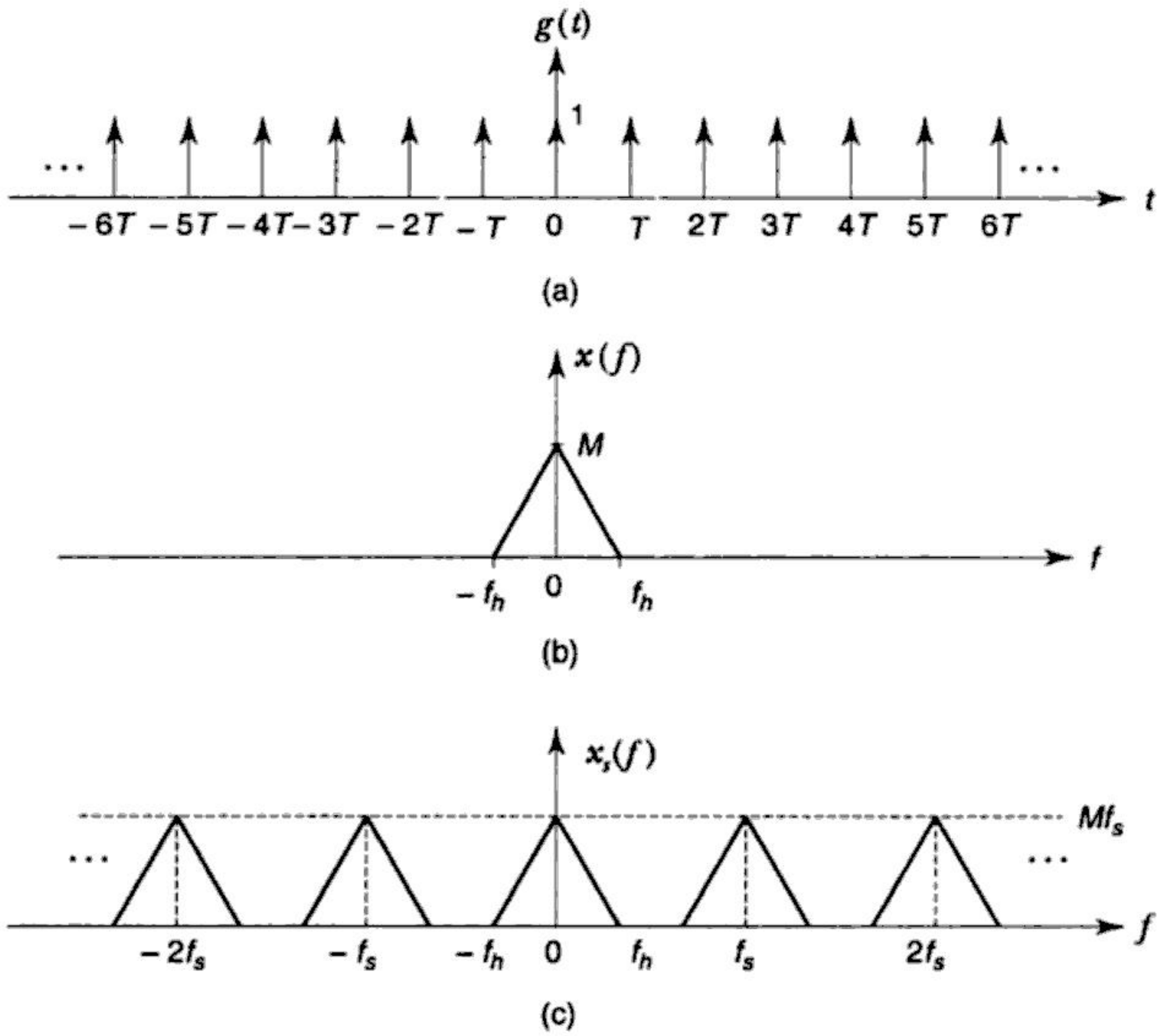

Fig. 1.14 *(a) Impulse Sampling Function (b) Spectrum of the Signal x(t) (c) Spectrum of Impulse Sampled Signal*

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0) \tag{1.51}$$

where $x(t)$ is continuous at $t = 0$. Using Eq. 1.51 in Eq. 1.50, we have

$$C_n = \frac{1}{T}e^0 = \frac{1}{T} = f_s \tag{1.52}$$

Thus C_n is same as the sampling frequency f_s, for all n. The spectrum of the impulse sampled signal, $x_s(t)$ is given by

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \tag{1.53}$$

The spectra of the signal $x(t)$ and the impulse sampled signal $X_s(t)$ are shown in Figs 1.14 (b) and (c). The effect of impulse sampling is same as sampling with a train of pulses. However, all the frequency translated spectra have the same amplitude. The original signal $X(f)$ can be reconstructed from $X_s(f)$ using a low-pass filter. Figure 1.15 shows the effect of sampling at a rate lower than the Nyquist rate.

Consider a bandlimited signal $x(t)$, with f_h as its highest frequency content, being sampled at a rate lower than the Nyquist rate, i.e., sampling frequency $f_s < 2f_h$. This results in overlapping of adjacent

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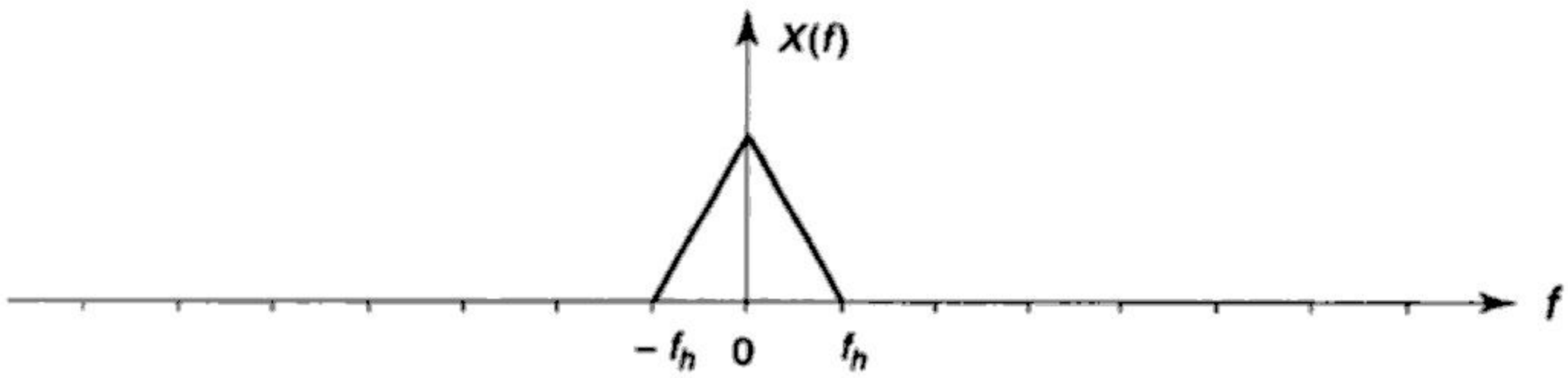
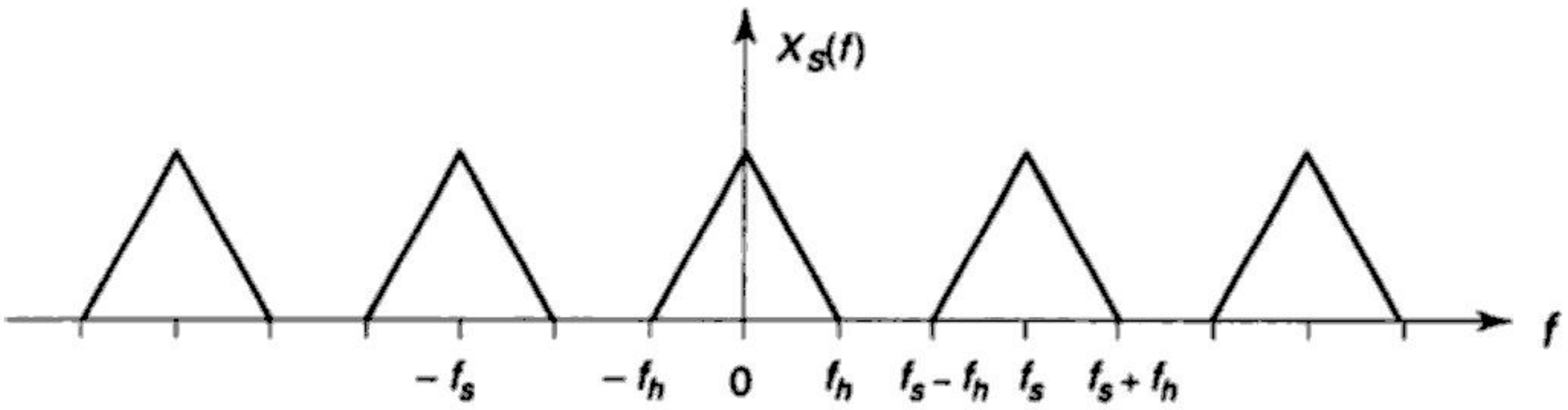
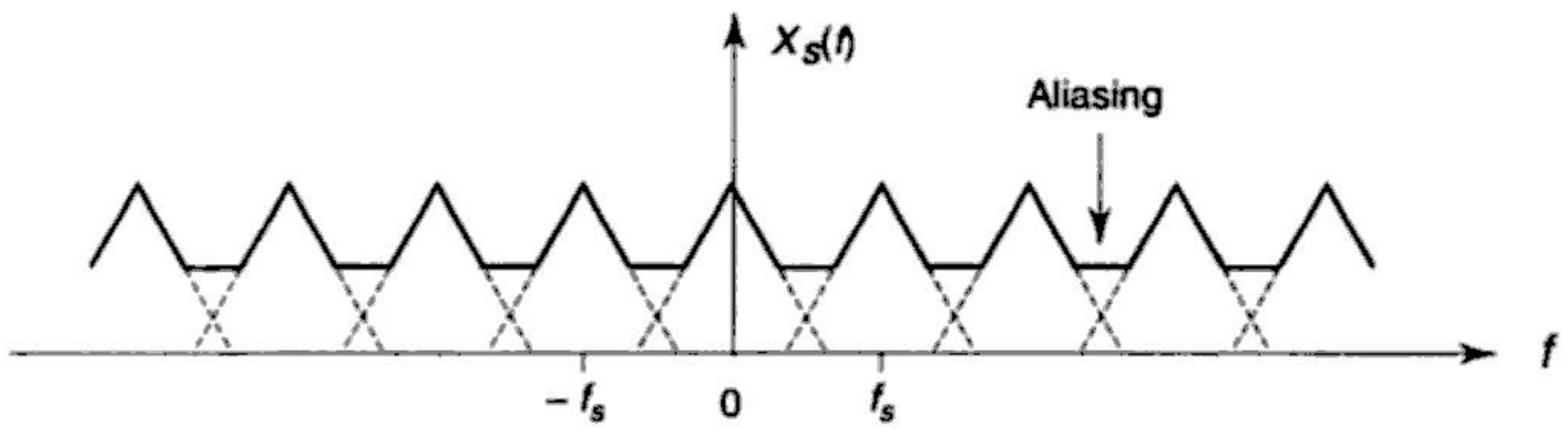


(a) Spectrum of the input signal



(b) Spectrum of the sampled signal's for $f_s > 2f_h$



(c) Sampled signal's spectrum for $f_s < 2f_h$

Fig. 1.15 *Illustration of Aliasing*

spectra i.e., higher frequency components of $X_s(f)$ get superimposed on lower frequency components as shown in Fig.1.15. Here, faithful reconstruction or recovery of the original continuous time signal from its sampled discrete-time equivalent by filtering is very difficult because portions of $X(f - f_s)$ and $X(f + f_s)$ overlap $X(f)$, and thus add to $X(f)$ in producing $X_s(f)$. The original shape of the signal is lost due to undersampling, i.e. down-sampling. This overlap is known as *aliasing* or *overlapping* or *fold over*. Aliasing, as the name implies, means that a signal can be impersonated by another signal. In practice, no signal is strictly bandlimited but there will be some frequency beyond which the energy is very small and negligible. This frequency is generally taken as the highest frequency content of the signal.

To prevent aliasing, the sampling frequency f_s should be greater than two times the frequency f_h of the sinusoidal signal being sampled. The condition to be satisfied by the sampling frequency to prevent aliasing is called the *sampling theorem*. In some applications, an analog anti-aliasing filter is placed before sample/hold circuit in order to prevent the aliasing effect.

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A useful application of aliasing due to undersampling arises in the *sampling oscilloscope*, which is meant for observing very high frequency waveforms.

1.8.2 Signal Reconstruction

Any signal $x(t)$ can be faithfully reconstructed from its samples if these samples are taken at a rate greater than or equal to the Nyquist rate. It can be seen from the spectrum of the sampled signal, $X_s(t)$ that it consists of the spectra of the signal and its frequency translated harmonics. Thus, if the spectrum of the signal alone can be separated from that of the harmonics then the original signal can be obtained. This can be achieved by filtering the sampled signal using a low-pass filter with a bandwidth greater than f_h and less than $f_s - f_h$. hertz.

If the sampling function is an impulse sequence, we note from Eq.1.53 that the spectrum of the sampled signal has an amplitude equal to $f_s = 1/T$. Therefore, in order to remove this scaling constant, the low-pass filter must have an amplitude response of $1/f_s = T$. Assuming that sampling has been done at the Nyquist rate, i.e. $f_s = 2f_h$, the bandwidth of the low-pass filter will be $f_h = \frac{f_s}{2}$. Therefore, the unit impulse response of an ideal filter for this bandwidth is

$$h(t) = T \int_{-f_s/2}^{f_s/2} e^{j2\pi ft}\, df \tag{1.54}$$

That is

$$h(t) = \frac{T}{j2\pi t}\left(e^{j\pi f_s t} - e^{-j\pi f_s t}\right)$$

The above expression can be alternatively written as

$$h(t) = Tf_s \frac{\sin \pi f_s t}{\pi f_s t} = \text{sinc}\, f_s\, t \tag{1.55}$$

The ideal reconstruction filter is shown in Fig.1.16a. The input to this filter is the sampled signal $x(nT)$ and the output of the filter is the reconstructed signal $x(t)$. The output signal $x(t)$ is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)$$

Using Eq.1.55, we get

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \text{sinc}\, f_s(t - nT) \tag{1.56}$$

The above expression is a ***convolution*** expression and the signal $x(t)$ is reconstructed by convoluting its samples with the unit-impulse response of the filter. Eq. 1.56 can also be interpreted as follows. The

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original signal can be reconstructed by weighting each sample by a sinc function and adding them all. This process is shown in Fig. 1.16b.

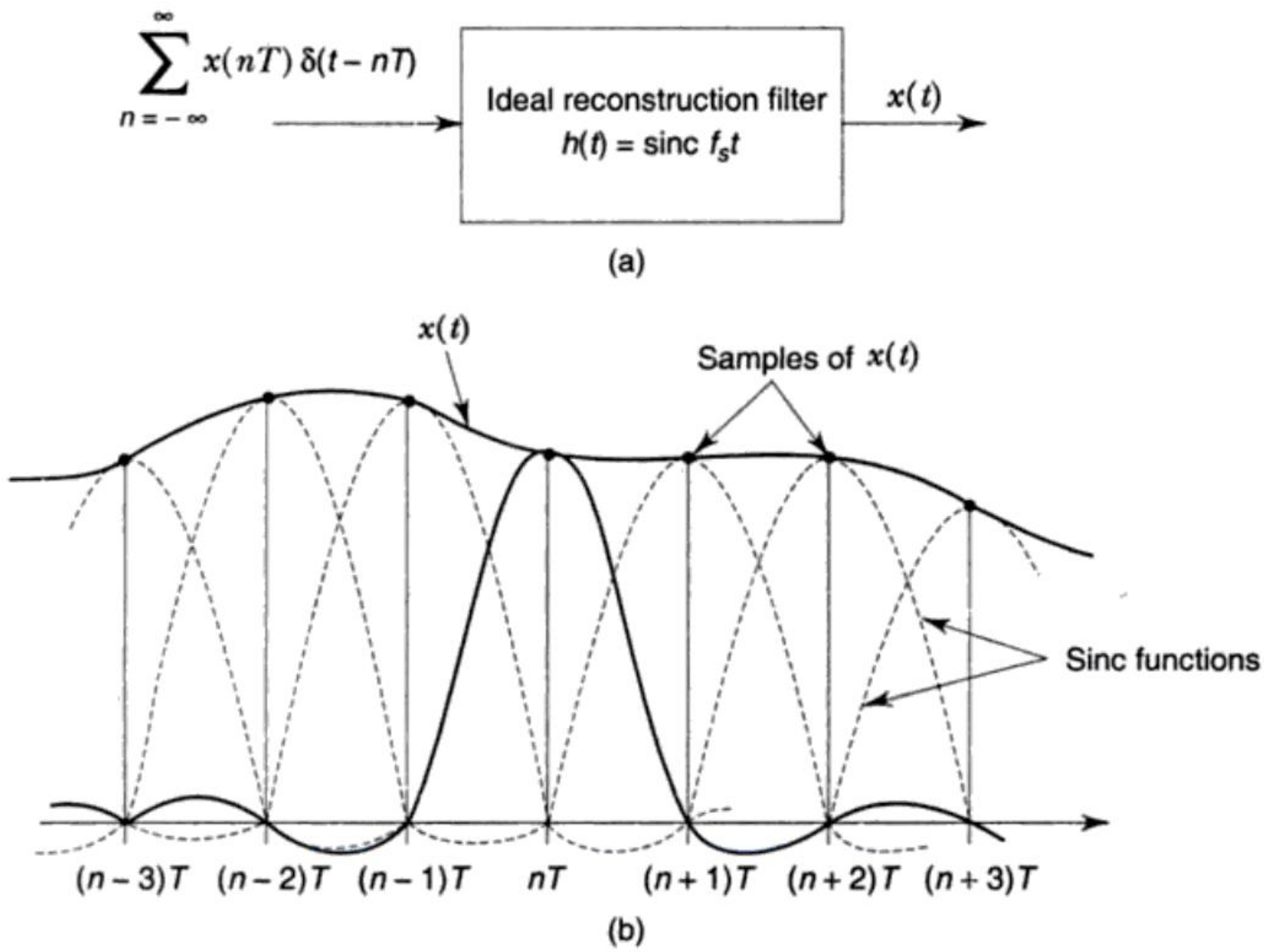

Fig. 1.16 *Signal Reconstruction (a) Reconstruction Filter (b) Time Domain Representation*

1.8.3 Signal Quantisation and Encoding

A discrete-time signal with continuous-valued amplitudes is called a *sampled data signal*, whereas a continuous-time signal with discrete-valued amplitudes is referred to as a quantised boxcar signal. Quantisation is a process by which the amplitude of each sample of a signal is rounded off to the nearest permissible level. That is, quantisation is conversion of a discrete-time continuous-amplitude signal into a discrete-time, discrete-valued signal. Then encoding is done by representing each of these permissible levels by a digital word of fixed wordlength.

The process of quantisation introduces an error called *quantisation error* and it is simply the difference between the value of the analog input and the analog equivalent of the digital representation. This error will be small if there are more permissible levels and the width of these quantization levels is very small. In the analog-to-digital conversion process, the only source of error is the quantiser. Even if there are more quantisation levels, error can occur if the signal is at its maximum or minimum value for significant time intervals. Figure 1.17 shows how a continuous-time signal is quantised in a quantiser that has 16 quantising levels.

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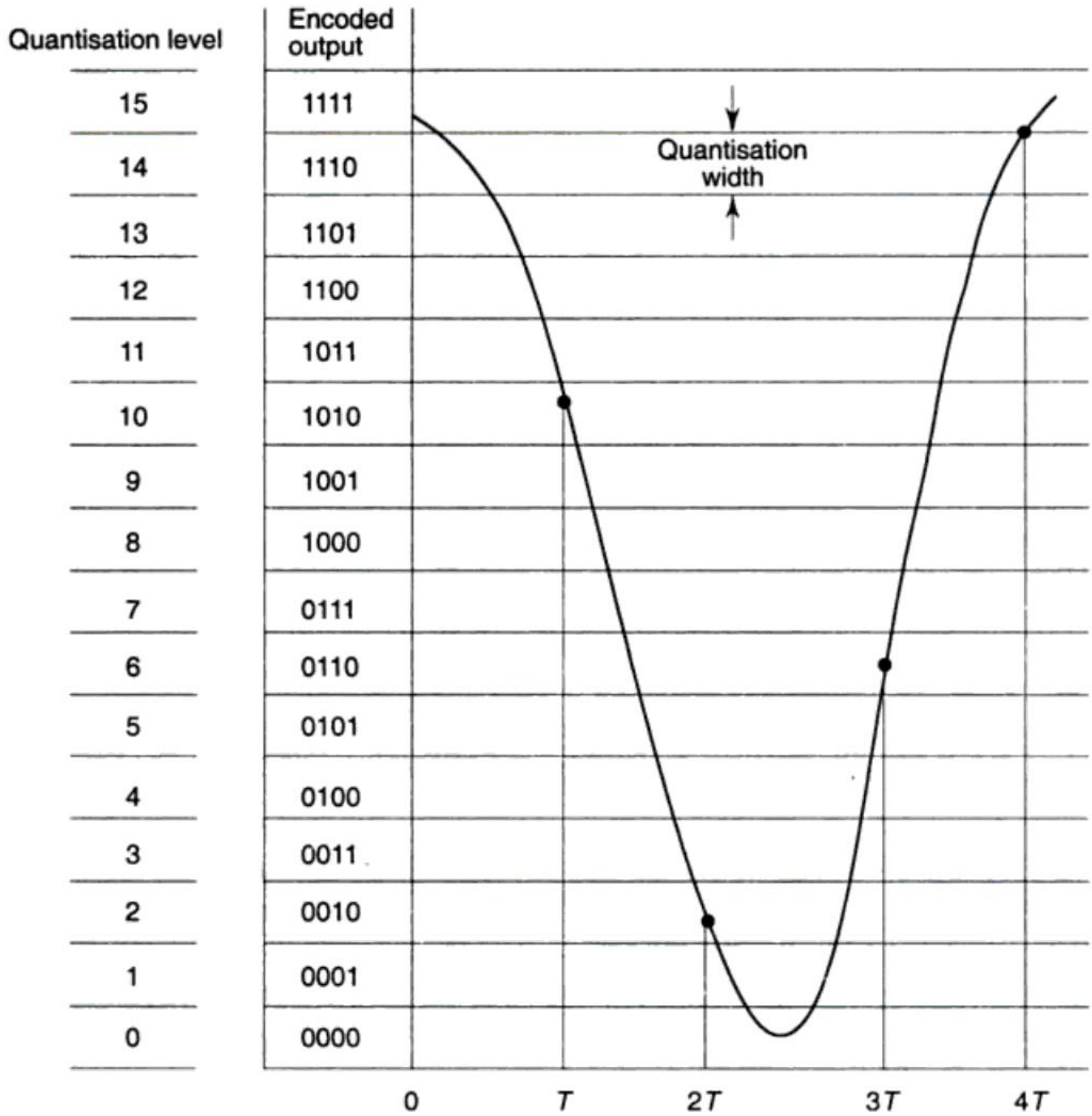

Fig. 1.17 *Quantizing and Encoding*

REVIEW QUESTIONS

1.1 What are the major classifications of signals?

1.2 With suitable examples distinguish a deterministic signal from a random signal.

1.3 What are periodic signals? Give examples.

1.4 Describe the procedure used to determine whether the sum of two periodic signals is periodic or not.

1.5 Determine which of the following signals are periodic and determine the fundamental period.

(a) $x_1(t) = 10 \sin 25\pi t$ (b) $x_2(t) = 10 \sin \sqrt{5}\,\pi t$

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(c) $x_3(t) = \cos 10\pi t$ (d) $x_4(t) = x_1(t) + x_2(t)$

(e) $x_5(t) = x_1(t) + x_3(t)$ (f) $x_4(t) = x_2(t) + x_3(t)$

1.6 *What are even signals? Give examples.*

1.7 *What are odd signals? Give examples.*

1.8 *What is energy signal?*

1.9 *What is power signal?*

1.10 *What are singularity functions?*

1.11 *Define unit-impulse function?*

1.12 *What is unit-step function? How it can be obtained from an unit-impulse function?*

1.13 *What is unit-ramp function? How it can be obtained from an unit-impulse function?*

1.14 *What is pulse function?*

1.15 *Evaluate*

(a) $\int_{-\infty}^{\infty} e^{-at^2}\,\delta(t-10)\,dt$ (b) $\int_{-\infty}^{\infty} e^{-2t^2}\,\delta(t+5)\,dt$

(c) $\int_{-\infty}^{\infty} 40\,e^{-2t^2}\,\delta(t-10)\,dt$ and (d) $\int_{-\infty}^{\infty} e^{-2t^2}\,\delta(t-10)\,dt$

Ans (a) e^{-100a} (b) 0 (c) $40\,e^{-200}$ (d) e^{-200}

1.16 *Explain the terms single-sided spectrum and double-sided spectrum with respect to a signal.*

1.17 *Sketch the single-sided and double-sided frequency spectra of the signals*

(a) $x_1(t) = 10 \sin\left(10\pi t - \frac{2\pi}{3}\right), -\infty < t < \infty$

(b) $x_2(t) = 25 \cos\left(5\pi t - \frac{\pi}{2}\right), -\infty < t < \infty$

(c) $x_3(t) = 100 \sin\left(10\pi t - \frac{2\pi}{3}\right) + 50 \cos\left(25\pi t - \frac{\pi}{3}\right), -\infty < t < \infty$

1.18 *How are systems classified?*

1.19 *Distinguish static systems from dynamic systems.*

1.20 *What is linear system ?*

1.21 *Determine whether the following systems are linear*

(a) $\frac{dy(t)}{dt} + 5y(t) + 2 = x(t)$ (b) $5\frac{dy(t)}{dt} + y(t) = 5x(t)$

(c) $\frac{dy(t)}{dt} + y(t) + 5 = 10x(t)$

1.22 *What is LTI system?*

1.23 *What is a causal system? Why are non-causal systems unrealisable?*

1.24 *What is BIBO stability?*

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1.25 What are the conditions for BIBO stability?
1.26 With illustrations, explain shifting, folding and time scaling operations on discrete-time signals.
1.27 What are the different ways of representing a system?
1.28 Explain how difference/differential equations are used to model a system.
1.29 Explain how impulse response can model a system.
1.30 Discuss the state-variable modelling of a system.
1.31 Explain the terms. (i) state variable (ii) state space (iii) state vector (iv) trajectory (v) state equations and (vi) output equations.
1.32 With a block diagram explain the process of analog-to-digital conversion.
1.33 What is meant by sampling? State the sampling theorem.
1.34 Explain how sampling can be done with an impulse function.
1.35 Draw the spectrum of a sampled signal and explain aliasing.
1.36 Explain the process of reconstruction of the signal from its samples. Obtain the impulse response of an ideal reconstruction filter.
1.37 What is meant by quantization and encoding?
1.38 What is a quantized boxcar signal?

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Chapter 2

Fourier Analysis of Periodic and Aperiodic Continuous-Time Signals and Systems

2.1 INTRODUCTION

A signal which is repetitive is a periodic function of time. Any periodic function of time $f(t)$ can be represented by an infinite series called the ***Fourier Series***. A function of time $f(t)$ is said to be periodic of period T if $f(t) = f(t + T)$ for all t. For example, the periodic waveforms of sinusoidal and exponential forms are shown in Fig. 2.1.

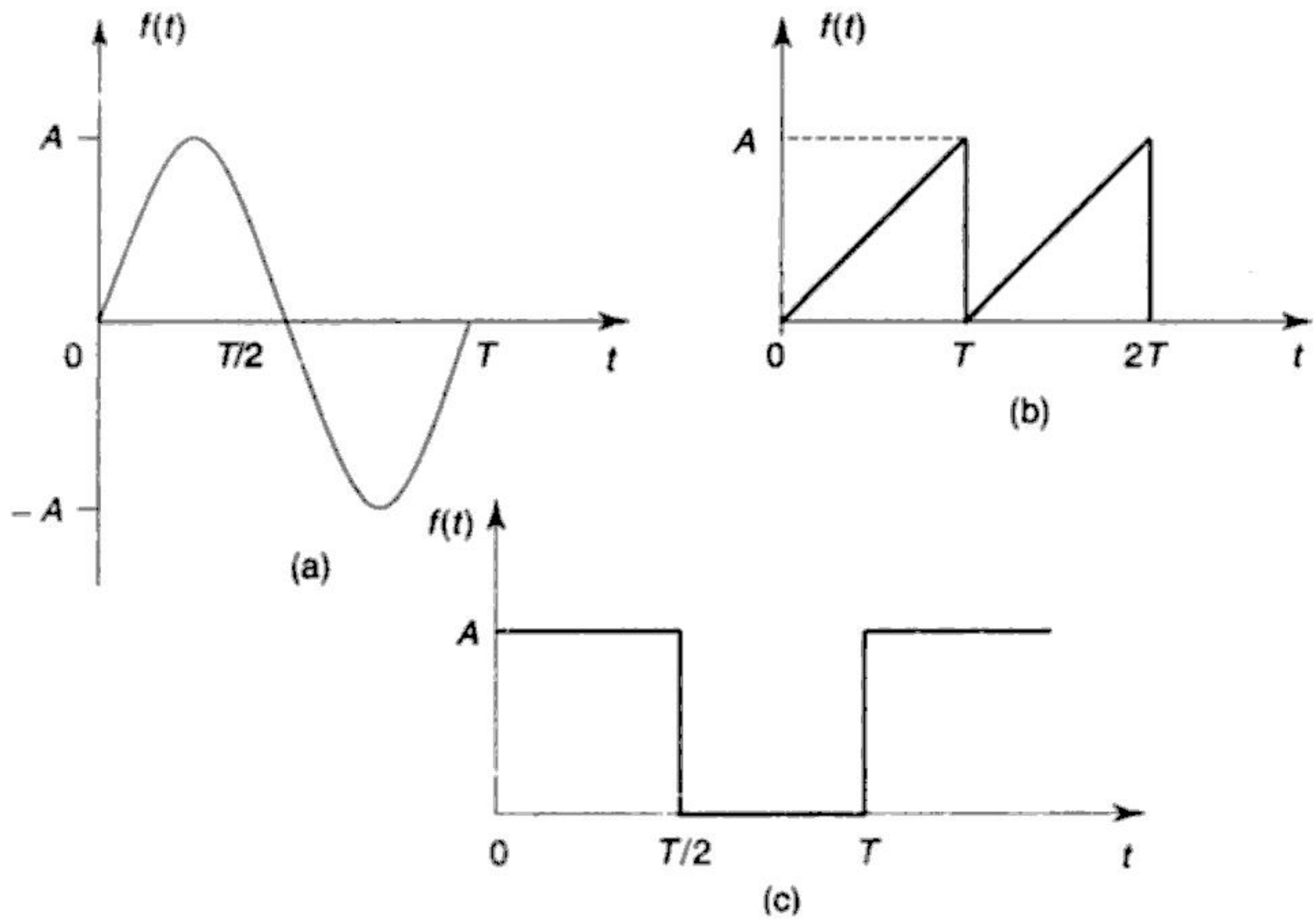

Fig. 2.1 *Waveforms Representing Periodic Functions*

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Examples of periodic processes are the vibration of a tuning fork, oscillations of a pendulum, conduction of heat, alternating current passing through a circuit, propagation of sound in a medium, etc. Fourier series may be used to represent either functions of time or functions of space co-ordinates. In a similar manner, functions of two and three variables may be represented as double and triple Fourier series respectively. Periodic waveforms may be expressed in the form of Fourier series. Non-periodic waveforms may be expressed by Fourier transforms.

2.2 TRIGONOMETRIC FOURIER SERIES

A periodic function $f(t)$ can be expressed in the form of trigonometric series as

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \ldots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \ldots \qquad (2.1)$$

where $\omega_0 = 2\pi f = \frac{2\pi}{T}$, f is the frequency and a's and b's are the coefficients. The Fourier series exists only when the function $f(t)$ satisfies the following three conditions called ***Dirichlet's conditions***.

(i) $f(t)$ is well defined and single-valued, except possibly at a finite number of points, i.e.
$f(t)$ has a finite average value over the period T.

(ii) $f(t)$ must posses only a finite number of discontinuities in the period T.

(iii) $f(t)$ must have a finite number of positive and negative maxima in the period T.

Equation 2.1 may be expressed by the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \qquad (2.2)$$

where a_n and b_n are the coefficients to be evaluated.

Integrating Eq. 2.2 for a full period, we get

$$\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}a_0 \int_{-T/2}^{T/2} dt + \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\,dt$$

Integration of cosine or sine function for a complete period is zero.

Therefore, $$\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}a_0 T$$

Hence, $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \qquad (2.3)$$

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or, equivalently $a_0 = \frac{2}{T}\int_0^T f(t)\,dt$

Multiplying both sides of Eq. 2.2 by $\cos m\omega_0 t$ and integrating, we have

$$\int_{-T/2}^{T/2} f(t)\cos m\omega_0 t\,dt = \frac{1}{2}\int_{-T/2}^{T/2} a_0 \cos m\omega_0 t\,dt +$$

$$\int_{-T/2}^{T/2}\sum_{n=1}^{\infty} a_n \cos n\omega_0 t \cos m\omega_0 t\,dt + \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} b_n \sin n\omega_0 t \cos m\omega_0 t\,dt$$

Here, $\frac{1}{2}\int_{-T/2}^{T/2} a_0 \cos m\omega_0 t\,dt = 0$

$$\int_{-T/2}^{T/2} a_n \cos n\omega_0 t \cos m\omega_0\,dt = \frac{a_n}{2}\int_{-T/2}^{T/2}[\cos(m+n)\omega_0 t + \cos(m-n)\omega_0 t]\,dt$$

$$= \begin{cases} 0, & \text{for } m \neq n \\ \frac{T}{2}a_n, & \text{for } m = n \end{cases}$$

$$\int_{-T/2}^{T/2} b_n \sin n\omega_0 t \cos m\omega_0 t\,dt = \frac{bn}{2}\int_{-T/2}^{T/2}[\sin(m+n)\omega_0 t - \sin(m-n)\omega_0 t]\,dt$$

$$= 0$$

Therefore, $\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt = \frac{T a_n}{2}$, for $m = n$

Hence,
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt \tag{2.4}$$

or, equivalently $a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$

Similarly, multiplying both sides of Eq. 2.2 by $\sin m\omega_0 t$ and integrating, we get

$$\int_{-T/2}^{T/2} f(t)\sin m\omega_0 t\,dt = \frac{1}{2}\int_{-T/2}^{T/2} a_0 \sin m\omega_0 t\,dt$$

$$+ \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} a_n \cos n\omega_0 t \sin m\omega_0 t\,dt + \int_{-T/2}^{T/2}\sum_{n=1}^{\infty} b_n \sin n\omega_0 t \sin m\omega_0 t\,dt$$

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Here, $\frac{1}{2}\int_{-T/2}^{T/2} a_0 \sin m\omega_0 t \, dt = 0$

$$\int_{-T/2}^{T/2} a_n \cos n\omega_0 t \sin m\omega_0 t \, dt = 0$$

$$\int_{-T/2}^{T/2} b_n \sin n\omega_0 t \sin m\omega_0 t \, dt = \begin{cases} 0, & \text{for } m \neq n \\ \frac{T}{2} b_n, & \text{for } m = n \end{cases}$$

Therefore, $\int_{-T/2}^{T/2} f(t) \sin m\omega_0 t \, dt = \frac{T}{2} b_n$, for $m = n$

Hence, $$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt \tag{2.5}$$

or, equivalently, $b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega_0 t \, dt$

The number $n = 1, 2, 3, \ldots$ gives the values of the harmonic frequencies.

Symmetry Conditions

(i) If the function $f(t)$ is even, then $f(-t) = f(t)$. For example, $\cos t$, t^2, $t \sin t$, are all even. The cosine is an even function, since it may be expressed as the power series

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

The waveforms representing the even functions of t are shown in Fig. 2.2. Geometrically, the graph of an even function will be symmetrical with respect to the y-axis and only cosine terms are present (d.c. term optional). When $f(t)$ is even,

$$\int_{-a}^{a} f(t) \, dt = 2\int_0^a f(t) \, dt$$

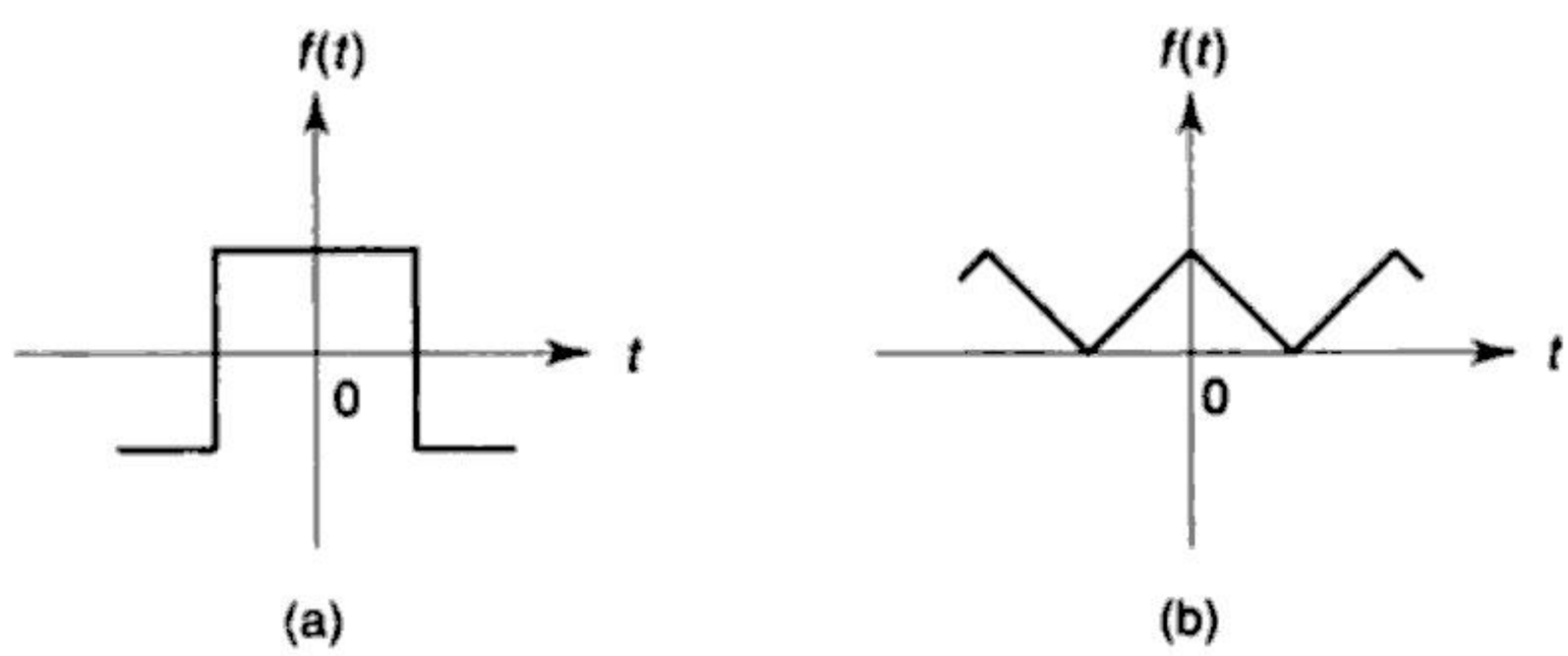

Fig. 2.2 *(Contd.)*

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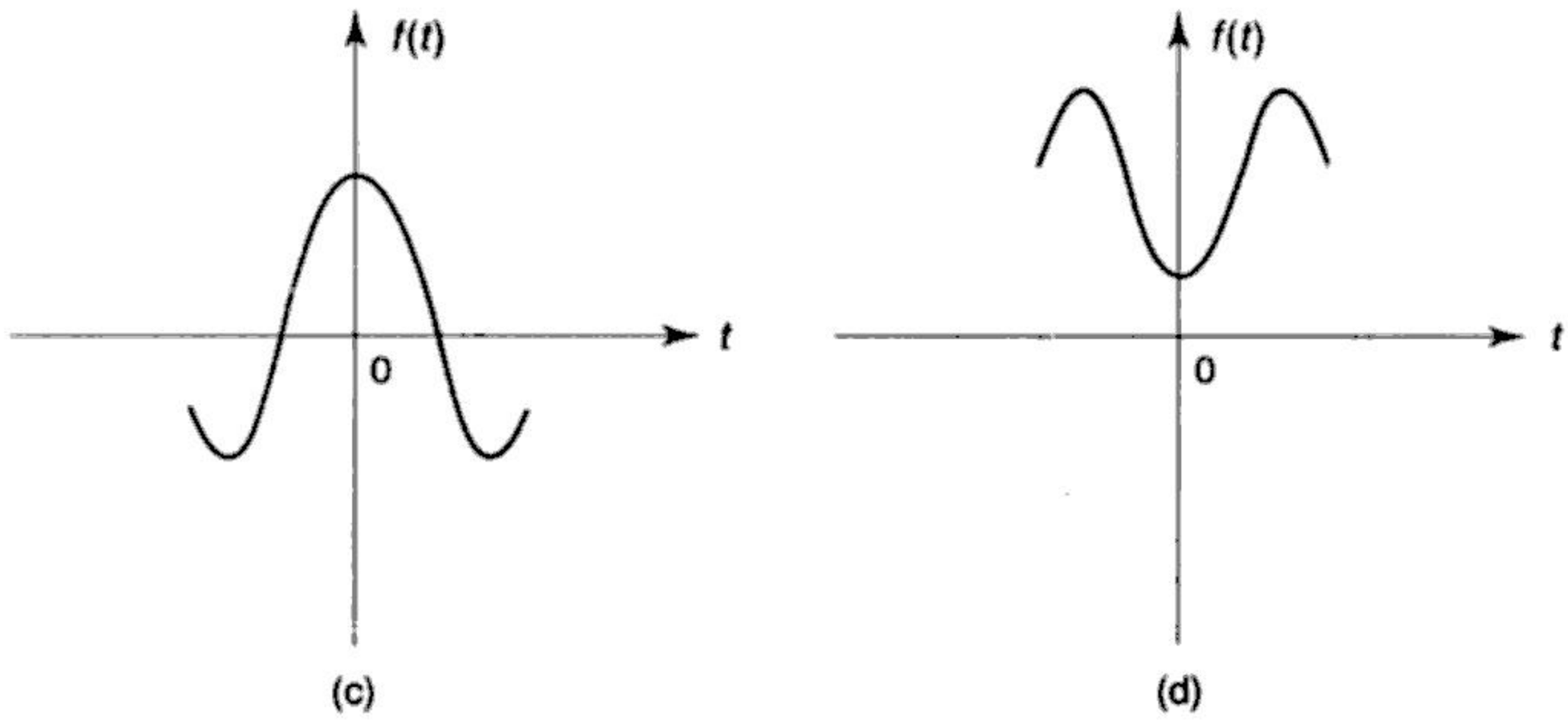

Fig. 2.2 *Waveforms Representing Even Functions*

The sum or product of two or more even functions is an even function.

(ii) If the function $f(t)$ is odd, then $f(-t) = -f(t)$ and only sine terms are present (d.c. term optional). For example, $\sin t$, t^3, $t \cos t$ are all odd. The waveforms shown in Fig. 2.3 represent odd functions of t. The graph of an odd function is symmetrical about the origin. If $f(t)$ is odd, $\int_{-a}^{a} f(t)\, dt = 0$. The sum of two or more odd functions is an odd function and the product of two odd functions is an even function.

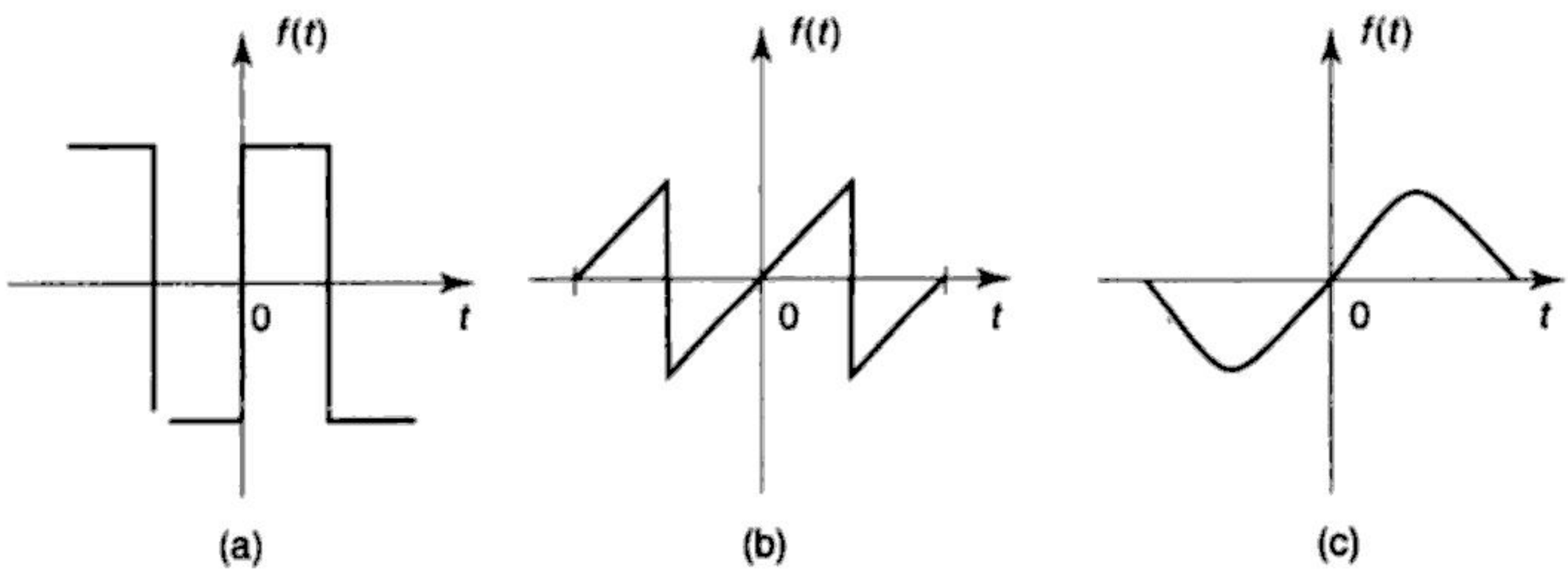

Fig. 2.3 *Waveforms Representing Odd Functions*

(iii) If $f(t + T/2) = f(t)$, only even harmonics are present.

(iv) If $f(t + T/2) = -f(t)$, only odd harmonics are present and hence the waveform has half-wave symmetry.

Example 2.1 Obtain the Fourier components of the periodic square wave signal which is symmetrical with respect to the vertical axis at time $t = 0$, as shown in Fig. E2.1.

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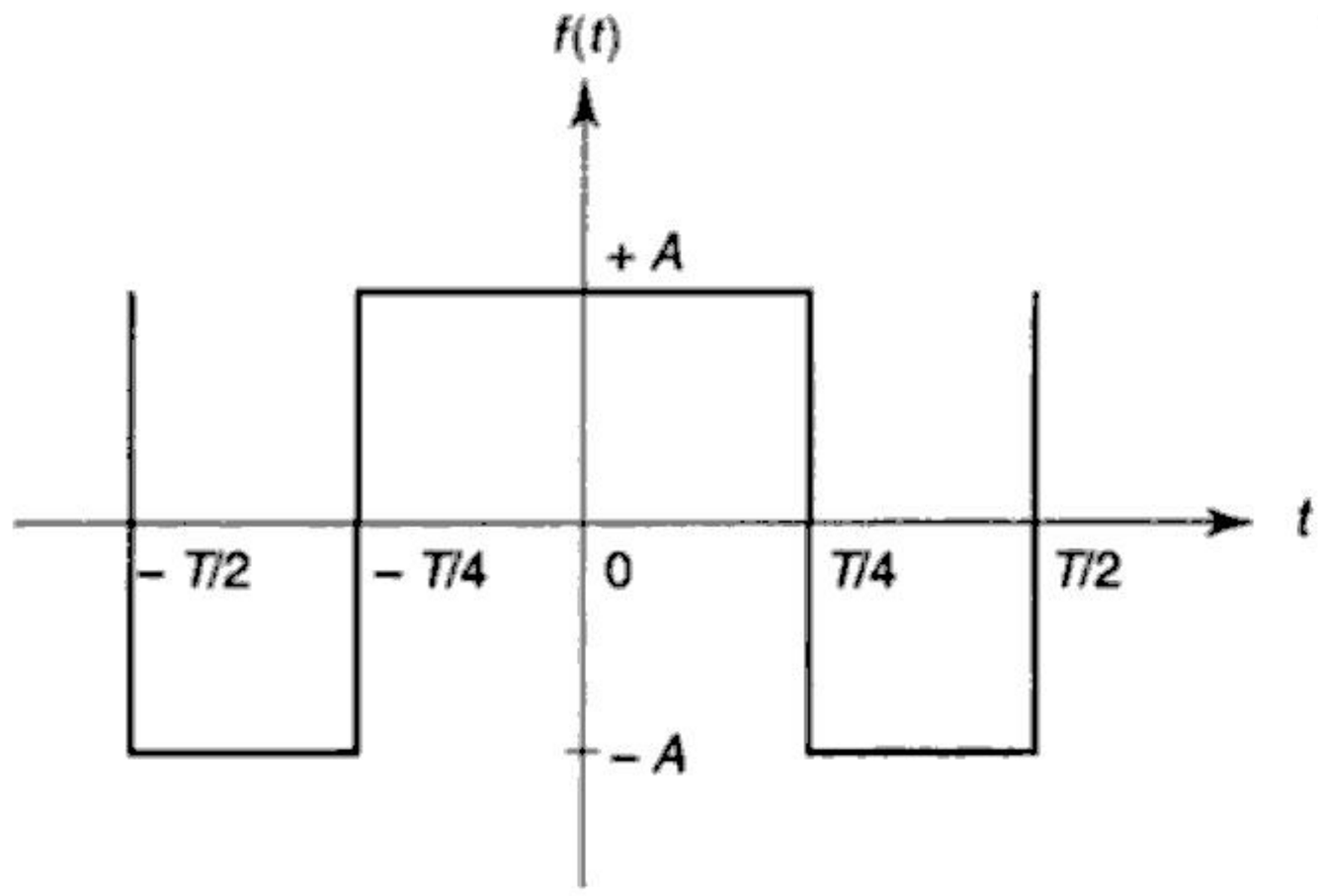

Fig. E2.1

Solution Since the given waveform is symmetrical about the horizontal axis, the average area is zero and hence the d.c. term $a_0 = 0$. In addition, $f(t) = f(-t)$ and so only cosine terms are present, i.e., $b_n = 0$.

Now, $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0 t\,\mathrm{d}t$$

where $$f(t) = \begin{cases} -A, & \text{from} \quad -T/2 < t < -T/4 \\ +A, & \text{from} \quad -T/4 < t < +T/4 \\ -A, & \text{from} \quad +T/4 < t < +T/2 \end{cases}$$

Therefore ,

$$a_n = \frac{2A}{T}\left[\int_{-T/2}^{-T/4}(-\cos n\omega_0 t)\mathrm{d}t + \int_{-T/4}^{T/4}\cos n\omega_0 t\,\mathrm{d}t + \int_{T/4}^{T/2}(-\cos n\,\omega_0\,t)\,\mathrm{d}t\right]$$

$$= \frac{2A}{T}\left[\left(\frac{-\sin n\omega_0 t}{n\,\omega_0}\right)_{-T/2}^{-T/4} + \left(\frac{\sin n\,\omega_0 t}{n\,\omega_0}\right)_{-T/4}^{T/4} + \left(\frac{-\sin n\,\omega_0 t}{n\,\omega_0}\right)_{T/4}^{T/2}\right]$$

$$= \frac{2A}{n\,\omega_0 T}\left[-\sin\left(\frac{-n\,\omega_0 T}{4}\right) + \sin\left(\frac{-n\,\omega_0 T}{2}\right) + \sin\left(\frac{n\,\omega_0 T}{4}\right)\right.$$

$$\left. -\sin\left(\frac{-n\,\omega_0 T}{4}\right) - \sin\left(\frac{n\,\omega_0 T}{2}\right) + \sin\left(\frac{n\,\omega_0 T}{4}\right)\right]$$

$$= \frac{8A}{n\,\omega_0 T}\sin\left(\frac{n\,\omega_0 T}{4}\right) - \frac{4A}{n\,\omega_0 T}\sin\left(\frac{n\,\omega_0 T}{2}\right)$$

When $\omega_0\,T = 2\pi$, the second term is zero for all integer values of n. Hence,

$$a_n = \frac{8A}{2n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_0 = 0 \text{ (d.c. term)}$$

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$$a_1 = \frac{4A}{\pi}\sin\left(\frac{\pi}{2}\right) = \frac{4A}{\pi}$$

$$a_2 = \frac{4A}{\pi}\sin(\pi) = 0$$

$$a_3 = \frac{4A}{3\pi}\sin\left(\frac{3\pi}{2}\right) = -\frac{4A}{3\pi}$$

. . . . . .

. . . . . .

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{4A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \cdots\right]$$

Example 2.2 Obtain the Fourier Components of the periodic rectangular waveform shown in Fig. E2.2.

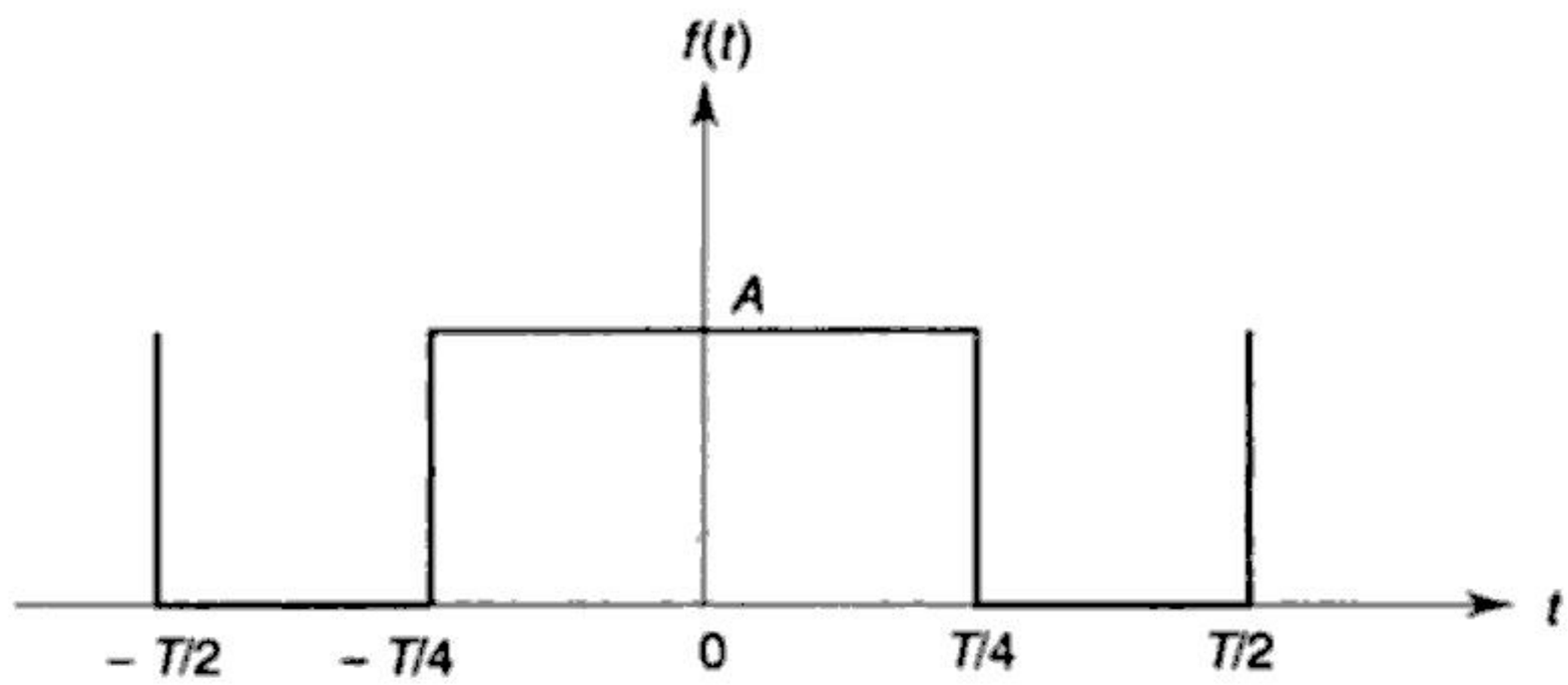

Fig. E2.2

Solution The given waveform for one period can be written as

$$f(t) = \begin{cases} 0, & \text{for} \quad -T/2 < t < -T/4 \\ A, & \text{for} \quad -T/4 < t < T/4 \\ 0, & \text{for} \quad T/4 < t < T/2 \end{cases}$$

For the given waveform, $f(-t) = f(t)$ and hence it is an even function and has $b_n = 0$.

The value of the d.c.term is

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\,dt = \frac{2A}{T} \times \frac{T}{2} = A$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt$$

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$$= \frac{2}{T}\int_{-T/4}^{T/4} A\cos n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2A}{T}\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-T/4}^{T/4}$$

$$= \frac{4A}{n\omega_0 T}\sin(n\omega_0 T/4)$$

When $\omega_0 T = 2\pi$, we have

$$a_n = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= 0, \quad \text{for } n = 2, 4, 6, \ldots$$

$$= \frac{2A}{n\pi}, \quad \text{for } n = 1, 5, 9, 13, \ldots$$

$$= -\frac{2A}{n\pi}, \quad \text{for } n = 3, 7, 11, 15, \ldots$$

Substituting the values of the coefficients in Eq. 2.2, we obtain

$$f(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right)$$

Example 2.3 **Obtain the trigonometric Fourier series for the half-wave rectified sine wave shown in Fig. E2.3.**

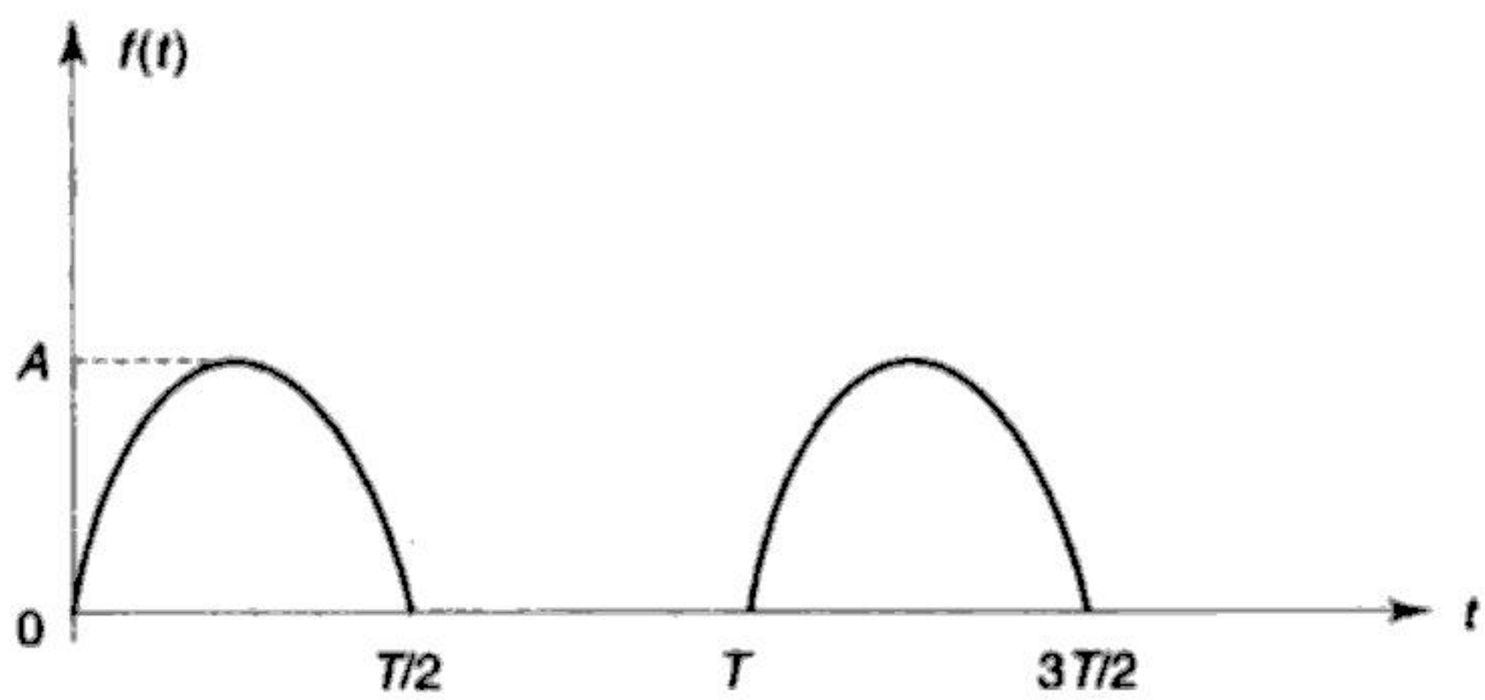

Fig. E2.3

Solution As the waveform shows no symmetry, the series may contain both sine and cosine terms. Here, $f(t) = A\sin\omega_0 t$

To evaluate a_0:

$$a_0 = \frac{2}{T}\int_0^T A\sin\omega_0 t\,\mathrm{d}t$$

$$= \frac{2}{T}\int_0^{T/2} A\sin\omega_0 t\,\mathrm{d}t$$

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$$= \frac{2A}{\omega_0 T}[-\cos \omega_0 t]_0^{T/2} = \frac{2A}{\omega_0 T}[-\cos(\omega_0 T/2) + 1]$$

Substituting $\omega_0 T = 2\pi$, we have $a_0 = \dfrac{2A}{\pi}$.

To evaluate a_n:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2}{T}\int_0^{T/2} A\sin\omega_0 t\cos n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2A}{\omega_0 T}\left[\frac{-n\sin\omega_0 t\sin n\omega_0 t - \cos n\omega_0 t\cos\omega_0 t}{-n^2+1}\right]_0^{T/2}$$

Substituting $\omega_0 T = 2\pi$, we have

$$a_n = \frac{A}{\pi(1-n^2)}[\cos n\pi + 1]$$

Hence, $\quad a_n = \dfrac{2A}{\pi(1-n^2)}$, for n even

$= 0$, for n odd

For $n = 1$, this expression is infinite and hence we have to integrate separately to evaluate a_1.

Therefore, $\quad a_1 = \dfrac{2}{T}\displaystyle\int_0^{T/2} A\sin\omega_0 t\cos\omega_0 t\,\mathrm{d}t$

$$= \frac{A}{T}\int_0^{T/2}\sin 2\omega_0 t\,\mathrm{d}t$$

$$= \frac{A}{2\omega_0 T}[-\cos 2\omega_0 t]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $a_1 = 0$.

To find b_n:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2}{T}\int_0^{T/2} A\sin\omega_0 t\sin n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2A}{\omega_0 T}\left[\frac{n\sin\omega_0 t\cos n\omega_0 t - \sin n\omega_0 t\cos\omega_0 t}{-n^2+1}\right]_0^{T/2}$$

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When $\omega_0 T = 2\pi$, we have $b_n = 0$.
For $n = 1$, the expression is infinite and hence b_1 has to be calculated separately.

$$b_1 = \frac{2A}{T}\int_0^{T/2} \sin^2 \omega_0 t \, dt$$

$$= \frac{2A}{\omega_0 T}\left[\frac{\omega_0 t}{2} - \frac{\sin 2\omega_0 t}{4}\right]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $b_1 = \dfrac{A}{2}$.
Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{A}{\pi}\left\{1 + \frac{\pi}{2}\sin \omega_0 t - \frac{2}{3}\cos 2\omega_0 t - \frac{2}{15}\cos 4\omega_0 t - \cdots\right\}$$

Example 2.4 Obtain the trigonometric Fourier series of the triangular waveform shown in Fig. E2.4.

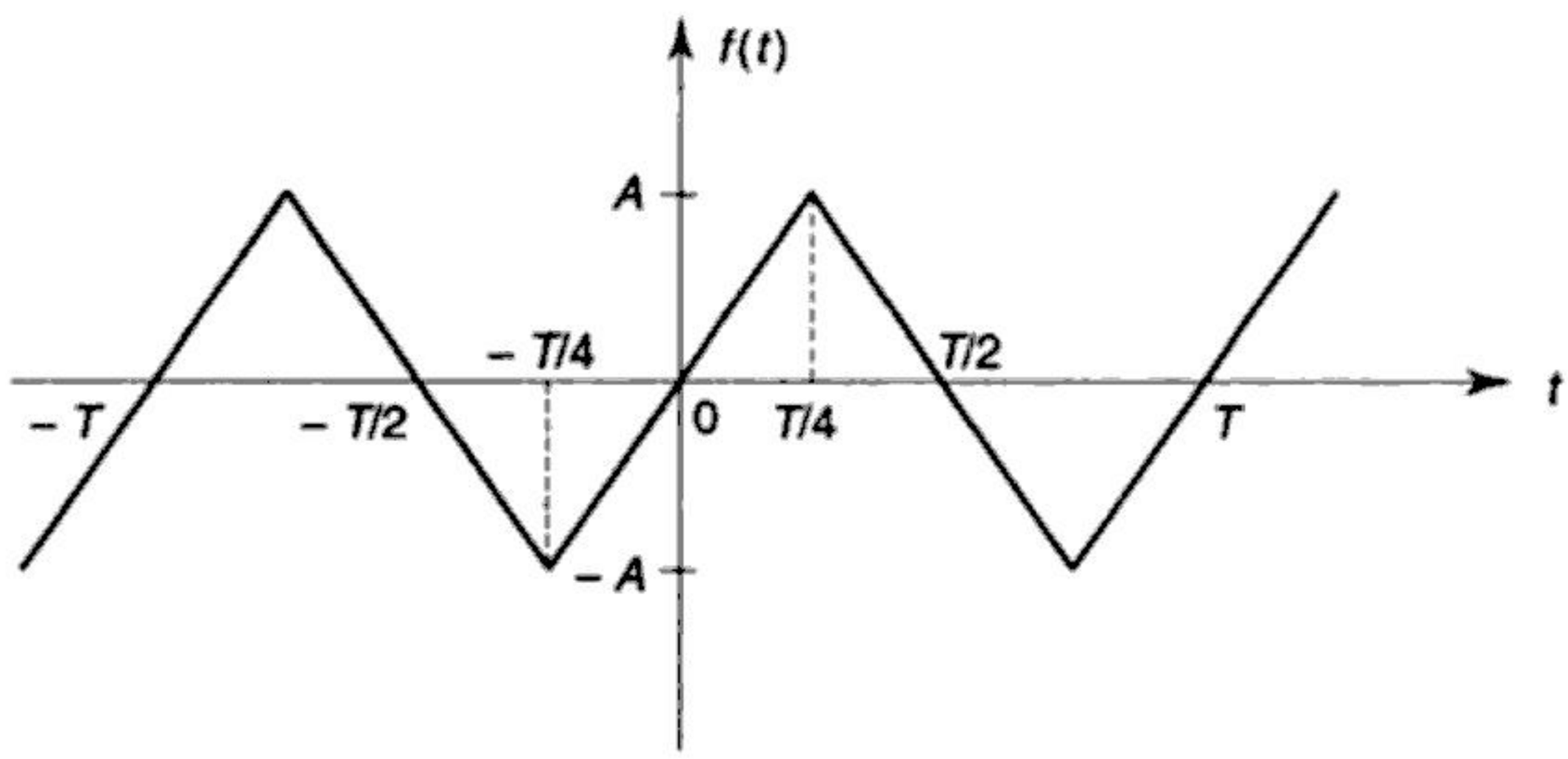

Fig. E2.4

Solution

(i) As the waveform has equal positive and negative area in one cycle, the average value of $a_0 = 0$.

(ii) As $f(t) = -f(t)$, it is an odd function and hence $a_n = 0$

and $b_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$

(iii) Here, $f(t \pm T/2) = -f(t)$. Hence it has half-wave odd symmetry and $a_n = b_n = 0$ for n even.

(iv) To find $f(t)$ for the given waveform

The equation of a straight line is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

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For the region $0 < t < \frac{T}{4}$

$$\frac{f(t)-0}{t-0} = \frac{0-A}{0-T/4}$$

Therefore, $f(t) = \frac{4A}{T}t$

For the region $\frac{T}{4} < t < \frac{T}{2}$,

$$\frac{f(t)-A}{t-\frac{T}{4}} = \frac{A-0}{\frac{T}{4}-\frac{T}{2}}$$

$$f(t) - A = -\frac{4A}{T}\left(t - \frac{T}{4}\right) = -\frac{4A}{T}t + A$$

Therefore, $f(t) = -\frac{4A}{T}t + 2A$.

Now, $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$

$$= \frac{4}{T}\int_0^{T/4}\left(\frac{4A}{T}\right)t\sin n\omega_0 t\,dt + \frac{4}{T}\int_{T/4}^{T/2}\left(-\frac{4A}{T}t+2A\right)\sin n\omega_0 t\,dt$$

$$= \frac{16A}{T^2}\int_0^{T/4} t\sin n\omega_0 t\,dt - \frac{16A}{T^2}\int_{T/4}^{T/2} t\sin n\omega_0 t\,dt + \frac{8A}{T}\int_{T/4}^{T/2}\sin n\omega_0 t\,dt$$

$$= \frac{16A}{T^2}\left[\left\{t\frac{\cos n\omega_0 t}{-n\omega_0}\right\}_0^{T/4} - \int_0^{T/4}\frac{\cos n\omega_0 t}{-n\omega_0}dt\right]$$

$$- \frac{16A}{T^2}\left[\left\{t\frac{\cos n\omega_0 t}{-n\omega_0}\right\}_{T/4}^{T/2} - \int_{T/4}^{T/2}\frac{\cos n\omega_0 t}{-n\omega_0}dt\right]$$

$$+ \frac{8A}{T}\left\{\frac{\cos n\omega_0 t}{-n\omega_0}\right\}_{T/4}^{T/2}$$

$$= \frac{16A}{T^2}\left[\frac{T}{4}\frac{\cos n\omega_0 T/4}{(-n\omega_0)} + \left\{\frac{\sin n\omega_0 t}{n^2\omega_0^2}\right\}_0^{T/4}\right]$$

$$- \frac{16A}{T^2}\left[\frac{T}{2}\frac{\cos n\omega_0 T/2}{(-n\omega_0)} + \frac{T}{4}\frac{\cos n\omega_0 T/4}{n\omega_0} + \left\{\frac{\sin n\omega_0 t}{n^2\omega_0^2}\right\}_{T/4}^{T/2}\right]$$

$$+ \frac{8A}{T}\left[\frac{\cos n\omega_0 T/2}{(-n\omega_0)} + \frac{\cos n\omega_0 T/4}{n\omega_0}\right]$$

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Substituting $\omega_0 = \dfrac{2\pi}{T}$, we have

$$b_n = \frac{16\,A}{T^2}\,\frac{2\sin n\,\pi/2}{n^2\,4\pi^2/T^2} + \frac{16\,A}{T^2}\cdot\frac{T}{2}\,\frac{\cos n\,\pi}{n.2\pi/T} - \frac{8\,A}{T}\,\frac{\cos n\,\pi}{n\,2\,\pi/T}$$

Simplifying, we get

$$b_n = \frac{8\,A}{n^2\pi^2}\sin(n\pi/2)$$

$$b_1 = \frac{8\,A}{\pi^2}\sin(\pi/2) = \frac{8\,A}{\pi^2}$$

$$b_3 = \frac{8\,A}{3^2\pi^2}\sin\frac{3\pi}{2} = -\frac{8\,A}{3^2\pi^2}$$

$$b_5 = \frac{8\,A}{5^2\pi^2}\sin\frac{5\pi}{2} = \frac{8\,A}{5^2\pi^2}$$

.....

.....

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{8\,A}{\pi^2}\left[\sin\omega_0 t - \frac{1}{3^2}\sin 3\,\omega_0 t + \frac{1}{5^2}\sin 5\,\omega_0 t + \cdots\right]$$

Example 2.5 Deduce the Fourier series for the waveform of a positive going rectangular pulse train shown in Fig. E2.5.

Fig. E2.5

Solution The periodic function of the Fourier series for the given pulse train is expressed by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0\,t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

where $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$

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$$= \frac{2}{T}\int_{-d/2}^{d/2} A\,dt = \frac{2A}{T}[t]_{-d/2}^{d/2} = \frac{2Ad}{T}$$

Here, since the choice of $t = 0$ is at the centre of a pulse, the b_n coefficients are zero.

Therefore, $a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_{-d/2}^{d/2}\cos n\omega_0 t\,dt$

$$= \frac{2A}{T}\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-d/2}^{d/2} = \frac{2A}{n\omega_0 T}\left[\sin\left(\frac{n\omega_0 d}{2}\right) - \sin\left(\frac{-n\omega_0 d}{2}\right)\right]$$

$$= \frac{4A}{n\omega_0 T}\sin\frac{n\omega_0 d}{2}$$

Hence, $f(t) = \frac{Ad}{T} + \frac{2Ad}{T}\sum_{n=1}^{\infty}\frac{\sin(n\omega_0 d/2)}{n\omega_0 d/2}\cos n\omega_0 t$

2.3 COMPLEX OR EXPONENTIAL FORM OF FOURIER SERIES

From Eq. 2.2, the trigonometric form of the Fourier series is

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t)$$

An alternative but convenient way of writing the periodic function $f(t)$ is in exponential form with complex quantities. Since

$$\cos n\,\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$\sin n\,\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

Substituting these quantities in the expression for the Fourier series gives

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + \sum_{n=1}^{\infty}b_n\left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{(a_n - jb_n)e^{jn\omega_0 t}}{2}\right) + \left(\frac{(a_n + jb_n)e^{-jn\omega_0 t}}{-jb}\right)$$

Here, taking $c_n = \frac{1}{2}(a_n - jb\,n)$

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$$c_{-n} = \frac{1}{2}(a_n + jb_n) \tag{2.6}$$

$$c_0 = a_0$$

Where c_{-n} is the complex conjugate of c_n. Substituting expressions for the coefficients a_n and b_n from Eqs 2.4 and 2.5 gives

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)[\cos n\,\omega_0 t - j \sin n\,\omega_0 t]\,\mathrm{d}t$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,\mathrm{d}t \tag{2.7}$$

and $$c_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,[\cos n\,\omega_0 t + j \sin n\,\omega_0 t]\,\mathrm{d}t$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}\,\mathrm{d}t \tag{2.8}$$

with $$f(t) = c_0 + \sum_{n=1}^{\infty} c_n\, e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n\, e^{jn\omega_0 t} \tag{2.9}$$

where the values of n are negative in the last term and are included under the Σ sign. Also, c_0 may be included under the Σ sign by using the value of $n = 0$. Therefore,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t} \tag{2.10}$$

It is clear from the result given in Eq. 2.10 that the periodic function $f(t)$ may be expressed mathematically by an infinite set of positive and negative frequency components. The negative frequencies have not only mathematical significance, but also physical significance, since a positive frequency may be associated with an anti-clockwise rotation and a negative frequency with a clockwise rotation.

The complex Fourier series furnishes a method of decomposing a signal in terms of a sum of elementary signals of the form $\{e^{jn\omega_0 t}\}$. This representation may be used for signals $f(t)$ that are

(i) Periodic, $f(t) = f(t + T)$, in which case the representation is valid on $(-\infty, \infty)$

(ii) Aperiodic, in which case the representation is valid on a finite interval (t_1, t_2). The periodic extension of $f(t)$ is obtained outside of (t_1, t_2).

Note that similar to the evaluation of integrals a_n and b_n, the limits of integration in Eq. 2.7 may be the end points of any convenient full period and not essentially 0 to T or 0 to 2π. For $f(t)$ to be real, $C_{-n} = C_n^*$, so that only positive value of n are considered in Eq. 2.7. Also, we have

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$$a_n = 2\text{Re}\,[c_n] \quad \text{and} \quad b_n = -\,2\,\text{Im}\,[c_n] \tag{2.11}$$

For an even waveform, the trigonometric Fourier series has only cosine terms and hence, by Eq. 2.6, the exponential Fourier series coefficients will be pure real numbers. Similarly, for an odd waveform, the trigonometric Fourier series contains only sine terms and hence the exponential Fourier series coefficients will be pure imaginary.

Example 2.6

(a) Find the trigonometric Fourier series of the waveform shown in Fig. E2.6 and

(b) Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.

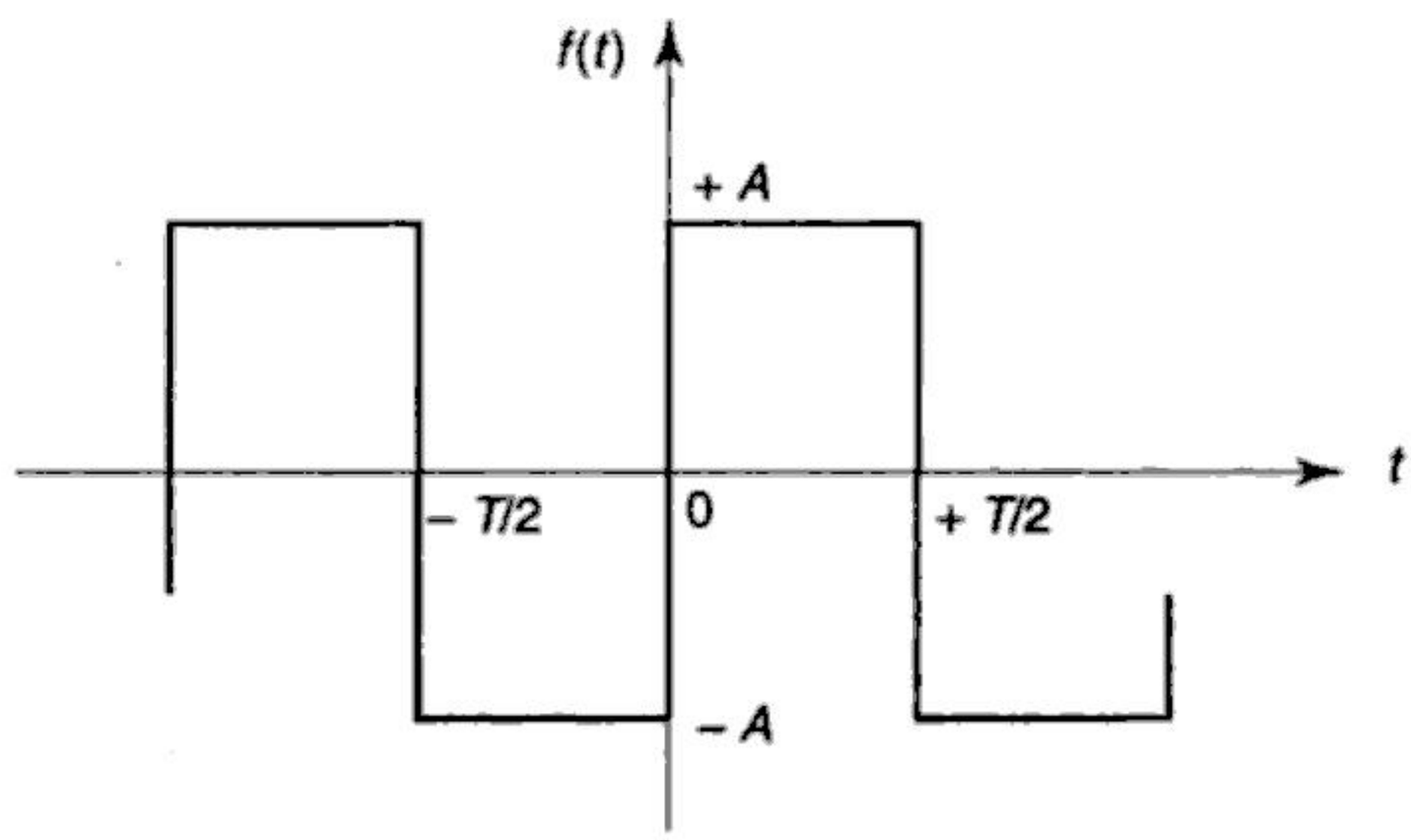

Fig. E2.6

Solution The function of the given waveform for one period can be written as

$$f(t) = \begin{cases} -A, & \text{for} \quad -T/2 < t < 0 \\ +A, & \text{for} \quad 0 \quad\quad < t < T/2 \end{cases}$$

As the waveform is symmetrical about the origin, the function of the waveform is odd and hence $a_0 = a_n = 0$, and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0}(-A\sin n\omega_0 t)\,\mathrm{d}t + \int_{0}^{T/2} A\sin n\omega_0 t\,\mathrm{d}t\right]$$

$$= \frac{2A}{T}\left\{\left[\frac{\cos n\omega_0 t}{n\omega_0}\right]_{-T/2}^{0} + \left[\frac{-\cos n\omega_0 t}{n\omega_0}\right]_{0}^{T/2}\right\}$$

$$= \frac{2A}{n\omega_0 T}\{[1-\cos(n\omega_0 T/2)] + [1-\cos(n\omega_0 T/2)]\}$$

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$$= \frac{4A}{n\,\omega_0\,T}\,[1 - \cos(n\,\omega_0\,T/2)]$$

When $\omega_0 = \dfrac{2\pi}{T}$, we have

$$b_n = \frac{4A}{n \cdot 2\pi}\left[1 - \cos n\left(\frac{2\pi}{T} \cdot T/2\right)\right]$$

$$= \frac{2A}{n\pi}[1 - \cos n\pi]$$

$$b_n = \begin{cases} 0 \quad , & \text{if } n \text{ is even} \\ \dfrac{4A}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Substituting the values of the coefficients in Eq. 2.2, we obtain

$$f(t) = \frac{4A}{\pi}\left[\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right], \text{ where } \omega_0 = \frac{2\pi}{T}$$

(b) To determine exponential Fourier series

Here $c_0 = \left|\dfrac{1}{2}a_0\right| = 0$

To evaluate c_n

Since the wave is odd, c_n consists of pure imaginary coefficients. From Eq. 2.7, we have

$$c_n = \frac{1}{T}\int_0^T f(t)\,e^{-jn\omega_0 t}\,\mathrm{d}t$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0}(-A)\,e^{-jn\omega_0 t}\,\mathrm{d}t + \int_0^{T/2} A\,e^{-jn\omega_0 t}\,\mathrm{d}t\right]$$

$$= \frac{A}{T}\left\{\left[(-1)\frac{1}{(-jn\omega_0)}e^{-jn\omega_0 t}\right]_{-T/2}^{0} + \left[\frac{1}{(-jn\omega_0)}e^{-jn\omega_0 t}\right]_0^{T/2}\right\}$$

$$= \frac{A}{T}\cdot\frac{1}{(-jn\omega_0)}\left\{-e^0 + e^{jn\omega_0(T/2)} + e^{-jn\omega_0(T/2)} - e^0\right\}$$

When $\omega_0 = \dfrac{2\pi}{T}$, we get

$$c_n = \frac{A}{T}\cdot\frac{T}{-jn2\pi}\left\{-e^0 + e^{jn(2\pi/T)(T/2)} + e^{-jn(2\pi/2)(T/2)} - e^0\right\}$$

$$= \frac{A}{(-j2\pi n)}\left\{-e^0 + e^{jn\pi} + e^{-jn\pi} - e^0\right\} = j\frac{A}{n\pi}(e^{jn\pi} - 1)$$

Here, $e^{jn\pi} = +1$ for even n and $e^{jn\pi} = -1$ for odd n

Therefore, $c_n = -j\left(\dfrac{2A}{n\pi}\right)$ for odd n only.

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Hence, the exponential Fourier series is

$$f(t) = \ldots + j\frac{2A}{3\pi}e^{-j3\omega_0 t} + j\frac{2A}{\pi}e^{-j\omega_0 t} - j\frac{2A}{\pi}e^{j\omega_0 t} - j\frac{2A}{3\pi}e^{j3\omega_0 t}$$

By using Eq. 2.11, the trigonometric Fourier series coefficients a_n and b_n can be evaluated as

$$a_n = 2\,\mathrm{Re}[c_n] = 2\,|c_n| = 0 \text{ and } b_n = -2\,\mathrm{Im}\,[c_n] = \frac{4A}{n\pi} \text{ for odd } n \text{ only.}$$

These coefficients are the same as the coefficients obtained in the trigonometric Fourier series.

Example 2.7

(a) Find the trigonometric Fourier series of the waveform shown in Fig. E2.7 and

(b) Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.

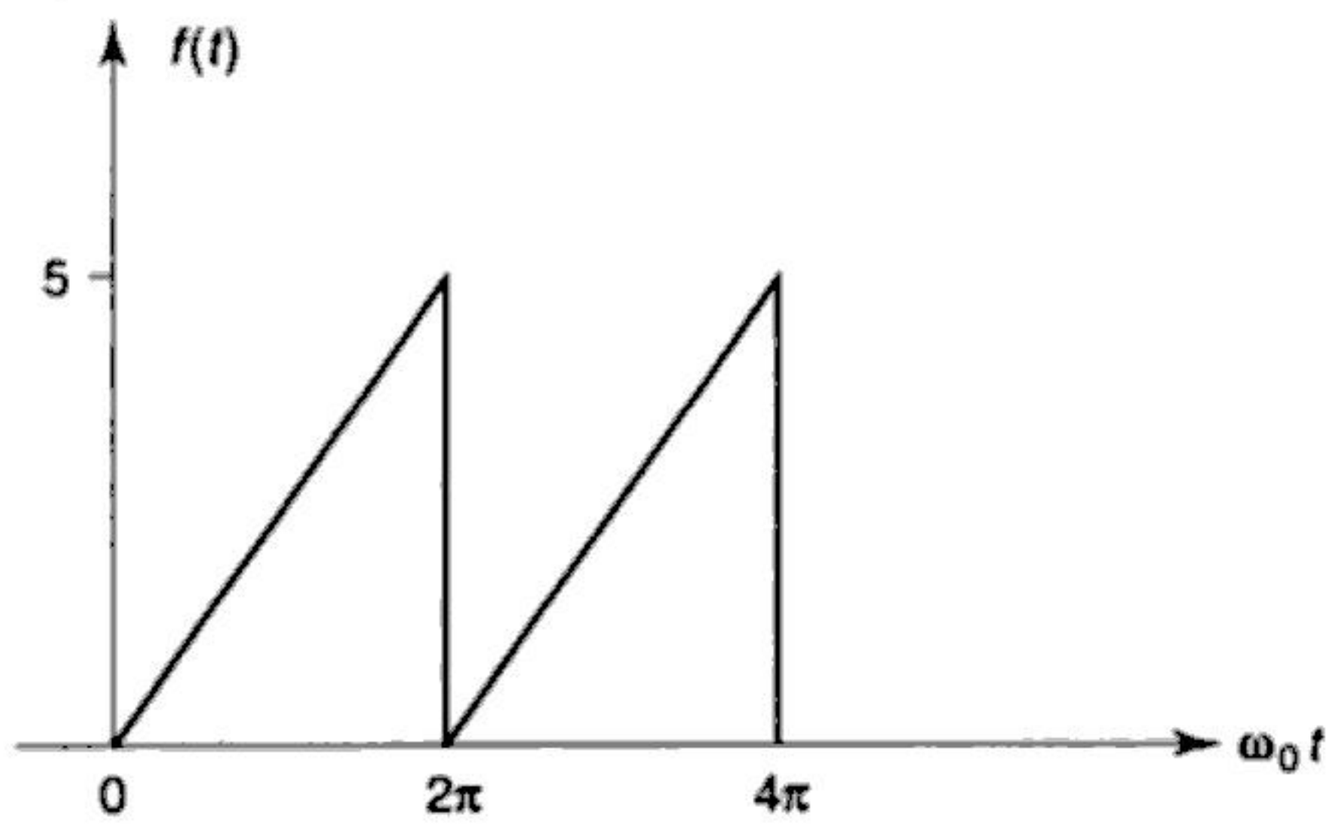

Fig. E2.7

Solution (a) As the waveform is periodic with period 2π in $\omega_0 t$ and continuous for $0 < \omega_0 t < 2\pi$, with discontinuities at $\omega_0 t = n\,(2\pi)$, where $n = 0, 1, 2, \ldots$, the Dirichlet conditions are satisfied.

To find f(t) for the given waveform of region $0 < \omega_0 t < 2\pi$:

The equation of the straight line is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

Substituting $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2\pi, 5)$, we get

$$\frac{f(t) - 0}{\omega_0 t - 0} = \frac{0 - 5}{0 - 2\pi}$$

Therefore, $f(t) = \left(\dfrac{5}{2\pi}\right)\omega_0 t$

To find Fourier coefficients

Using Eq. 2.3, we obtain the average term,

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$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$

$$= \frac{2}{2\pi}\int_{0}^{2\pi} \frac{5}{2\pi}\,\omega_0 t\,\mathrm{d}(\omega_0 t)$$

$$= \frac{10}{(2\pi)^2}\left[\frac{(\omega_0 t)^2}{2}\right]_0^{2\pi}$$

$$= \frac{10}{(2\pi)^2}\,\frac{(2\pi)^2}{2} = 5$$

Using Eq. 2.4, we obtain $a_n = \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} f(t)\cos n\,\omega_0 t\,\mathrm{d}t$

$$= \frac{2}{2\pi}\int_{0}^{2\pi}\left(\frac{5}{2\pi}\right)\omega_0 t\cos n\,\omega_0 t\,\mathrm{d}(\omega_0 t)$$

$$= \frac{5}{2\pi^2}\left[\frac{\omega_0 t}{n}\sin n\,\omega_0 t + \frac{1}{n^2}\cos n\,\omega_0 t\right]_0^{2\pi}$$

$$= \frac{5}{2\pi^2 n^2}(\cos n\,2\pi - \cos 0) = 0$$

Hence, the series contains no cosine terms.

Using Eq. 2.5, we obtain $b_n = \dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2} f(t)\sin n\,\omega_0 t\,\mathrm{d}t$

$$= \frac{1}{\pi}\int_{0}^{2\pi}\left(\frac{5}{2\pi}\right)\omega_0 t\sin n\,\omega_0 t\,\mathrm{d}(\omega_0 t)$$

$$= \frac{5}{2\pi^2}\left[-\frac{\omega_0 t}{n}\cos n\,\omega_0 t + \frac{1}{n^2}\sin n\,\omega_0 t\right]_0^{2\pi}$$

$$= -\frac{5}{n\pi}$$

Combining the average term and the sine-term coefficients, the series becomes

$$f(t) = \frac{1}{2}a_0 + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \ldots$$

$$= \frac{5}{2} - \frac{5}{\pi}\sin\omega_0 t - \frac{5}{2\pi}\sin 2\omega_0 t - \frac{5}{3\pi}\sin 3\omega_0 t - \ldots$$

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$$= \frac{5}{2} - \frac{5}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega_0 t}{n}$$

(b) *To determine exponential Fourier series*

Here, $c_0 = \left| \frac{1}{2} a_0 \right| = \frac{5}{2}$

To evaluate c_n:

From Eq. 2.7, we have

$$c_n = \frac{1}{T} \int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{5}{2\pi}\right) \omega_0 t\, e^{-jn\omega_0 t}\, d(\omega_0 t)$$

$$= \frac{5}{(2\pi)^2} \left[\frac{e^{-jn\omega_0 t}}{(-jn)^2} (-jn\omega_0 t - 1) \right]_0^{2\pi} = j \frac{5}{2\pi n}$$

Substituting the coefficients c_n in Eq. 2.9, the exponential Fourier series is

$$f(t) = \ldots - j\frac{5}{4\pi} e^{-j2\omega_0 t} - j\frac{5}{2\pi} e^{-j\omega_0 t} + \frac{5}{2} + j\frac{5}{2\pi} e^{j\omega_0 t} + j\frac{5}{4\pi} e^{-j2\omega_0 t} + \ldots$$

By using Eq. 2.11, the trigonometric Fourier series coefficients a_n and b_n can be evaluated as

$$a_n = 2\text{Re}[c_n] = 2|c_n| = 0 \quad \text{and} \quad b_n = -2\,\text{Im}\,[c_n] = -\frac{5}{n\pi}$$

Hence, $f(t) = \frac{5}{2} - \frac{5}{\pi} \sin \omega_0 t - \frac{5}{2\pi} \sin 2\omega_0 t - \frac{5}{3}\pi \sin 3\omega_0 t - \ldots$

This result is the same as that of the trigonometric Fourier series method.

2.4 PARSEVAL'S IDENTITY FOR FOURIER SERIES

A periodic function $f(t)$ with a period T is expressed by the Fourier series as

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Now, $$[f(t)]^2 = \frac{1}{2} a_0 f(t) + \sum_{n=1}^{\infty} [a_n f(t) \cos n\omega_0 t + b_n f(t) \sin n\omega_0 t]$$

Therefore, $$\frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2\, dt = \frac{(a_0/2)}{T} \int_{-T/2}^{T/2} [f(t)]\, dt$$

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$$+\frac{1}{T}\sum_{n=1}^{\infty}\left[a_n\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,\mathrm{d}t + b_n\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t\right]$$

From Eqns 2.2, 2.3 and 2.4, we have

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t$$

Therefore, substituting all these values, we get

$$\frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2\,\mathrm{d}t = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{2.12}$$

This is the ***Parseval's identity.***

2.5 POWER SPECTRUM OF A PERIODIC FUNCTION

The power of a periodic signal spectrum $f(t)$ in the time domain is defined as

$$P = \frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2\,\mathrm{d}t$$

The Fourier series for the signal $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

According to Parseval's relation, we have

$$P_{\text{av}} = \frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2\,\mathrm{d}t$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}\,\mathrm{d}t$$

$$= \sum_{n=-\infty}^{\infty} c_n\,\frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{jn\omega_0 t}\,\mathrm{d}t$$

$$= \sum_{n=-\infty}^{\infty} c_n\, c_{-n}$$

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$$= \sum_{n=-\infty}^{\infty} |c_n|^2, \text{ watts}$$

From Eq. 2.12, the above equation becomes

$$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \sum_{n=0}^{\infty} |c_n|^2$$

Here, $$c_0 = \left|\frac{a_0}{2}\right| \quad \text{and} \quad c_n = \sqrt{a_n^2 + b_n^2}, \quad (n \geq 1) \tag{2.13}$$

Thus the power in $f(t)$ is

$$P = \ldots + |c_{-n}|^2 + \ldots + |c_{-1}|^2 + |c_0|^2 + |c_1|^2 + \ldots + |c_n|^2 + \ldots \tag{2.14}$$

$$P = |c_0|^2 + 2|c_1|^2 + 2|c_2|^2 + \ldots + |c_n|^2 + \ldots$$

Hence, for a periodic function, the power in a time waveform $f(t)$ can be evaluated by adding together the powers contained in each harmonic, i.e. frequency component of the signal $f(t)$.

The power for the nth harmonic component at $n\,\omega_0$ radians per sec is $|c_n|^2$ and that of $-n\,\omega_0$ is $|c_{-n}|^2$. For the single real harmonic, we have to consider both the frequency components $\pm\, n\,\omega_0$.

Here, $c_n = c_{-n}^*$ and hence $|c_n|^2 = |c_{-n}|^2$. The power for the nth real harmonic $f(t)$ is

$$P_n = |c_n|^2 + |c_{-n}|^2 = 2|c_n|^2$$

The effective or RMS value of $f(t)$

Using Eqns 2.12, 2.13 and 2.14, the RMS value of the function $f(t)$ expressed by Eq. 2.1 is

$$F_{\text{rms}} = \sqrt{\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 + \ldots + \frac{1}{2}b_1^2 + \frac{1}{2}b_2^2 + \ldots}$$

$$= \sqrt{c_0^2 + \frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 + \ldots} \tag{2.15}$$

Example 2.8 The complex exponential Fourier representation of a signal $f(t)$ over the interval $(0, T)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4 + (n\pi)^2} e^{jn\pi t}$$

(a) What is the numerical value of T ?
(b) One of the components $f(t)$ is $A \cos 3\pi t$. Determine the value of A.
(c) Determine the minimum number of terms which must be retained in the representation of $f(t)$ in order to include 99.9% of the energy in the interval.

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Note: $\sum_{n=-\infty}^{\infty} \left| \frac{3}{4+(n\pi)^2} \right| \approx 0.669$

Solution The complex exponential Fourier transform representation of a signal $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

The given signal $f(t)$ over the interval $(0, T)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t}$$

(a) Comparing the above two equations, we get

$$c_n = \frac{3}{4+(n\pi)^2} \text{ and}$$

$$e^{jn\frac{2\pi}{T}t} = e^{jn\pi t}$$

Hence, $\frac{2\pi}{T} = \pi$, i.e. $T = 2$

(b) When $n = 3$, the component of $f(t)$ will be

$$c_3 = \frac{3}{4+(3\pi)^2} e^{j3\pi t} = \frac{3}{4+(3\pi)^2} [\cos 3\pi t + j \sin 3\pi t]$$

Similarly, when $n = -3$, the component will be

$$c_{-3} = \frac{3}{4+(-3\pi)^2} e^{-j3\pi t} = \frac{3}{4+(3\pi)^2} [\cos 3\pi t - j \sin 3\pi t]$$

Therefore, $c_3 + c_{-3} = \frac{6}{4+(3\pi)^2} \cos 3\pi t$

Hence, when one of the components of $f(t)$ is $A \cos 3\pi t$, the value of A is

$$A = \frac{6}{4+(3\pi)^2}$$

(c) Total (maximum) power $P_t = \sum_{n=-\infty}^{\infty} \left| \frac{3}{4+(n\pi)^2} \right| \approx 0.669$

The power in $f(t)$ is

$$P = |c_0|^2 + 2\left[|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2\right]$$

$$= \left|\frac{3}{4}\right|^2 + 2\left[\left|\frac{3}{4+(\pi)^2}\right|^2 + \left|\frac{3}{4+(2\pi)^2}\right|^2 + \left|\frac{3}{4+(3\pi)^2}\right|^2 + \left|\frac{3}{4+(4\pi)^2}\right|^2\right]$$

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$$= 0.5625 + 0.0935 + 9.52 \times 10^{-3} + 2.088 \times 10^{-3} + 6.866 \times 10^{-4}$$

$$= 0.66836$$

Therefore, energy contained in the four terms is

$$\frac{P}{P_t} \times 100 = \frac{0.66836}{0.669} \times 100 = 99.9\%$$

Hence, the first four terms include 99.9% of the total energy.

2.6 FOURIER TRANSFORM

The plot of amplitudes at different frequency components for a periodic wave is known as discrete (line) frequency spectrum because amplitude values have significance only at discrete values of $n\,\omega_0$ where $\omega_0 = 2\pi/T$ is the separation between two adjacent (consecutive) harmonic components. If the repetition period T increases, ω_0 decreases .

Hence, when the repetition period T becomes infinity, i.e. $T \to \infty$, the wave $f(t)$ will become non-periodic, the separation between two adjacent harmonic components will be zero, i.e. $\omega_0 = 0$. Therefore, the discrete spectrum will become a continuous spectrum. When $T \to \infty$, the adjacent pulses virtually never occur and the pulse train reduces to a single isolated pulse. The exponential form of the Fourier series given in Eq. 2.10 can be extended to aperiodic waveforms such as single pulses or single transients by making a few changes.

Assuming $f(t)$ is initially periodic, from Eq. 2.10, we have,

$$f(t) = \sum_{-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

where $$c_n = 1/T \int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, \mathrm{d}t$$

In the limit, for a single pulse, we have

$$T \to \infty,\ \omega_0 = 2\pi/T \to \mathrm{d}\omega \quad \text{(a small quantity)}$$

or $$1/T = \omega_0/2\pi \to \mathrm{d}\omega/2\pi$$

Furthermore, the nth harmonic in the Fourier series is $n\,\omega_0 \to n\,d\,\omega$. Here n must tend to infinity as ω_0 approaches zero, so that the product is finite, i.e. $n\,\omega_0 \to \omega$.

In the limit, the Σ sign leads to an integral and we have

$$c_n = \frac{\mathrm{d}\omega}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t$$

and, $$f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t \right] e^{j\omega t}$$

When evaluated, the quantity in bracket is a function of frequency only and is denoted as $F(j\,\omega)$ where

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$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{2.16}$$

It is called the *Fourier transform* of $f(t)$.
Substituting for $f(t)$ above, We obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

or, equivalently,

$$f(t) = \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, df \tag{2.17}$$

which is called the *inverse* Fourier *transform*. Now the time function $f(t)$ represents the expression for a single pulse or transient only. Equations 2.16 and 2.17 constitute a Fourier *transform pair*.

From Eqns 2.16 and 2.17, it is apparent that the Fourier transform and inverse Fourier transform are similar, except for a sign change on the exponential component.

2.6.1 Energy Spectrum for a Non-Periodic Function

For a non-periodic energy signal, such as a single pulse, the total energy in $(-\infty, \infty)$ is finite, whereas the average power, i.e. energy per unit time, is zero because $\frac{1}{T}$ tends to zero as T tends to infinity. Hence, the total energy associated with $f(t)$ is given by

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt$$

Since, $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$, we obtain

$$E = \int_{-\infty}^{\infty} f(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \left[\int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, F(-j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, F^*(j\omega)\, d\omega$$

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$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

$$= \int_{-\infty}^{\infty} |F(f)|^2 \, df, \text{ joules}$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(f)|^2 \, df \tag{2.18}$$

This result is called ***Rayleigh's energy theorem or Parseval's theorem*** for Fourier transform. The quantity $|F(f)|^2$ is referred to as the *energy spectral density*, $S(f)$, which is equal to the energy per unit frequency.

The integration in Eq. 2.18 is carried out over positive and negative frequencies. If $f(t)$ is real, then $|F(j\omega)| = |F(-j\omega)|$, then the Eq. 2.18 becomes,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} |F(j\omega)|^2 \, d\omega = \int_{0}^{\infty} S(\omega) \, d\omega$$

Here the integration is carried out over only positive frequencies. The quantity $S(\omega) = |F(j\omega)|^2/\pi$ is called the energy spectral density.

2.7 PROPERTIES OF FOURIER TRANSFORM

Table 2.1 presents important properties of the Fourier transform.

Table 2.1 *Important properties of the Fourier transform*

Operation	*f(t)*	*F(jω)*
Transform	$f(t)$	$\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} \, dt$
Inverse transform	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t} \, d\omega$	$F(j\omega)$
Linearity	$af_1(t) + bf_2(t)$	$aF_1(j\omega) + bF_2(j\omega)$
Time-reversal	$f(-t)$	$F(-j\omega) = F^*(j\omega)$, $f(t)$ real
Time-shifting (Delay)	$f(t - t_0)$	$F(j\omega)\, e^{-j\omega t_0}$
Time-Scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{j\omega}{a}\right)$
Time-differentiation	$\frac{d^n}{dt^n} f(t)$	$(j\omega)^n F(j\omega)$
Frequency-differentiation	$(-jt)\, f(t)$	$\frac{dF(j\omega)}{d\omega}$

(Contd.)

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Operation	*f(t)*	*F(j ω)*
Time-integration	$\int_{-\infty}^{t} f(\tau)\,d\tau$	$\frac{1}{j\omega}F(j\omega) + \pi F(0)\,\delta(\omega)$
Frequency-integration	$\frac{1}{(-jt)} f(t)$	$\int^{\omega} f(j\omega')\,d\omega'$
Time convolution	$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\,f_2(t-\tau)\,d\tau$	$F_1(j\omega)\,F_2(j\omega)$
Frequency convolution (Multiplication)	$f_1(t)\cdot f_2(t)$	$\frac{1}{2\pi}[F_1(j\omega) * F_2(j\omega)]$
Frequency shifting (Modulation)	$f(t)\,e^{j\omega_0 t}$	$F(j\omega - j\omega_0)$
Symmetry	$F(jt)$	$2\pi f(-\omega)$
Real-time function	$f(t)$	$F(j\omega) = F^*(-j\omega)$ $\mathrm{Re}[F(j\omega)] = \mathrm{Re}[F(-j\omega)]$ $\mathrm{Im}[F(j\omega)] = -\mathrm{Im}[F(-j\omega)]$ $\lvert F(j\omega)\rvert = \lvert F(-j\omega)\rvert$ $\Phi f(j\omega) = -\Phi f(j\omega)$
Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert f(t)\rvert^2\,dt$	$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(j\omega)\rvert^2\,d\omega$
Duality	If $f(t) \Leftrightarrow g(j\omega)$, then $g(t) \Leftrightarrow 2\pi f(-j\omega)$	

2.7.1 Linearity

The Fourier transform is a linear operation. Therefore, if

$$f_1(t) \Leftrightarrow F_1(j\omega)$$
$$f_2(t) \Leftrightarrow F_2(j\omega)$$

then, $af_1(t) + bf_2(t) \Leftrightarrow aF_1(j\omega) + bF_2(j\omega)$

where a and b are arbitrary constants.

2.7.2 Symmetry

If $f(t) \Leftrightarrow F(j\omega)$

then, $F(jt) \Leftrightarrow 2\pi f(-\omega)$

Proof

Since
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\,e^{j\omega t}\,d\omega$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega')\,e^{-j\omega' t}\,d\omega'$$

where the dummy variable ω is replaced by ω'.

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Now if t is replaced by ω, we have

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(j\omega')\, e^{-j\omega'\omega}\, d\omega'$$

Finally, ω' is replaced by t to obtain a more recognisable form and we have

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)\, e^{-j\omega t}\, dt = \mathcal{F}[F(jt)]$$

Therefore, $F(jt) \Leftrightarrow 2\pi f(-\omega)$
If $f(t)$ is an even function, $f(t) = f(-t)$.
Hence, $\mathcal{F}[F(jt)] = 2\pi f(\omega)$

2.7.3 Scaling

If $\quad f(t) \Leftrightarrow F(j\omega)$,

then, $\quad f(at) \Leftrightarrow \dfrac{1}{|a|} F\left(\dfrac{j\omega}{a}\right)$

Proof

If $a > 0$, then the transform of $f(at)$ is

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt$$

Putting $x = at$, we have $dx = a dt$. Substituting in the above equation, we get

$$\mathcal{F}[f(at)] = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{\frac{-j\omega x}{a}}\, \frac{dx}{a} = \frac{1}{a} F\left(\frac{j\omega}{a}\right)$$

If $a < 0$, then $\mathcal{F}[f(at)] = -\dfrac{1}{a} F\left(\dfrac{j\omega}{a}\right)$

Combining these two results, we get

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{j\omega}{a}\right)$$

We conclude that larger the duration of the time function, smaller is the bandwidth of its spectrum by the same scaling factor. Conversely, smaller the duration of the time function, larger is the bandwidth of its spectrum. This scaling property provides an *inverse relationship* between time-duration and bandwidth of a signal i.e. the time-bandwidth product of an energy signal is a constant.

2.7.4 Convolution

Convolution is a powerful way of characterising the input-output relationship of time-invariant linear systems. There are two convolution

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theorems, one for the time domain and another for the frequency domain.

Time Convolution

If $\quad x(t) \Leftrightarrow X(j\omega)$ and $h(t) \Leftrightarrow H(j\omega)$,

then $\quad y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau \Leftrightarrow Y(j\omega) = X(j\omega)\, H(j\omega)$$

Proof

$$\mathcal{F}\,[y(t)] = Y(j\,\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau\right]\mathrm{d}t$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, \mathrm{d}t\right]\mathrm{d}\tau$$

Putting $a = t - \tau$, then $t = a + \tau$ and $\mathrm{d}a = \mathrm{d}t$

$$\text{Therefore, } Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(a)\, e^{-j\omega(a+\tau)}\, \mathrm{d}a\right]\mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, \mathrm{d}\tau \int_{-\infty}^{\infty} h(a)\, e^{-j\omega a}\, \mathrm{d}a$$

$$= X(j\,\omega)\, H(j\,\omega)$$

Hence, the convolution of the signals in the time domain is equal to the multiplication of their individual Fourier transforms in the frequency domain.

Example 2.9 In the system shown in Fig. E2.9 determine the output response of the low-pass RC network for an input signal $x(t) = e^{-\frac{t}{RC}}$

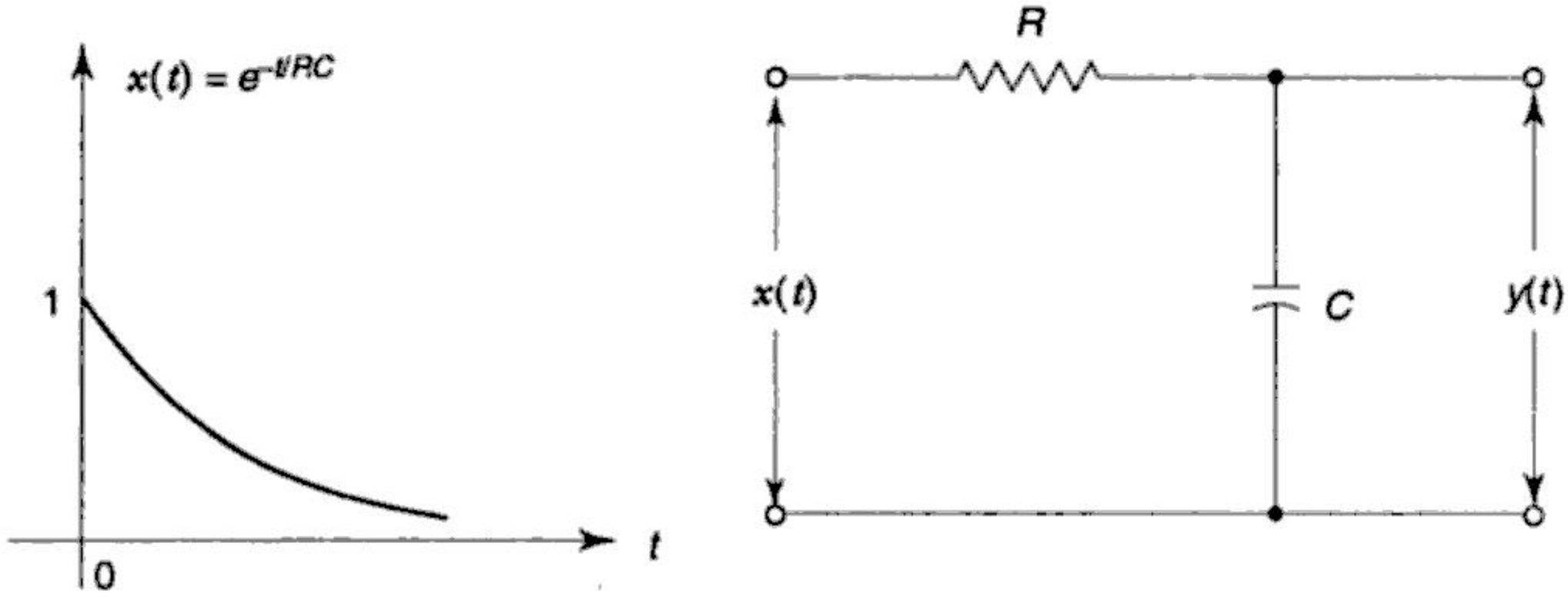

Fig. E2.9

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Solution The input signal $x(t) = e^{-\frac{t}{RC}}$
Using the convolution theorem, we can find $y(t)$

$$y(t) = x(t) * h(t) = \mathcal{F}^{-1}[X(j\omega)\,H(j\omega)]$$

$$X(j\omega) = \mathcal{F}\,[x(t)] = \int_0^\infty e^{-\frac{t}{RC}}\, e^{-j\omega t}\, dt$$

$$= \int_0^\infty e^{-\left(\frac{1}{RC} + j\omega\right)t}\, dt$$

$$= \frac{1}{j\omega + \frac{1}{RC}}$$

Similarly, the transfer function of the network is

$$H(j\omega) = \frac{1/j\omega C}{\left(R + \frac{1}{j\omega C}\right)} = \frac{1}{(j\omega RC + 1)}$$

$$= \frac{1}{RC} \cdot \frac{1}{\left(j\omega + \frac{1}{RC}\right)}$$

Hence, $Y(j\omega) = X(j\omega)\,H(j\omega) = \dfrac{1}{RC} \cdot \dfrac{1}{\left(j\omega + \frac{1}{RC}\right)^2}$

$$y(t) = \mathcal{F}^{-1}\,[Y(j\omega)] = \frac{1}{RC} t e^{-\frac{t}{RC}}\, u(t)$$

Example 2.10 Determine the output response of the low-pass RC network due to an input $x(t) = t\,e^{-t/RC}$ by convolution.

Solution The transfer function of the network is

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{(1 + j\omega RC)}$$

The given input time function is $x(t) = te^{-t/RC}$

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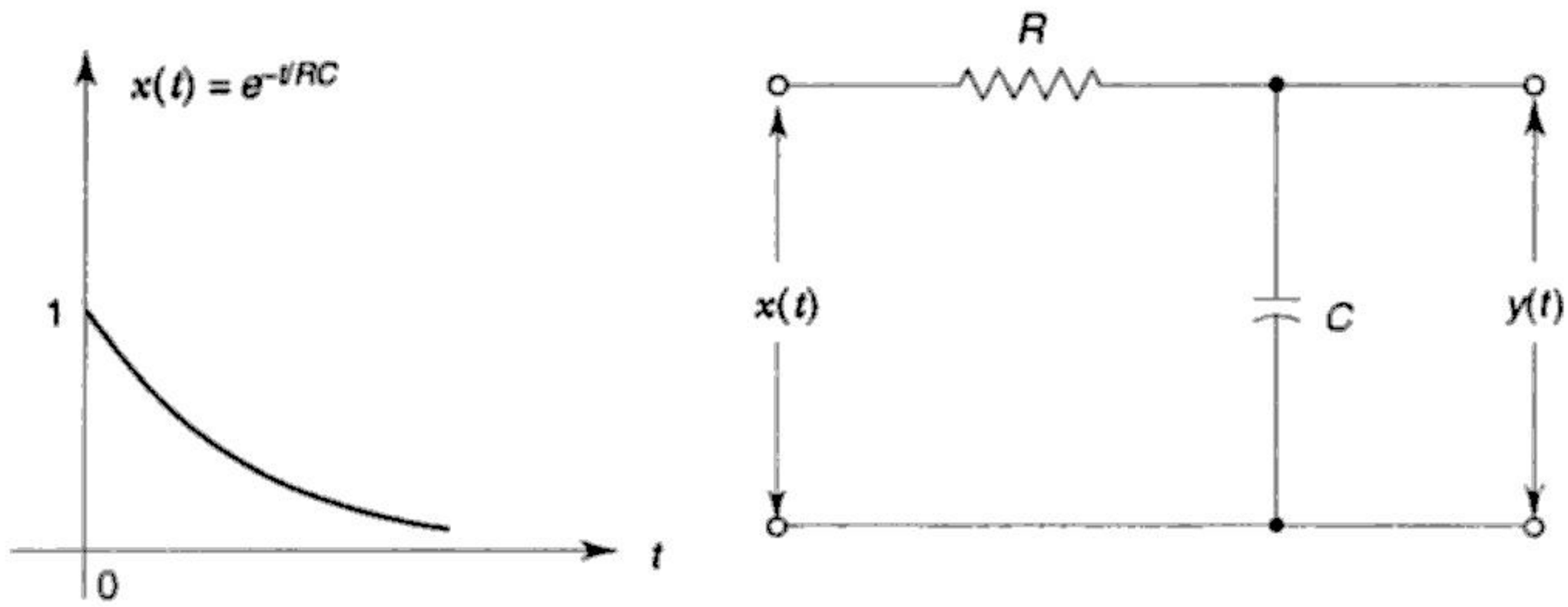

Fig. E2.9

Therefore, $X(j\omega) = \mathcal{F}[te^{-t/RC}] = \dfrac{1}{\left(\dfrac{1}{RC} + j\omega\right)^2} = \dfrac{(RC)^2}{(1 + j\omega RC)^2}$

We know that $Y(j\omega) = X(j\omega)\, H(j\omega)$

Hence, $Y(j\omega) = \dfrac{(RC)^2}{(1 + j\omega RC)^2} \cdot \dfrac{1}{(1 + j\omega RC)} = \dfrac{(RC)^2}{(1 + j\omega RC)^3}$

Therefore, $y(t) = \mathcal{F}^{-1}\left[\dfrac{(RC)^2}{(1 + j\omega RC)^3}\right] = \mathcal{F}^{-1}\left[\dfrac{1}{RC} \cdot \dfrac{1}{\left(\dfrac{1}{RC} + j\omega\right)^3}\right]$

$$= \frac{1}{RC}\,\frac{t^2 e^{-t/RC}}{2}\, u(t)$$

Hence, $\quad y(t) = \dfrac{1}{RC}\,\dfrac{t^2 e^{-t/RC}}{2}\, u(t)$

Example 2.11 The input signal $x(t) = e^{-at}\, u(t)$, a > 0 is applied to the system whose transfer function is $h(t) = e^{-bt}\, u(t)$, $b > 0$. Determine the output signal $y(t)$ when $b \neq a$ and $b = a$.

Solution The Fourier transform of $x(t)$ and $h(t)$ are

$$X(j\omega) = \frac{1}{(a + j\omega)} \text{ and}$$

$$H(j\omega) = \frac{1}{(b + j\omega)}$$

Therefore, $Y(j\omega) = X(j\omega)\, H(j\omega) = \dfrac{1}{(a + j\omega)(b + j\omega)}$

Expanding the function $Y(j\omega)$ in partial fractions, we get

$$Y(j\omega) = \frac{A}{(a + j\omega)} + \frac{B}{(b + j\omega)}$$

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$$A = Y(j\omega)(a + j\omega)\big|_{j\omega = -a} = \frac{1}{(b-a)}$$

$$B = Y(j\omega)(b + j\omega)\big|_{j\omega = -b} = \frac{1}{(a-b)}$$

Hence, $Y(j\omega) = \dfrac{1}{(b-a)}\left[\dfrac{1}{(a+j\omega)} - \dfrac{1}{(b+j\omega)}\right]$

Taking inverse Fourier transform, we get

$$y(t) = \frac{1}{(b-a)}\left[e^{-at}\,u(t) - e^{-bt}\,u(t)\right]$$

When $b = a$, the partial fraction expansion is invalid. Hence,

$$Y(j\omega) = \frac{1}{(a+j\omega)^2}$$

$$= j\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{1}{a+j\omega}\right]$$

Using dual of the differentiation property,

$$e^{-at}\,u(t) \Leftrightarrow \frac{1}{a+j\omega}$$

$$te^{-at}\,u(t) \Leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{a+j\omega}\right) = \frac{1}{(a+j\omega)^2}$$

Therefore, $y(t) = te^{-at}\,u(t)$

2.7.5 Frequency Convolution

If $f(t) \Leftrightarrow F(j\omega)$ and $g(t) \Leftrightarrow G(j\omega)$,

then $f(t)\,g(t) \Leftrightarrow \dfrac{1}{2\pi}F(j\omega) * G(j\omega)$

Proof

The inverse transform of $[F(j\omega) * G(j\omega)]/2\pi$ is

$$\mathcal{F}^{-1}\left[\frac{F(j\omega) * G(j\omega)}{2\pi}\right] = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{j\omega t} \int_{-\infty}^{\infty} F(ju)\,G(j\omega - ju)\,\mathrm{d}u\,\mathrm{d}\omega$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} F(ju)\int_{-\infty}^{\infty} G(j\omega - ju)\,e^{j\omega t}\,\mathrm{d}\omega\,\mathrm{d}u$$

Putting $x = \omega - u$, then $\omega = x + u$ and $\mathrm{d}x = \mathrm{d}\omega$

Therefore

$$\mathcal{F}^{-1}\left[\frac{F(j\omega) * G(j\omega)}{2\pi}\right] = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} F(ju)\int_{-\infty}^{\infty} G(jx)\,e^{j(x+u)t}\,\mathrm{d}x\,\mathrm{d}u$$

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$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(ju)\, e^{jut}\, du \frac{1}{(2\pi)} \int_{-\infty}^{\infty} G(jx)\, e^{jxt}\, dx$$

$$= f(t) \cdot g(t)$$

Therefore, $f(t) \cdot g(t) \Leftrightarrow \frac{1}{2\pi} F(j\omega) * G(j\omega)$.

Hence, the multiplication of two signals in the time domain is equal to the convolution of their individual Fourier transforms in the frequency domain. This property is called the ***multiplication theorem***.

2.7.6 Time Shifting (Delay)

If $f(t) \Leftrightarrow F(j\omega)$,
then $f(t - t_0) \Leftrightarrow F(j\omega)\, e^{-j\omega t_0}$

Proof

The Fourier transform of $f(t - t_0)$ is given by

$$\mathcal{F}[f(t - t_0)] = \int_{-\infty}^{\infty} f(t - t_0)\, e^{-j\omega t}\, dt$$

Putting $x = t - t_0$, then $t = t_0 + x$ and $dx = dt$,

$$\mathcal{F}[f(t - t_0)] = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega(t_0 + x)}\, dx$$

$$= e^{-j\omega t_0} F(j\omega)$$

Therefore, $f(t - t_0) \Leftrightarrow F(j\omega)\, e^{-j\omega t_0}$

When the function $f(t)$ is delayed by t_0 the original spectrum is multiplied by $e^{-j\omega t_0}$. Here there is no change in the amplitude spectrum and each frequency component is shifted in phase by an amount $-\omega t_0$.

Example 2.12 Find the amplitude and phase spectrum of the time shifted impulse signal $f(t) = 10\,\delta(t - 2)$.

Solution $f(t) = 10\,\delta(t - 2)$

Using the shifting theorem, $f(t - t_0) \Leftrightarrow F(j\omega)\, e^{-j\omega t_0}$, we get

$$F(j\omega) = \mathcal{F}[10\,\delta(t - 2)] = 10\, e^{-j2\omega}$$

Hence, the magnitude $|F(j\omega)| = 10$
and the phase $\angle F(j\omega) = -2\omega$

Table E2.12 *Phase spectrum*

ω	$\angle F(j\omega)$
0.25	−0.5
0.50	−1.0
1.0	−2.0
−0.25	+0.5
−0.50	+1.0
−1.0	+2.0

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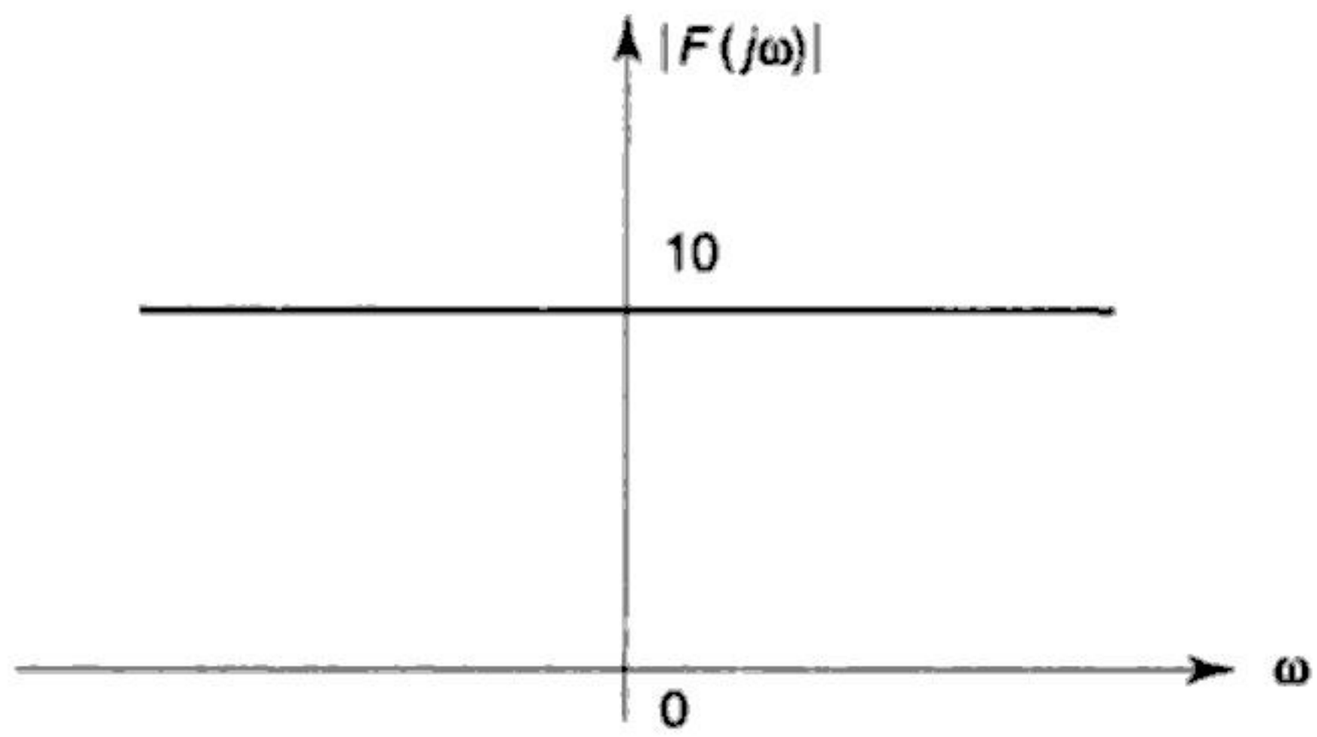

Fig. E2.12(a) *Amplitude Spectrum*

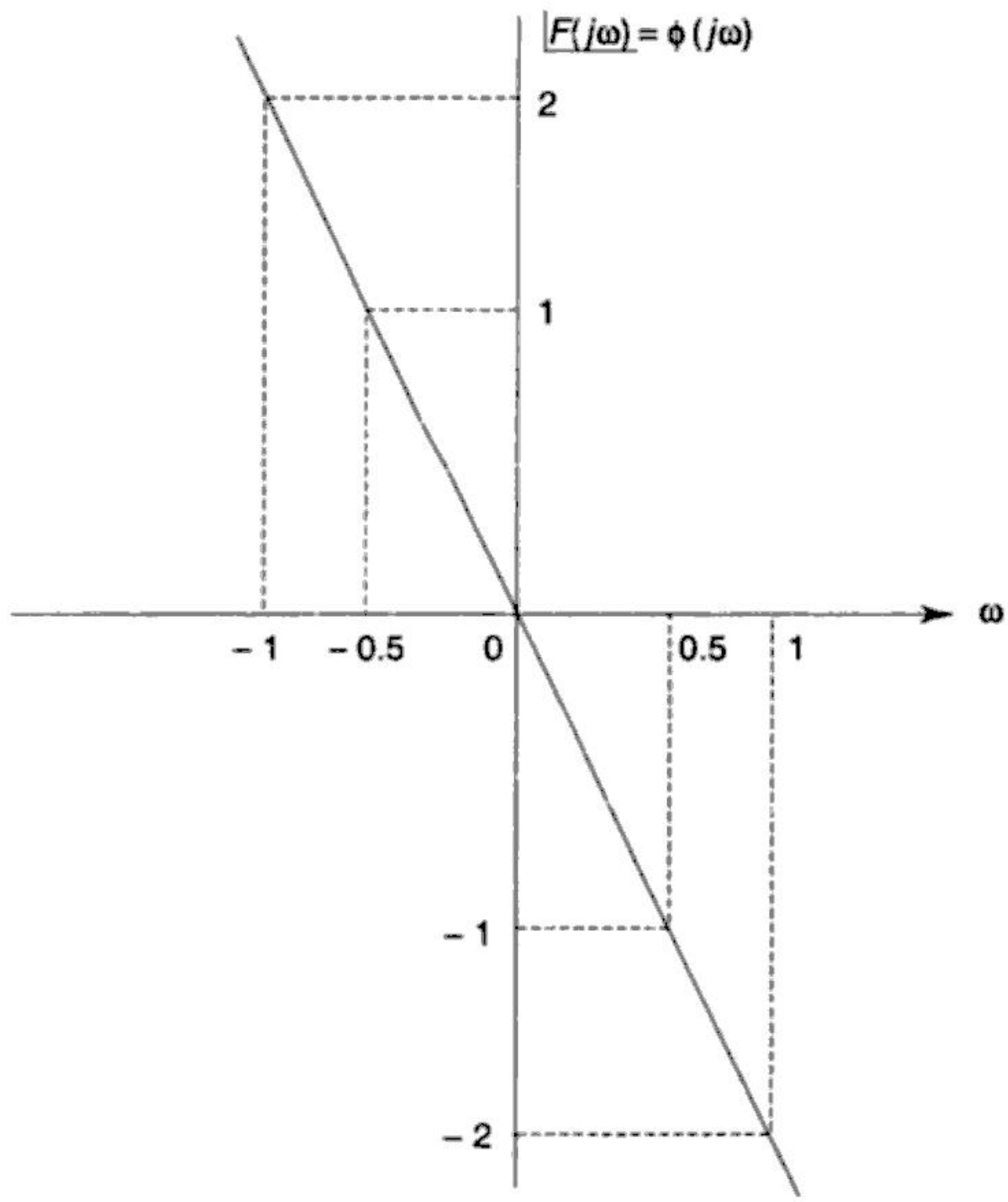

Fig. E2.12(b) *Phase Spectrum*

2.7.7 Frequency Shifting : (Modulation)

If $f(t) \Leftrightarrow F(j\omega)$,

then $f(t)\, e^{j\omega_0 t} \Leftrightarrow F(j\omega - j\omega_0)$

Proof

The transform of $f(t)\, e^{j\omega_0 t}$ is by definition

$$\mathcal{F}[f(t)\, e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

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$$= \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= F(j\omega - j\omega_0)$$

Hence, $f(t)\, e^{j\omega_0 t} \Leftrightarrow F(j\omega - j\omega_0)$

This theorem is very important for understanding the modulation in telecommunication systems.

Example 2.13 Determine the spectrum of $f(t) \cos \omega_0 t$ by using the frequency shifting theorem.

Solution The spectrum of $f(t) \cos \omega_0 t$ is

$$\mathcal{F}[f(t) \cos \omega_0 t] = \mathcal{F}\left[f(t)\left\{\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right\}\right]$$

$$= \frac{1}{2}\mathcal{F}\left[f(t)\, e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[f(t)\, e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2}\{F(j\omega - j\omega_0) + F(j\omega + j\omega_0)\}, \text{ (using the shifting theorem)}$$

Thus, due to the product of $f(t)$ and $\cos \omega_0 t$, the original spectrum is shifted in such a way that half the original spectrum is centred about ω_0 and the other half is centred about $-\omega_0$.

2.7.8 Time Differentiation

If $f(t) \Leftrightarrow F(j\omega)$, then $\dfrac{d\, f(t)}{dt} \Leftrightarrow j\omega F(j\omega)$

and $\dfrac{d^n}{dt^n} f(t) \Leftrightarrow (j\omega)^n F(j\omega)$

2.7.9 Frequency Differentiation

If $f(t) \Leftrightarrow F(j\omega)$, then

$$(-jt)\, f(t) \Leftrightarrow \frac{dF(j\omega)}{d\omega}$$

2.7.10 Time Integration

If $f(t) \Leftrightarrow F(j\omega)$, then

$$\int_{-\infty}^{t} f(\tau)\, d\tau \Leftrightarrow \frac{F(j\omega)}{j\omega} + \pi F(0)\, \delta(\omega)$$

2.7.11 Frequency Integration

$$\frac{f(t)}{-jt} \Leftrightarrow \int^{\omega} F(j\omega')\, d\omega'$$

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2.7.12 Time Reversal

If $f(t) \Leftrightarrow F(j\omega)$, then $f(-t) \Leftrightarrow F(-j\omega)$

It is clear that the time-reversal theorem is similar to the scaling theorem with $a = -1$. When the signal is real, time-reversal affects only the phase spectrum because the amplitude spectrum is an even function of frequency.

2.7.13 Complex Conjugation

If $f(t) \Leftrightarrow F(j\omega)$, then $f^*(t) \Leftrightarrow F^*(-j\omega)$

2.7.14 Duality

If $f(t) \Leftrightarrow g(j\omega)$, then $g(t) \Leftrightarrow 2\pi f(-j\omega)$

2.7.15 Area Under *f* (t)

If $f(t) \Leftrightarrow F(f)$, then $\int_{-\infty}^{\infty} f(t)\,dt = F(0)$

Thus, the area under a function $f(t)$ is equal to the value of its Fourier transform $F(f)$ at $f = 0$.

The result can be obtained by substituting $f = 0$ in the formula defining the Fourier transform of the function $f(t)$.

2.7.16 Area Under *F* (*f*)

If $f(t) \Leftrightarrow F(f)$, then $\int_{-\infty}^{\infty} F(f)\,df = f(0)$

Thus, the value of a function $f(t)$ at $t = 0$ is equal to the area under its Fourier transform $F(f)$.The result can be obtained by substituting $t = 0$ in the formula defining the inverse Fourier transform of $F(f)$.

Example 2.14 A certain function of time $f(t)$ has the following Fourier transform

$$F(j\omega) = \frac{1}{\omega^2 + 1} e^{-2\omega^2/(\omega^2+1)}$$

Using the properties of the Fourier transform, write the Fourier transforms of

(a) $f(2t)$, (b) $f(t-2)e^{jt}$, (c) $4\frac{d}{dt} f(t)$ and (d) $\int_{-\infty}^{t} f(\tau)\,d\tau$

In each case state clearly the properties you will use.

Solution

$$F(j\omega) = \frac{1}{\omega^2 + 1} e^{-2\omega^2/(\omega^2+1)}$$

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(a) **To find** $f(2t)$

Using the scaling property $f(at) = \frac{1}{|a|} F\left(\frac{j\omega}{a}\right)$, we obtain

$$f(2t) = \frac{1}{|2|} F\left(\frac{j\omega}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{\left[\left(\frac{\omega}{2}\right)^2 + 1\right]} \cdot e^{-2(\omega/2)^2/[(\omega/2)^2 + 1]}$$

$$= \frac{2}{(\omega^2 + 4)} e^{-2\omega^2/(\omega^2 + 4)}$$

(b) **To find** $f(t-2)\, e^{jt}$

Using the time shifting property $f(t - t_0) \Leftrightarrow F(j\omega)\, e^{-j\omega t_0}$, we get

$$f(t-2) = F(j\omega)\, e^{-j2\omega}$$

Therefore, $f(t-2) = \frac{1}{(\omega^2 + 1)} e^{-2\omega^2/(\omega^2 + 1)}\, e^{-j2\omega}$

Using the frequency shifting property $f(t)\, e^{j\omega_0 t} \Leftrightarrow F(j\omega - j\omega_0)$, we get

$$f(t-2)\, e^{jt} = F[j(\omega - 1)]\, e^{-j2(\omega - 1)}$$

$$= \frac{1}{[(\omega - 1)^2 + 1]} \cdot e^{-2(\omega - 1)^2/[(\omega - 1)^2 + 1]}\, e^{-j2(\omega - 1)}$$

(c) **To find** $4 \frac{d}{dt} f(t)$

Using the differentiation property , $\frac{df(t)}{dt} \Leftrightarrow j\omega F(j\omega)$, we get

$$4 \frac{d}{dt} f(t) = j\, 4\, \omega F(j\omega) = j\, 4\omega \frac{1}{\omega^2 + 1} e^{-2\omega^2/(\omega^2 + 1)}$$

(d) **To find** $\int_{-\infty}^{t} f(\tau)\, d\tau$

Using the integration property

$$\int_{-\infty}^{t} f(\tau)\, d\tau \Leftrightarrow \frac{F(j\omega)}{j\omega} + \pi F(0)\, \delta(\omega), \text{ we get}$$

$$\Leftrightarrow \frac{1}{j\omega} \frac{1}{(\omega^2 + 1)} e^{-2\omega^2/(\omega^2 + 1)} + \pi\delta(\omega), \quad [\text{since } F(0) = 1]$$

2.8 FOURIER TRANSFORM OF SOME IMPORTANT SIGNALS

Table 2.2 presents the Fourier transform of some important signals.

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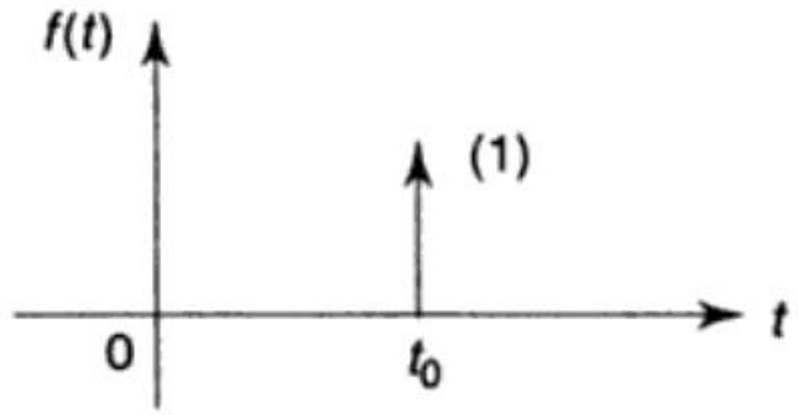
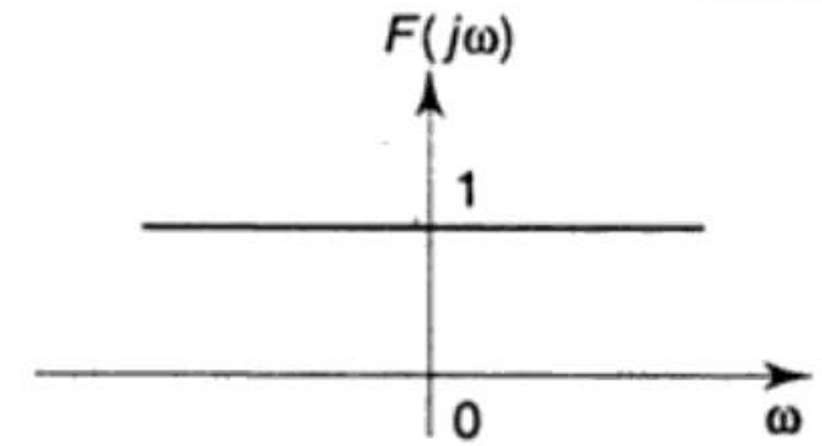
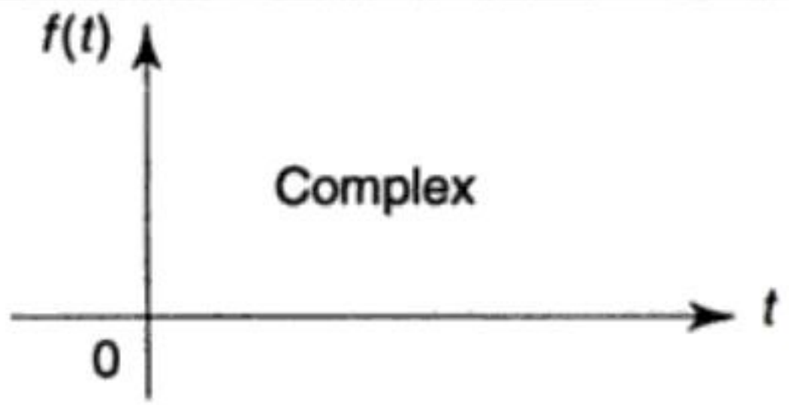
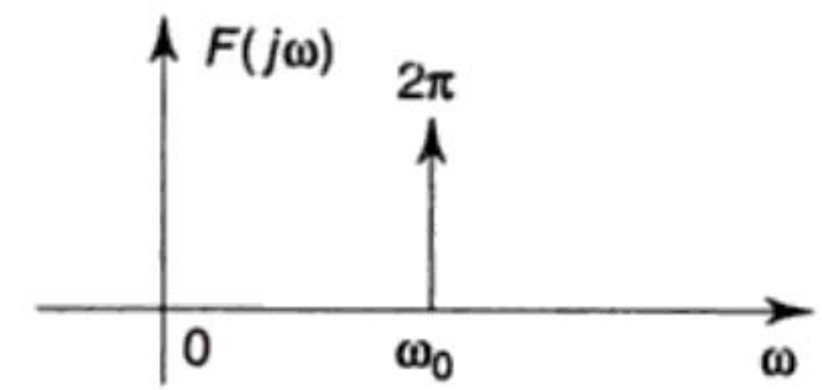
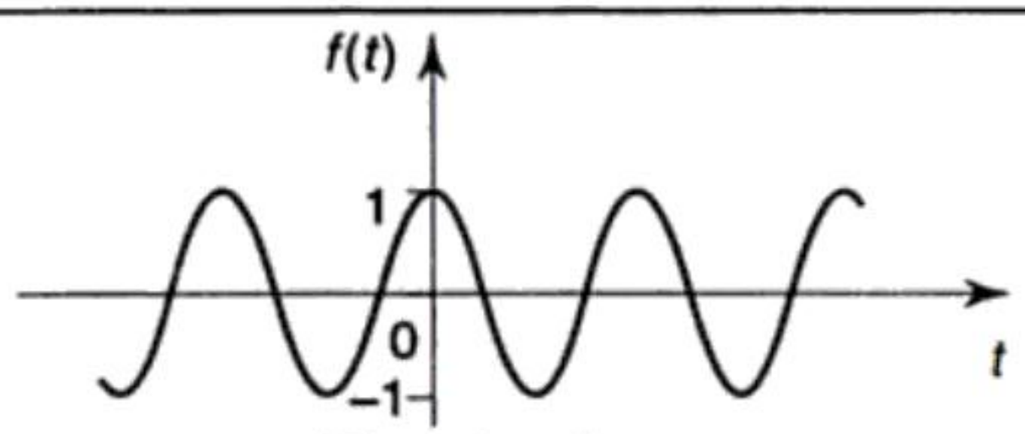
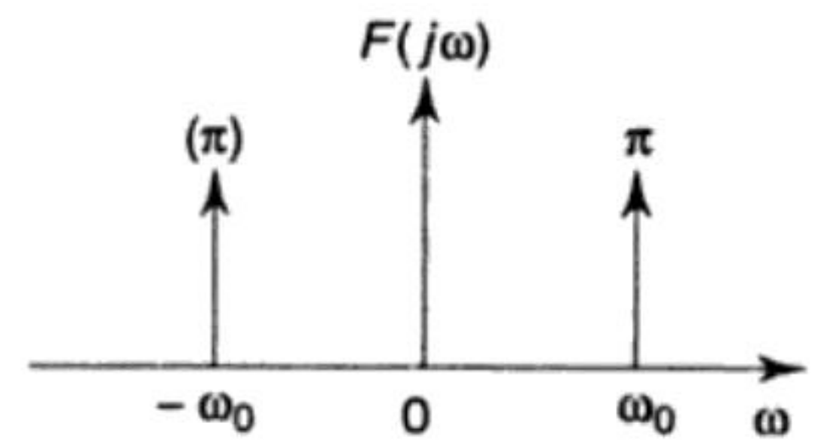
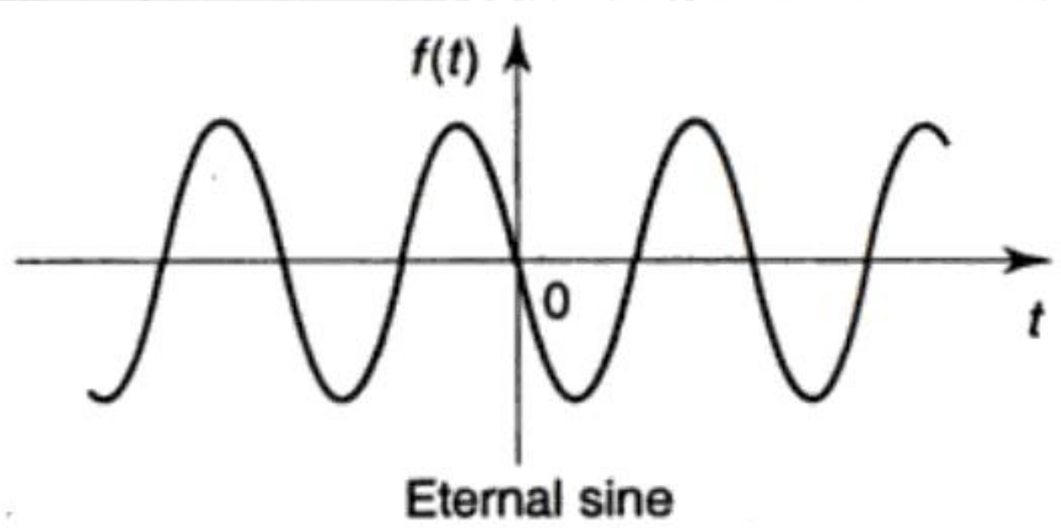
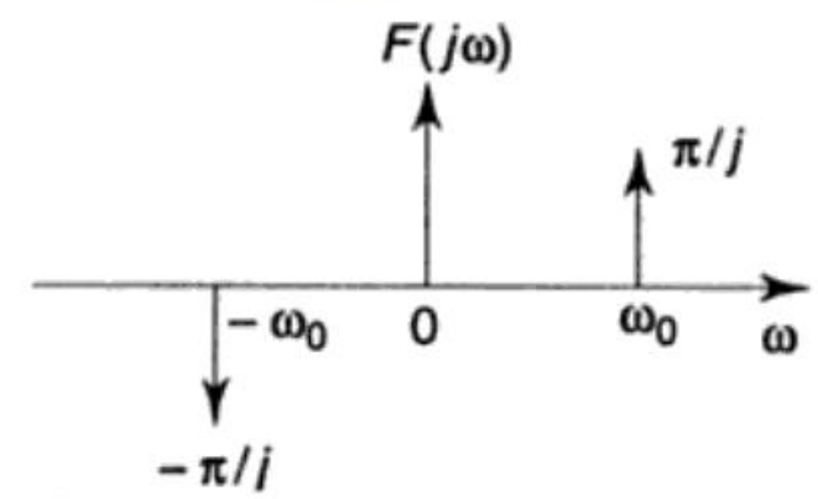
Table 2.2 *Fourier transform of some important signals*

Sl. No.	Time domain $f(t)$		Frequency domain $F(j\omega)$	
1.		$\delta(t - t_0)$	$e^{-j\omega t_0}$	
2.		$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
3.		$\cos \omega_0 t$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$	
4.		$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	

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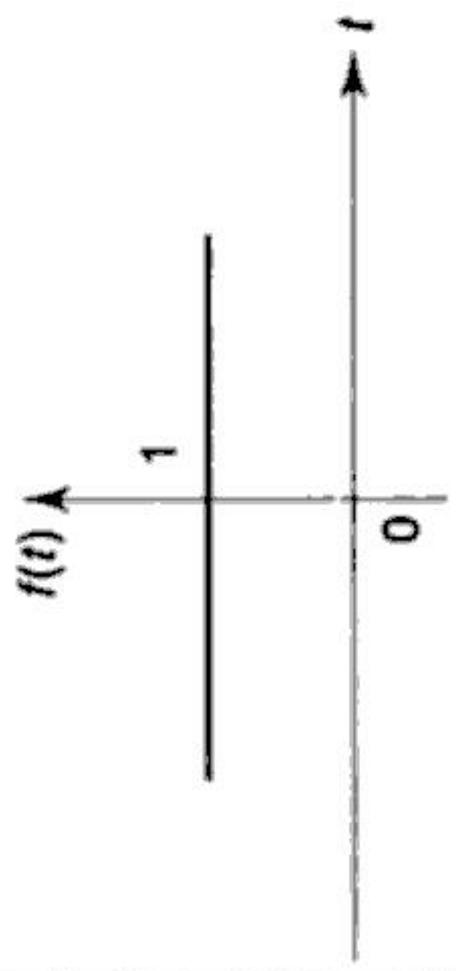
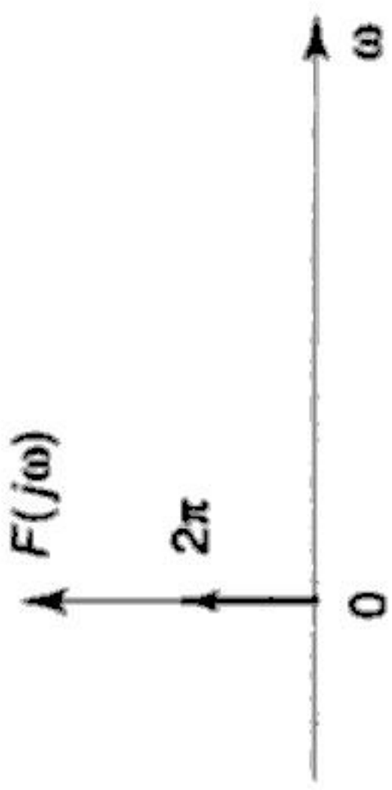
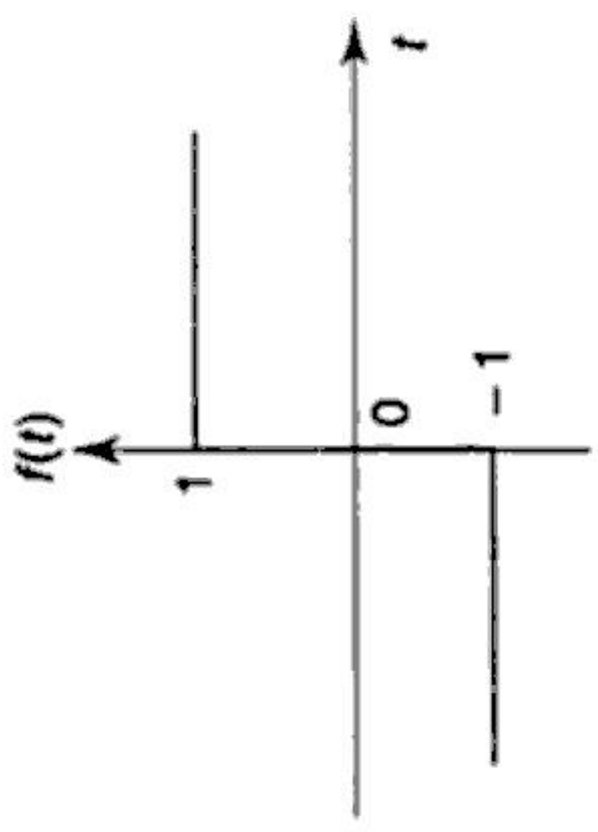
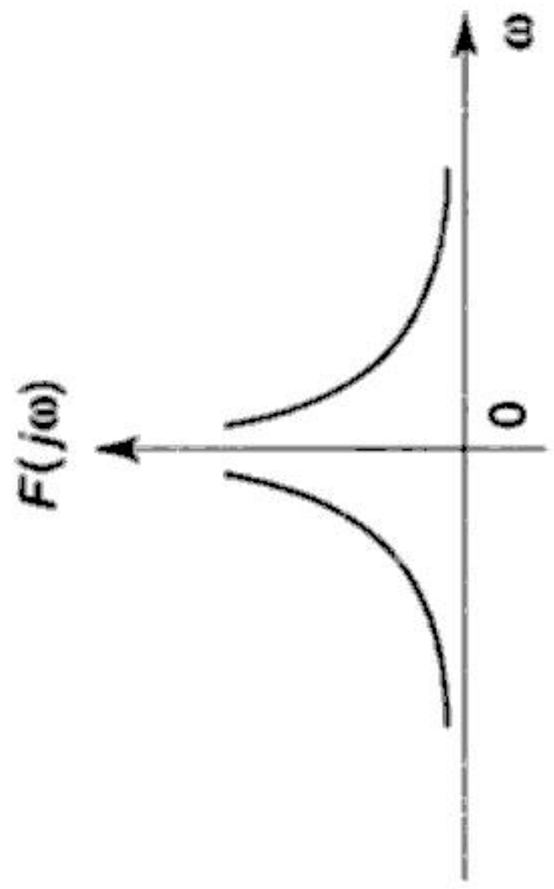
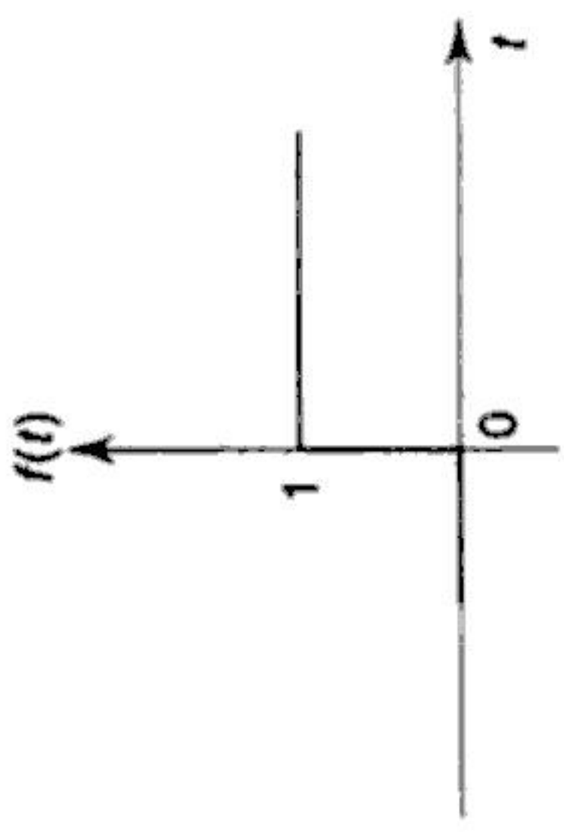
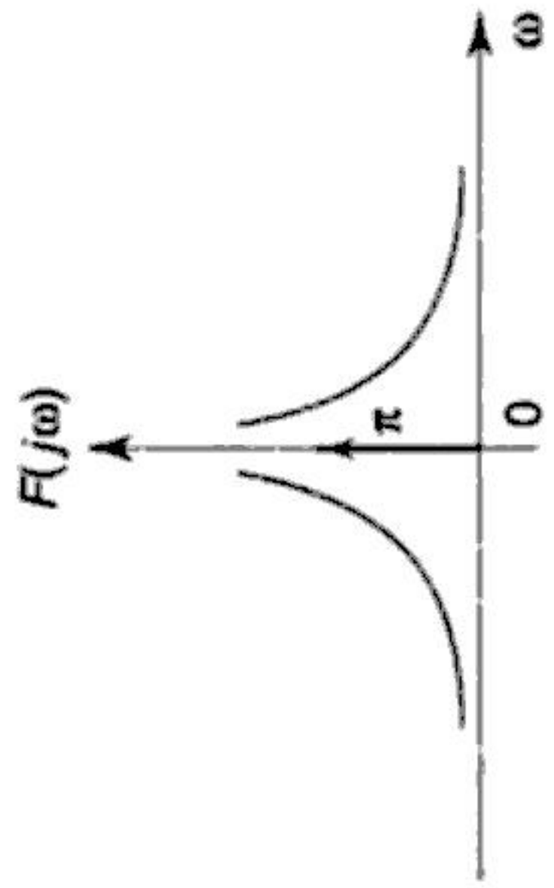
Sl. No.	*Time domain f (t)*	*Frequency domain F (jω)*	
5.		1	$2\pi\,\delta(\omega)$ 
6.		$f(t) = \text{sgn}(t) = \frac{t}{\lvert t \rvert}$	$\frac{2}{j\omega}$ 
7.		$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$ 

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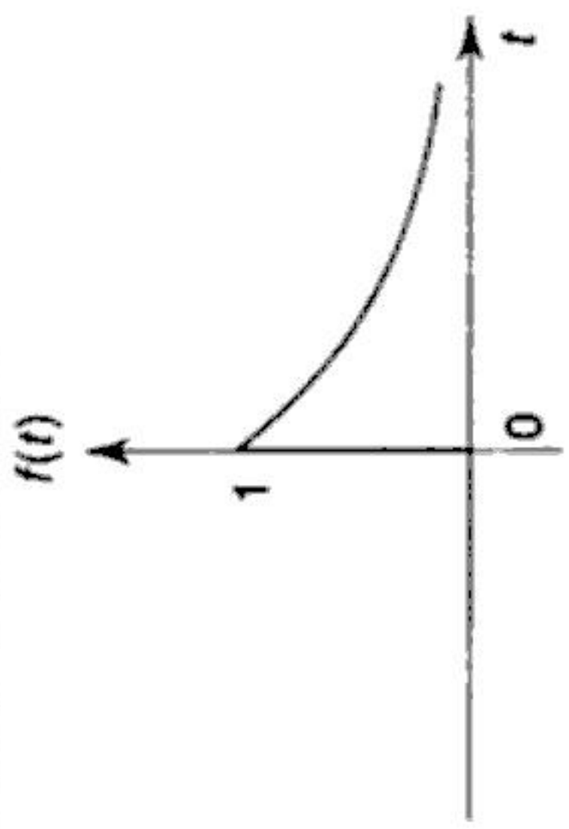
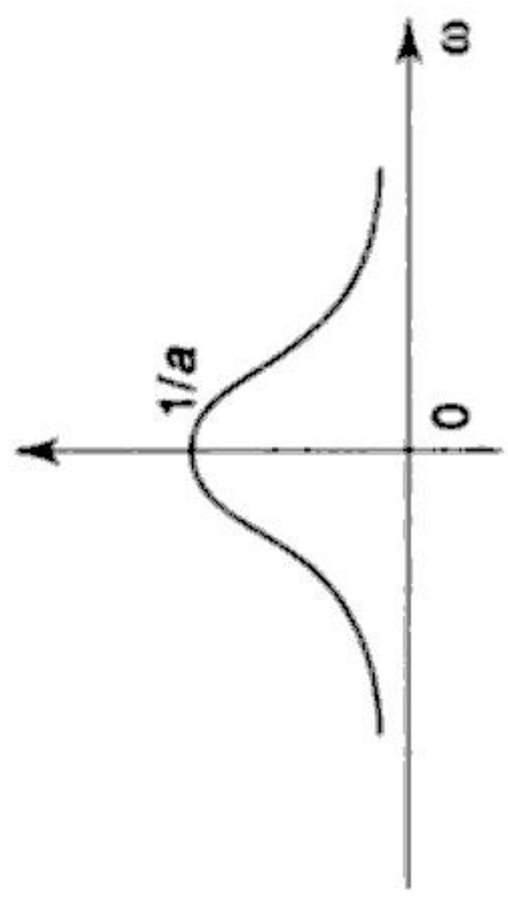
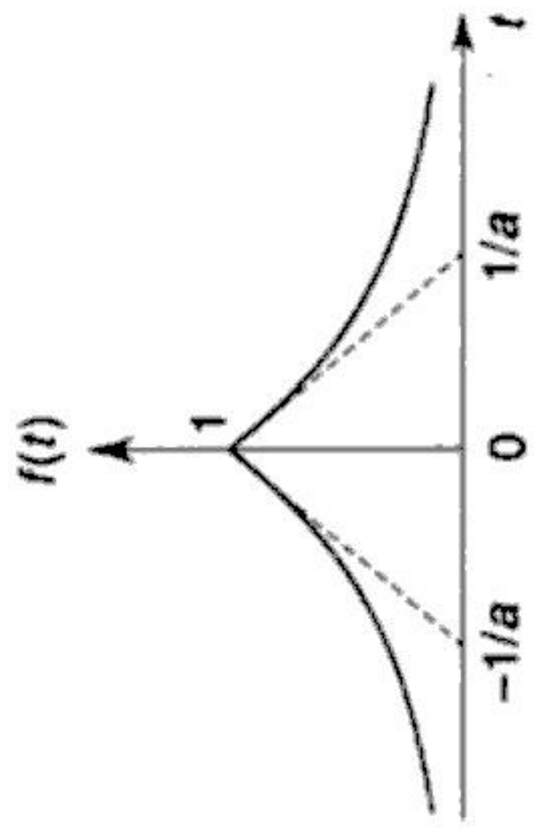
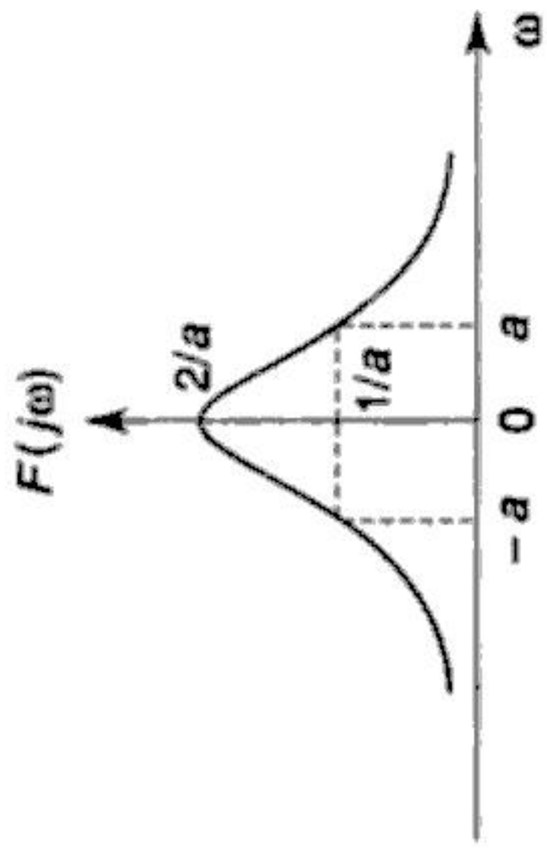
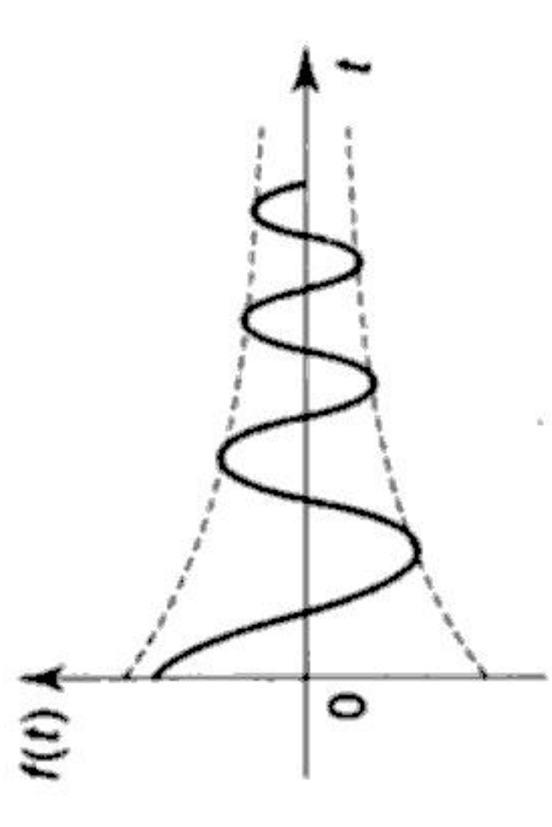
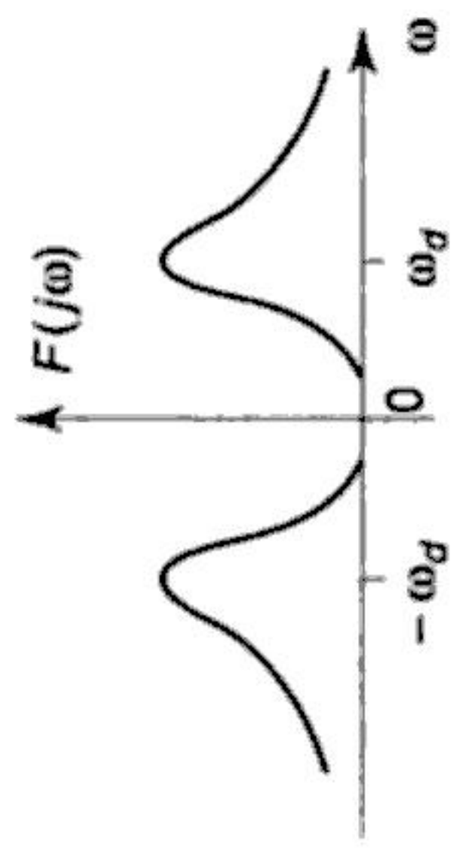
Sl. No.	Time domain $f(t)$	Frequency domain $F(j\omega)$
8.	 $e^{-at}\,u(t)$	$\dfrac{1}{a+j\omega}$ 
9.	 Double exponential $e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$ 
10.	 $e^{-at}\cos\omega_0 t\cdot u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$ 

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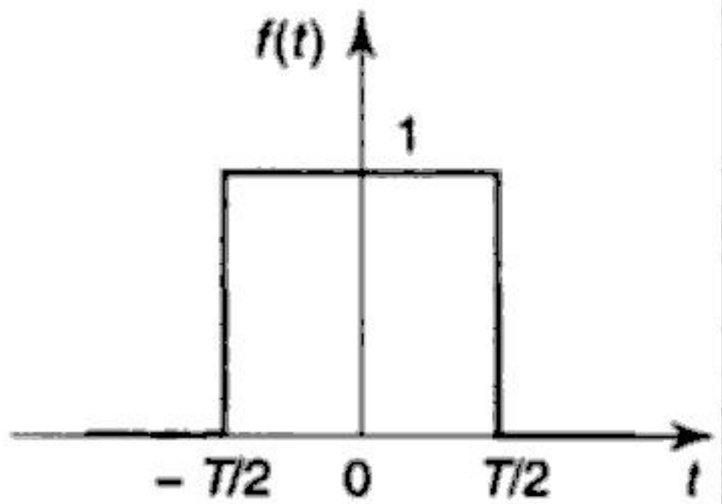
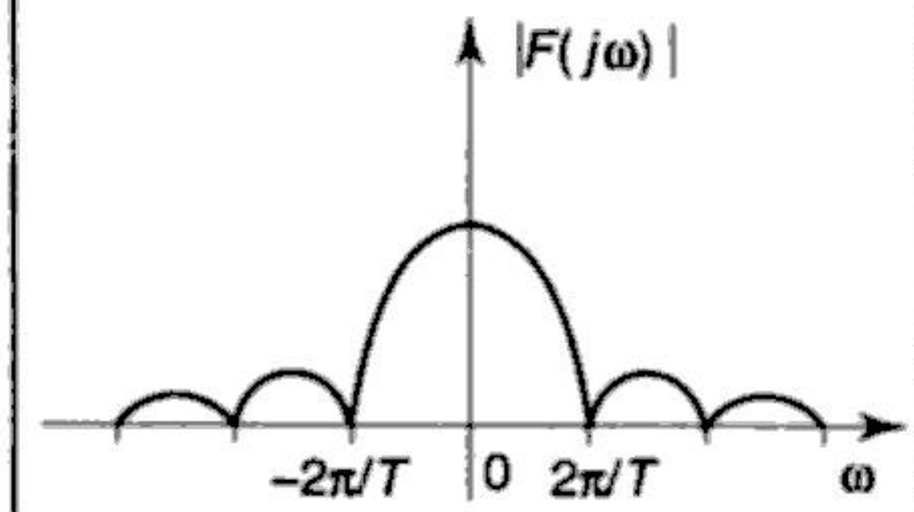
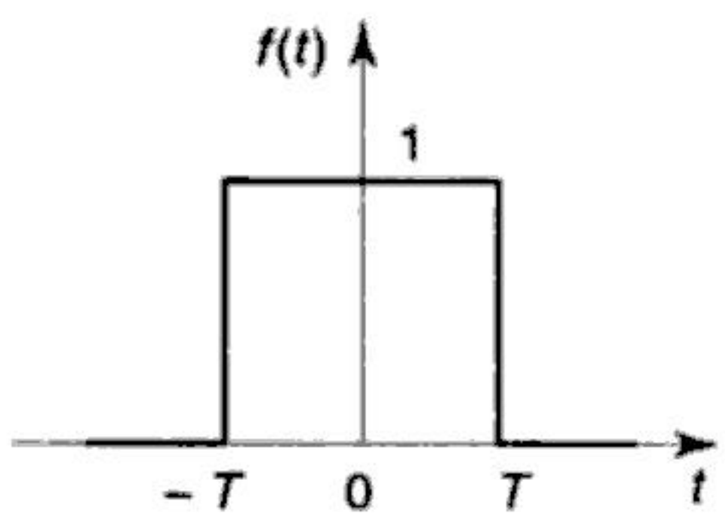
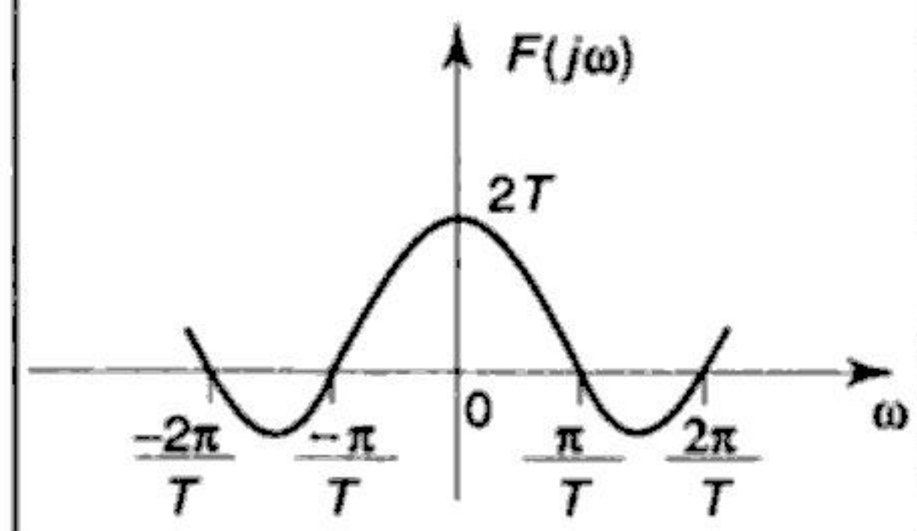
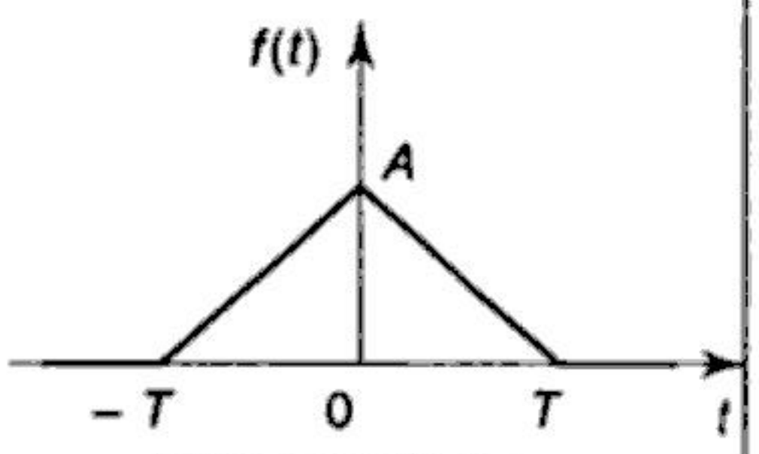
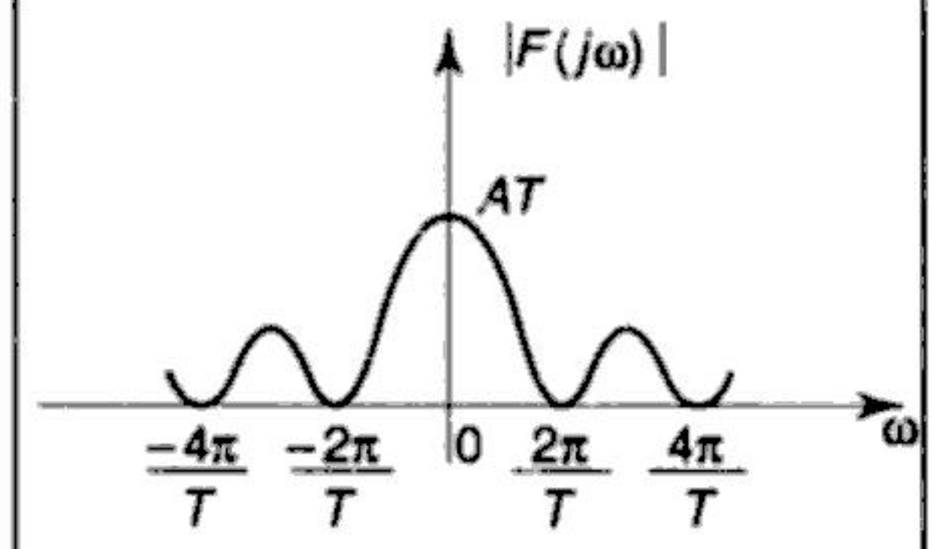
Sl. No.	Time domain $f(t)$		Frequency domain $F(j\omega)$	
11.		$u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)$	$T\dfrac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$	
12.		$f(t) = 1$, for $\|t\| \le T$ $= 0$, otherwise $f(t) = u(t+T) - u(t-T)$	$2T\dfrac{\sin(\omega T)}{\omega T} = 2T\,\text{sinc}(\omega T)$	
13.	 Triangular Pulse	$f(t) = A\left[1-\frac{\|t\|}{T}\right], \|t\| \le T$ $= 0$, elsewhere	$AT\left[\dfrac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right]^2 = AT\,\text{sinc}^2\left(\dfrac{\omega T}{2}\right)$	

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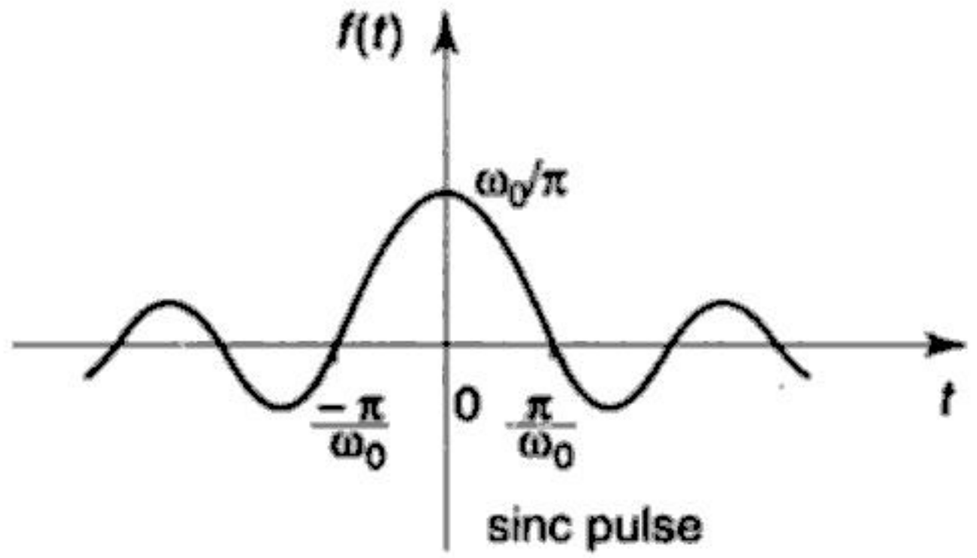
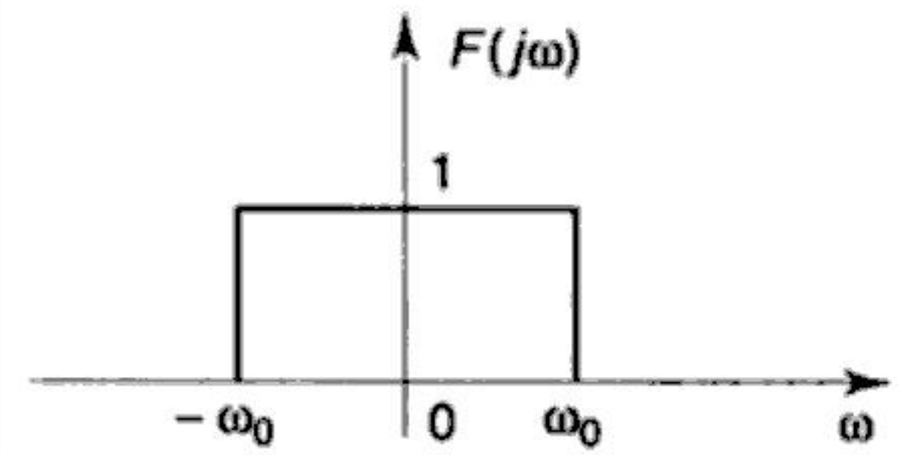
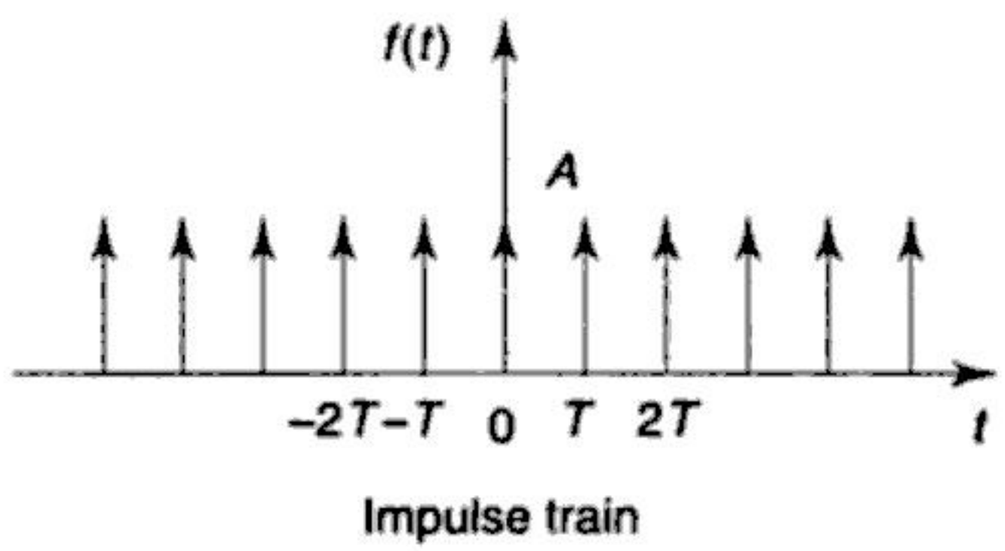
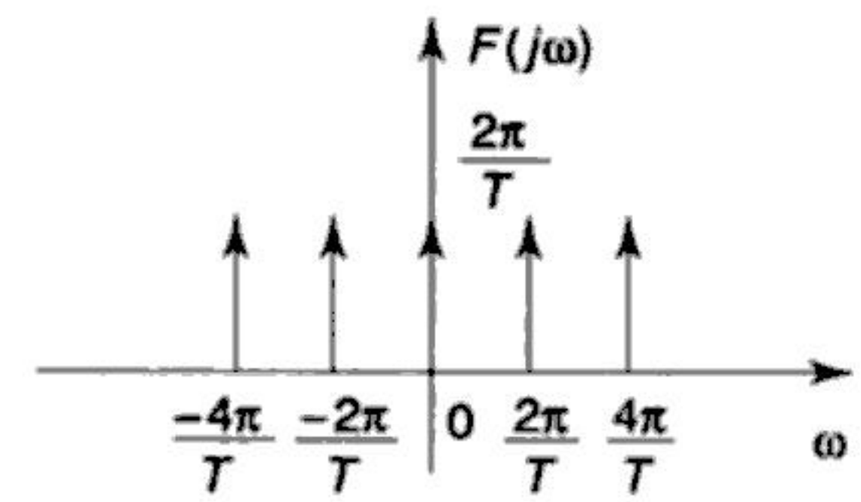
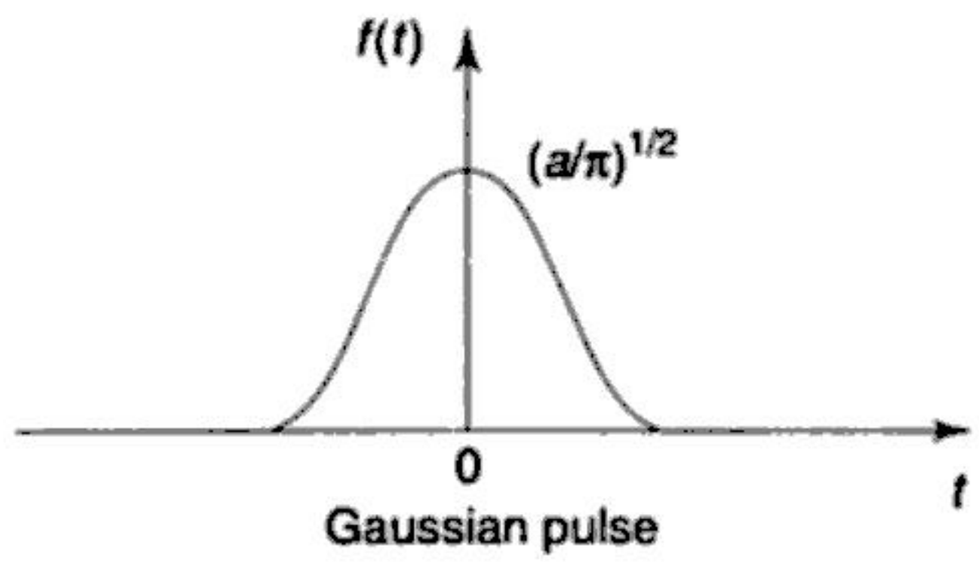
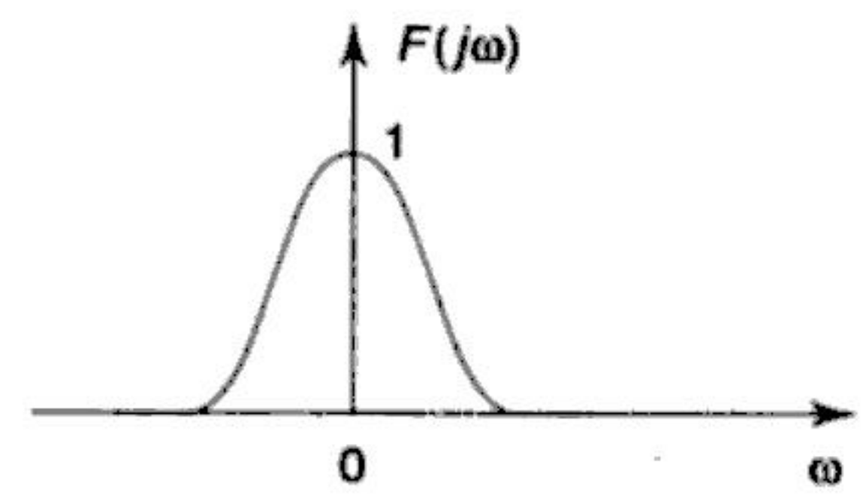
Sl. No.	Time domain $f(t)$		Frequency domain $F(j\omega)$	
14.		$\frac{\omega_0}{\pi}\,\text{sinc}\left(\frac{\omega_0 t}{\pi}\right)$	$u(\omega+\omega_0)-u(\omega-\omega_0)$	
15.		$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T}\right)$	
16.		$\left(\frac{a}{\pi}\right)^{\frac{1}{2}}\exp(-at^2)$	$\exp\left(\frac{-\pi^2 f^2}{a}\right)$	

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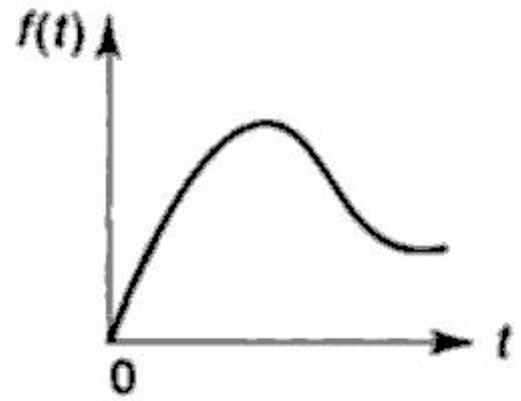
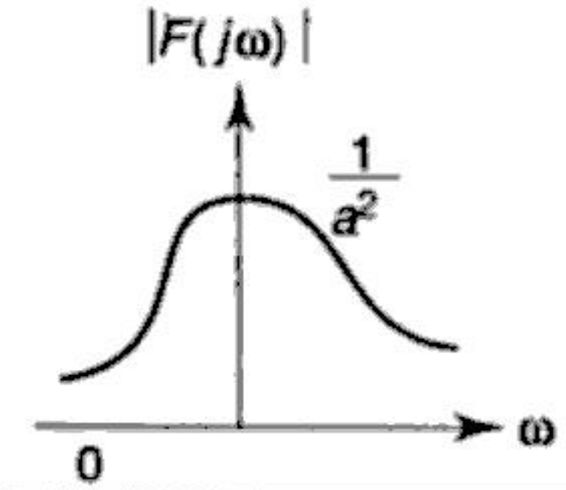
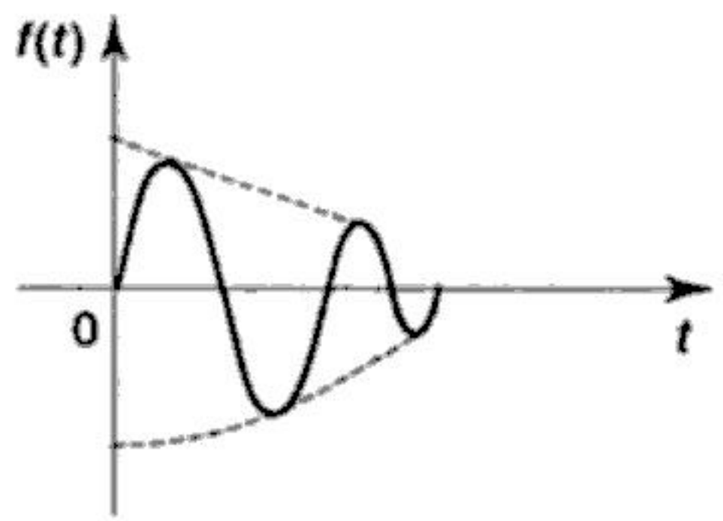
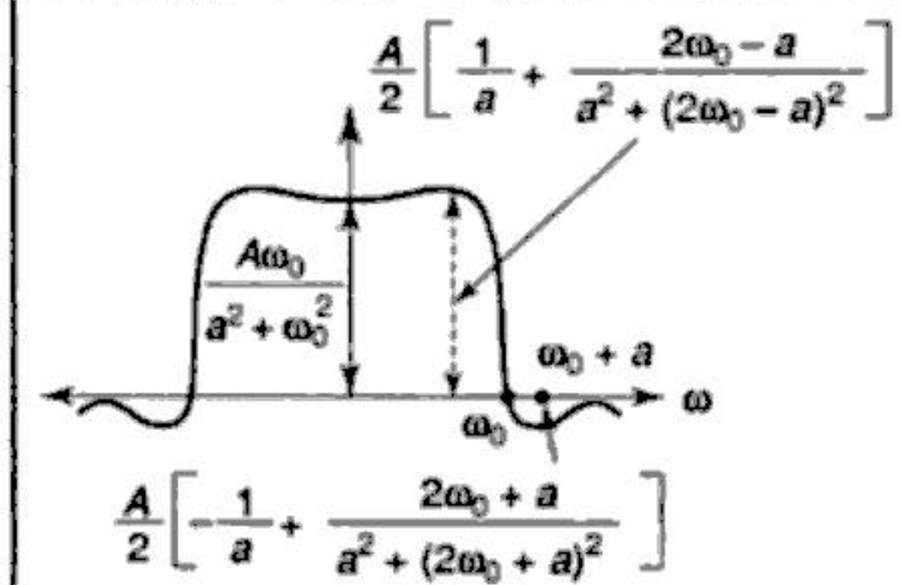
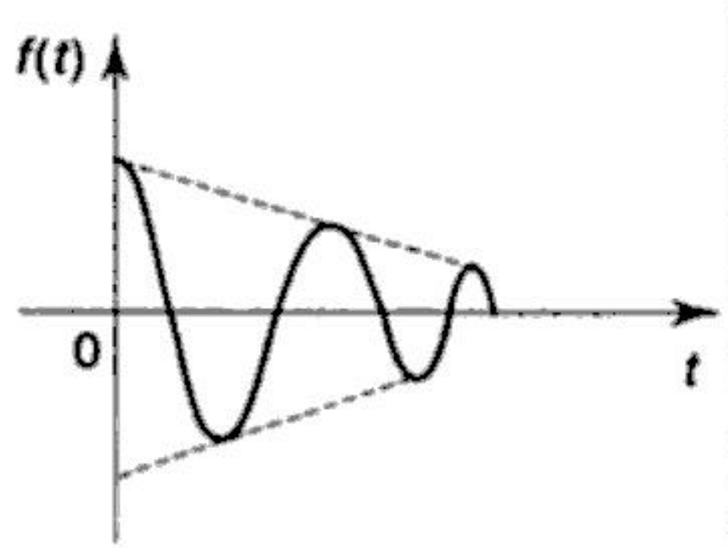
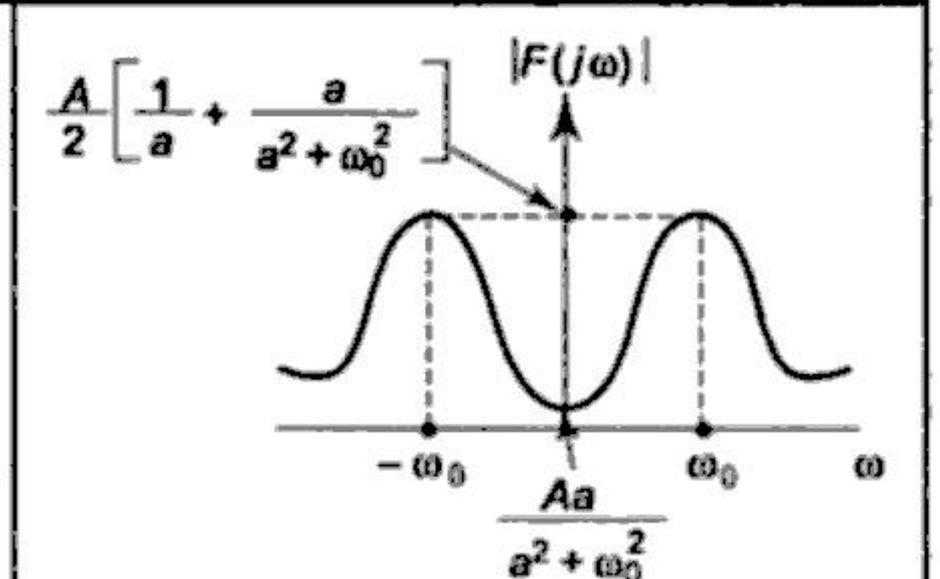
Sl. No.	Time domain $f(t)$		Frequency domain $F(j\omega)$	
17.		$t \exp(-at)\, u(t)$	$\dfrac{1}{(a+j\omega)^2}$	
18.		$\dfrac{t^{n-1}}{(n-1)!} \exp(-at)\, u(t)$	$\dfrac{1}{(a+j\omega)^n}$	
19.		$A \exp(-at) \sin(\omega_0 t)\, u(t)$	$\dfrac{A\omega_0}{(a+j\omega)^2+\omega_0^2}$	
20.		$A \exp(-at) \cos(\omega_0 t)\, u(t)$	$A \cdot \dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	
21.		$\cos\omega_0 t\, [u(t+T)-u(t-T)]$	$T\left[\dfrac{\sin(\omega-\omega_0)T}{(\omega-\omega_0)T}+\dfrac{\sin(\omega+\omega_0)T}{(\omega+\omega_0)T}\right]$	

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2.8.1 Gate Function

Let us consider the single gate function (rectangular pulse) shown in Fig. 2.4. It has the analytic expression given by

$$f(t) = \begin{cases} 1, & \text{for } -T/2 < t < T/2 \\ 0, & \text{otherwise} \end{cases}$$

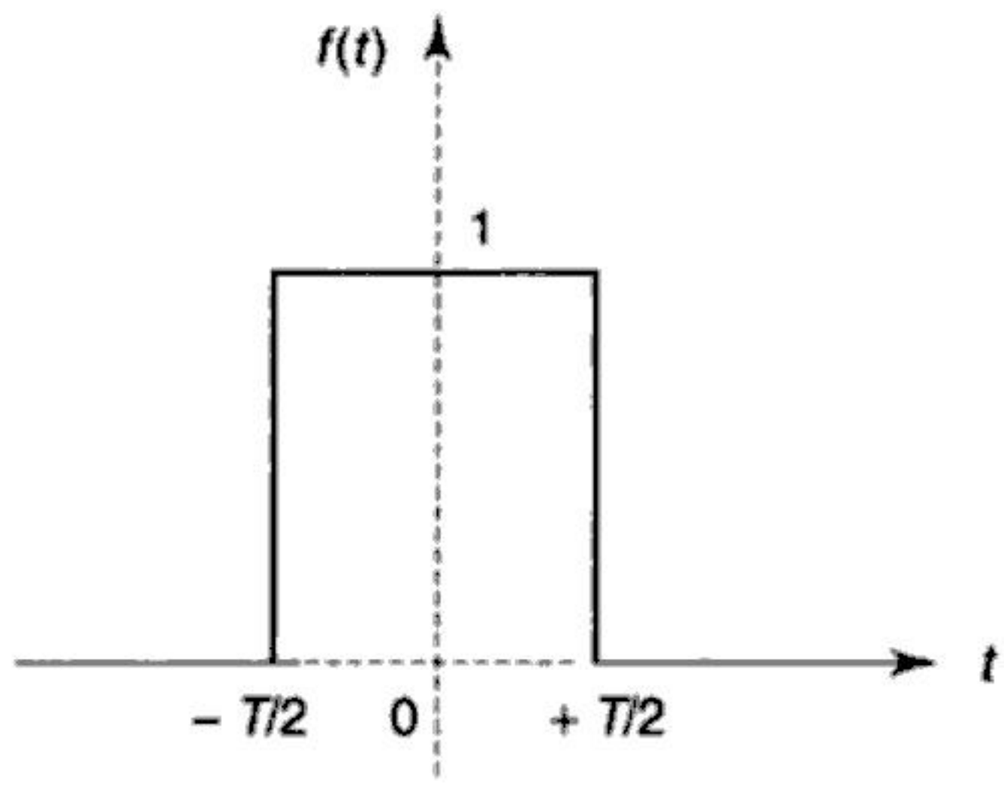

Fig. 2.4 *Single Gate Function*

The Fourier transform of $f(t)$ is

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

$$= T \cdot \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

Hence, the amplitude spectrum is

$$|F(j\omega)| = T\left|\operatorname{sinc}\left(\frac{\omega T}{2}\right)\right|$$

and the phase spectrum is $\angle F(\omega) = \begin{cases} 0, & \operatorname{sinc}\left(\frac{\omega T}{2}\right) > 0 \\ \pi & \operatorname{sinc}\left(\frac{\omega T}{2}\right) < 0 \end{cases}$

The amplitude and phase spectra are shown in Fig. 2.5.

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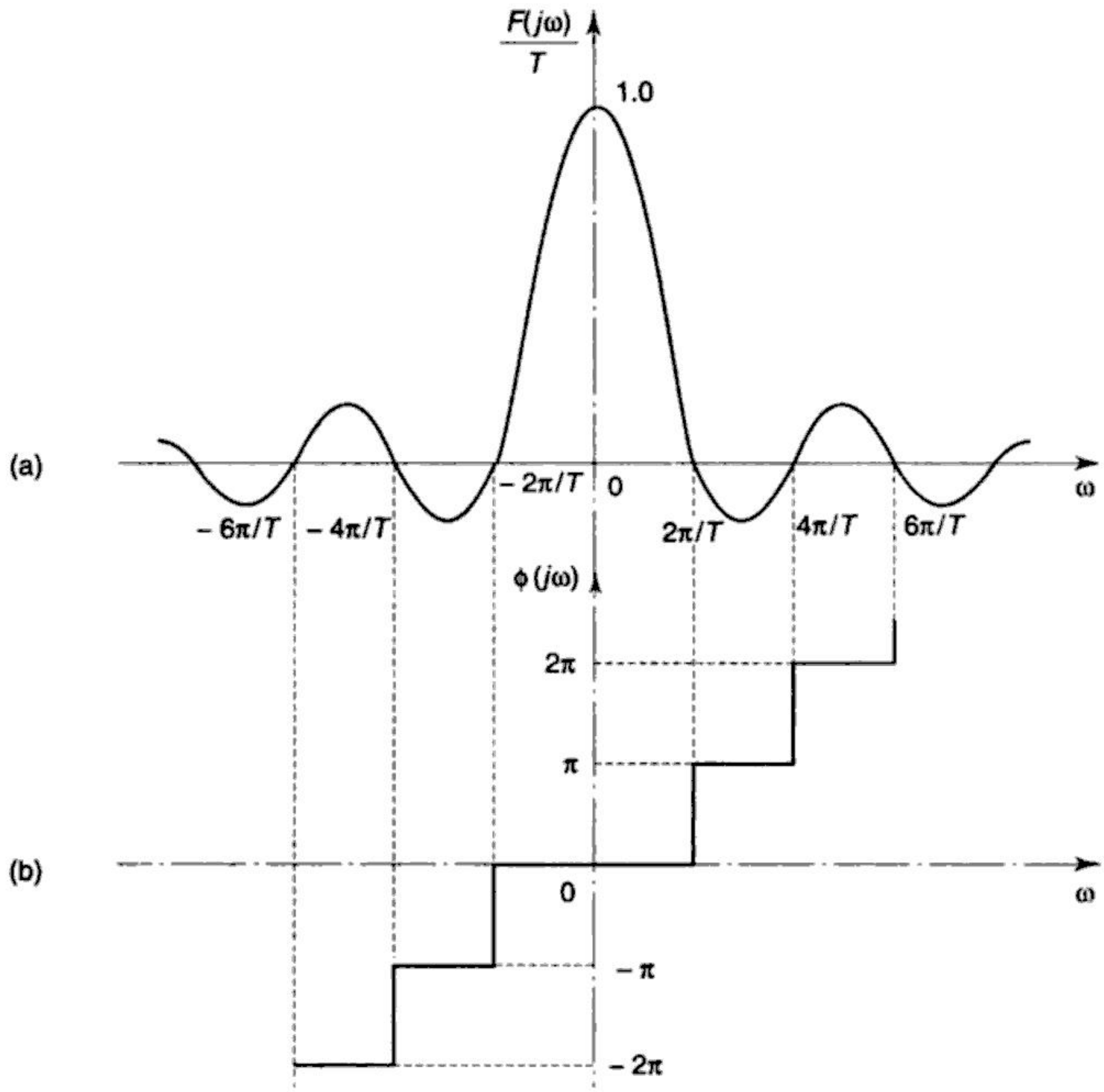

Fig. 2.5 (a) *Amplitude Spectrum and*
(b) *Phase Spectrum of the Single Gate Function*

Comments

(1) $\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$, where $x = \dfrac{\omega T}{2}$. The phase value of the amplitude is one.

(2) The zeros occur when $\dfrac{\sin x}{x} = 0$.

Therefore, $\sin x = 0$ (or) $x = n\pi$ where $n = 1, 2$, etc...

$$\text{and } x = \frac{\omega T}{2} = \pi, 2\pi, \ldots \quad \text{(or)} \quad f = \frac{1}{T}, \frac{2}{T}, \ldots$$

(3) There will be a shift in phase by π radians when the phase curve changes polarity from positive to negative or vice-versa.

Example 2.15 Determine the Fourier transform of the double gate function shown in Fig. E2.15, using time-shifting theorem.

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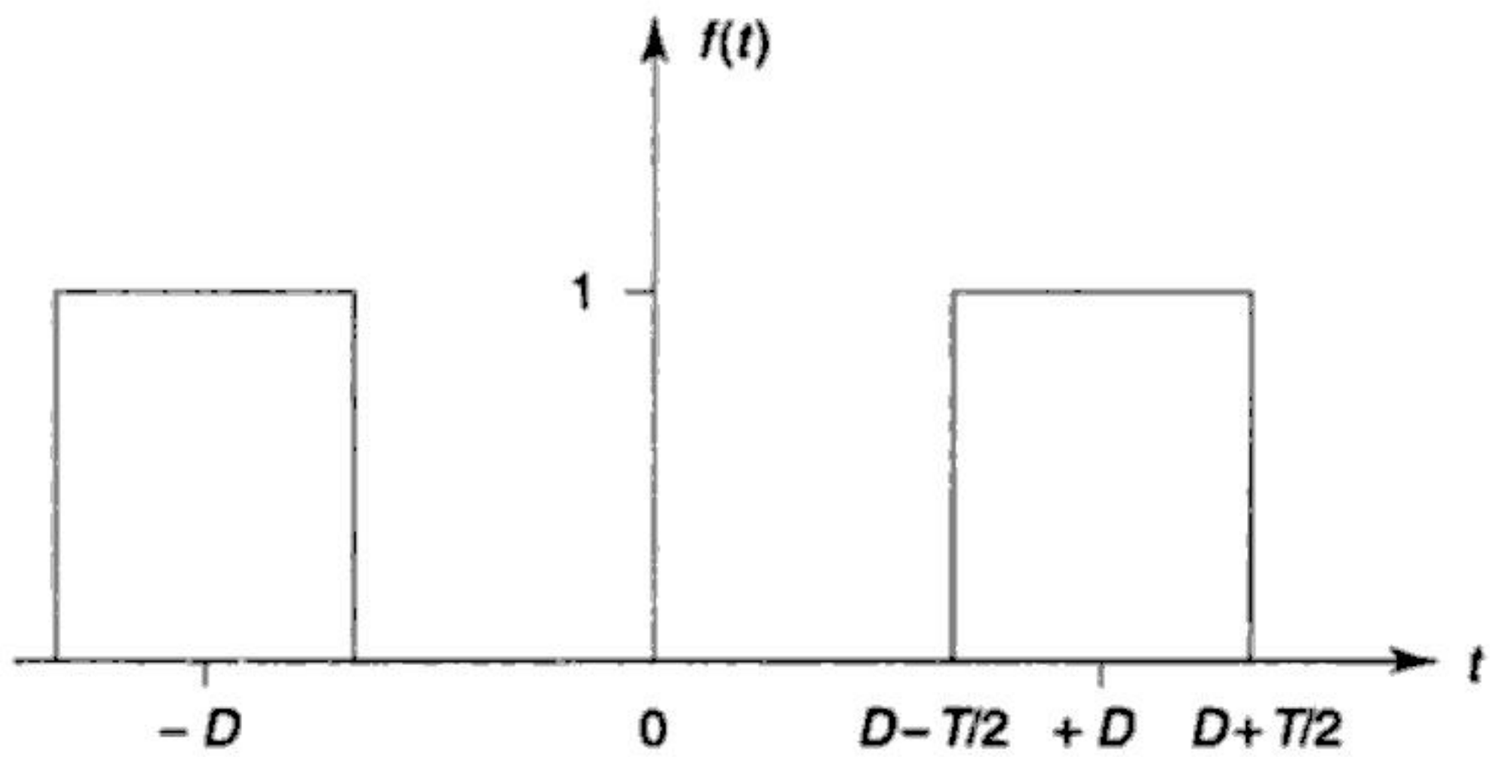

Fig. E2.15

Solution We know that the Fourier transform of the single gate function is $T \sin c \dfrac{\omega T}{2}$. Here in the double gate function, the single gate function is shifted by $+D$ and $-D$ at both sides, i.e. one gate function is centered about $+D$ and the other is centered about $-D$. Hence, by applying the shifting theorem of Fourier transform, we get

$$\mathcal{F}[f(t)] = e^{-j\omega D}\, T \operatorname{sinc}\left(\frac{\omega T}{2}\right) + e^{j\omega D}\, T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

$$= 2T \cos(\omega D) \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

2.8.2 Rectangular Pulse

Consider the rectangular pulse shown in Fig. 2.6. The analytic expression for the given pulse is

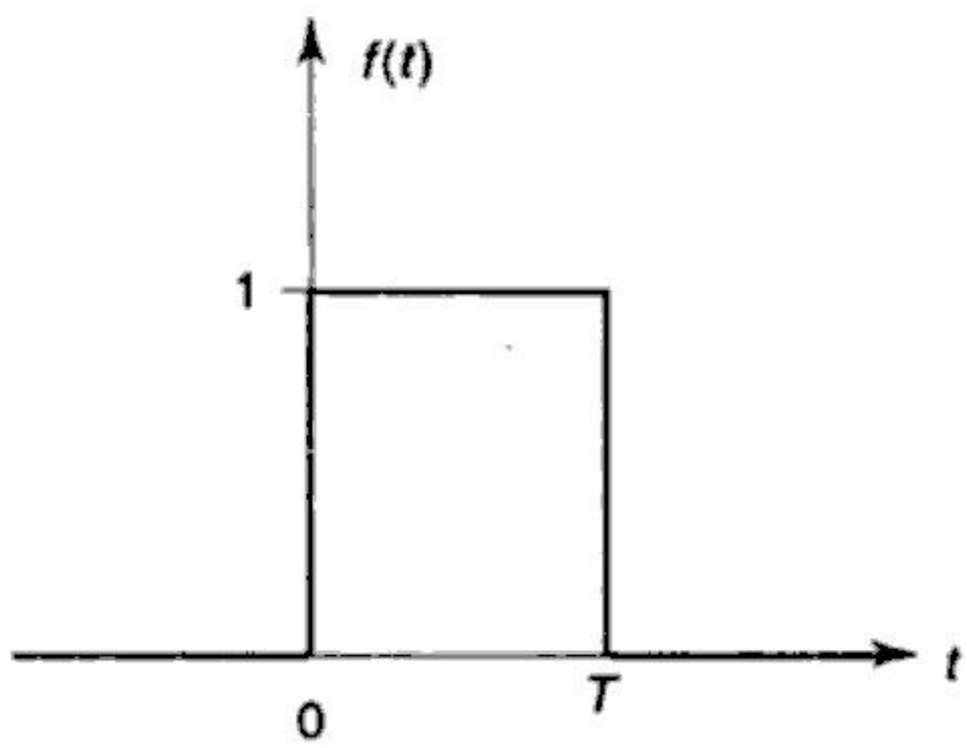

Fig. 2.6

$$f(t) = \begin{cases} 1, & \text{for } 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

The Fourier transform of $f(t)$ becomes

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

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$$= \int_0^T e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T = \frac{e^{-j\omega T} - 1}{-j\omega}$$

$$= \frac{e^{-j\omega T/2}}{-j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

$$= \frac{2e^{-j\omega T/2}}{\omega}\left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right]$$

$$= Te^{-j\omega T/2}\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}\right]$$

$$= Te^{-j\omega T/2}\,\text{sinc}\left(\frac{\omega T}{2}\right)$$

Its amplitude and phase spectra are plotted in Fig. 2.7.

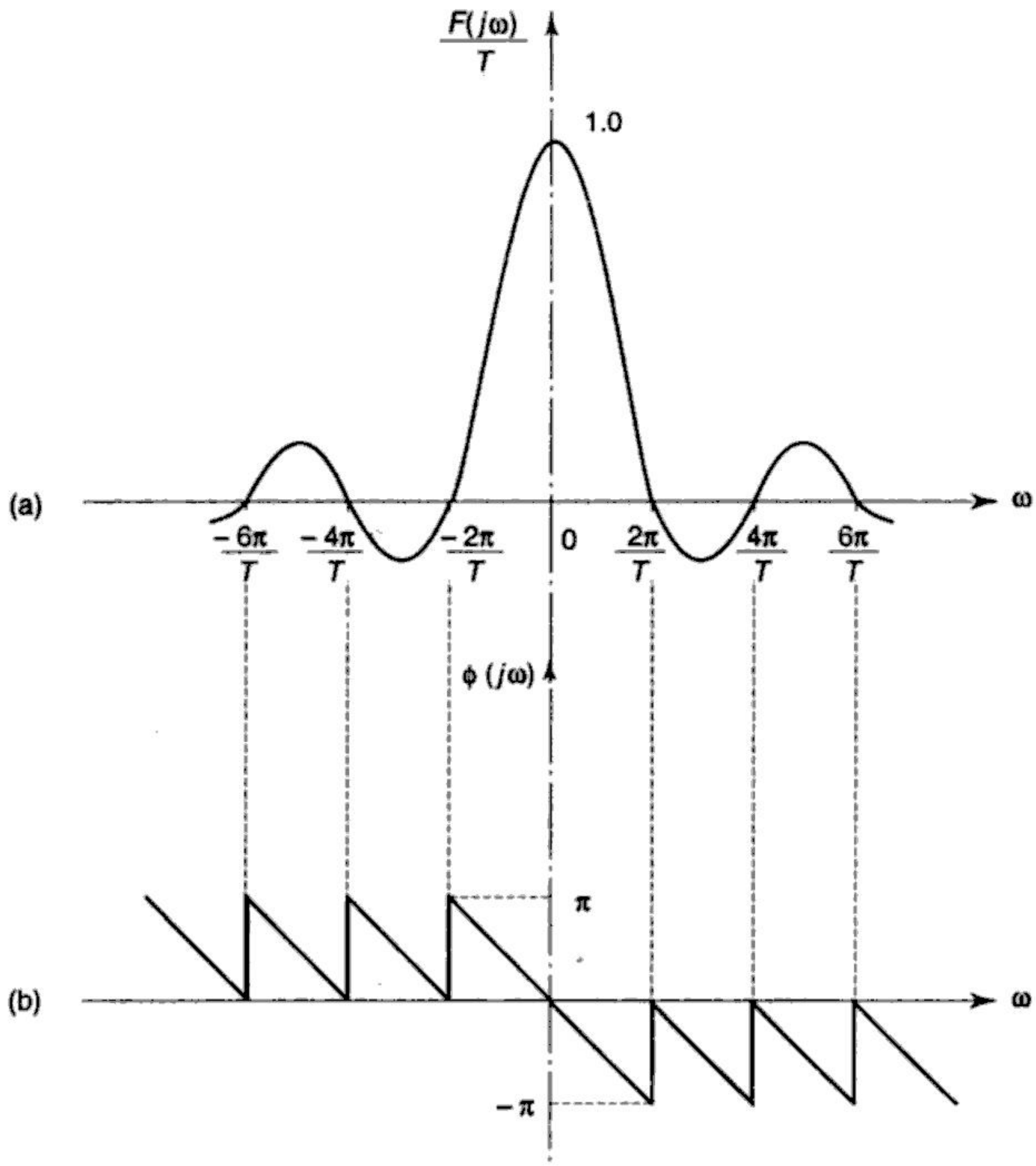

Fig. 2.7 *(a) Amplitude Spectrum and (b) Phase Spectrum of the Rectangular Pulse*

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Comments

(1) The phase of the amplitude spectrum is exactly the same as that in the previous case given in Fig. 2.7.

(2) There is an additional uniform phase shift factor $e^{-j\omega T/2}$ which changes the phase spectrum of the previous case.

(3) By using the time-shift theorem, $\mathcal{F}\left[f\left(t-\frac{T}{2}\right)\right] = F(j\omega)\, e^{-j\omega T/2}$

where $F(j\omega) = T \text{ sinc}\left(\frac{\omega T}{2}\right)$, the above result can readily be obtained.

Example 2.16 Find the Fourier transform of a rectangular pulse 2 seconds long with a magnitude of 10 volts as shown in Fig. E2.16.

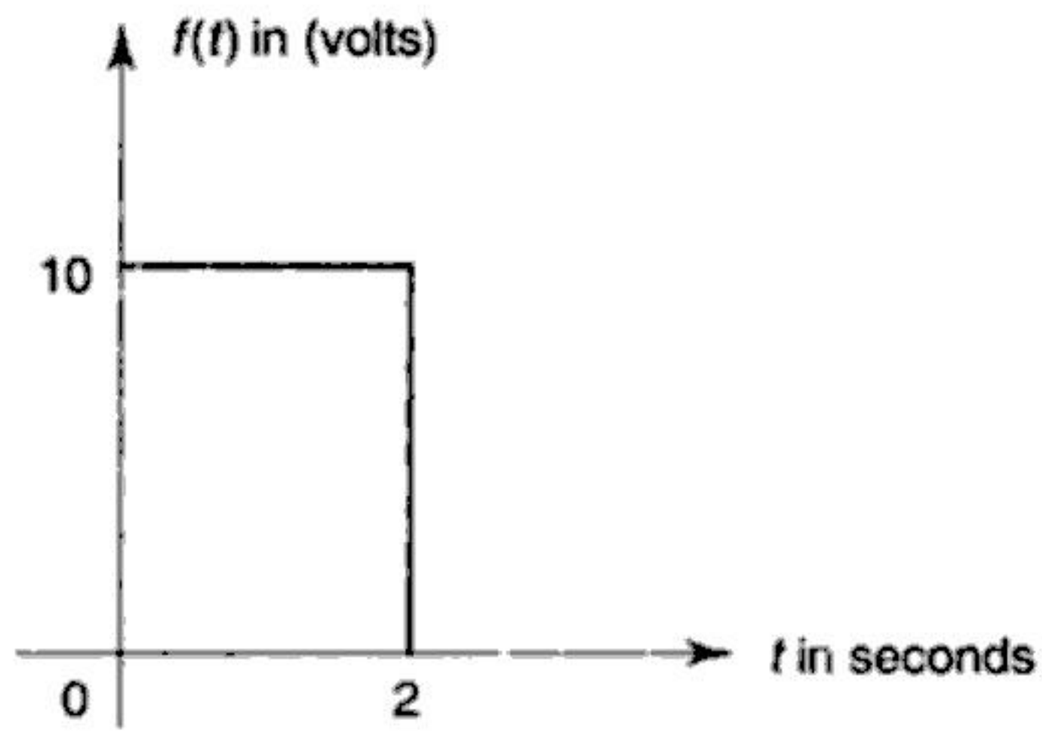

Fig. E2.16

Solution Fourier transform $F(j\omega)$ of the given pulse is given by

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{2} 10\, e^{-j\omega t}\, dt = 10\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^2$$

$$= 10\left(\frac{e^{-j2\omega}-1}{-j\omega}\right)$$

$$= 10\,\frac{e^{-j\omega}}{-j\omega}\left[e^{-j\omega} - e^{j\omega}\right]$$

$$= 20\,\frac{e^{-j\omega}}{\omega}\left[\frac{e^{j\omega}-e^{-j\omega}}{2j}\right]$$

$$= 20\, e^{-j\omega}\,\frac{\sin\omega}{\omega}$$

$$= 20\, e^{-j\omega}\, \text{sinc}\,\omega$$

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Alternate Method

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{2} 10\, e^{-j\omega t}\, dt = 10\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^2$$

$$= 10\left(\frac{e^{-j2\omega} - 1}{-j\omega}\right) = j\,10\left(\frac{e^{-j2\omega} - 1}{\omega}\right)$$

$$= j\,10\left(\frac{\cos 2\omega - j\sin 2\omega - 1}{\omega}\right)$$

$$= \frac{10}{\omega}\,[\sin 2\omega + j(\cos 2\omega - 1)]$$

Therefore, $|F(j\omega)| = \dfrac{10}{\omega}\sqrt{\sin^2 2\omega + (\cos 2\omega - 1)^2}$

$$\angle F(j\omega) = \Phi(j\omega) = \tan^{-1}\left(\frac{\cos 2\omega - 1}{\sin 2\omega}\right)$$

2.8.3 Exponential Pulse

The analytic expression for the exponential pulse shown in Fig. 2.8 is

$$f(t) = e^{-at}, \text{ for } t \geq 0$$
$$= 0, \quad \text{for } t < 0$$

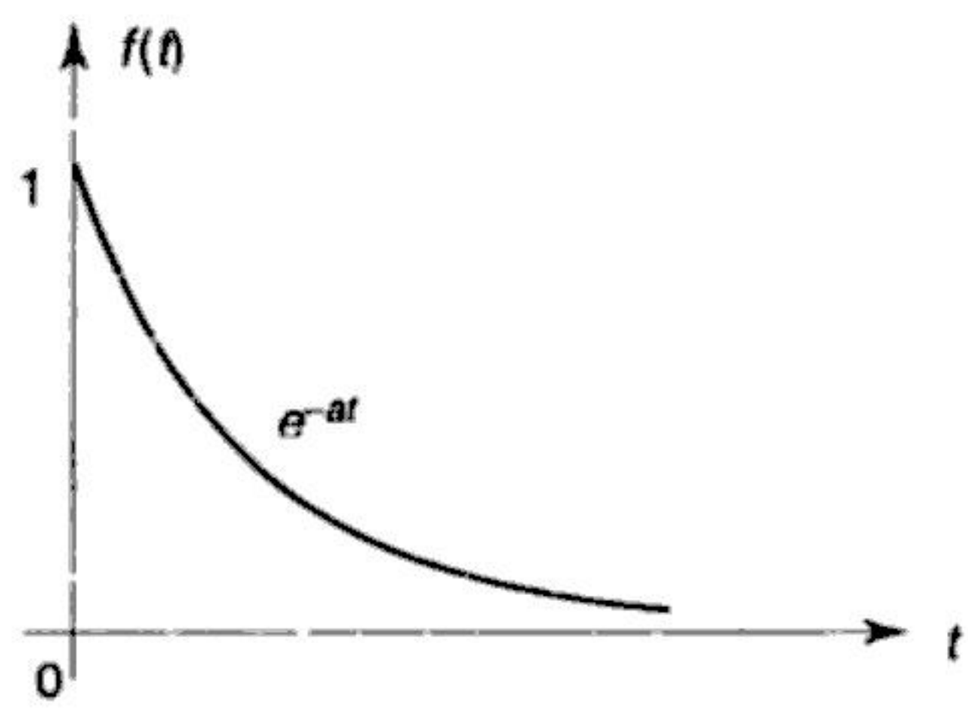

Fig. 2.8

Its Fourier transform is

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt, \text{ since } f(t) = 0 \text{ for } t < 0$$

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$$= \int_0^\infty e^{-(a+j\omega)t}\, dt = \frac{1}{a+j\omega}$$

$$= \frac{1}{\sqrt{a^2+\omega^2}} \Big\lfloor -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Therefore, $|F(j\omega)| = \dfrac{1}{(a^2+\omega^2)^{1/2}}$ and $\Phi(j\omega) = -\tan^{-1}\left(\dfrac{\omega}{a}\right)$

The amplitude and phase spectra of the exponential pulse are shown in Figs. 2.9(a) and (b).

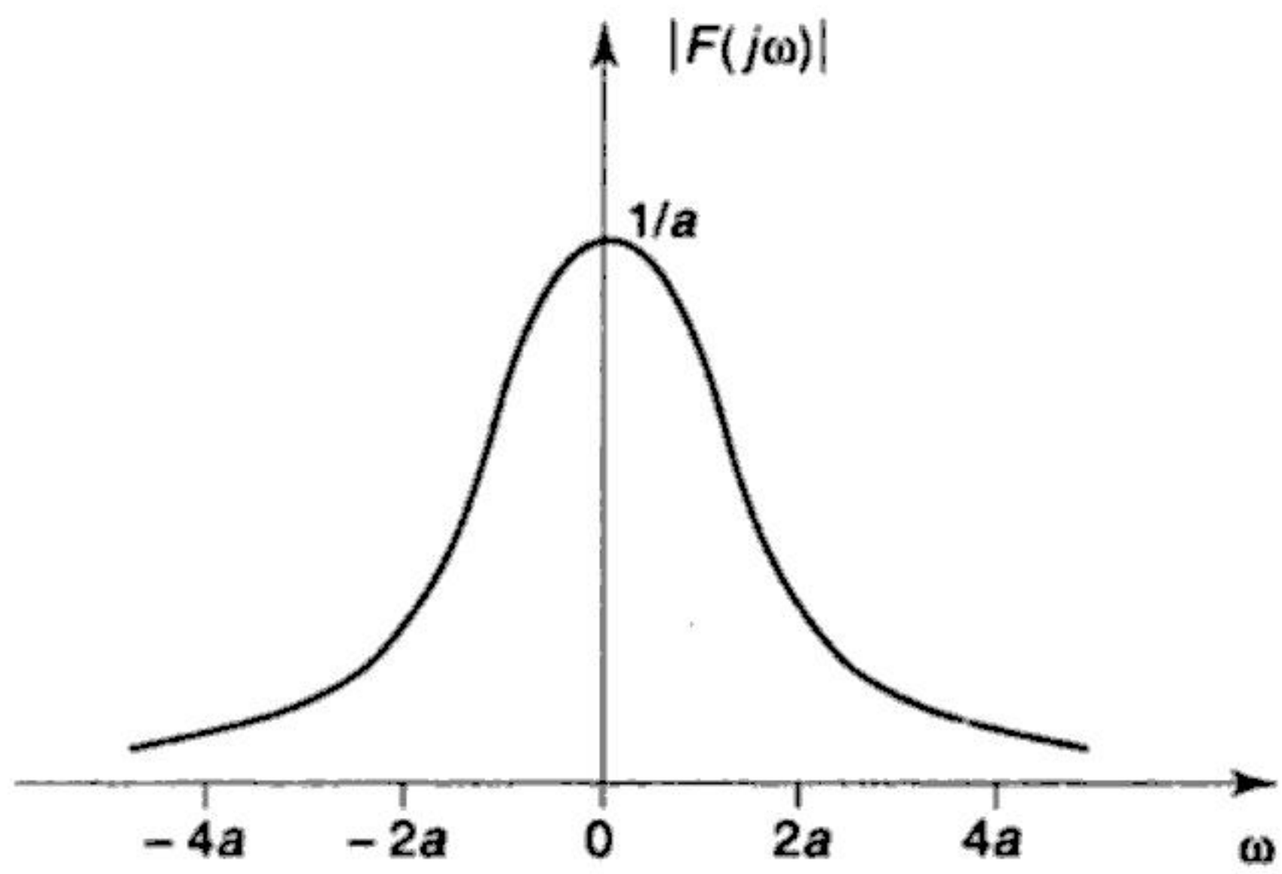

Fig. 2.9(a) *Amplitude Spectrum of the Exponential Pulse*

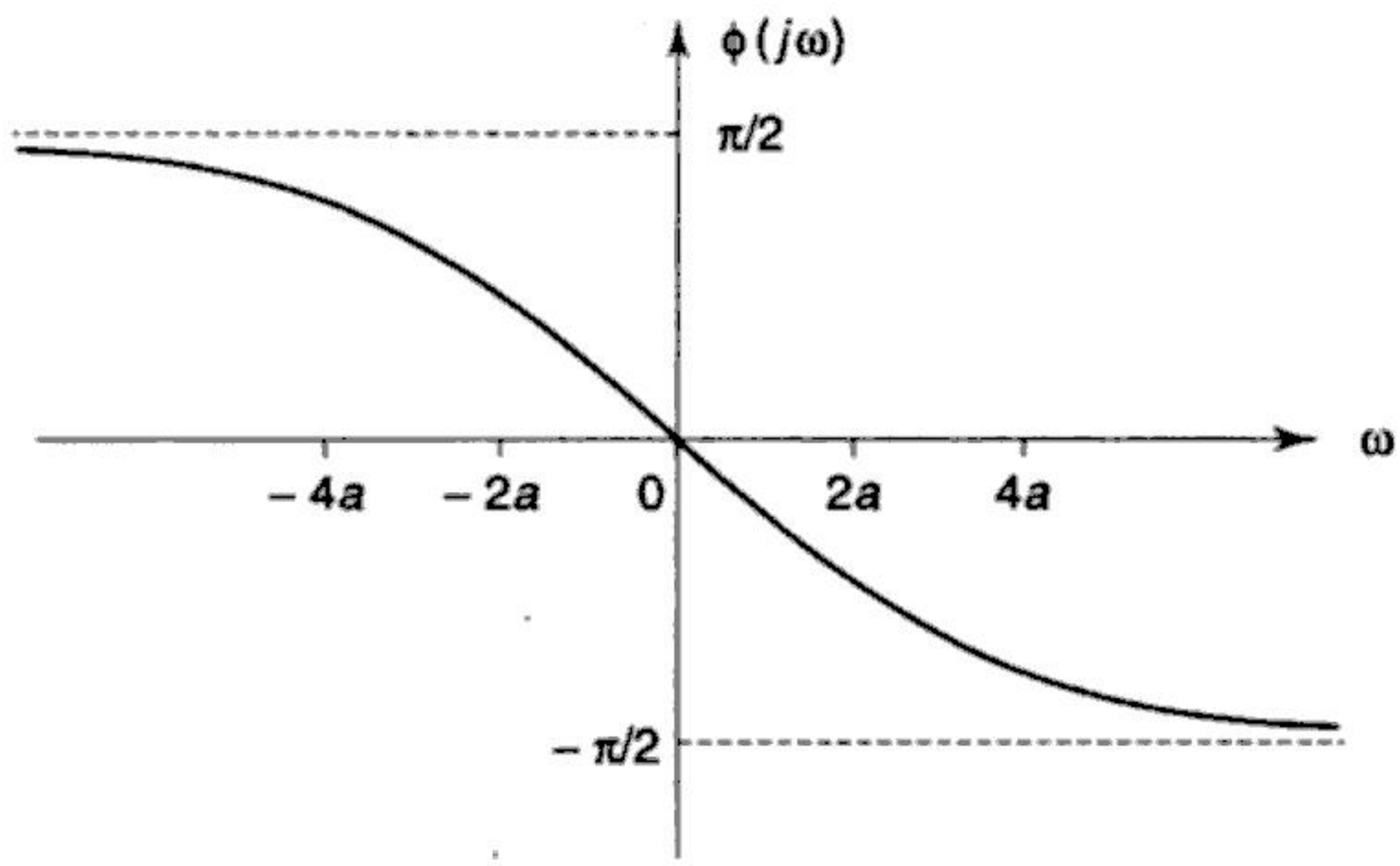

Fig. 2.9(b) *Phase Spectrum of the Exponential Pulse*

The energy spectral density is $S(\omega) = \dfrac{1}{\pi}|F(j\omega)|^2$

$$= \frac{1}{\pi}\left|\frac{1}{a+j\omega}\right|^2 = \frac{1}{\pi(a^2+\omega^2)}$$

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Example 2.17 Determine the Fourier transform for the double exponential pulse shown in Fig. E2.17(a) whose function is given by $f(t) = e^{-a|t|}$.

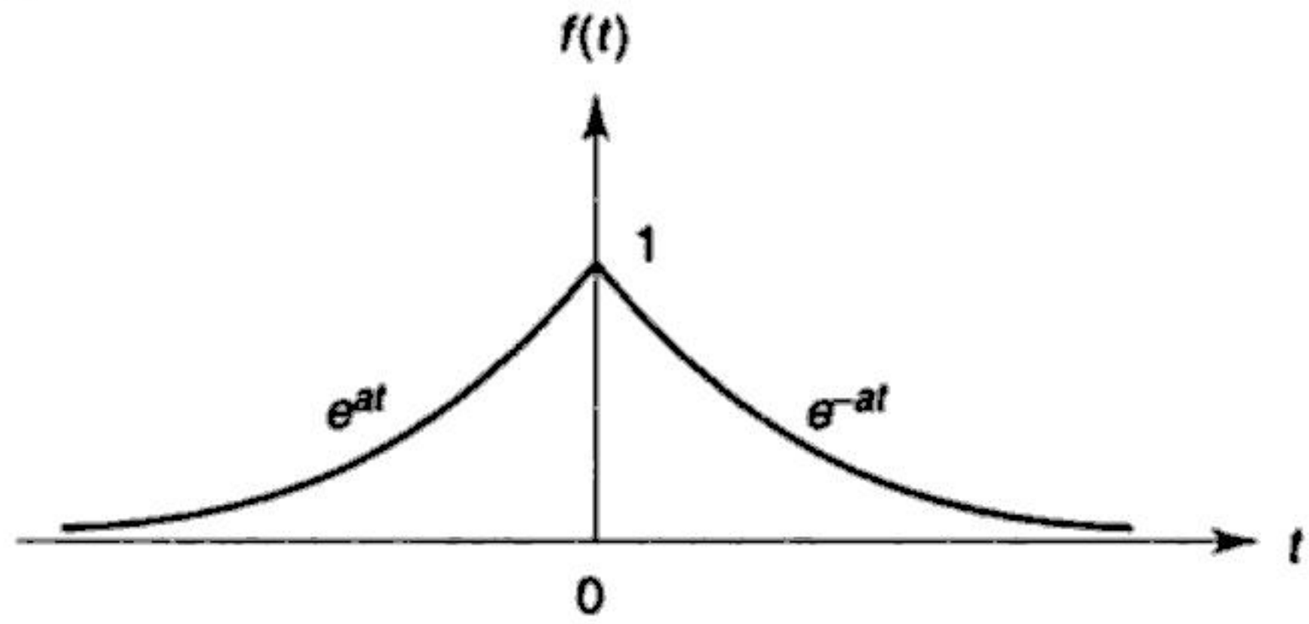

Fig. E2.17(a) *Double Exponential Pulse*

Solution The expression for the double exponential pulse is $f(t) = e^{-a|t|}$

Hence, $f(t) = e^{at}$ for $t < 0$

$= e^{-at}$ for $t \geq 0$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

Therefore, $F(j\omega) = \dfrac{2a}{a^2+\omega^2}$.

The spectrum of the double exponential pulse is shown in Fig. E2.17(b).

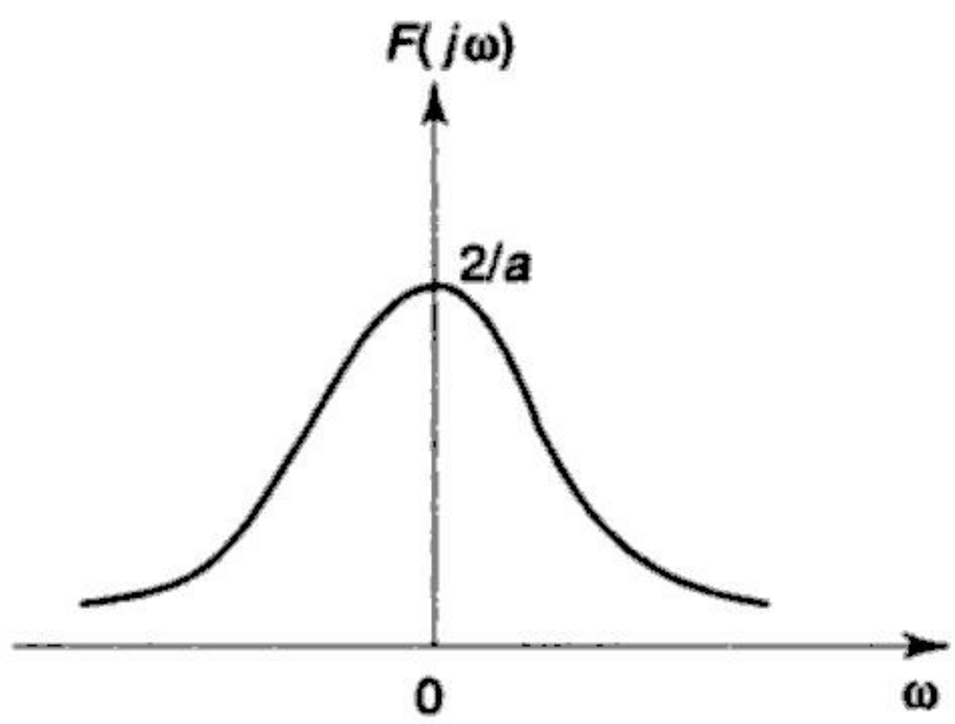

Fig. E2.17 (b) *Spectrum of the Double Exponential Pulse*

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2.8.4 Triangular Pulse

Consider the triangular pulse shown in Fig. 2.10.

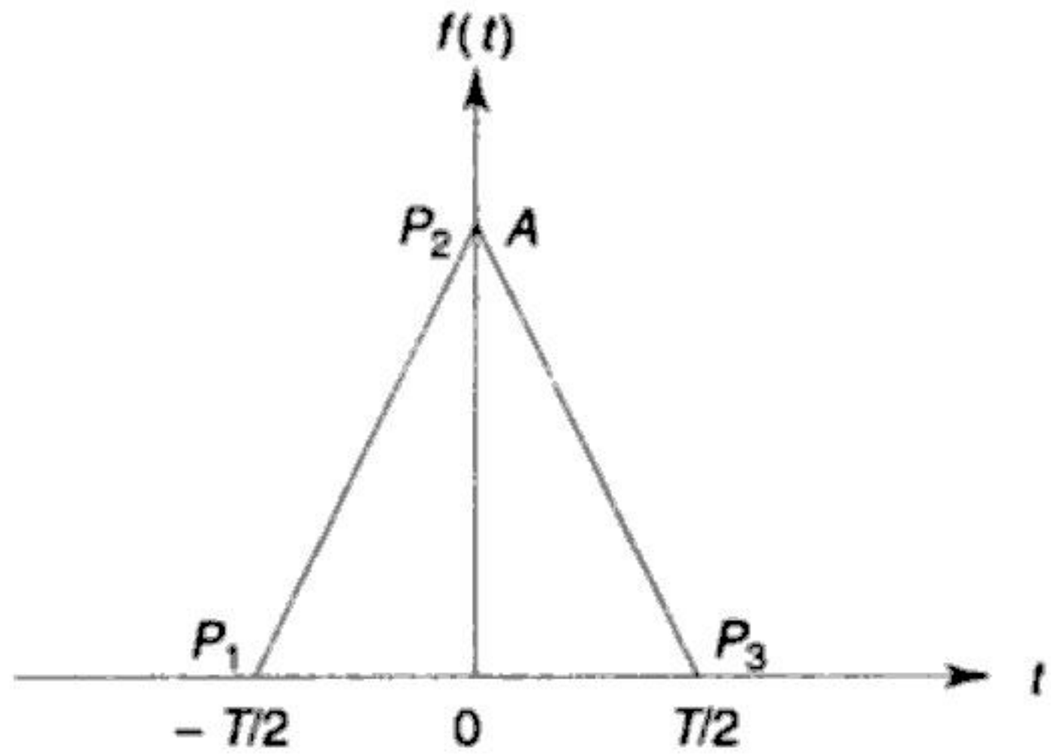

Fig. 2.10 *Triangular Pulse*

Equation of line P_1P_2 is

$$f(t)_{P_1P_2} = \frac{A}{T/2}t + A = A\left(1 + \frac{2}{T}t\right)$$

Equation of line P_2P_3 is

$$f(t)_{P_2P_3} = -\frac{A}{T/2}t + A = A\left(1 - \frac{2}{T}t\right)$$

$$\text{Therefore } f(t) = A\left(1 + \frac{2}{T}t\right) \quad \text{for} \quad -\frac{T}{2} < t \le 0$$

$$= A\left(1 - \frac{2}{T}t\right) \quad \text{for} \quad 0 \le t \le \frac{T}{2}$$

Now,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{T/2} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{0} A\left(1 + \frac{2}{T}t\right) e^{-j\omega t}\, dt + \int_{0}^{T/2} A\left(1 - \frac{2}{T}t\right) e^{-j\omega t}\, dt$$

$$= A\int_{-T/2}^{0} e^{-j\omega t}\, dt + A\int_{0}^{T/2} e^{-j\omega t}\, dt$$

$$+ \frac{2A}{T}\int_{-T/2}^{0} t\, e^{-j\omega t}\, dt - \frac{2A}{T}\int_{0}^{T/2} t\, e^{-j\omega t}\, dt$$

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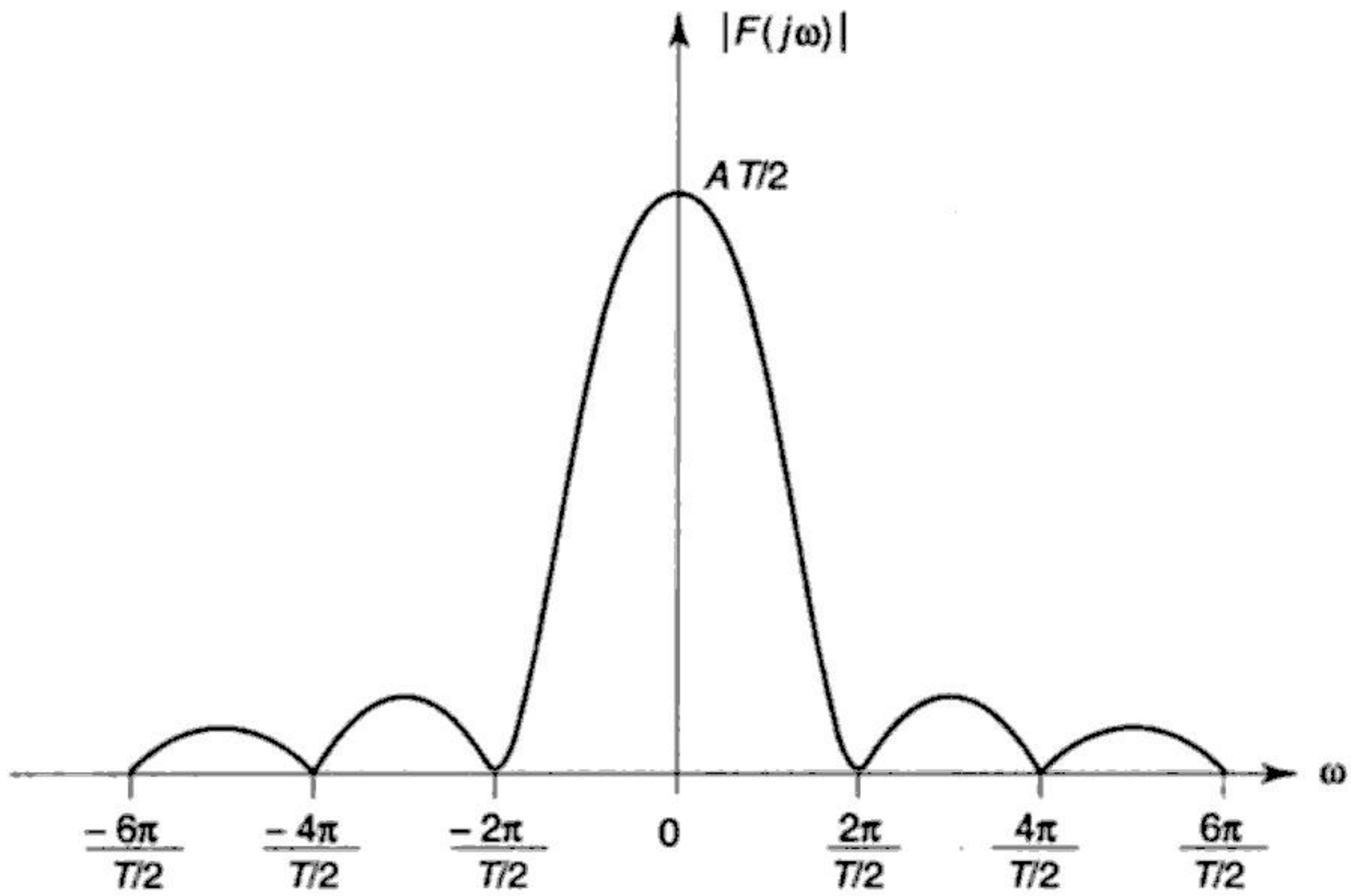

Fig. 2.11 *The Frequency Spectrum of a Single Triangular Pulse*

Changing t to $-t$ in the first and third integrals, we obtain

$$F(j\omega) = A\left[\int_0^{T/2} e^{j\omega t}\,dt + \int_0^{T/2} e^{-j\omega t}\,dt\right] - \frac{2A}{T}\left[\int_0^{T/2} t\,e^{j\omega t}\,dt + \int_0^{T/2} t\,e^{-j\omega t}\,dt\right]$$

$$= A\int_0^{T/2} 2\cos\omega t\,dt - \frac{2A}{T}\int_0^{T/2} 2t\cos\omega t\,dt$$

$$= 2A\frac{\sin\frac{\omega T}{2}}{\omega} - \frac{4A}{T}\left[\frac{T}{2}\frac{\sin\frac{\omega T}{2}}{\omega} - \int_0^{T/2}\frac{\sin\omega t}{\omega}\,dt\right]$$

$$= 2A\frac{\sin\frac{\omega T}{2}}{\omega} - \frac{4A}{T}\left[\frac{T}{2}\frac{\sin\frac{\omega T}{2}}{\omega} + \left\{\frac{\sin\omega t}{\omega^2}\right\}_0^{T/2}\right]$$

$$= 2A\frac{\sin\frac{\omega T}{2}}{\omega} - \frac{4A}{T}\left[\frac{T}{2}\frac{\sin\frac{\omega T}{2}}{\omega} + \frac{\cos\frac{\omega T}{2}}{\omega^2} - \frac{1}{\omega^2}\right]$$

$$= \frac{2A}{\omega}\sin\frac{\omega T}{2} - \frac{2A}{\omega}\sin\frac{\omega T}{2} - \frac{4A}{T\omega^2}\left(\cos\frac{\omega T}{2} - 1\right)$$

$$= \frac{4A}{T\omega^2}\left(1 - \cos\frac{\omega T}{2}\right)$$

$$= \frac{4A}{T\omega^2}\left[2\sin^2\frac{\omega T}{4}\right] = \frac{8A}{T\omega^2}\sin^2\frac{\omega T}{4}$$

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$$= \frac{8A}{T\omega^2}\left(\frac{\omega T}{4}\right)^2 \frac{\sin^2 \frac{\omega T}{4}}{\left(\frac{\omega T}{4}\right)^2}$$

$$= \frac{AT}{2}\,\text{sinc}^2\left(\frac{\omega T}{4}\right)$$

The frequency spectrum of the single triangular pulse is shown in Fig. 2.11.

2.8.5 Gaussian Pulse

The pulse defined by $f(t) = e^{-\pi t^2}$ is called ***Gaussian Pulse***, which is plotted in Fig. 2.12.

The Fourier transform of this function is

$$F[e^{-\pi t^2}] = \int_{-\infty}^{\infty} e^{-\pi t^2}\, e^{-j\omega t}\, \mathrm{d}t$$

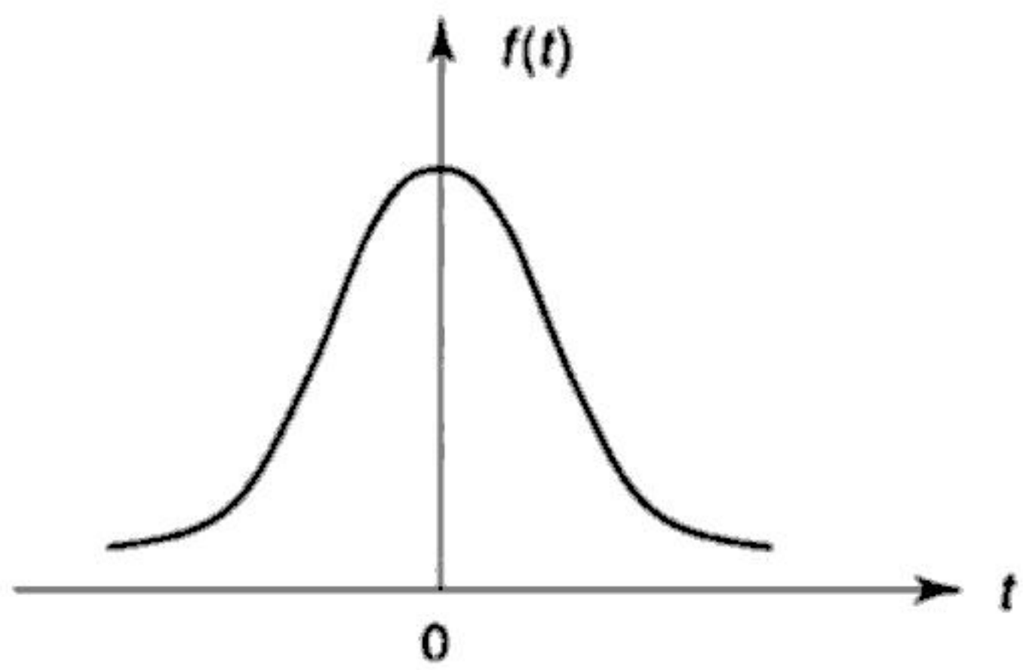

Fig. 2.12 *Gaussian Pulse*

$$= \int_{-\infty}^{\infty} e^{-(\pi t^2 + j\omega t)}\, \mathrm{d}t$$

Substituting $\pi t^2 + j\,\omega t = \left(\sqrt{\pi}\,t + \frac{j\omega}{2\sqrt{\pi}}\right)^2 + \frac{\omega^2}{4\pi}$, we get

$$\mathcal{F}[e^{-\pi t^2}] = \int_{-\infty}^{\infty} e^{-\left(\sqrt{\pi}\,t + \frac{j\omega}{2\sqrt{\pi}}\right)^2}\, e^{-\frac{\omega^2}{4\pi}}\, \mathrm{d}t$$

$$= e^{-\frac{\omega^2}{4\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\pi}\,t + \frac{j\omega}{2\sqrt{\pi}}\right)^2}\, \mathrm{d}t$$

Putting $u = \sqrt{\pi}\,t + \frac{j\omega}{2\sqrt{\pi}}$, $\mathrm{d}u = \sqrt{\pi}\ \mathrm{d}t$ and hence, $\mathrm{d}t = \frac{\mathrm{d}u}{\sqrt{\pi}}$

$$= e^{-\frac{\omega^2}{4\pi}}\, 2\int_{0}^{\infty} e^{-u^2}\, \frac{\mathrm{d}u}{\sqrt{\pi}}$$

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$$= e^{-\frac{\omega^2}{4\pi}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}, \quad \text{since} \int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

$$= e^{-\frac{\omega^2}{4\pi}} = e^{-\pi f^2}$$

Hence, $\mathcal{F}\left[e^{-\pi t^2}\right] = e^{-\pi f^2}$

The area under the Gaussian pulse is

$$\int_{-\infty}^{\infty} f(t)\, dt = \int_{-\infty}^{\infty} e^{-\pi t^2}\, dt = F(0) = \left[e^{-\pi f^2}\right]_{at\, f=0} = 1$$

When both the area under the curve and the central ordinate of a pulse are unity, then we say that the pulse is ***normalised***. Hence, we conclude that the normalised Gaussian pulse is its own Fourier transform, as derived by

$$e^{-\pi t^2} \Leftrightarrow e^{-\pi f^2}$$

Example 2.18 Find the Fourier transform of the Gaussian pulse $f(t) = e^{-a^2 t^2}$.

Solution

$$\mathcal{F}\left[e^{-a^2 t^2}\right] = \int_{-\infty}^{\infty} e^{-a^2 t^2}\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-(a^2 t^2 + j\omega t)}\, dt$$

Substituting $a^2 t^2 + j\,\omega\, t = \left(at + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}$, we get

$$\mathcal{F}\left[e^{-a^2 t^2}\right] = \int_{-\infty}^{\infty} e^{-\left(at + \frac{j\omega}{2a}\right)^2}\, e^{-\frac{\omega^2}{4a^2}}\, dt$$

$$= e^{-\frac{\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(at + \frac{j\omega}{2a}\right)^2}\, dt$$

Putting $u = at + \frac{j\omega}{2a}$, $du = a\, dt$ and so $dt = \frac{du}{a}$

$$= e^{-\frac{\omega^2}{4a^2}}\, 2\int_0^{\infty} e^{-u^2}\, \frac{du}{a}. \quad \text{Here} \int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

$$= e^{-\frac{\omega^2}{4a^2}}\, \frac{2}{a}\, \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{a}\, e^{-\frac{\omega^2}{4a^2}} = \frac{\sqrt{\pi}}{a}\, e^{-\left(\frac{\omega}{2a}\right)^2}$$

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Therefore,

$$\mathcal{F}\left[e^{-a^2t^2}\right] = \frac{\sqrt{\pi}}{a} e^{-\left(\frac{\pi f}{a}\right)^2}$$

2.8.6 Impulse Function (Unit Impulse)

The impulse function is called the **Dirac delta function** $\delta(t)$ which has an infinite amplitude and is infinitely narrow. This is defined as

(i) $\delta(t) = 0$ for all values except at $t = 0$.

(ii) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$, i.e. the area within the pulse is unity.

The Fourier transform of the impulse function $\delta(t)$ is obtained as

$$F(j\omega) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$

Hence, we have the pair $\delta(t) \Leftrightarrow 1$.

The frequency spectrum of the impulse function $\delta(t)$ shown in Fig. 2.13 (a) has a constant amplitude and extends over positive and negative frequencies.

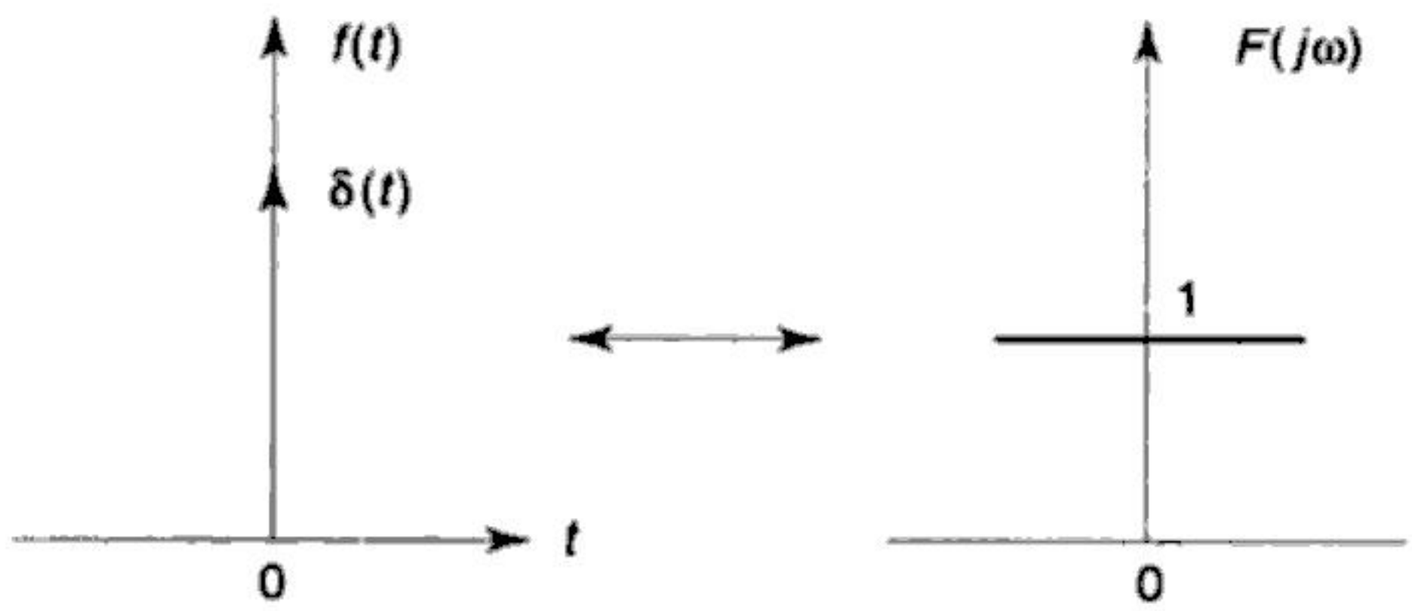

Fig. 2.13 (a) *Impulse Function and its Spectrum*

Using the time-shift theorem, we get

$$\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}$$

The shifted impulse and its amplitude and phase spectra are shown in Fig. 2.13 (b)

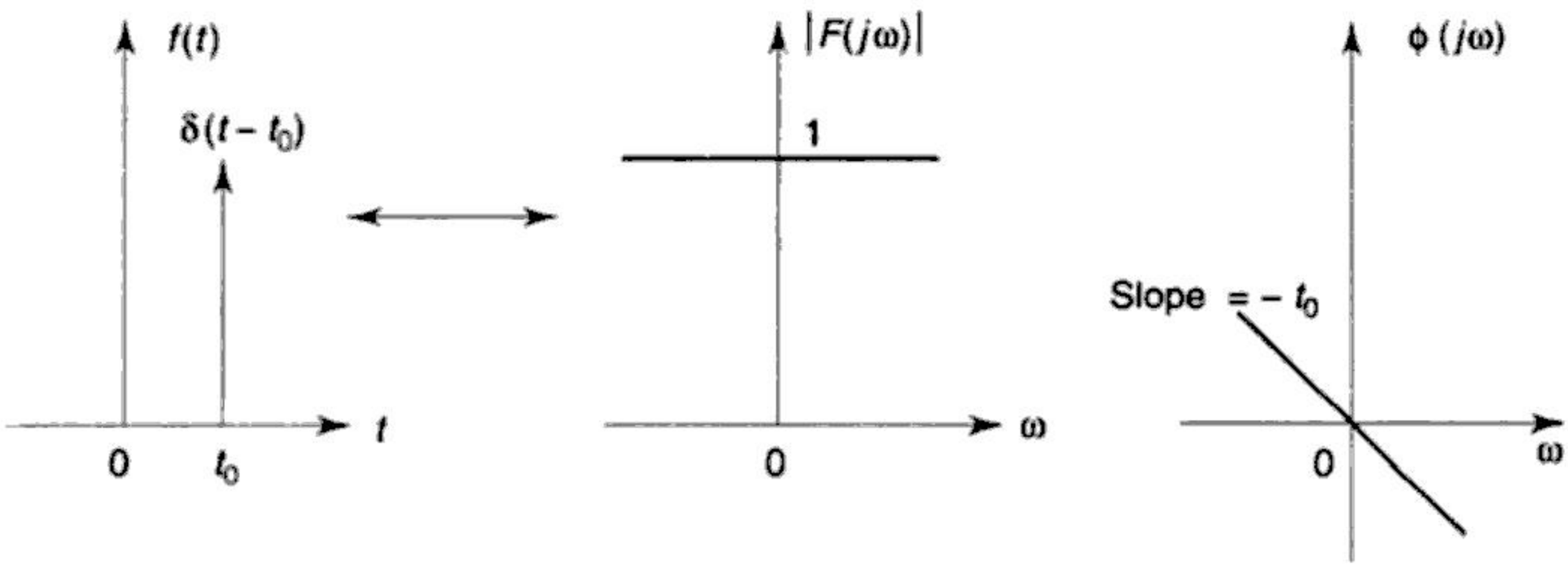

Fig. 2.13 (b) *Shifted Impulse and its Spectrum*

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The inverse Fourier transform of the unit impulse is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{e^{j\omega t}}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, d\omega\, \frac{e^{j\omega t}}{2\pi}$$

$$\left[\text{since } \int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1\right]$$

$$= \frac{1}{2\pi}\angle \omega t$$

or $|f(t)| = \dfrac{1}{2\pi}$. Therefore $\dfrac{1}{2\pi} \Leftrightarrow \delta(\omega)$ or $1 \Leftrightarrow 2\pi\delta(\omega)$

Hence, the Fourier transform of the constant function is an impulse at the origin with an area equal to 2π, as illustrated in Fig. 2.13 (c).

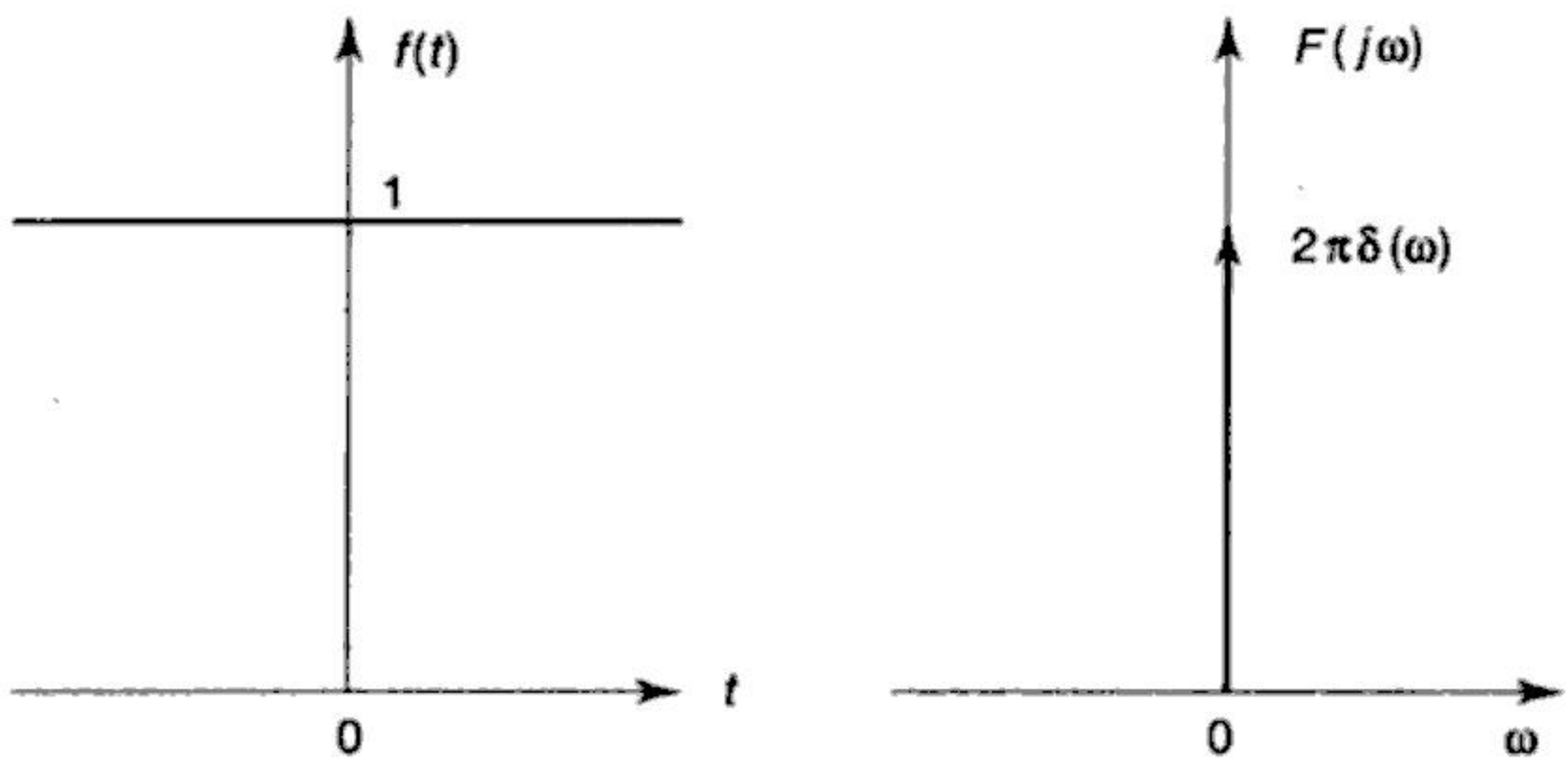

Fig. 2.13 (c) *Constant Function and its Spectrum*

2.8.7 Unit Step Function

The unit step function is obtained by suddenly closing a switch of a d.c. circuit. For easier analysis, the waveform of the unit step function is split into two component waveforms as shown in Fig. 2.14.

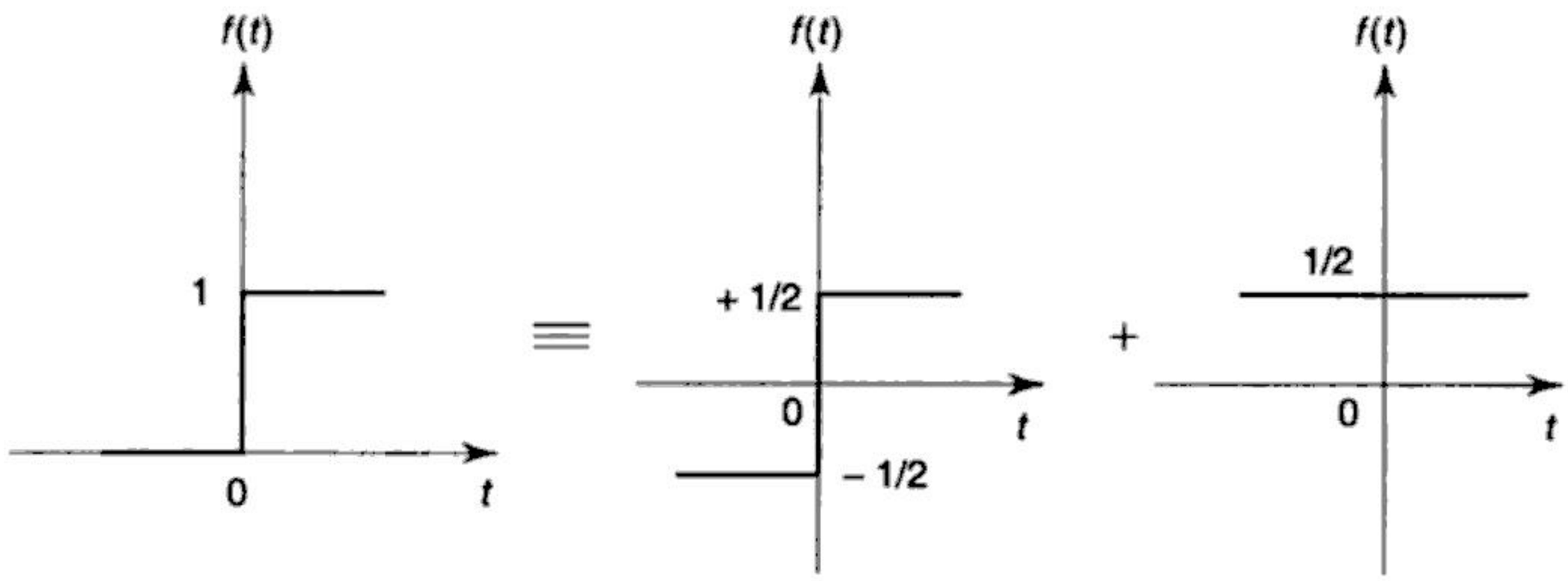

Fig. 2.14 *Unit Step Function*

The first waveform is similar to the signum function with half amplitude. Therefore, the Fourier transform function is given by

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$$= -\int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= -\left[\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= -\left[\frac{1}{a-j\omega}\right] + \left[0 + \frac{1}{a+j\omega}\right]$$

$$= \left[\frac{1}{j\omega - a} + \frac{1}{j\omega + a}\right] = \frac{j2\omega}{(j\omega)^2 - a^2}$$

$$= \frac{-j2\omega}{a^2+\omega^2}$$

Hence, $F(j\omega) = \mathcal{F}\left[e^{-a|t|}\,\text{sgn}(t)\right] = \dfrac{-j2\omega}{a^2+\omega^2}$

It is a pure imaginary function of ω.

Since $F(-j\omega) = \dfrac{-j2(-\omega)}{a^2+(-\omega)^2} = \dfrac{j2\omega}{a^2+\omega^2} = -F(j\omega)$, it is also an odd function of ω. Hence, this is proved that if $f(t)$ is real and odd, then $F(j\omega)$ is pure imaginary and odd.

The phase of $F(j\omega)$ is 90 for all $\omega < 0$ and $-90°$ for all $\omega > 0$.

The amplitude of $F(j\omega)$ is $|F(j\omega)| = \dfrac{2|\omega|}{a^2+\omega^2}$

2.8.10 Pulse Train Signal or Periodic Gate Function

From Eq. 2.10, the Fourier series representation for this pulse train signal shown in Fig. 2.19 is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

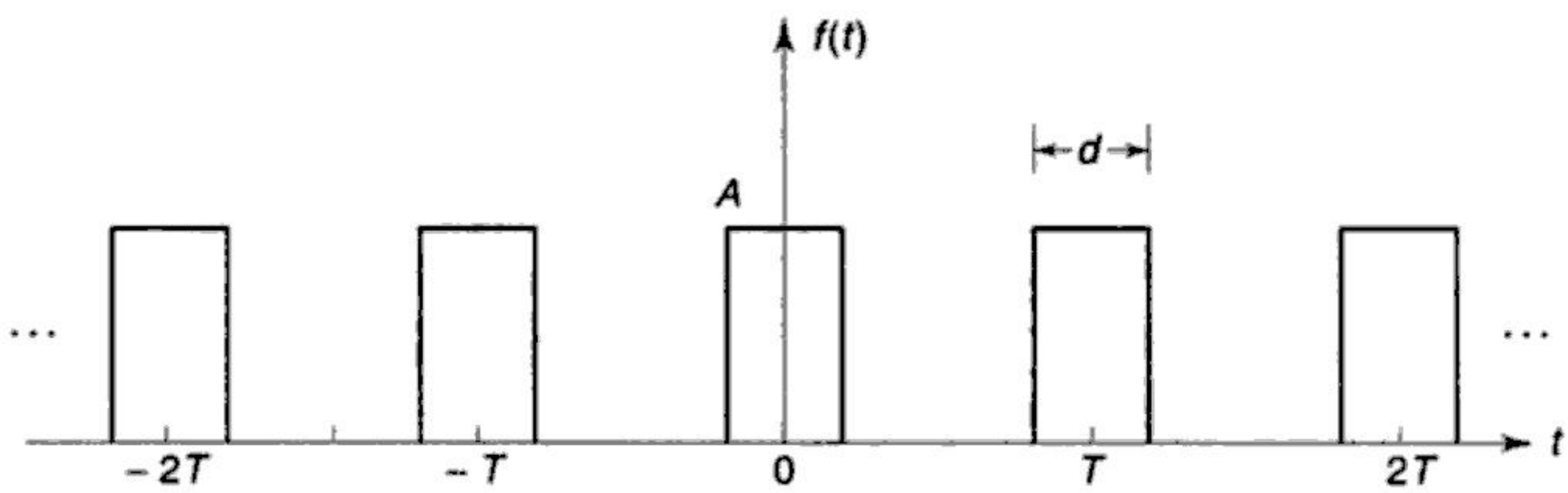

Fig. 2.19 *Pulse Train Signal*

To find the spectrum, we calculate the Fourier series coefficients as

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\,dt$$

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Comparing the powers P and P_{fzc}, we see that $(0.181/0.2) \times 100 =$ 90.5% of the total power in $f(t)$ is contained within the first zero crossing of the spectrum for $f(t)$.

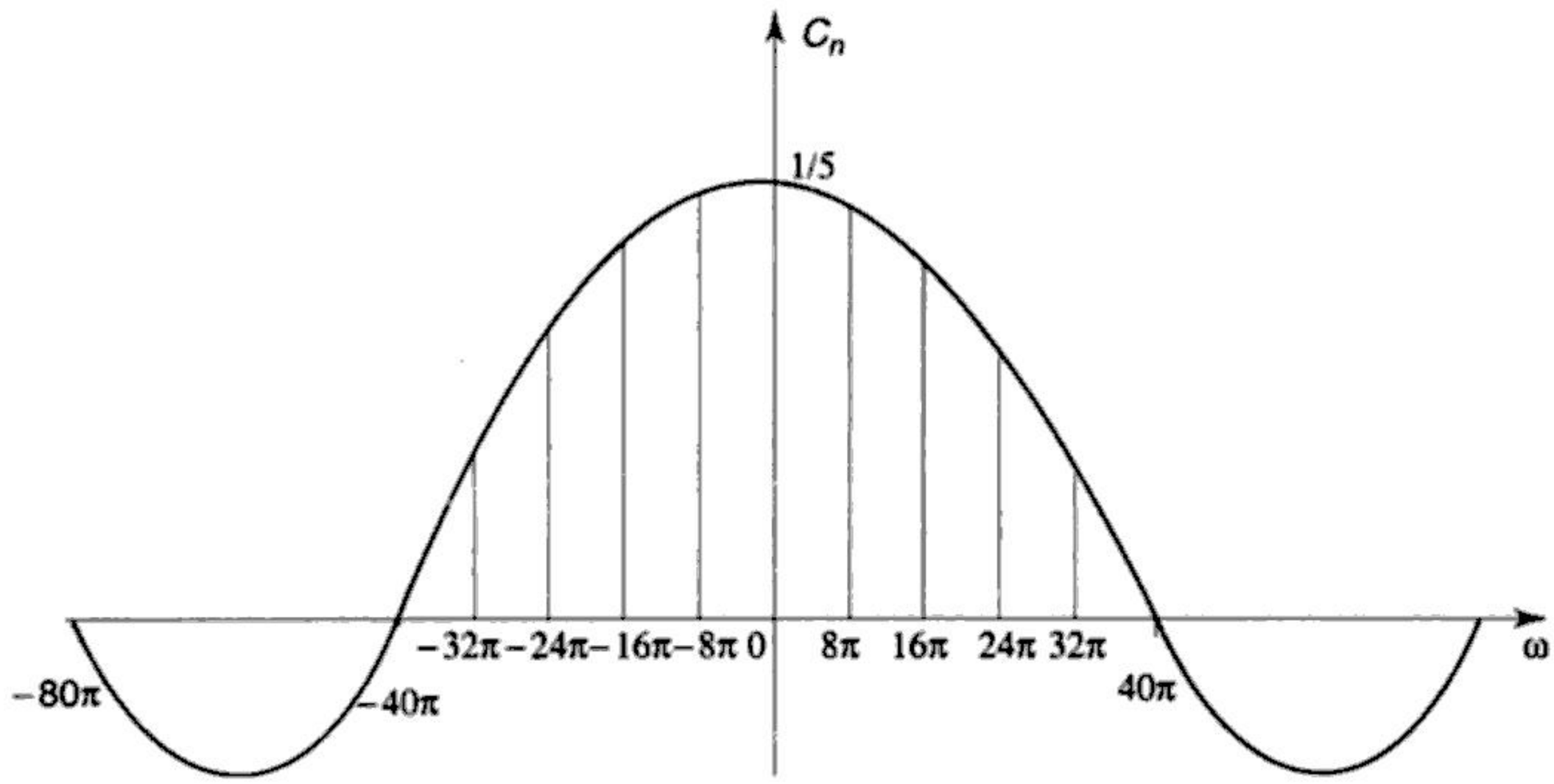

Fig. E2.20(b) *Spectrum of the Function*

2.8.11 Unit Impulse Train

Since $f(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ is a periodic function, as shown in Fig. 2.20, the Fourier series representation of this unit impulse train is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where,
$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} \, dt = \frac{1}{T}$$

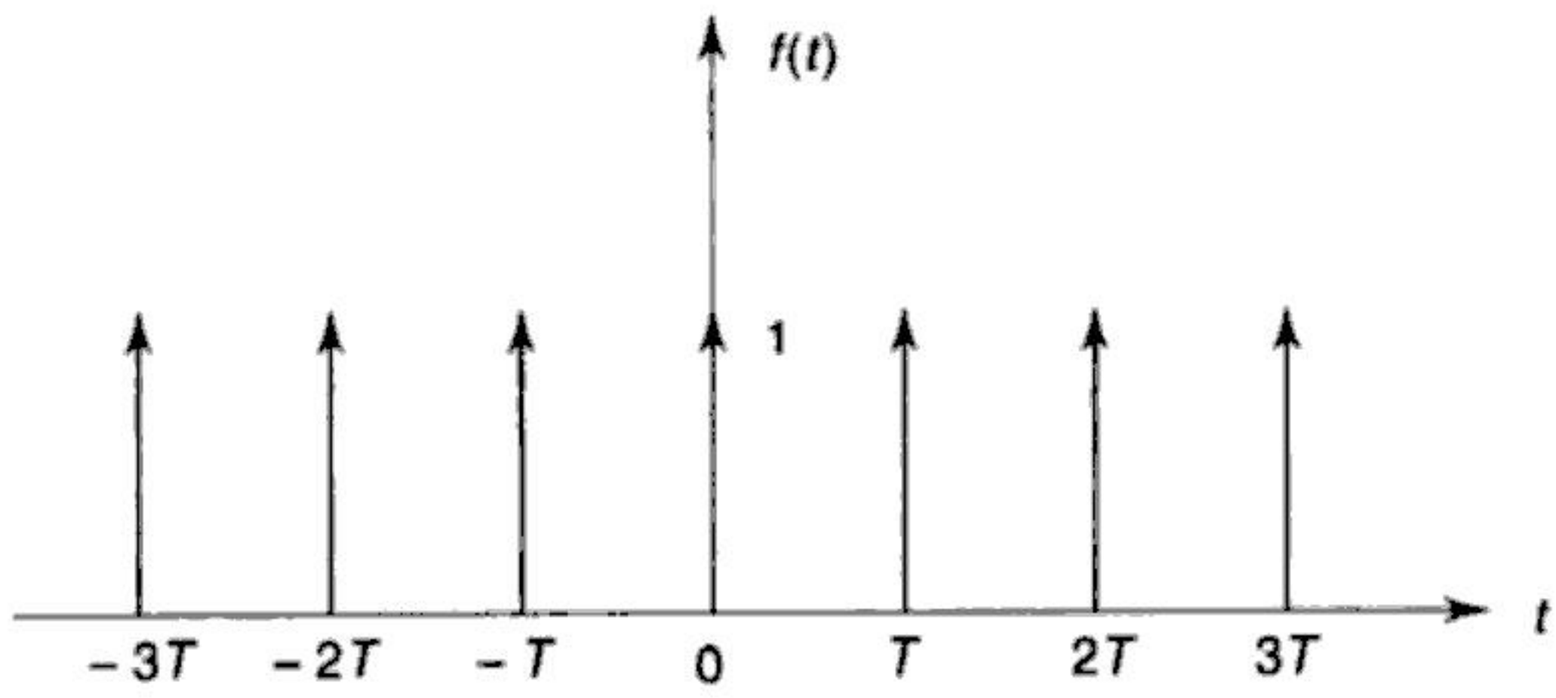

Fig. 2.20 *Unit Impulse Train*

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Substituting this result in the above Fourier series representation, we obtain

$$f(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Therefore, $F(j\omega) = \mathcal{F}[f(t)] = \mathcal{F}\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\right]$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \text{ where } \omega_0 = \frac{2\pi}{T}$$

Hence, $\sum_{k=-\infty}^{\infty}\delta(t - kT) \Leftrightarrow \omega_0 \sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$

Each unit impulse of the unit-impulse train in the time domain has a transform of an impulse train in the frequency domain. The locations of the impulses are at $n\,\omega_0 = n2\pi/T$ where $n = 0, \pm 1, \pm 2, \ldots$ The area for each impulse in the frequency domain is ω_0.

2.9 FOURIER TRANSFORM OF POWER AND ENERGY SIGNALS

2.9.1 Power Signal

The average power of a signal $x(t)$ over a single period $(t_1, t_1 + T)$ is given by

$$P_{av} = \frac{1}{T}\int_{t_2}^{t_1+T} |x(t)|^2 \, dt$$

where $x(t)$ is a complex periodic signal.

A signal $f(t)$ is called a power signal if the average power is expressed by

$$P_x = \underset{t\to\infty}{\text{Lt}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2$$

$$P_{av} = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt, \text{ for a continuous periodic signal with period } T;$$

or $P_{av} = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$, for a digital signal with $x(n) = 0$ for $n < 0$;

is equal to a finite value, i.e. equal to the average power over a single period. If $x(t)$ *is bounded,* P_∞ *is finite. Hence, every bounded and periodic signal is a power signal. But it is true that a power signal is not necessarily a bounded and periodic signal.*

If the signal $x(t)$ contains finite signal power, i.e. $0 < P_x < \infty$, then $x(t)$ is called a power signal.

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2.9.2 Energy Signal

A signal $x(t)$ is called an ***energy signal*** if its total energy over the interval $(-\infty, \infty)$ is finite, that is

$$E_x = \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} |x(t)|^2 \, dt < \infty$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

If the signal $x(t)$ contains finite signal energy, i.e. $0 < E_x < \infty$, then $x(t)$ is called an energy signal.

For a digital signal, the energy is defined by

$$E = \sum_n |x(n)|^2$$

As $n \to \infty$, the energy of periodic signals becomes infinite, whereas the energy of aperiodic pulse-like signals have a finite value. So, these aperiodic pulse-like signals are called *energy signals*. Since the signal energy of a periodic signal is always either zero or infinite, any periodic signal cannot be an energy signal.

Note : There are signals for which neither P_∞ nor E_∞ are finite. If the signals contain infinite energy and zero power or infinite energy and infinite power, then the signals like impulse functions and $x(t) = t$ are neither energy nor power signals.

Example 2.21 Determine the signal energy and signal power for

(a) $f(t) = e^{-3|t|}$ and (b) $f(t) = e^{-3t}$.

Solution

(a)
$$E_\infty = \int_{-\infty}^{\infty} \left[e^{-3|t|}\right]^2 dt$$

$$= \int_{-\infty}^{0} e^{6t} \, dt + \int_{0}^{\infty} e^{-6t} \, dt$$

$$= 2 \int_{0}^{\infty} e^{-6t} \, dt$$

$$= -\frac{2}{6}\left[e^{-6t}\right]_{t=0}^{\infty} = -\frac{2}{6}[0 - 1] = \frac{1}{3}$$

The signal power $P_\infty = 0$ since E_∞ is finite. Hence, the signal $f(t)$ is an energy signal.

(b)
$$E_T = \int_{-T}^{T} \left(e^{-3t}\right)^2 dt$$

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$$= \int_{-T}^{T} e^{-6t}\, dt$$

$$= -\frac{1}{6}\left[e^{-6T} - e^{6T}\right]$$

As $T \to \infty$, E_T approaches infinity.
Its average power is

$$P_\infty = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} E_T = \underset{T\to\infty}{\text{Lt}} \frac{e^{6T} - e^{-6T}}{8T} = \infty \text{ [Using L' Hospital's rule]}$$

Hence, e^{-3t} is neither an energy signal nor a power signal.

Example 2.22 Compute the signal energy for $x(t) = e^{-4t}\, u\,(t)$

$$u(t) = \begin{cases} 1, & \text{for} \quad t \geq 0 \\ 0, & \text{for} \quad t < 0 \end{cases}$$

Solution Since the value of the given function is zero between $-T$ to 0, the limits must be applied between 0 to T instead of $-T$ to T.

$$E_x = \underset{T\to\infty}{\text{Lt}} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \int_{0}^{T} \left|e^{-4t}\right|^2 dt$$

$$= \underset{T\to\infty}{\text{Lt}} \int_{0}^{T} e^{-8t}\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \left[\frac{e^{-8t}}{-8}\right]_0^T$$

$$= \underset{T\to\infty}{\text{Lt}} \left[\frac{e^{-8T}}{-8} + \frac{1}{8}\right] = \frac{1}{8}$$

$$P_x = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$= \underset{T\to\infty}{\text{Lt}} \frac{1}{2T}\left[\frac{e^{-8t}}{-8}\right]_{-T}^{T} = \underset{T\to\infty}{\text{Lt}} \frac{\left(e^{-8T} - 1\right)}{-16T} = 0.$$

Here, as the signal energy E_x is finite and the signal power $P_x = 0$, the signal is an energy signal.

Example 2.23 Determine the signal energy and signal power for the following complex-valued signal and determine whether it is an energy signal or a power signal.

$$x(t) = Ae^{j2\pi\alpha t}$$

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Solution Here, the signal period is $T = \frac{1}{\alpha}$. Since $x(t)$ is a periodic signal, it cannot be an energy signal. Therefore, the power signal is evaluated as

$$P_x = \frac{1}{T}\int_{t_1}^{t_1+T} |x(t)|^2 \, dt$$

$$P_x = \alpha \int_{t_1}^{t_1+\left(\frac{1}{\alpha}\right)} \left|A e^{2\pi\alpha t}\right|^2 dt$$

$$= \alpha \int_{t_1}^{t_1+\left(\frac{1}{\alpha}\right)} A^2 \, dt$$

$$= \alpha \left[A^2 t\right]_{t_1}^{t_1+\frac{1}{\alpha}} = A^2$$

Since the signal has finite power, it is a power signal and $E_x = \infty$.

Example 2.24 Determine the magnitude and phase spectrum of the pulse shown in Fig. E 2.24(a).

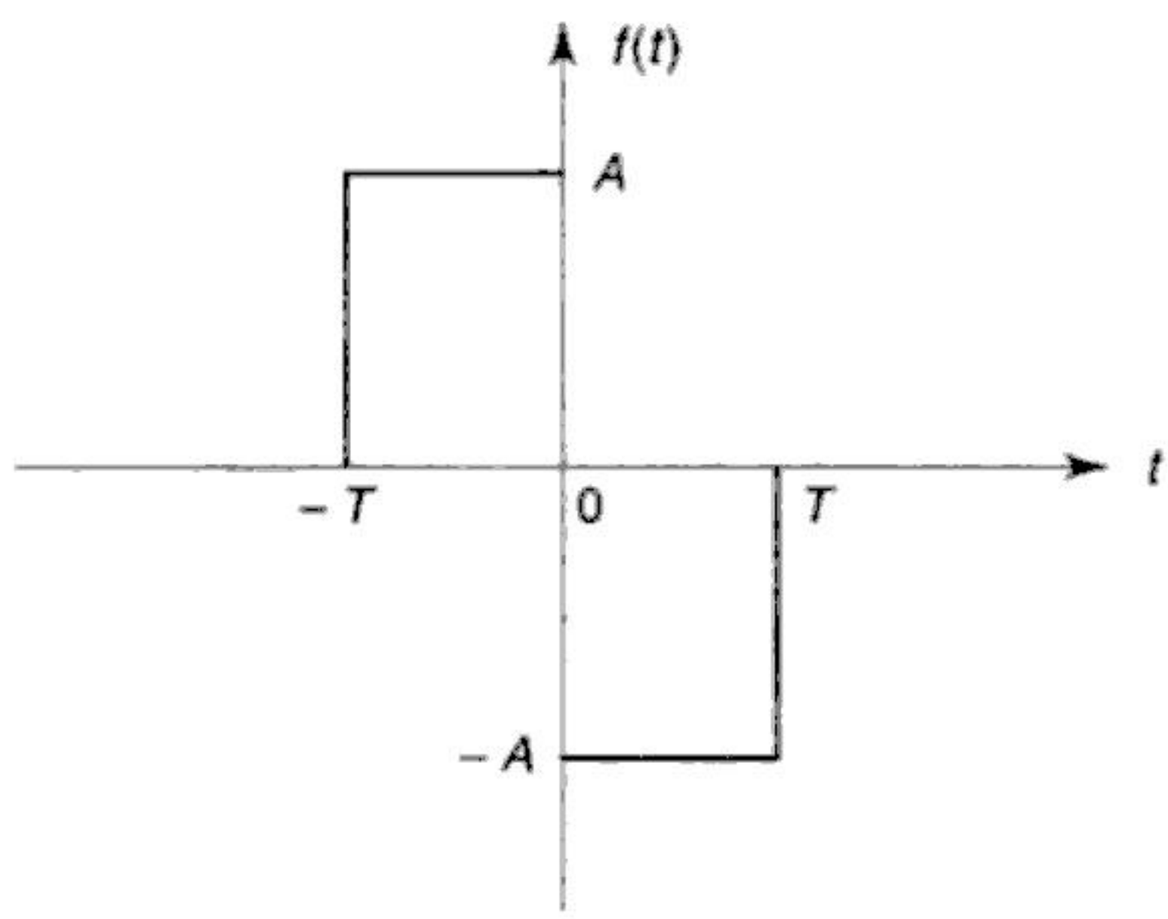

Fig. E 2.24(a)

Solution

Here
$$f(t) = A, \quad \text{for } -T \le t \le 0$$
$$= -A, \text{ for } 0 \le t \le T$$
$$= 0, \quad \text{otherwise}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{0} f(t)\, e^{-j\omega t}\, dt + \int_{0}^{T} f(t)\, e^{-j\omega t}\, dt$$

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$$= \int_{-T}^{0} A\, e^{-j\omega t}\, \mathrm{d}t + \int_{0}^{T} (-A)\, e^{-j\omega t}\, \mathrm{d}t$$

Changing t by $-t$ in the first integral, we get

$$F(j\omega) = \int_{0}^{T} A\, e^{j\omega t}\, \mathrm{d}t - \int_{0}^{T} A\, e^{-j\omega t}\, \mathrm{d}t$$

$$= A \int_{0}^{T} \left(e^{j\omega t} - e^{-j\omega t}\right) \mathrm{d}t$$

$$= j2A \int_{0}^{T} \sin \omega t\, \mathrm{d}t$$

$$= j2A \left(\frac{\cos \omega t}{-\omega}\right)_{0}^{T}$$

$$= j2A \left(\frac{\cos \omega T}{-\omega} + \frac{1}{\omega}\right)$$

$$= \frac{j2A}{\omega} [1 - \cos \omega T]$$

Therefore , $|F(j\omega)| = \dfrac{2A}{\omega} (1 - \cos \omega T)$

Therefore, the magnitude is zero when $(1 - \cos \omega T) = 0$

i.e. $\omega T = 2n\pi$, or $\omega = n \dfrac{2\pi}{T}$

The magnitude spectrum is shown in Fig. E2.24(b).

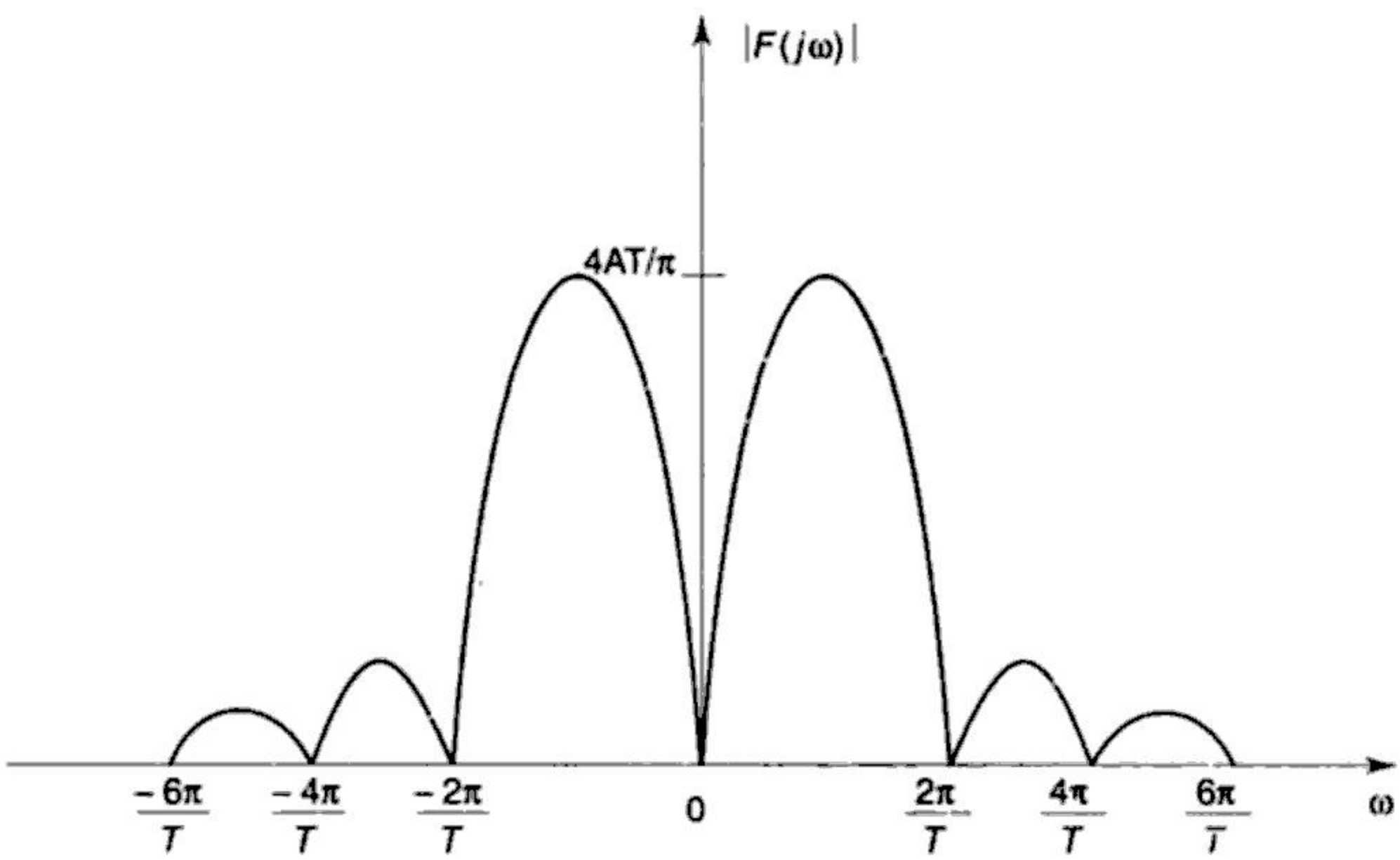

Fig. E2.24 (b) *Magnitude Spectrum*

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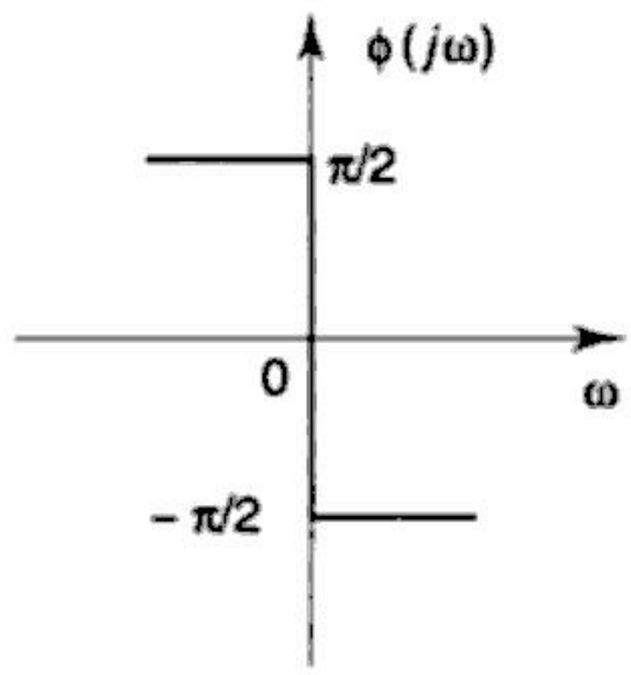

Fig. E2.26 (b) *Phase Spectrum*

Solution The magnitude $|F(j\omega)| = \pi$, for $-T \le \omega \le T$

$$\Phi(\omega) = \angle F(j\omega) = \pi/2, \text{ for } \omega < 0$$
$$= -\pi/2, \text{ for } \omega > 0$$
$$= 0, \qquad \text{for } \omega = 0$$

For the limits $-T \le \omega \le 0$, $F(j\omega) = \pi e^{j\pi/2}$
For the limits $0 \le \omega \le T$, $F(j\omega) = \pi e^{-j\pi/2}$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-T}^{0} \pi e^{j\pi/2}\, e^{j\omega t}\, d\omega + \int_{0}^{T} \pi e^{-j\pi/2}\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2} e^{j\pi/2}\left[\frac{e^{j\omega t}}{jt}\right]_{-T}^{0} + \frac{1}{2} e^{-j\pi/2}\left[\frac{e^{j\omega t}}{jt}\right]_{0}^{T}$$

$$= \frac{1}{2}\frac{e^{j\pi/2}}{jt}\left[1 - e^{-jTt}\right] + \frac{1}{2}\frac{e^{-j\pi/2}}{jt}\left[e^{jTt} - 1\right]$$

$$= \frac{1}{j2t}\left[e^{j\pi/2} - e^{j\pi/2} e^{-jTt} + e^{-j\pi/2} e^{jTt} - e^{-j\pi/2}\right]$$

$$= \frac{1}{t}\left[\frac{e^{j\pi/2} - e^{-j\pi/2}}{j2}\right] - \frac{1}{t}\left[\frac{e^{j(\pi/2 - Tt)} - e^{-j(\pi/2 - Tt)}}{j2}\right]$$

$$= \frac{1}{t}\left[\sin(\pi/2) - \sin(\pi/2 - Tt)\right]$$

$$= \frac{1}{t}\left[1 - \cos Tt\right] = \frac{2\sin^2\left(\frac{Tt}{2}\right)}{t} = \frac{T^2 t}{2}\operatorname{sinc}^2\left(\frac{Tt}{2}\right)$$

Example 2.27 Obtain the Fourier transform of the trapezoidal pulse shown in Fig. E2.27.

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$$= \frac{1}{2}\int_0^\infty \left[e^{-(a-jb+j\omega)t} + e^{-(a+jb+j\omega)t}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(a+j\omega-jb)t}}{-(a+j\omega-jb)} + \frac{e^{-(a+j\omega+jb)t}}{-(a+j\omega+jb)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(a+j\omega-jb)} + \frac{1}{(a+j\omega+jb)}\right]$$

$$= \frac{1}{2}\left[\frac{a+j\omega+jb+a+j\omega-jb}{(a+j\omega)^2+b^2}\right]$$

$$= \frac{a+j\omega}{(a+j\omega)^2+b^2}$$

Example 2.30 Find the Fourier transform of $f(t) = t\cos at$

Solution

$$\mathcal{F}[t\cos at] = \mathcal{F}\left[t\left\{\frac{e^{jat}+e^{-jat}}{2}\right\}\right]$$

$$= \int_0^\infty t\left[\frac{e^{jat}+e^{-jat}}{2}\right]e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\int_0^\infty \left(te^{-j(-a+\omega)t} + te^{-j(a+\omega)t}\right)dt$$

$$= \frac{1}{2}\left[t\left\{\frac{e^{-j(-a+\omega)t}}{-j(-a+\omega)}\right\} - \frac{e^{-j(-a+\omega)t}}{[-j(-a+\omega)]^2}\right]_0^\infty$$

$$+ \frac{1}{2}\left[t\left\{\frac{e^{-j(a+\omega)t}}{-j(a+\omega)}\right\} - \frac{e^{-j(a+\omega)t}}{[-j(a+\omega)]^2}\right]_0^\infty$$

$$= \frac{1}{2}\left[-\frac{1}{(-j)^2(-a+\omega)^2} + \frac{1}{(-j)^2(a+\omega)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{(-a+\omega)^2} + \frac{1}{(a+\omega)^2}\right]$$

$$= -\frac{1}{2}\frac{a^2+\omega^2+2a\omega+a^2+\omega^2-2a\omega}{(\omega^2-a^2)^2}$$

$$= -\frac{\omega^2+a^2}{(\omega^2-a^2)^2} \quad \text{or} \quad = -\frac{a^2+\omega^2}{(a^2-\omega^2)^2}$$

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Example 2.33 Determine the Fourier transform of the sinusoidal pulse shown in Fig. E2.33.

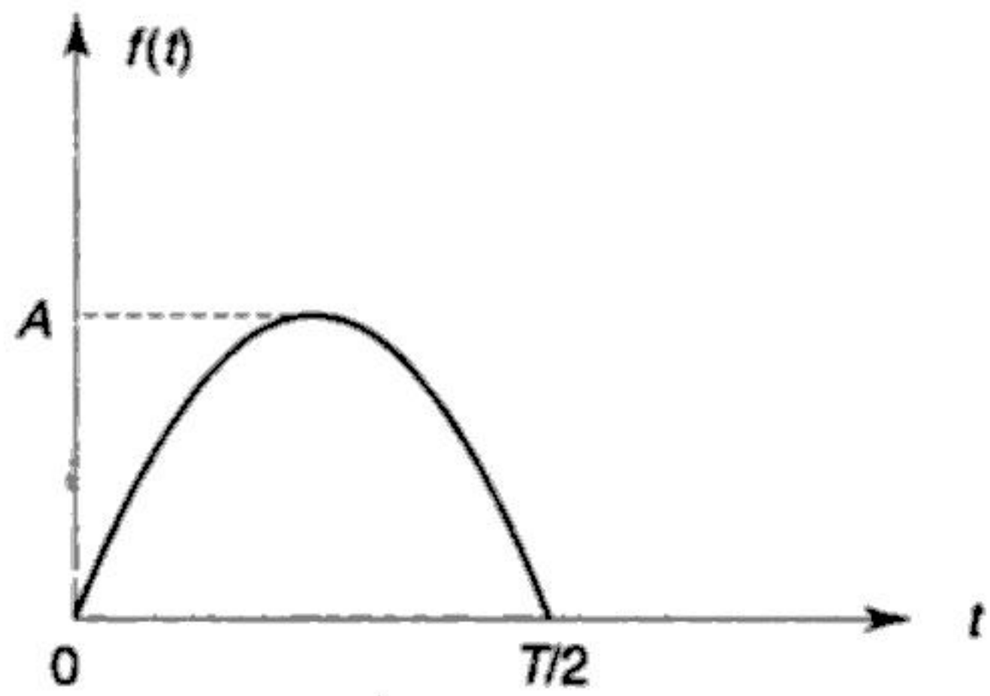

Fig. E2.33

Solution

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t$$

Here $f(t) = A \sin \omega_0 t$ for $0 < t \leq = T/2$ where $\omega_0 = 2\pi/T$.

$= 0$, otherwise

$$F(j\omega) = \int_{0}^{T/2} A \sin \omega_0 t\, e^{-j\omega t}\, \mathrm{d}t$$

$$= A \int_{0}^{T/2} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right) e^{-j\omega t}\, \mathrm{d}t$$

$$= \frac{A}{2j} \int_{0}^{T/2} \left[e^{j(\omega_0 - \omega)t} - e^{-j(\omega_0 + \omega)t} \right] \mathrm{d}t$$

$$= \frac{A}{2j} \left[\frac{e^{j(\omega_0 - \omega)t}}{j(\omega_0 - \omega)} - \frac{e^{-j(\omega_0 + \omega)t}}{-j(\omega_0 + \omega)} \right]_0^{T/2}$$

$$= -\frac{A}{2} \left[\frac{e^{j(\omega_0 - \omega)T/2} - 1}{(\omega_0 - \omega)} + \frac{e^{-j(\omega_0 + \omega)T/2} - 1}{(\omega_0 + \omega)} \right]$$

$$= -\frac{A}{2} \left[\frac{\left(e^{j(\omega_0 - \omega)T/2} - 1\right)(\omega_0 + \omega) + \left(e^{-j(\omega_0 + \omega)T/2} - 1\right)(\omega_0 - \omega)}{\left(\omega_0^2 - \omega^2\right)} \right]$$

$$= \frac{-A}{2\left(\omega_0^2 - \omega^2\right)} \left[\omega_0 \left(e^{j(\omega_0 - \omega)T/2} + e^{-j(\omega_0 + \omega)T/2} \right) \right.$$

$$\left. + \omega \left(e^{j(\omega_0 - \omega)T/2} - e^{-j(\omega_0 + \omega)T/2} \right) - 2\omega_0 \right)\Big]$$

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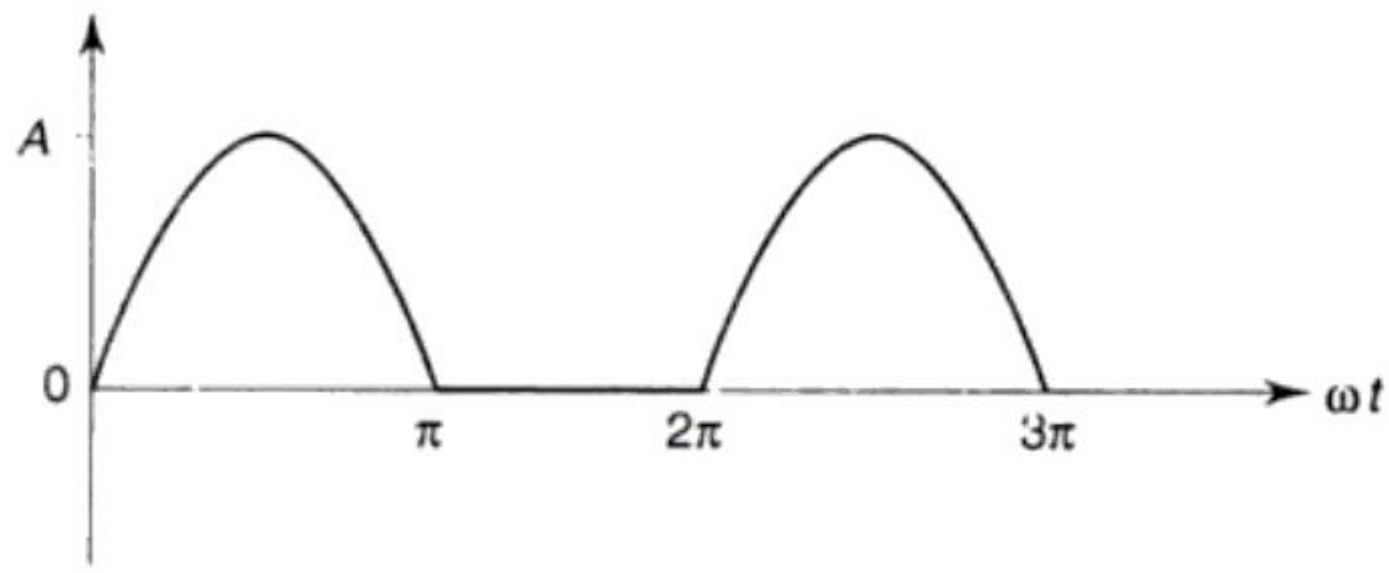

Fig. Q2.12

2.13 Obtain the Fourier series for the full wave rectified sine waves shown in Figs Q2.13 (a) and (b).

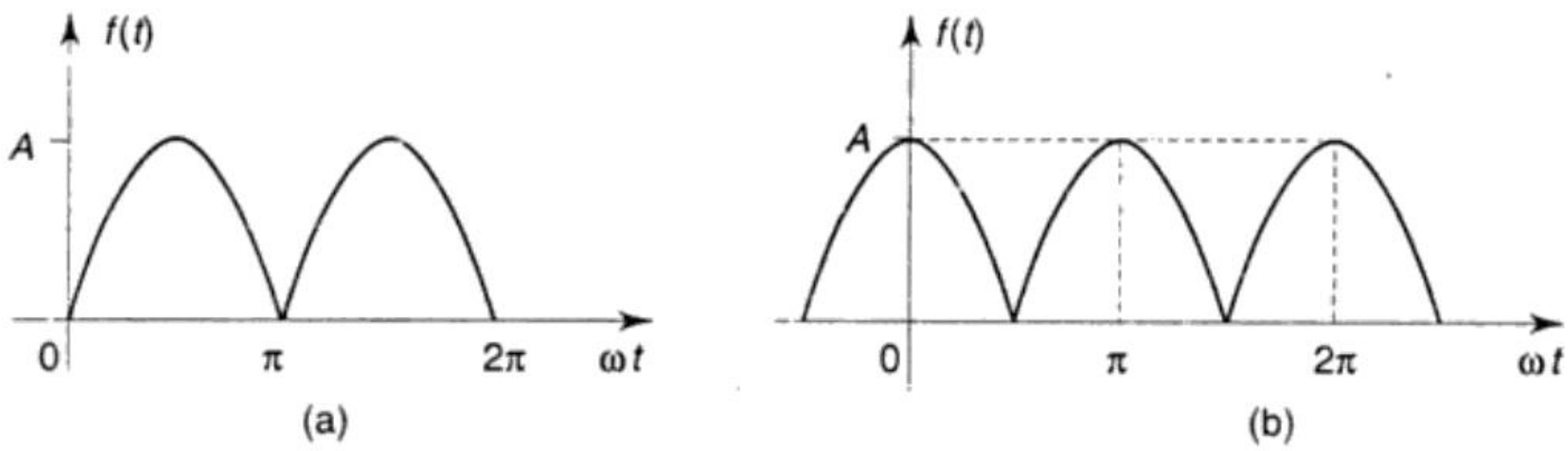

Fig. Q2.13

$$Ans: (a)\ f(t) = \frac{2A}{\pi}\left(1 - \frac{2}{3}\cos 2\,\omega t - \frac{2}{15}\cos 4\,\omega t - \frac{2}{35}\cos 6\,\omega t - \cdots\right)$$

$$(b)\ f(t) = \frac{2A}{\pi}\left(1 + \frac{2}{3}\cos 2\,\omega t - \frac{2}{15}\cos 4\,\omega t + \frac{2}{35}\cos 6\,\omega t - \cdots\right)$$

2.14 Obtain the Trigonometric Fourier series expansion of the periodic signal

$$x(t) = \begin{cases} A & for \quad kT < t \le (k+1)T \\ -A & for \quad (k+1)T < t \le -kT \end{cases}$$

with k taking the values 0, 2, 4, 6, . . .

2.15 A periodic triangular waveform starts at the origin with zero value and increases linearly with respect to time. After a time T, it becomes zero. Obtain its Fourier series.

2.16 With regard to Fourier series representation, justify the following statement :

(i) Odd functions have only sine terms

(ii) Even functions have no sine terms

(iii) Functions with half-wave symmetry have only odd harmonics.

2.17 Obtain the exponential Fourier series for the waveforms shown in Figs Q17 (a) and (b).

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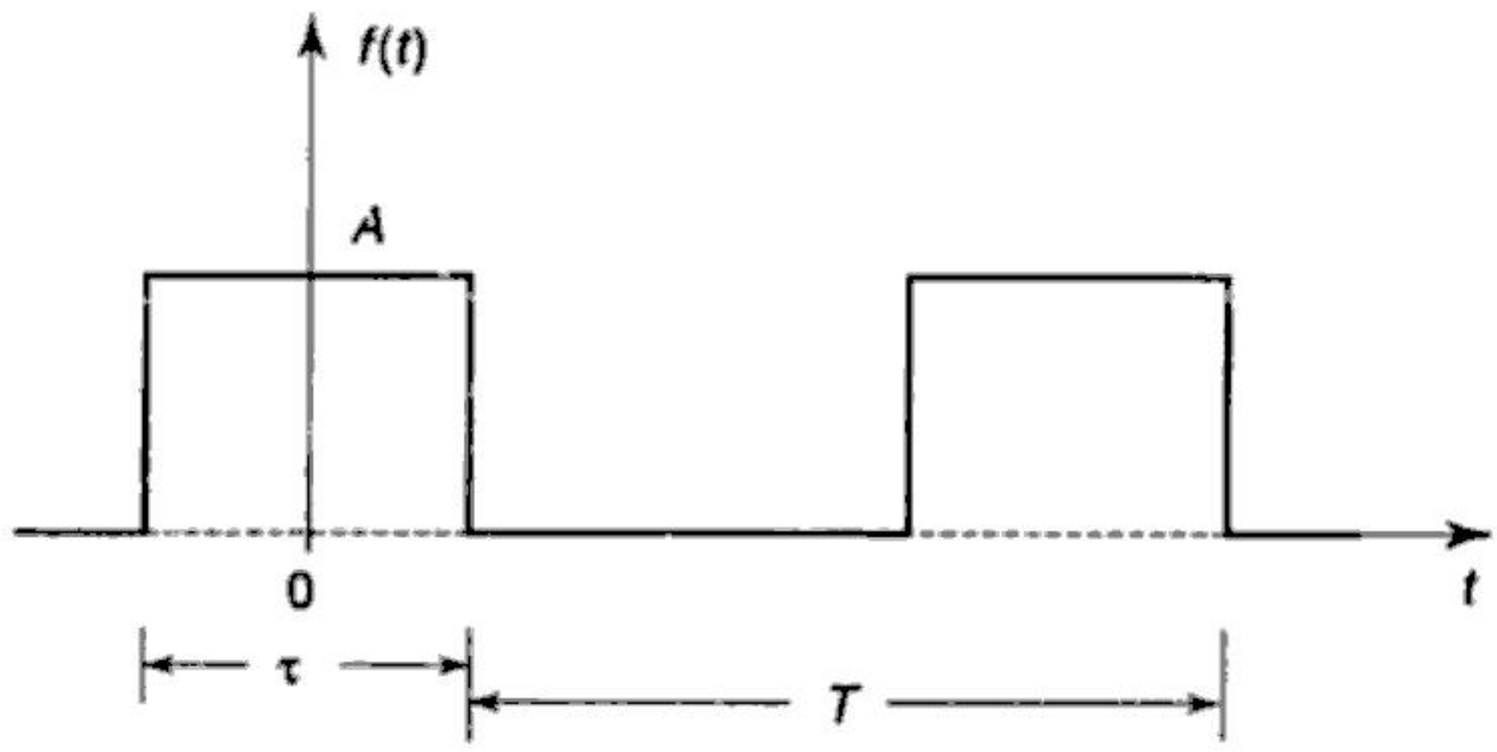

Fig. Q2.43

2.44 Defining x(t) and y(t) as

$$x(t) = \begin{cases} e^{-t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \text{ and}$$

$$y(t) = \begin{cases} \alpha e^{-\alpha t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Find $f(t) = x(t) * y(t)$.

2.45 With suitable waveform, explain the convolution of x(t), the input and h(t), the system transfer function to get y(t), the output.

2.46 Evaluate the Fourier transform of a single unit pulse of 1 volt.

2.47 Obtain the Fourier transform of a single symmetrical triangular pulse.

2.48 Show that a time shift in the time domain is equal to a phase shift in the frequency domain.

2.49 Find the Fourier transform of $x(t) = Ae^{-t/T}u(t)$ and sketch its magnitude and phase as functions of frequency.

2.50 Determine the Fourier transform of a two-sided exponential pulse $x(t) = e^{-|t|}$.

2.51 Find the Fourier transform of $x(t) = A\cos(\omega_c t + \theta)$.

2.52 Find the Fourier transform of $f(t) = e^{-at} \sin bt$

2.53 Find Fourier transform of the following functions

(a) $5 \sin^2 3t$ *(b) $\cos(8t + 0.1\pi)$*

Ans: (a) $2.5\pi[2\,\delta(\omega) + \delta(\omega + 6) + \delta(\omega - 6)$

(b) $18.85\angle 18°\ \delta(\omega - 8) + 18.85\ \angle -18°\ \delta(\omega + 8)$

2.54 Find the Fourier transform of the single triangular pulse with period T = 8 sec and amplitude A = 10 V.

Ans: $40\ \text{sinc}^2 2\omega$

2.55 Determine the Fourier transform of a one-cycle sine wave.

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$$= -\frac{A}{a+s}\left[e^{-(a+s)t}\right]_0^{\infty} = \frac{A}{(s+a)}$$

Hence, $\mathcal{L}\{Ae^{-at}\} = \dfrac{A}{(s+a)}$ (3.4)

3. Sine Function

$$f(t) = \sin \omega_0 t$$

Using Euler's identity, we have

$$\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

Hence, $\mathcal{L}\{\sin \omega_0 t\} = \dfrac{1}{2j}\left[\mathcal{L}(e^{j\omega_0 t}) - \mathcal{L}(e^{-j\omega_0 t})\right]$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right] = \frac{\omega_0}{s^2 + \omega_0^2}$$

Hence, $\mathcal{L}\{\sin \omega_0 t\} = \dfrac{\omega_0}{s^2 + \omega_0^2}$ (3.5)

4. Cosine Function

$$f(t) = \cos \omega_0 t$$

We know that $\cos \omega_0 t = \dfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

$$\mathcal{L}\{\cos \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{j\omega_0 t}) + \mathcal{L}(e^{-j\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] = \frac{s}{s^2 + \omega_0^2}$$

Hence, $\mathcal{L}\{\cos \omega_0 t\} = \dfrac{s}{s^2 + \omega_0^2}$ (3.6)

5. Hyperbolic Sine and Cosine Functions

$$\sinh \omega_0 t = \frac{1}{2}\left[e^{\omega_0 t} - e^{-\omega_0 t}\right]$$

$$\cosh \omega_0 t = \frac{1}{2}\left[e^{\omega_0 t} + e^{-\omega_0 t}\right]$$

$$\mathcal{L}\{\sinh \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{\omega_0 t}) - \mathcal{L}(e^{-\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - \omega_0} - \frac{1}{s + \omega_0}\right] = \frac{\omega_0}{s^2 - \omega_0^2}$$

$$\mathcal{L}\{\sinh \omega_0 t\} = \frac{\omega_0}{s^2 - \omega_0^2} \quad (3.7)$$

Similarly, $\mathcal{L}\{\cosh \omega_0 t\} = \dfrac{1}{2}\left[\mathcal{L}(e^{\omega_0 t}) + \mathcal{L}(e^{-\omega_0 t})\right]$

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$$= \left[-\log\left(1-\frac{1}{s}\right)\right]_s^{\infty}$$

$$= \log\left(1-\frac{1}{s}\right) = \log\left(\frac{s-1}{s}\right)$$

(f) The given impulse function is $f(t) = \delta(t^2 - 3t + 2)$

$$= \delta[(t-1)(t-2)]$$

$$= \delta(t-1)\,u(t-1) + \delta(t-2)\,u(t-2)$$

$$= \delta(t-1) + \delta(t-2)$$

Therefore, $F(s) = \mathcal{L}\,[\delta(t-1) + \mathcal{L}\,[\delta(t-2)]$

$$= e^{-s} + e^{-2s}$$

Example 3.2 Determine the Laplace transform of the rectangular pulse shown in Fig. **E3.2.**

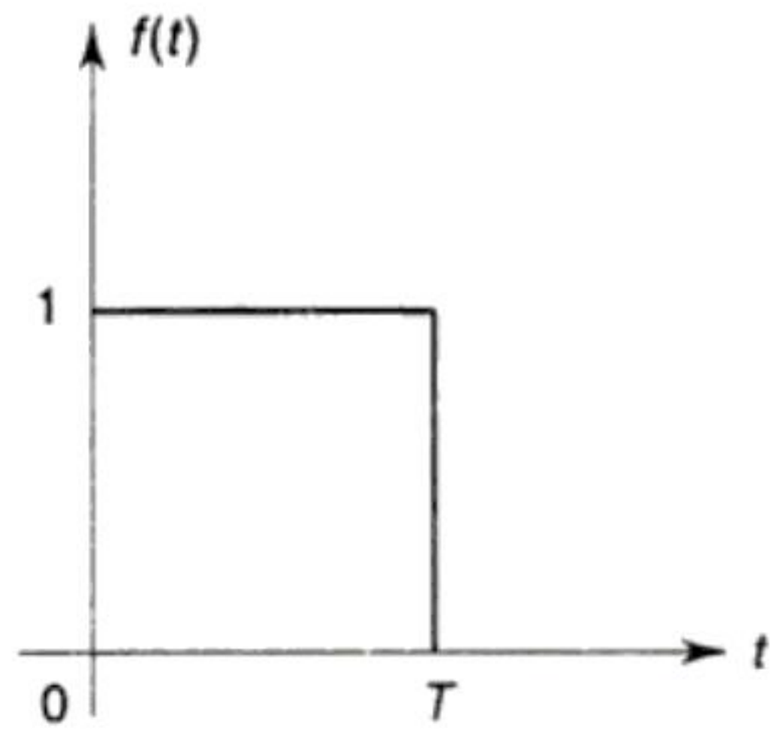

Fig. E3.2

Solution $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$

$$= \int_0^{T} 1 \cdot e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{T}$$

$$= \frac{1}{-s}\,[e^{-sT} - 1]$$

$$= \frac{1}{s}\,[1 - e^{-sT}]$$

Alternate method

The given pulse is represented in terms of the step function as

$$f(t) = u(t) - u(t-T)$$

Taking Laplace transform, we obtain

$$F(s) = \frac{1}{s}\left[1 - e^{-sT}\right]$$

Example 3.3 Find the Laplace transform of a single sawtooth pulse shown in Fig. E3.3.

Solution

The function for the given wave form is

$$f(t) = t \quad \text{for } 0 < t < 1$$
$$= 0 \quad \text{for } t > 1$$

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$$\underset{t \to 0^+}{\text{Lt}} f(t) = \underset{s \to \infty}{\text{Lt}} sF(s)$$

Proof We know that

$$\mathcal{L}\{f'(t)\} = s[\mathcal{L}(f(t))] - f(0)$$

By taking the limit $s \to \infty$ on both sides

$$\underset{s \to \infty}{\text{Lt}} \mathcal{L}\{f'(t)\} = \underset{s \to \infty}{\text{Lt}} [sF(s) - f(0)]$$

$$\underset{s \to \infty}{\text{Lt}} \int_0^\infty f'(t)\, e^{-st}\, dt = \underset{s \to \infty}{\text{Lt}} [sF(s) - f(0)]$$

As $s \to \infty$, the integration of LHS becomes zero

i.e.
$$\int_0^\infty \underset{s \to \infty}{\text{Lt}} [f'(t)\, e^{-st}] dt = 0$$

$$\underset{s \to \infty}{\text{Lt}}\, sF(s) - f(0) = 0$$

Therefore,
$$\underset{s \to \infty}{\text{Lt}}\, sF(s) = f(0) = \underset{t \to 0+}{\text{Lt}} f(t)$$

3.5.2 Final Value Theorem

If $f(t)$ and $f'(t)$ are Laplace transformable, then

$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}}\, sF(s) \tag{3.15}$$

Proof We know that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Taking the limit $s \to 0$ on both sides, we get

$$\underset{s \to 0}{\text{Lt}} \mathcal{L}\{f'(t)\} = \underset{s \to 0}{\text{Lt}} [sF(s) - f(0)]$$

$$\underset{s \to 0}{\text{Lt}} \int_0^\infty f'(t)\, e^{-st}\, dt = \underset{s \to 0}{\text{Lt}}\, sF(s) - f(0)]$$

Therefore,
$$\int_0^\infty f'(t) dt = \underset{s \to 0}{\text{Lt}} [sF(s) - f(0)]$$

$$[f(t)]_0^\infty = \underset{t \to \infty}{\text{Lt}} f(t) - \underset{t \to 0}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}}\, sF(s) - f(0)$$

Since $f(0)$ is not a function of s, it gets cancelled from both sides of the above equation.

Therefore,
$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}}\, sF(s)$$

3.6 CONVOLUTION INTEGRAL

If $X(s)$ and $H(s)$ are the Laplace transforms of $x(t)$ and $h(t)$, then the product of $X(s)H(s) = Y(s)$, where $Y(s)$ is the Laplace transform of $y(t)$ given by

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$$= \frac{1}{RC} \cdot \frac{1}{\left(s + \frac{1}{RC}\right)^3}$$

Taking inverse Laplace transform, we obtain

$$y(t) = \frac{1}{RC} \frac{t^2 e^{-t/RC}}{2} u(t)$$

3.7 TABLE OF LAPLACE TRANSFORMS

Table 3.1 presents some functions and their corresponding Laplace transforms. Table 3.2 lists the properties of the Laplace transform. Table 3.3 gives the elements needed to develop the s-domain image of a given time domain circuit.

Table 3.1 *Laplace transforms pairs*

S.No.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$\delta(t-a)$	e^{-as}
3.	$u(t)$	$\frac{1}{s}$
4.	$u(t-a)$	$\frac{e^{-as}}{s}$
5.	$\frac{t^n}{n!}u(t)$, n positive interger	$(-1)^n \frac{1}{s^{n+1}}$
6.	$e^{-at}\, u(t)$	$\frac{1}{s+a}$
7.	$\frac{t^n e^{-at}}{n!} u(t)$	$\frac{1}{(s+a)^{n+1}}$
8.	$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
9.	$\cos(\omega_0 t)\, u(t)$	$\frac{s}{s^2+\omega_0^2}$
10.	$t\cos(\omega_0 t)\, u(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$
11.	$t\sin(\omega_0 t)\, u(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$
12.	$e^{-at}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

(Contd.)

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the frequency response used in Fourier analysis. There is a considerable similarity between the transfer functions of Laplace transform and Fourier transform.

The main advantage of transformed functions is that the time-domain convolution is replaced by frequency-domain multiplication. Hence,

$$Y(s) = X(s)\,H(s) \tag{3.19}$$

If the Laplace transform $Y(s)$ of the output signal is determined, then its equivalent time determined function $y(t)$ can be determined by taking the inverse Laplace transform.

3.9.1 Step and Impulse Responses

We know that the Laplace transform of a unit impulse $\delta(t)$ is unity, i.e., $\mathcal{L}\{\delta(t)\} = 1$. If the unit impulse is given as the system excitation, i.e. $X(s) = 1$, then the output response will be

$$\begin{aligned} Y(s) &= X(s)\,H(s) \\ &= H(s) \end{aligned}$$

Thus, it is shown that the impulse response $h(t)$ and the transfer function of the system $H(s)$ constitute a transform pair, i.e.

$$\begin{aligned} \mathcal{L}\{h(t)\} &= H(s) \\ \mathcal{L}^{-1}\{H(s)\} &= h(t) \end{aligned}$$

This implies that the step and impulse responses can be directly obtained from the system function.

The step response is the integral of the impulse response. Hence, the integral property of the Laplace transform is used to obtain the step response $\alpha(t)$ as

$$\alpha(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$

The unit ramp response $\gamma(t)$ is obtained from the equation given by

$$\gamma(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s^2}\right\}$$

3.10 s-PLANE POLES AND ZEROS

The transfer function of a linear time-invariant system may be expressed as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0} \tag{3.20}$$

The system function is a rational function of s and is expressed as

$$H(s) = \frac{N(s)}{D(s)} = k\,\frac{(s - z_1)(s - z_2)(s - z_3)\cdots}{(s - p_1)(s - p_2)(s - p_3)\cdots} \tag{3.21}$$

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and Phase $\Phi(j2) = \dfrac{\angle 90^\circ}{\angle \tan^{-1}(1)\,\angle \tan^{-1}(3)} = \dfrac{\angle 90^\circ}{\angle 45^\circ\,\angle 71.8^\circ}$

$$= 90^\circ - 45^\circ - 71.8^\circ = -26.8^\circ$$

3.11 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

The time-shift theorem is useful in determining the transform of periodic time functions. Let function $f(t)$ be a causal periodic waveform which satisfies the condition $f(t) = f(t + nT)$ for all $t > 0$ where T is the period of the function and $n = 0, 1, 2,...$

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \cdots + \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt + \cdots$$

As $f(t)$ is periodic, the above equation becomes

$$= \int_0^T f(t)e^{-st}\,dt + e^{-sT}\int_0^T f(t)e^{-st}\,dt + \cdots + e^{-nsT}\int_0^T f(t)e^{-st}\,dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT} + \cdots\right]\int_0^T f(t)\,e^{-st}\,dt$$

$$= \left[1 + e^{-sT} + \left(e^{-sT}\right)^2 + \cdots + \left(e^{-sT}\right)^n + \cdots\right]F_1(s)$$

where $\quad F_1(s) = \int_0^T f(t)\,e^{-st}\,dt$

Here, $F_1(s) = \mathcal{L}\{[u(t) - u(t-T)]\,f(t)\}$, which is the transform of the first period of the time function, and $\{[u(t) - u(t-T)]\,f(t)\}$ has non-zero only in the first period of $f(t)$.

When we apply the binomial theorem to the bracketed expression, it becomes $1/(1 - e^{-sT})$

$$F(s) = \frac{1}{1 - e^{-sT}}\int_0^T f(t)\,e^{-st}\,dt = \frac{F_1(s)}{1 - e^{-sT}}$$

Example 3.13 Find the Laplace transform of the periodic rectangular waveform shown in Fig. E3.13.

Solution Here the period is $2T$

Therefore, $\mathcal{L}\{f(t)\} = \dfrac{1}{1 - e^{-2sT}}\left[\displaystyle\int_0^{2T} f(t)\,e^{-st}\,dt\right]$

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Solution Applying Kirchoff's voltage law, the loop equation can be written as

$$2i(t) + 1\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 10\sin 25t$$

Taking Laplace transform, we get

$$2I(s) + [sI(s) - i(0)] = 10 \times \frac{25}{s^2 + (25)^2}$$

where $i(0)$ is the initial current passing through the circuit. As the inductor does not allow sudden changes in currents, $i\,(0) = 0$.

Therefore, $$sI(s) + 2I(s) = \frac{10 \times 25}{s^2 + (25)^2}$$

$$I(s) = \frac{250}{\left(s^2 + 625\right)(s + 2)}$$

Using partial fractions, the above equation can be expanded as

$$I(s) = \frac{250}{(s + 2)(s + j25)(s - j25)}$$

$$I(s) = \left[\frac{A_1}{s + 2} + \frac{A_2}{s + j25} + \frac{A_3}{s - j25}\right]$$

where $$A_1 = (s + 2)\,I(s)\big|_{s = -2}$$

$$= \left.\frac{250}{\left[s^2 + (25)^2\right]}\right|_{s = -2}$$

$$= \frac{250}{629}$$

$$A_2 = (s + j25)\,I(s)\big|_{s = -j25}$$

$$= \left.\frac{250}{(s + 2)(s - j25)}\right|_{s = -j25}$$

$$= \frac{250}{(2 - j25)(-j50)} = \frac{-5}{(25 + j2)}$$

$$A_3 = (s - j25)\,I(s)\big|_{s = j25}$$

$$= \left.\frac{250}{(s + 2)(s + j25)}\right|_{s = j25}$$

$$= \frac{250}{(2 + j25)(j50)} = \frac{-5}{(25 - j2)}$$

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(vi) If $x(n)$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$. That is, the ROC includes the intersection of the ROC's of the components.

(vii) If $X(z)$ is rational, then the ROC extends to infinity, i.e. the ROC is bounded by poles.

(viii) If $x(n)$ is causal, then the ROC includes $z = \infty$.

(ix) If $x(n)$ is anti-causal, then the ROC includes $z = 0$.

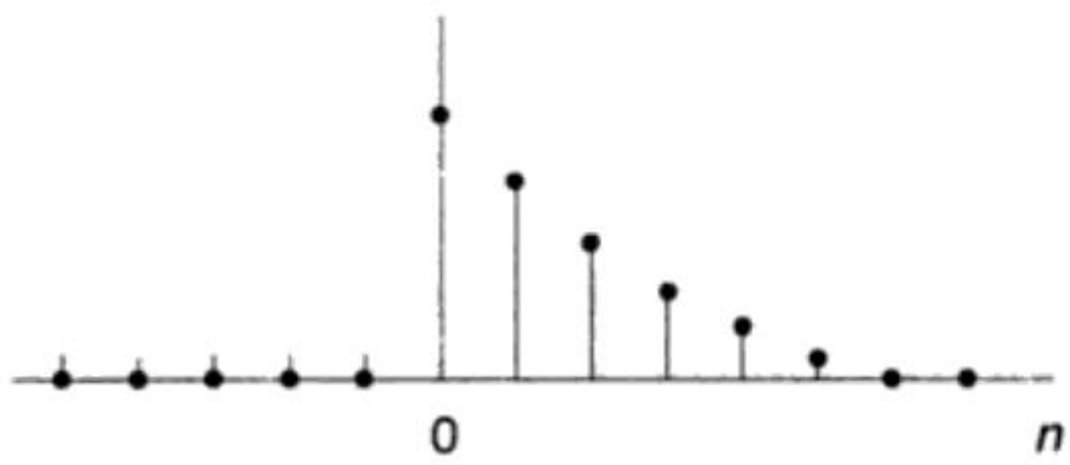
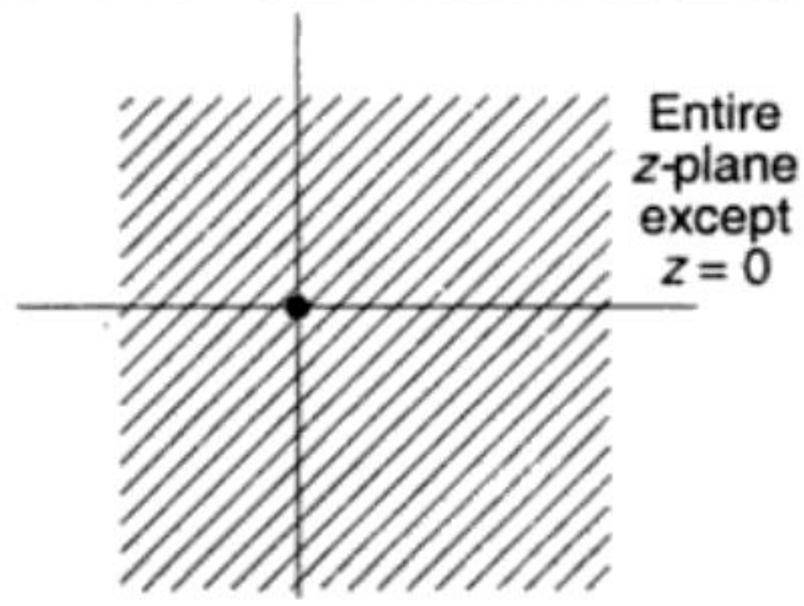
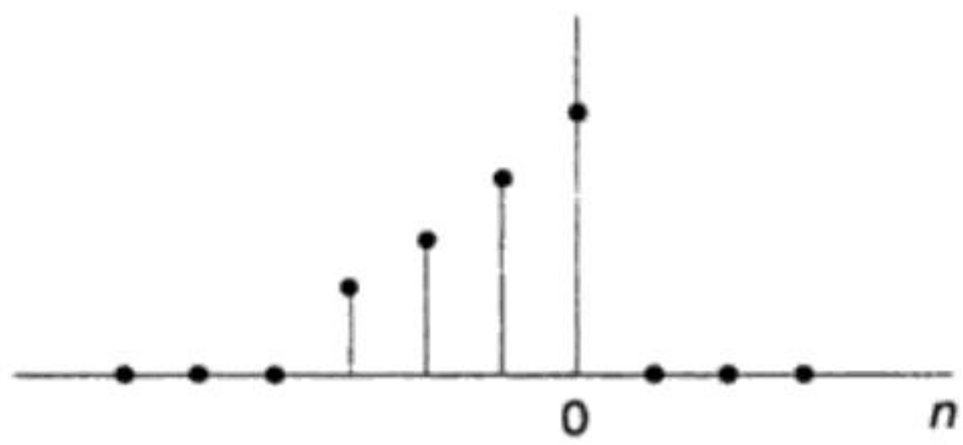
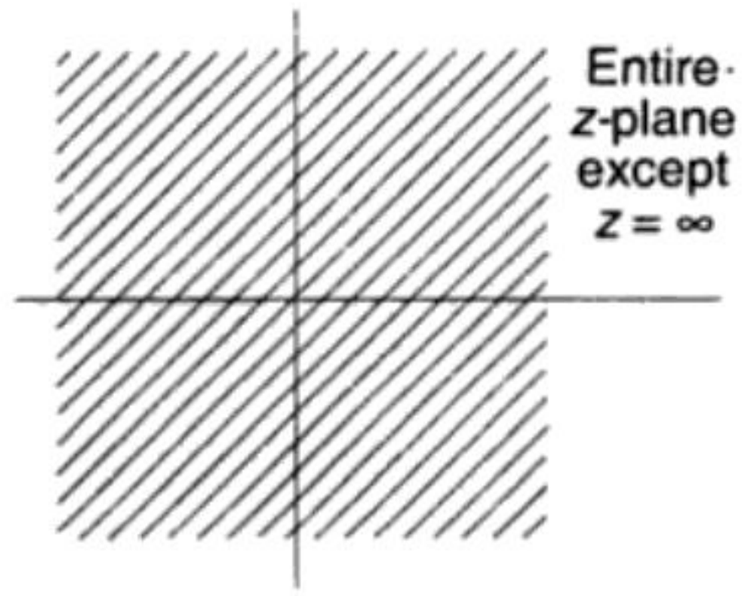
To determine the ROC for the series expressed by the Eq. 4.2, which is called a two-sided signal z-transform, this equation can be written as

$$\sum_{n=-\infty}^{\infty} x(n)\, r^{-n} = \sum_{n=-\infty}^{-1} x(n)\, z^{-n} + \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=1}^{\infty} x(-n)\, z^{+n} + \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

The first series, a non-causal sequence, converges for $|z| < r_2$, and the second series, a causal sequence, converges for $|z| > r_1$, resulting in an annular region of convergence. Then the Eq. 4.2 converges for $r_1 < |z| < r_2$. provided $r_1 < r_2$. The causal, anti-causal and two-sided signals with their corresponding ROCs are shown in Table 4.2. Some important commonly used z-transform pairs are given in Table 4.3.

Table 4.2 *The Causal, anti-causal and two-sided signals and their ROCs*

Signals	*ROCs*
(a) Finite duration signals	
Causal 	
Anti-causal 	

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(d) $x(n) = \{0, 0, 1, 2, 5, 4, 0, 1\}$
Taking z-transform, we get

$$X(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 4z^{-5} + z^{-7}.$$

ROC: Entire z-plane except $z = 0$.

(e) $x(n) = \delta(n)$, hence $X(z) = 1$, ROC: Entire z-plane.

(f) $x(n) = \delta(n - k)$, $k > 0$, hence $X(z) = z^{-k}$, ROC: Entire z-plane except $z = 0$

(g) $x(n) = \delta(n + k)$, $k > 0$, hence $X(z) = z^{k}$, ROC: Entire z-plane except $z = \infty$.

Example 4.3 Determine the z-transform including the region of convergence of

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution The z-transform for the given $x(n)$ is

$$X(z) = Z[a^n] = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

We know that $\sum_{0}^{\infty} a^n = \dfrac{1}{1-a}, |a| < 1$

Hence, $X(z) = \dfrac{1}{1 - az^{-1}} = \dfrac{z}{z - a}$

This converges when $|az^{-1}| < 1$ or $|z| > |a|$. Values of z for which $X(z) = 0$ are called zeros of $X(z)$, and values of z for which $X(z) \to \infty$ are called poles of $X(z)$.

Here the poles are at $z = a$ and zeros at $z = 0$. The region of convergence is shown in Fig. E 4.3.

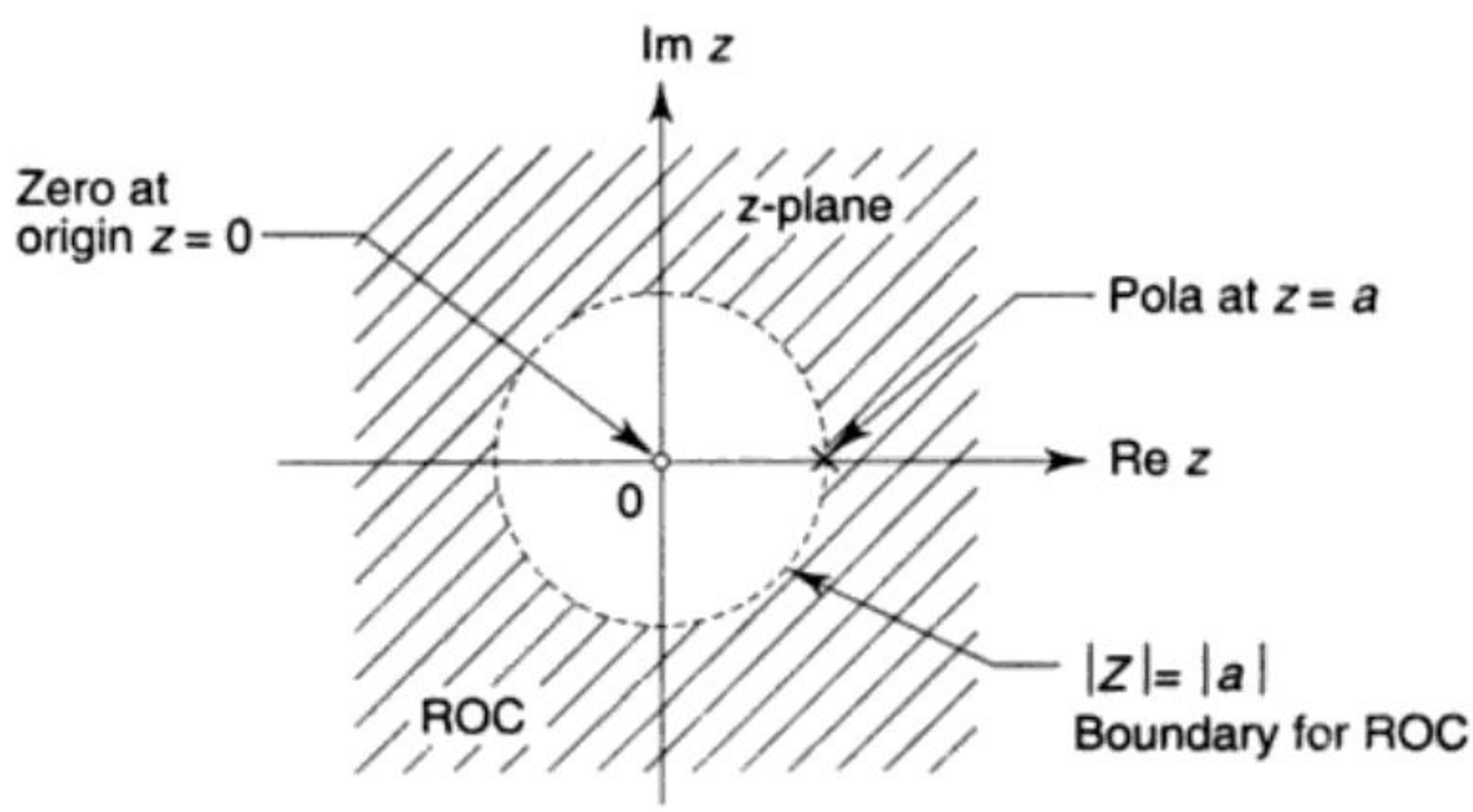

Fig. E 4.3 *ROC for the z-transform of $x(n) = a^n$.*

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$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots \\
\hline
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & 1 - 3z + 2z^2 \\
\hline
 & \quad 3z - 2z^2 \\
 & \quad 3z - 9z^2 + 6z^3 \\
\hline
 & \qquad 7z^2 - 6z^3 \\
 & \qquad 7z^2 - 21z^3 + 14z^4 \\
\hline
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad 15z^3 - 45z^4 + 30z^5 \\
\hline
 & \qquad\qquad 31z^4 - 30z^5
\end{array}
$$

Therefore, $X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots$

Taking inverse z-transform, we get $x(n) = \left\{ \cdots, 62, 30, 14, 6, 2, 0, 0, \underset{\uparrow}{} \right\}$

4.4.2 Partial Fraction Expansion

The partial fraction expansion method is useful to decompose a signal or a system transfer function into a sum of standard functions. First we have to factorise the denominator of the transfer function $H(z)$ into prime factors. Then, for simple poles, $H(z)$ can be expressed in the following form, with $m \le n$,

$$H(z) = \frac{a_0 z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

$$= A_0 + \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_n}{z - p_n}$$

where $$A_0 = \lim_{z \to \infty} H(z) = \begin{cases} a_0, & \text{if } m = n \\ 0, & \text{if } m < n \end{cases}$$

$$A_i = (z - p_i)H(z)\big|_{z = p_i} \quad \text{for } i = 1, 2, \ldots n.$$

With repeated linear roots, we have the terms

$$\frac{A_{i1}}{z - p_i} + \frac{A_{i2}}{(z - p_i)^2} + \cdots + \frac{A_{ir}}{(z - p_i)^r}$$

where $A_{ir} = (z - p_i)^r X(z)\big|_{z = p_i}$

$$A_{i(r-1)} = \frac{\mathrm{d}}{\mathrm{d}z}(z - p_i)^r X(z)\Big|_{z = p_i}$$

$$\vdots \qquad\qquad \vdots$$

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Hence, by multiplying $X(z)/z$ by z, we obtain

$$X(z) = \frac{2}{3} - \frac{z}{z - \frac{1}{2}} + \frac{z/3}{z - 3}$$

Here, the given function $X(z)$ has two poles, $p_1 = \frac{1}{2}$ and $p_2 = 3$ and the following three inverse transforms.

(a) In the region $|z| > 3$, all poles are interior, i.e. the signal $x(n)$ is causal, and therefore,

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}(3)^n u(n).$$

(b) In the region $|z| < \frac{1}{2}$, both the poles are exterior, i.e. $x(n)$ is anti-causal and hence

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(-n-1) - \frac{1}{3}(3)^n u(-n-1).$$

(c) In the region $\frac{1}{2} < |z| < 3$, $\left[\text{i.e.} |z| < 3 \text{ anti-causal and } |z| > \frac{1}{2} \text{ causal}\right]$, the pole $p_1 = \frac{1}{2}$ is interior and $p_2 = 3$ is exterior, and hence

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(n) - \frac{1}{3}(3)^n u(-n-1).$$

Example 4.24 Determine the inverse z-transform of

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

if the regions of convergence are (a) $|z| > 1$, (b) $|z| < \frac{1}{3}$ and (c) $\frac{1}{3} < |z| < 1$

Solution The partial fraction expansion of $X(z)$ yields

$$F(z) = \frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3(z-1)\left(z - \frac{1}{3}\right)}$$

$$= \frac{A_1}{(z-1)} + \frac{A_2}{\left(z - \frac{1}{3}\right)}$$

$$A_1 = F(z)(z-1)|_{z=1} = \frac{1}{3\left(z - \frac{1}{3}\right)}\Big|_{z=1} = \frac{1}{2}$$

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For poles at $z = 2$

$$\text{Residue} = \frac{1}{0!}\lim_{z \to 2}\left\{\frac{d^0}{dz^0}\left[(z-2)^1 \frac{z \cdot z^{n-1}}{(z-1)(z-2)}\right]\right\}$$

$$= \lim_{z \to 2}\left[\frac{z}{z-1} z^{n-1}\right] = 2 \cdot 2^{n-1} = 2^n$$

Therefore, $x(n) = (-1 + 2^n)\, u(n)$

Example 4.27 Using the residue method, find the inverse z-transform of

$$X(z) = \frac{1}{(z-0.25)(z-0.5)}, \text{ ROC: } |z| > 0.5$$

Solution

Let $\quad X_0(z) = X(z)z^{n-1}$

$$= \frac{z^{n-1}}{(z-0.25)(z-0.5)}$$

For convenience let us write this as

$$X_0(z) = \frac{z^n}{z(z-0.25)(z-0.5)}$$

This has poles at $z = 0$, $z = 0.25$ and $z = 0.5$

By the residue theorem

$$x(n) = \underset{z=0}{Res}\,[X_0(z)] + \underset{z=0.25}{Res}\,[X_0(z)] + \underset{z=0.5}{Res}\,[X_0(z)]$$

$$= \underset{z=0}{Res}\,\frac{z^n}{z(z-0.25)(z-0.5)} + \underset{z=0.25}{Res}\,\frac{z^n}{z(z-0.25)(z-0.5)}$$

$$+ \underset{z=0.5}{Res}\,\frac{z^n}{z(z-0.25)(z-0.5)}$$

(a) For pole at $z = 0$

$$\underset{z=0}{Res}\,X_0(z) = zX_0(z)\,|\,z = 0$$

$$= \left.\frac{z^n}{(z-0.25)(z-0.5)}\right|_{z=0}$$

$$= 0$$

(b) For pole at $z = 0.25$

$$\underset{z=0.25}{Res}\,X_0(z) = (z-0.25)X_0(z)\,|_{z=0.25}$$

$$= \left.\frac{(z-0.25)\,z^n}{z(z-0.25)(z-0.5)}\right|_{z=0.25}$$

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4.45 *Find the causal signal x(n) for* $X(z) = \dfrac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$

Ans: $x(n) = \sqrt{10}\left(\dfrac{1}{\sqrt{2}}\right)^n \cos\left(\dfrac{\pi n}{4} - 71.565°\right)u(n)$

4.46 *Find the inverse z-transform of*

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

where the ROC is (i) $|z| > 1$ *and (ii)* $|z| < \dfrac{1}{3}$ *using the long division method.*

Ans: (i) $x(n) = \left\{\underset{\uparrow}{0}, \dfrac{1}{3}, \dfrac{4}{9}, \dfrac{13}{27}, \dfrac{40}{81}, \cdots\right\}$

(ii) $x(n) = \left\{\cdots 121, 40, 13, 4, 1, \underset{\uparrow}{0}\right\}$

4.47 *Using long division, determine the inverse z-transform of*

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

if (a) x(n) is causal and (b) x(n) is anti-causal

Ans: (a) $x(n) = \left\{\underset{\uparrow}{1}, 4, 7, 10, 13, \cdots\right\}$

(b) $x(n) = \left\{\cdots 14, 11, 8, 5, 2, \underset{\uparrow}{0}\right\}$

4.48 *Determine the causal signal x(n) having the z-transform*

$$X(z) = \frac{z^2+z}{\left(z-\frac{1}{2}\right)^3\left(z-\frac{1}{4}\right)} \text{ for the region of convergence } |z| > \frac{1}{2}$$

Ans: $x(n) = \left\{80\left(\dfrac{1}{2}\right)^n - 20n\left(\dfrac{1}{2}\right)^{n-1} + 6[n(n-1)/2]\left(\dfrac{1}{2}\right)^{n-2} - 80\left(\dfrac{1}{4}\right)^n\right\}u(n)$

4.49 *Using (i) the long division method, (ii) partial fraction method and (iii) residue method, find x(n) and verify the results in each case for n in the range* $0 \le n \le 3$.

(a) $X(z) = \dfrac{z+3}{z-0.25}$

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This expression gives the output response $y(n)$ of the LTI system as a function of the input signal $x(n)$ and the unit impulse (sample) response $h(n)$ and is referred to as a **convolution sum**.

5.1.2 Unit Step Response [$u(n)$]

The unit step sequence $u(n)$ is defined by

$$u(n)=\begin{cases}0, & n<0\\ 1, & n\geq 0\end{cases} \tag{5.3}$$

The shifted unit step sequence $u(n-k)$ is given by

$$u(n-k)=\begin{cases}0, & n<k\\ 1, & n\geq k\end{cases}$$

The graphical representations of $u(n)$ and $u(n-2)$ are shown in Fig. 5.5.

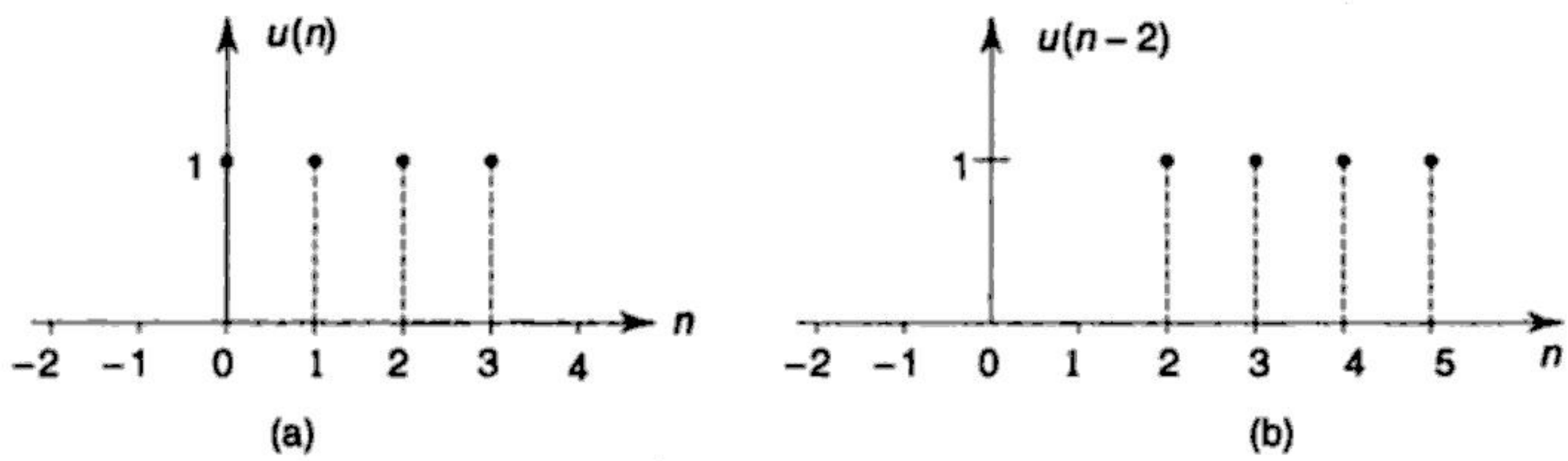

Fig. 5.5 *(a) The Unit-step Sequence u (n) and (b) The Shifted Unit-step Sequence u (n – 2).*

The step response can be obtained by exciting the input of the system by a unit-step sequence, i.e, $x(n) = u(n)$. Hence, the output response $y(n)$ is obtained by using the convolution formula as

$$y(n)=\sum_{k=-\infty}^{\infty} h(k)\,u(n-k)$$

Relation Between the Unit Sample and the Unit-step Sequences

The unit sample sequence $\delta(n)$ and the unit-step sequence $u(n)$ are related as

$$u(n)=\sum_{m=0}^{n}\delta(m),\ \delta(\mathrm{n})=u(n)-u(n-1) \tag{5.4}$$

5.2 PROPERTIES OF A DSP SYSTEM

The properties of linearity, time invariance, causality and stability of the difference equations are required for the DSP system to be practically realisable.

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Therefore,

$$aF[x_1(t)] + bF[x_2(t)] = a\frac{d^2y_1(t)}{dt^2} + b\frac{d^2y_2(t)}{dt^2} + ay_1(t)\frac{dy_1(t)}{dt} + by_2(t)\frac{dy_2(t)}{dt} + ay_1(t) + by_2(t)$$

$$F[a\,x_1(t) + b\,x_2(t)] = \frac{d^2}{dt^2}[a\,y_1(t) + b\,y_2(t)] + [a\,y_1(t) + b\,y_2(t)]\left[\frac{d}{dt}(a\,y_1(t) + b\,y_2(t)) + 1\right]$$

Here $aF[x_1(t)] + bF[x_2(t)] \neq F[ax_1(t) + bx_2(t)]$ and hence the system is non-linear.

5.2.2 Time- Invariance

A DSP system is said to be time-invariant if the relationship between the input and output does not change with time. It is mathematically defined as

$$\text{if } y(n) = F[x(n)], \text{ then } y(n-k) = F[x(n-k)] = z^{-k}F[x(n)] \quad (5.6)$$

for all values of k. This is true for all possible excitations. The operator z^{-k} represents a signal delay of k samples.

Example 5.8 Determine whether the DSP systems described by the following equations are time invariant.

(a) $y(n) = F[x(n)] = a\,n\,x(n)$.

(b) $y(n) = F[x(n)] = a\,x(n-1) + b\,x(n-2)$

Solution

(a) The response to a delayed excitation is

$$F[x(n-k)] = an\,[x(n-k)]$$

The delayed response is $y(n-k) = a(n-k)[x(n-k)]$

Here $F[x(n-k)] \neq y(n-k)$ and hence the system is not time invariant, i.e. the system is time dependent.

(b) Here, $F[x(n-k)] = ax[(n-k)-1] + bx[(n-k)-2]$

$$= y(n-k)$$

Hence the system is time invariant.

Example 5.9 Check whether the following systems are linear and time invariant.

(a) $F[x(n)] = n\,[x(n)]^2$

(b) $F[x(n)] = a\,[x(n)]^2 + b\,x(n)$

Solution

(a) (i) $F[x(n)] = n\,[x(n)]^2$

Here, $F[x_1(n)] = n\,[x_1(n)]^2$ and

$F[x_2(n)] = n\,[x_2(n)]^2$

Therefore, $F[x_1(n)] + F[x_2(n)] = n\,[\{x_1(n)\}^2 + \{x_2(n)\}^2]$

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$$= 2\,[(\infty - \tan^{-1}\infty) - (0 - \tan^{-1} 0)]$$
$$= 2\,[\infty - \pi/2] = \infty$$

Since the Paley-Wiener criterion is not satisfied, the amplitude function is not a suitable amplitude response for a causal LTI system.

Example 5.13 Using Paley-Wiener criterion, determine whether the magnitude function $|H(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}}$ is realisable.

Solution According to Paley-Wiener criterion, the magnitude function $|H(j\omega)|$ is realisable only when the following condition is satisfied.

$$\int_{-\infty}^{\infty} \frac{|ln\,|H(j\omega)||}{1+\omega^2}\,d\omega < \infty$$

Here, the given magnitude function is $|H(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}}$

$$ln\left(\frac{1}{\sqrt{1+\omega^2}}\right) = ln\ 1 - \log(1+\omega^2)^{1/2}$$
$$= -\frac{1}{2}\log(1+\omega^2)$$
$$\left|ln\left(\frac{1}{\sqrt{1+\omega^2}}\right)\right| = \frac{1}{2}\log(1+\omega^2)$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\left|ln\left(\frac{1}{\sqrt{1+\omega^2}}\right)\right|}{1+\omega^2}\,d\omega = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(1+\omega^2)}{1+\omega^2}\,d\omega$$
$$= 2\cdot\frac{1}{2}\int_{0}^{\infty} \frac{\log(1+\omega^2)}{1+\omega^2}\,d\omega$$
$$= \int_{0}^{\infty} \frac{\log(1+\omega^2)}{1+\omega^2}\,d\omega$$

Substituting $\omega = \tan\theta$, we get

$$1 + \omega^2 = 1 + \tan^2\theta = \sec^2\theta,$$
$$d\omega = \sec^2\theta\,d\theta \quad \text{and} \quad \theta \text{ becomes } 0 \text{ to } \pi/2$$
$$= \int_{0}^{\pi/2} \frac{\log\sec^2\theta}{\sec^2\theta}\,.\,\sec^2\theta\,d\theta$$

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Example 5.16 Check the stability condition for the DSP systems described by the following equations .

(a) $y(n) = a^n u(n)$

(b) $y(n) = x(n) + e^a y(n-1)$

Solution

(a) $y(n) = a^n u(n)$

Taking z-transform, we have

$$Y(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Here the pole is at $z = a$ and hence for the system to be stable, $|a| < 1$.

(b) $y(n) = x(n) + e^a y(n-1)$

Taking z-transform, we have

$$Y(z) = X(z) + e^a z^{-1} Y(z)$$

$$Y(z)\,[1 - e^a z^{-1}] = X(z)$$

Therefore, $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - e^a z^{-1}} = \frac{z}{z - e^a}$$

Here the pole is at $z = e^a$ and hence $|e^a| < 1$, i.e. $a < 0$ for stability.

5.2.5 Bounded Input- Bounded Output (BIBO) Stability

Stability is of utmost importance in any system design. There are many definitions for stability. One of them is BIBO stability. A sequence $x(n)$ is bounded if there exists a finite M such that $|x(n)| < M$ for all n. Any system is said to be BIBO stable if and only if every bounded input gives a bounded output.

For any linear time invariant (LTI) system, the BIBO stability depends on the impulse response of that system. To obtain the necessary and sufficient condition for BIBO stability, consider the convolution property which relates the input and the output of a LTI system as

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

It follows that

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x(n-k)|\ |h(k)|$$

$$\leq M \sum_{k=-\infty}^{\infty} |h(k)| \quad [\text{since } |x(n)| < M \text{ for all } n]$$

where M is a finite constant.

Therefore, the output is bounded if and only if

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The necessary and sufficient condition for BIBO stability is

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Here, $\sum_{k=0}^{\infty} |h(k)| = |b|$

Hence, the given system is BIBO stable if $|b| < \infty$.

(e) $y(n) = ax(n) \,.\, x(n-1)$

If $x(n) = \delta(n)$, then $y(n) = h(n)$.

The above equation can be changed into $h(n) = a\delta(n) \,.\, \delta(n-1)$

When $n = 0$, $h(0) = a\delta(0) \,.\, \delta(-1) = 0$

When $n = 1$, $h(1) = a\delta(1) \,.\, \delta(0) = 0$

When $n = 2$, $h(2) = a\delta(2) \,.\, \delta(1) = 0$

The necessary and sufficient condition for BIBO stability is

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Here, $\sum_{k=0}^{\infty} |h(k)| = 0$.

So the given system is BIBO stable .

(f) $y(n) = \text{max. of } [x(n), x(n-1), x(n-2)]$.

If $x(n) = \delta(n)$, then $y(n) = h(n)$.

The above equation can be changed into

$h(n) = \text{max . of } [\delta(n), \delta(n-1), \delta(n-2)]$

When $n = 0$, $h(0) = \text{max . of } [\delta(0), \delta(-1), \delta(-2)] = 1$

When $n = 1$, $h(1) = \text{max . of } [\delta(1), \delta(0), \delta(-1)] = 1$

When $n = 2$, $h(2) = \text{max . of } [\delta(2), \delta(1), \delta(0)] = 1$

When $n = 3$, $h(3) = \text{max . of } [\delta(3), \delta(2), \delta(1)] = 0$

In general,

$$h(n) = 1, \quad \text{for } n = 0, 1, 2 \,.$$
$$= 0, \quad \text{otherwise } (i.e., n > 2) \,.$$

The necessary and sufficient condition for BIBO stability is

$$\sum_{k=0}^{\infty} |h(k)| < \infty.$$

Here, $\sum_{k=0}^{\infty} |h(k)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(k)| + \cdots$

$$= 1 + 1 + 1 + 0 + \cdots = 3.$$

So, the given system is BIBO stable.

(g) $y(n) = \text{average of } [x(n+1), x(n), x(n-1)]$.

If $x(n) = \delta(n)$, then $y(n) = h(n)$.

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$$= \omega - \pi + 2\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]$$

To find time delay $\tau(\omega)$

$$\text{i.e.}\quad \tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = -1 - 2\,\frac{(1 - a\cos\omega)\,a\cos\omega - (a\sin\omega)\,a\sin\omega}{(1 - a\cos\omega)^2 + (a\sin\omega)^2}$$

$$= -1 - \frac{a\cos\omega - a^2}{1 - 2\cos\omega + a^2} = \frac{-1 + a\cos\omega}{1 - 2a\cos\omega + a^2}$$

5.4.6 Geometrical Construction Method to Determine Magnitude and Phase of the Frequency Response of a Digital System

Like an analog filter, a digital filter can be represented in the frequency domain by a magnitude response and a phase response. The difference between analog and digital filters is that the transfer function of the analog filter is calculated on the imaginary axis of the s-plane whereas the transfer function of the digital filter is evaluated on the unit circle $|z| = 1$ of the z-plane.

The geometric evaluation of the magnitude and phase responses of the system with a greater number of poles and zeros can be done by determining the magnitude and phase angles of the vectors at a given frequency on the unit circle $|z| = |e^{j\omega T}| = 1$. The magnitude $M(\omega)$ and phase $\Phi(\omega)$ of the frequency response of a digital system can be determined using the geometrical construction method as explained below.

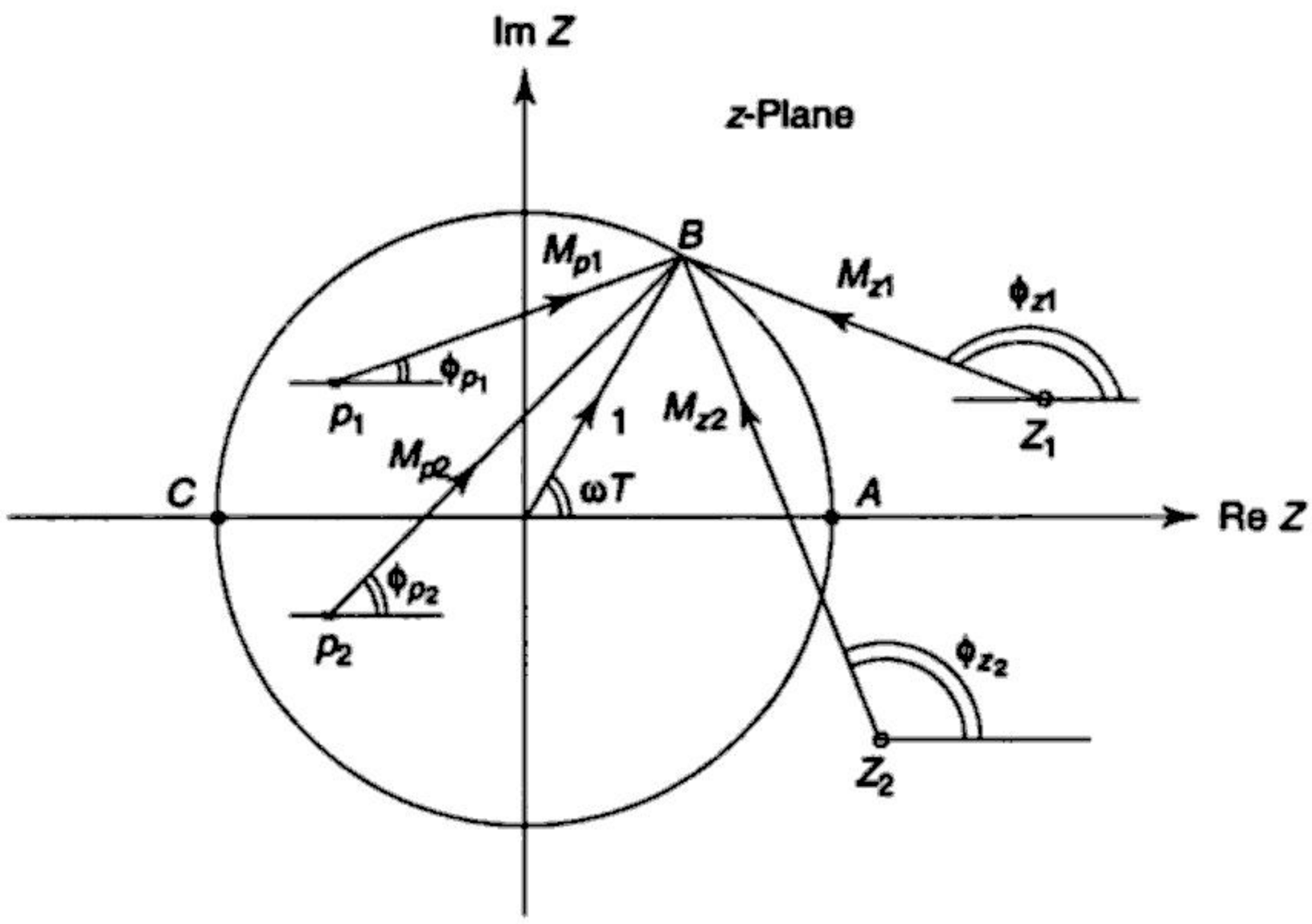

Fig. 5.8 *Geometric Representation for a Second Order Recursive Filter*

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5.5 Explain briefly the Paley-Wiener criterion.

5.6 Explain (i) Causality and (ii) stability of a linear time invariant system.

5.7 State the conditions for a digital filter to be causal and stable.

5.8 What is the necessary and sufficient condition for stability ?

5.9 What is the condition for a system to be BIBO stable?

5.10 Show that the necessary and sufficient condition for a LTI system to be stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \text{ where } h(n) \text{ is the impulse response.}$$

5.11 How will you test the stability of a digital filter .

5.12 How will you obtain the power spectral density of a digital filter.

5.13 Write the mathematical and graphical representation of a unit sample sequence.

5.14 Give the mathematical and graphical representation of a unit-step sequence.

5.15 State and explain the transfer function of an LTI system.

5.16 What is the impulse response of two LTI systems connected in cascade ?

5.17 Explain in detail the system transfer function.

5.18 Discuss briefly the frequency response of LTI systems.

5.19 Explain the method of obtaining the frequency response function of linear shift-invariant systems.

5.20 Describe the relationship between impulse response and frequency response of a discrete time system.

5.21 Discuss magnitude response, phase response and time delay.

5.22 Distinguish between IIR and FIR systems.

5.23 Determine the impulse response and the unit step response of the systems described by the difference equation

$$y(n) = 0.6\, y(n-1) - 0.08\, y(n-2) + x(n)$$

5.24 A system is characterised by

$$y(n) - b\, y(n-1) = x(n)$$

$$\text{where } x(n) = 1 \text{ for } n \geq 0$$

$$= 0 \text{ for } n < 0, \text{ and } y(0) = 1$$

Obtain the expression for step response of the system.

5.25 Verify whether the following impulse responses describe causal, stable or LTI systems. Give reasons for your answers.

(a) $h(n) = e^{-0.6n}\, u(n)$

(b) $h(n) = \sin(n)\, e^{n}\, u(n)$

(c) $h(n) = 4\,\delta(n-2) + 2\,\delta(n-4)$

(d) $h(n) = u(n-2) - u(n+3)$

(Contd.)

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processing because convolving two sequences in the time domain is equivalent to multiplying the sequences in the frequency-domain. Convolution finds its application in processing signals especially analysing the output of a system. Consider the signals $x_1(n)$ and $x_2(n)$. The convolution of these two signals is given by

$$x_3(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \tag{6.1}$$

6.2.1 Representation of Discrete-Time Signals in Terms of Impulse

An arbitrary signal $x(n)$ can be resolved into a sum of unit sample sequences. The signal $x(n)$ shown in Fig. 6.1 can be expressed in terms of unit sample sequences as

$$\begin{aligned} x(n) &= \sum_{k=-2}^{\infty} x(k)\,\delta(n-k) \\ &= x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) \\ &\qquad + x(2)\delta(n-2) + x(3)\,\delta(n-3) \\ &= (1)\,\delta(n+2) + (-2)\,\delta(n+1) + (2)\,\delta(n) + (1)\,\delta(n-1) + (-1) \\ &\qquad \delta(n-2) + (1)\,\delta(n-3) \end{aligned}$$

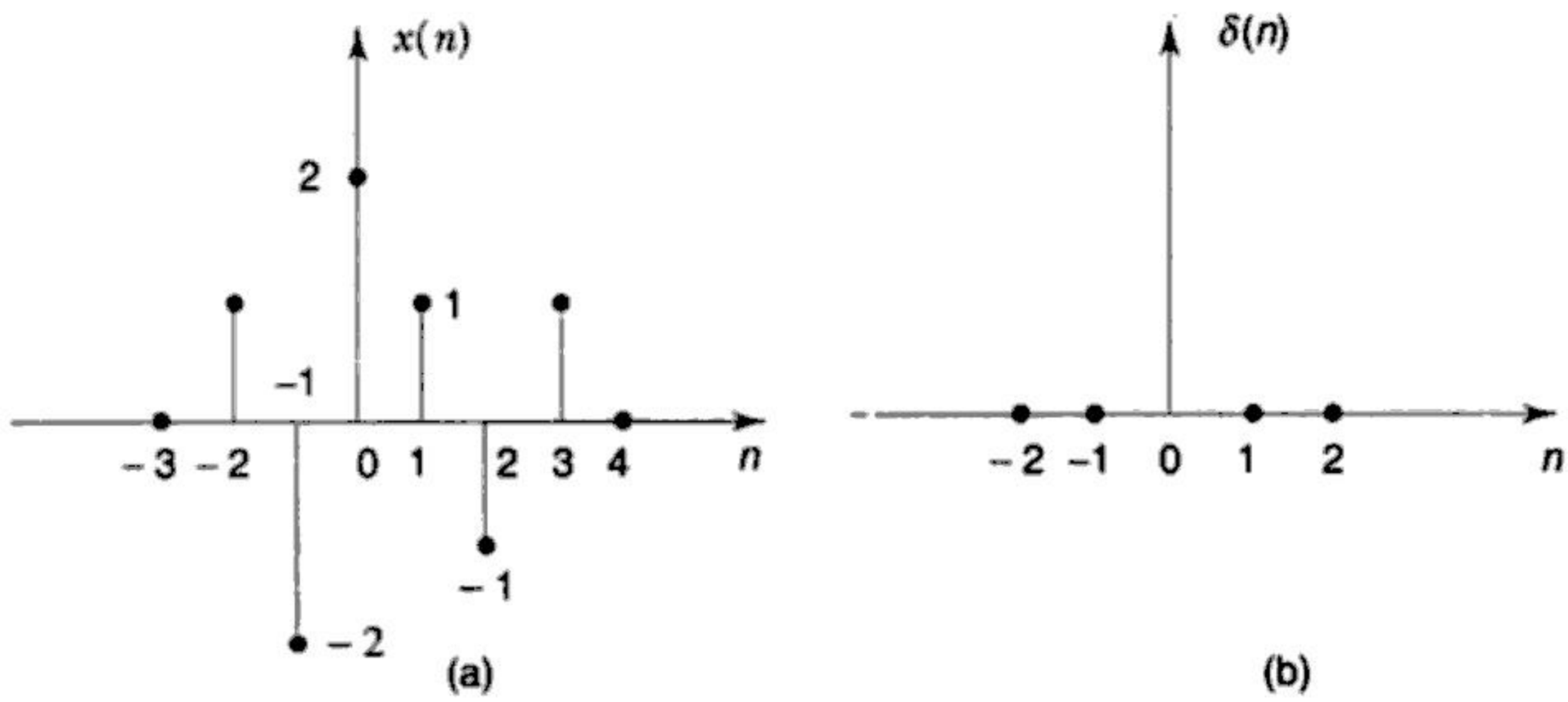

Fig. 6.1

Hence, any arbitrary signal can be represented in terms of the weighted sum of impulses.

6.2.2 Response of LTI Systems to Arbitrary Inputs (Convolution Sum)

Consider a linear system whose behaviour is specified by the impulse response $h(n)$. Let $x(n)$ be the input signal to this system and corresponding output signal be $y(n)$ as shown in Fig. 6.2.

To find the output signal $y(n)$, resolve the signal $x(n)$ into a weighted sum of impulses, and the linearity and time shift property of the LTI system can be used.

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Consider the term within the brackets. It has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a}, & \text{for } a \neq 1 \end{cases}$$

where, $a = e^{j2\pi(m-n-l)/N}$. When $(m-n-l)$ is a multiple of N, then $a = 1$, otherwise $a^N = 1$ for any value of $a \neq 0$.

Therefore, $\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN,\ N = (m-n)(\text{mod } N),\ p\text{: integer}) \\ 0, & \text{otherwise} \end{cases}$

$x_3(m)$ becomes

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n, (\text{mod } N)), \quad m = 0, 1, \ldots, N-1. \quad (6.4)$$

where $x_2((m-n) \text{ mod } N)$ is the reflected and circularly shifted version of $x_2(m)$ and n represents the number of indices that the sequence $x(n)$ is shifted to the right.

The above equation has a form similar to the convolution sum and it is called circular convolution. The circularly shifted versions of $x(m)$ are given below:

$$x(m) = [x(0), x(1) \ldots x(N-3), x(N-2), x(N-1)]$$
$$x(m-1, (\text{mod } N)) = [x(N-1), x(0), x(1) \ldots x(N-3), x(N-2)]$$
$$x(m-2, (\text{mod } N)) = [x(N-2), x(N-1), x(0), x(1) \ldots x(N-3)]$$

.

.

.

$$x(m-n, (\text{mod } N)) = [x(N-n), x(N-n+1), \ldots, x(N-n-1)]$$

.

.

.

$$x(m-N, (\text{mod } N)) = [x(0), x(1) \ldots x(N-3), x(N-2), x(N-1)]$$

It is noted that a shift of $n = N$ results in the original signal $x(m)$.

6.2.5 Linear Convolution vs Circular Convolution

Consider a finite duration sequence $x(n)$ of length N_1 which is given as input to an FIR system with impulse response $h(n)$ of length N_2, then the output is given by

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=0}^{N_1-1} x(k)\, h(n-k), \quad n = 0, 1, \ldots, N_1 + N_2 - 1$$

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(c) Here the DFT and IDFT are used for finding circular convolution

$$X_3(k) = X_1(k)\, X_2(k)$$

and $$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi nk/N}, \quad n = 0, 1, \ldots, N-1$$

(i) When $x_1(n) = [1,1,2,2]$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N-1$$

For $N = 4$

$$X_1(k) = \sum_{n=0}^{3} x_1(n)\, e^{-j2\pi nk/4}, \quad k = 0, 1, 2, 3$$

For $k = 0$

$$X_1(0) = \sum_{n=0}^{3} x_1(n)\, e^{-j2\pi n(0)/4} = 6$$

For $k = 1$

$$\begin{aligned} X_1(1) &= \sum_{n=0}^{3} x_1(n)\, e^{-j\pi n/2} \\ &= 1 + e^{-j\pi/2} + 2e^{-j\pi} + 2e^{-j3\pi/2} \\ &= 1 - j + 2(-1) + 2(j) \\ &= -1 + j \end{aligned}$$

For $k = 2$

$$\begin{aligned} X_1(2) &= \sum_{n=0}^{3} x_1(n)\, e^{-jn\pi} \\ &= 1 + e^{-j\pi} + 2e^{-2\pi} + 2e^{-j3\pi} \\ &= 1 - 1 + 2(1) + 2(-1) = 0 \end{aligned}$$

For $k = 3$

$$\begin{aligned} X_1(1) &= \sum_{n=0}^{3} x_1(n)\, e^{-j3n\pi/2} \\ &= 1 + e^{-j3\pi/2} + 2e^{-j3\pi} + 2e^{-j9\pi/2} \\ &= 1 + (j) + 2(-1) - j(2) \\ &= -1 - j \end{aligned}$$

$$X_1(k) = \{6, -1 + j, 0, -1 - j\}$$

(ii) When $x_2(n) = \{1, 2, 3, 4\}$

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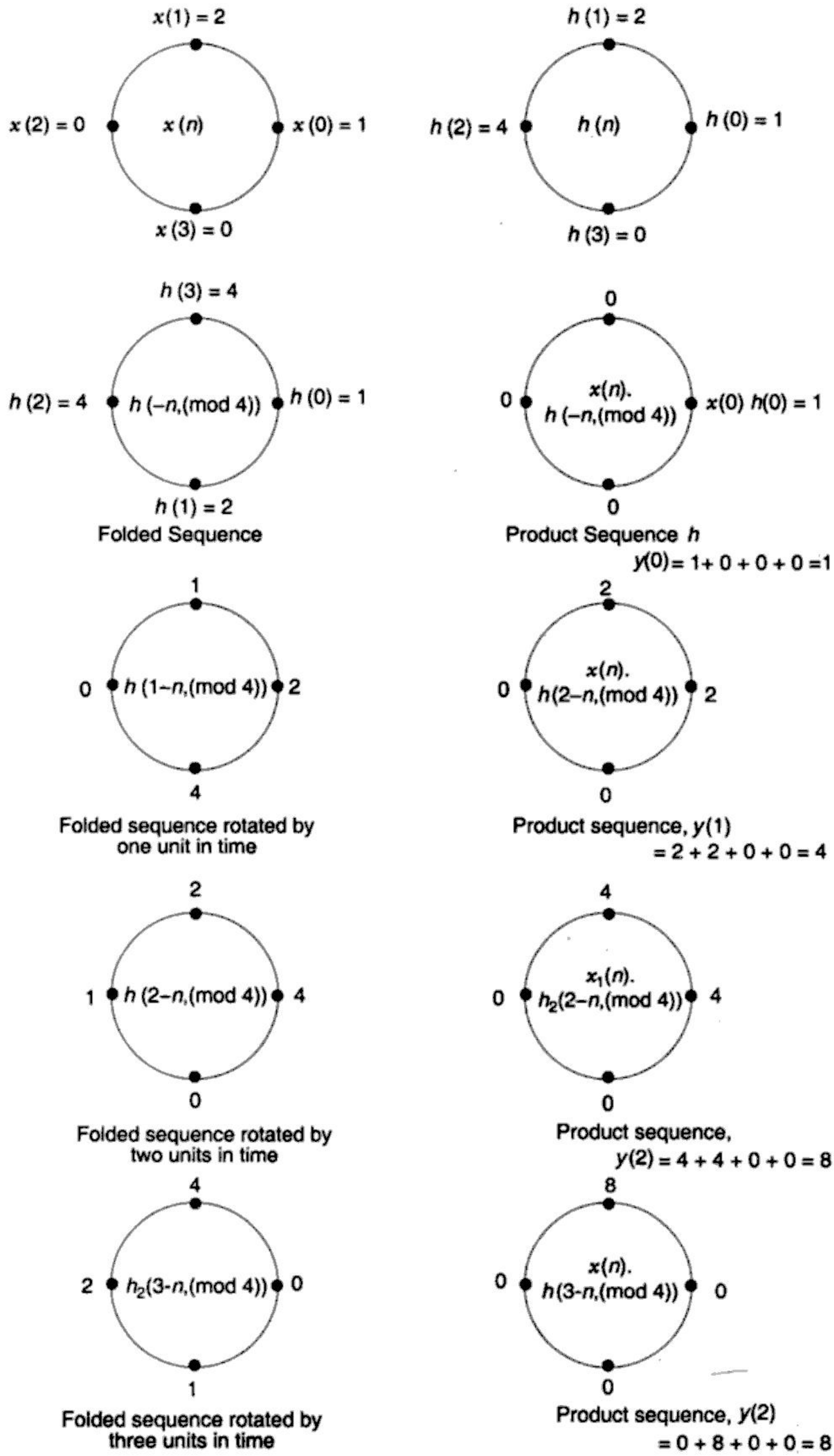

Fig. E6.2

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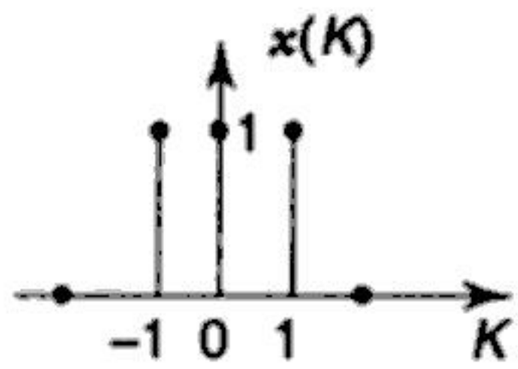
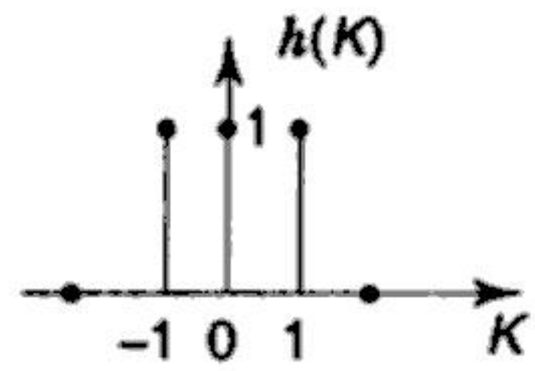
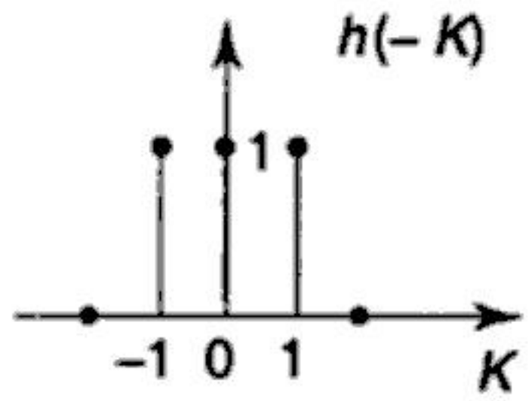
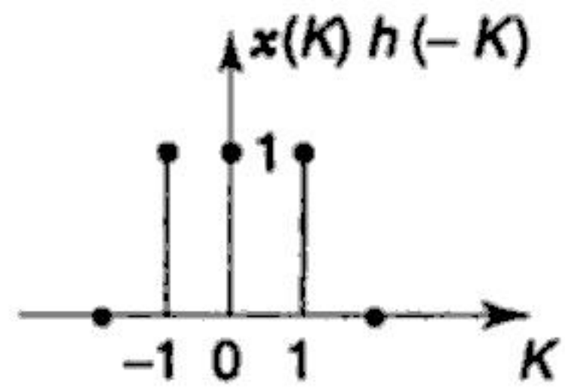




When $n = 0$

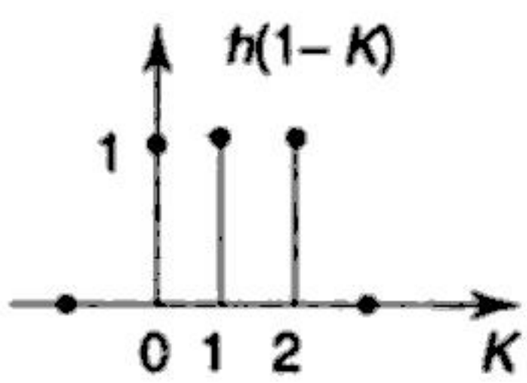


$h(-K) \Leftrightarrow$ fold $h(K)$

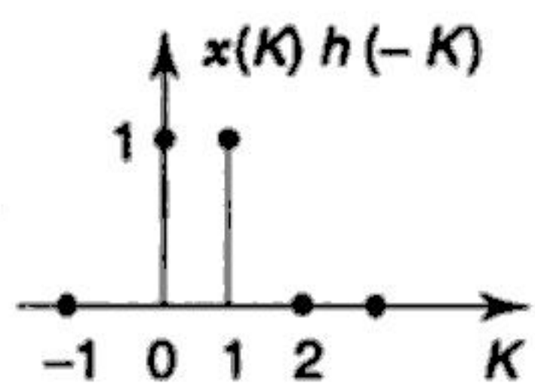


Product Sequence, $y_0(K) = x(K)\, h(-K)$

When $n = 1$

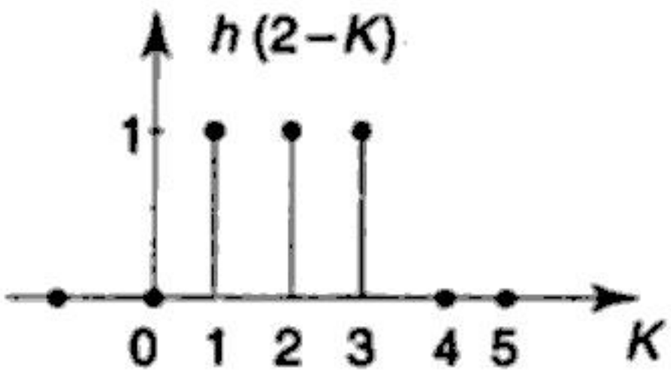


$h(1-K) \Leftrightarrow$ shift $h(-K)$ to right by one unit

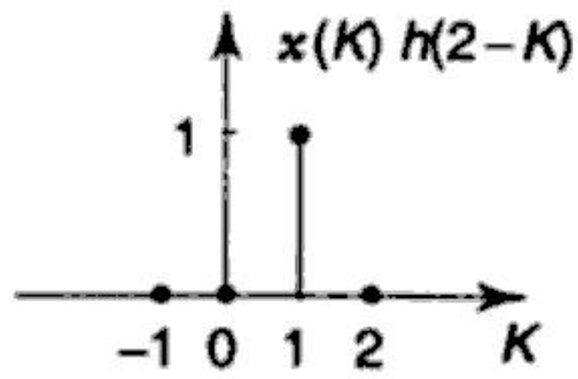


Product Sequence, $y_1(K) = x(K)\, h(1-K)$

When $n = 2$

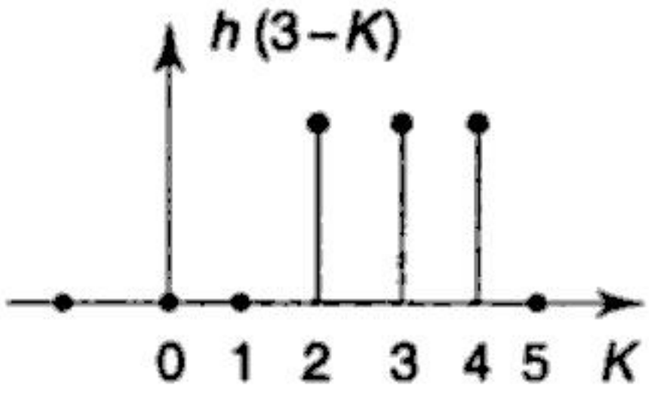


$h(2-K) \Leftrightarrow$ shift $h(1-K)$ to right by one unit

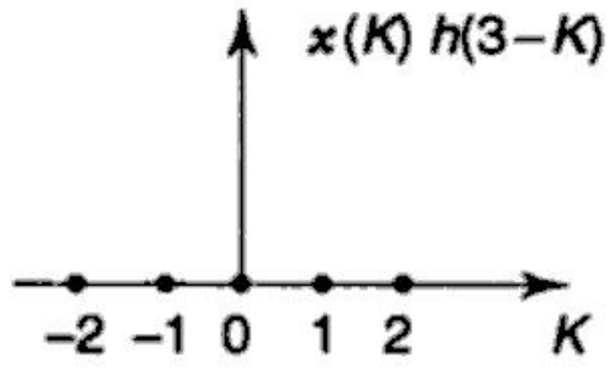


Product Sequence, $y_2(K) = x(K)\, h(2-K)$

When $n = 3$

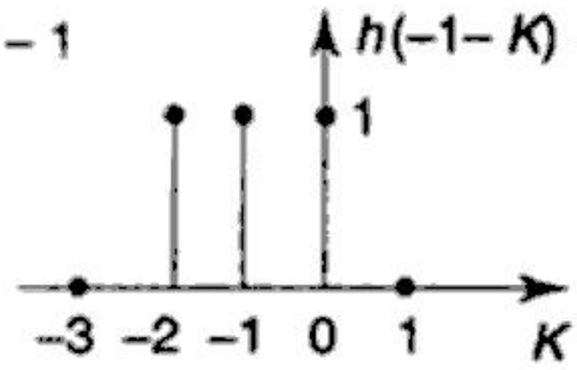


$h(3-K) \Leftrightarrow$ shift $h(2-K)$ to right by one unit

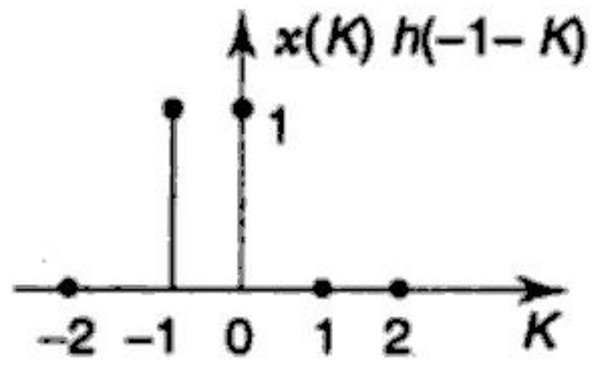


Product Sequence, $y_3(K) = x(K)\, h(3-K)$

When $n = -1$



$h(-1-K) \Leftrightarrow$ shift $h(-K)$ to left by one unit



Product Sequence, $y_{-1}(K) = x(K)\, h(-1-K)$

Fig. E6.3(c) *Graphical Method (Contd.)*

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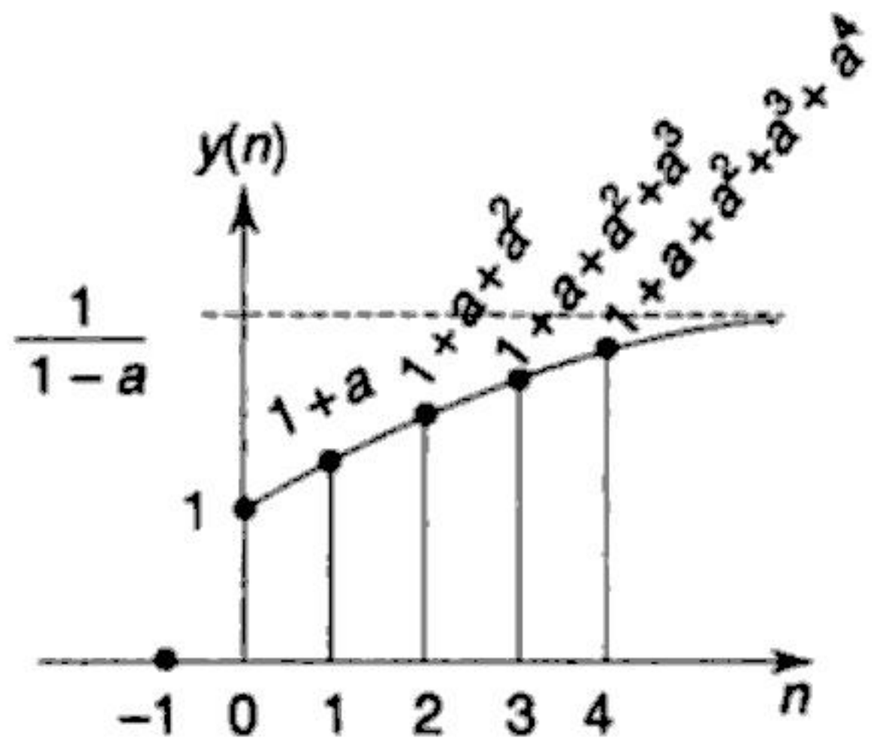

Fig. E6.4(c)

Example 6.5 Find the convolution of the two signals shown in Fig. E 6.5(a).

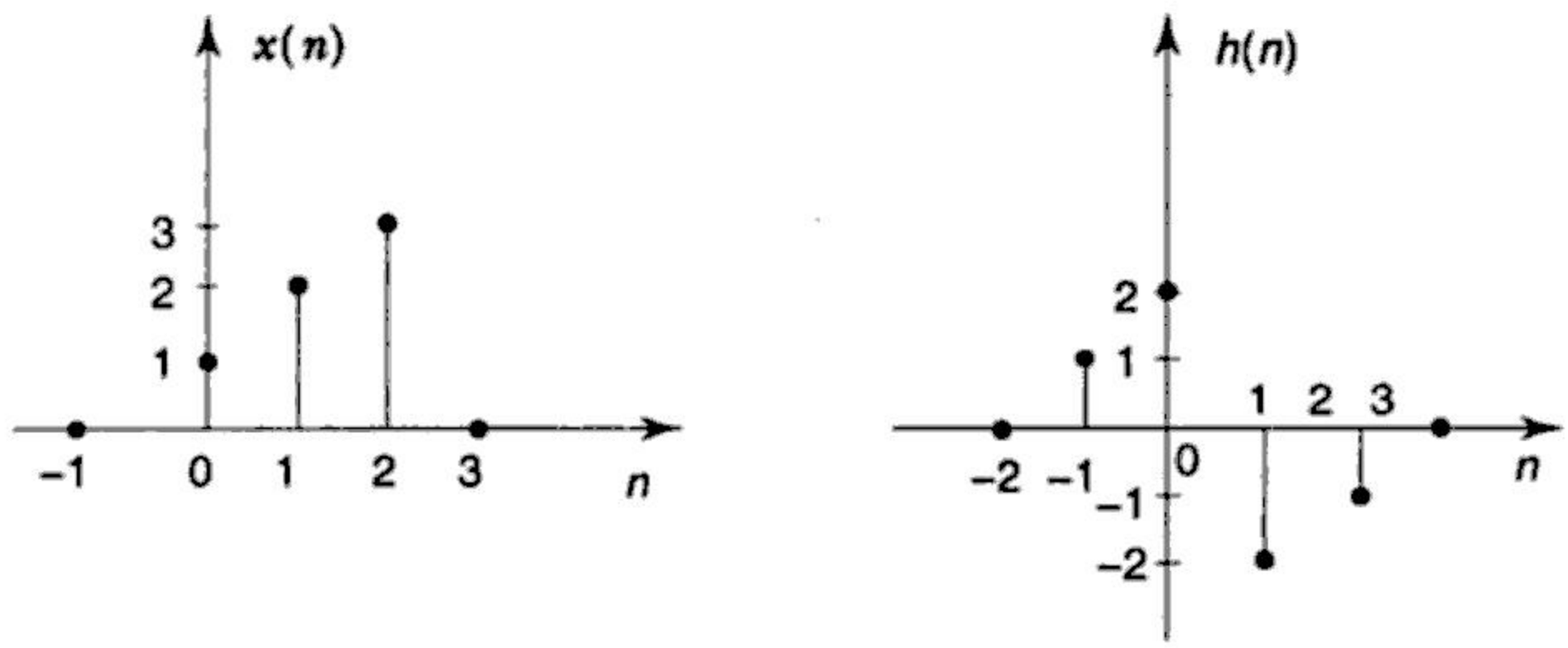

Fig. E6.5(a)

Solution Let $y(n)$ be the resultant convoluted signal. Then

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

From the graph,

$$x_l = 0, \quad x_r = 2, \quad h_l = -1, \; h_r = 2$$

Therefore, the left and right extremes of $y(n)$ is found to be

$$y_l = x_l + h_l = 0 + (-1) = -1$$
$$y_r = x_r + h_r = 2 + 2 = 4$$

When $\quad n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$
$$= \ldots + x(0)\, h(-1) + x(1)\, h(-2) + x(1)\, h(-3) + \ldots$$
$$= 0 + (1)(1) + (2)(0) + 0 \ldots = 1$$

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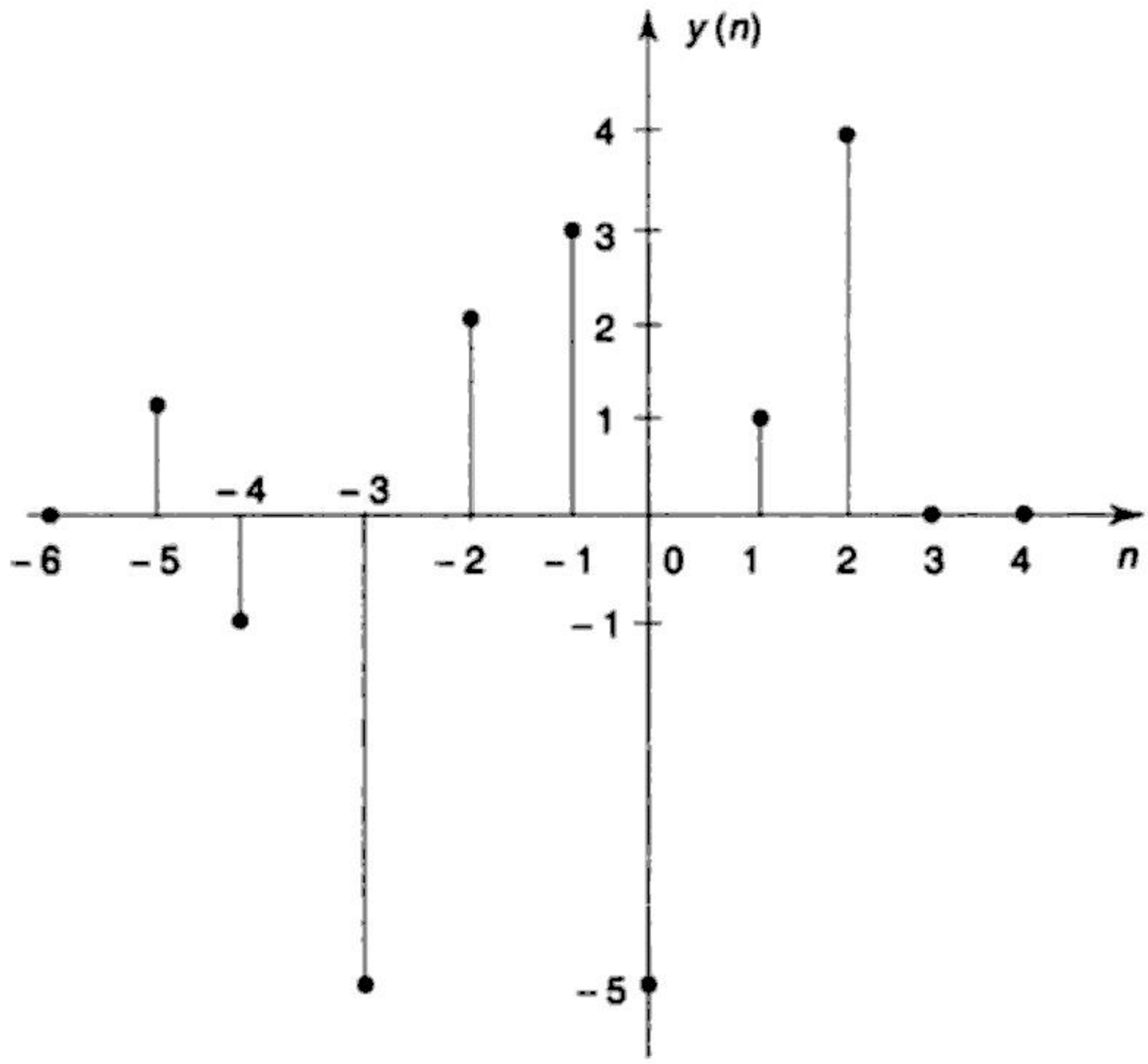

Fig. E6.6(b)

$$= 0 + (1)(4) + (1)(-3) + (0)(-2) + (1)(1) + (1)(0) + 0 \ldots = 2$$

When $n = -3$

$$y(-3) = \sum_{k=-\infty}^{\infty} h(k)\, x(-3-k)$$

$$= \ldots + x(-2)\, h(-1) + x(-1)\, h(-2) + x(0)\, h(-3) + x(1)\, h(-4) + x(2)\, h(-5) + \ldots$$

$$= 0 + (1)(-3) + (1)(-2) + (0)(1) + (1)(0) + (1)(0) + 0 \ldots = -5$$

When $n = -4$

$$y(-4) = \sum_{k=-\infty}^{\infty} h(k)\, x(-4-k)$$

$$= \ldots + x(-2)\, h(-2) + x(-1)\, h(-3) + x(0)\, h(-4) + x(1)\, h(-5) + x(2)\, h(-6) + \ldots$$

$$= 0 + (1)(-2) + (1)(1) + (0)(0) + (1)(0) + (1)(0) + 0 + \ldots = -1$$

When $n = -5$

$$y(-5) = \sum_{k=-\infty}^{\infty} h(k)\, x(-5-k)$$

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$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X(k)\,e^{j2\pi nk/N}\right]z^{-n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}X(k)\sum_{n=0}^{N-1}\left(e^{j2\pi k/N}z^{-1}\right)^n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}X(k)\,\frac{1-z^{-N}}{1-e^{j2\pi k/N}z^{-1}}$$

$$= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j2\pi k/N}z^{-1}}$$

The above equation is identical to that of frequency sampling form. When this is evaluated over a unit circle, then

$$X(e^{j\omega}) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j(\omega-2\pi k/N)}}$$

6.3.2 Properties of the DFT

The properties of the DFT are useful in the practical techniques for processing signals. The various properties are given below.

1. Periodicity

If $X(k)$ is an N-point DFT of $x(n)$, then

$$x(n+N) = x(n) \text{ for all } n$$
$$X(k+N) = X(k) \text{ for all } k$$

2. Linearity

If $X_1(k)$ and $X_2(k)$ are the N-point DFTs of $x_1(n)$ and $x_2(n)$ respectively, and a and b are arbitrary constants either real or complex-valued, then

$$a\,x_1(n) + b\,x_2(n) \xleftrightarrow[N]{DFT} a\,X_1(k) + b\,X_2(k)$$

3. Shifting Property

Let $x_p(n)$ is a periodic sequence with period N, which is obtained by extending $x(n)$ periodically, i.e.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x\,(n-lN)$$

Now, shift the sequence $x_p(n)$ by k units to the right. Let the resultant signal be expressed as $x_{p'}(n)$ which is given by

$$x_{p'}(n) = x_p\,(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

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$$= \frac{1}{2\pi}\left[\frac{e^{jn\omega}}{jn} + \frac{e^{j(n-1)\omega}}{j(n-1)} + \frac{e^{j(n-2)\omega}}{j(n-2)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{2\sin n\pi}{n} + \frac{2\sin((n-1)\pi)}{(n-1)} + \frac{2\sin((n-2)\pi)}{(n-2)}\right]$$

For $n = 0$

$$x(0) = \frac{1}{2\pi}\left[\frac{2\sin n\pi}{n}\right] = \frac{\sin n\pi}{n\pi} = 1$$

$$\left[\, L \text{ Hospital's rule gives } \lim_{n\to 0} \frac{\sin n\pi}{n\pi} = 1 \,\right]$$

Similarly,

For $n = 1 : x(1) = 1$

For $n = 2 : x(2) = 1$

For $n \geq 3 : x(n) = 0$

These values are identical to the defined sequence for $L = 3$ and $A = 1$V.

i.e. $$x(n) = \begin{cases} 1, & \text{for } 0 \leq n \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Example 6.8 Determine the DFT of the sequence

$$x(n) = \begin{cases} \frac{1}{4}, & \text{for } 0 \leq n \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Solution The N-point DFT of the sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, k = 0, 1, \ldots, N-1.$$

$$x(n) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

Therefore, $X(k) = \frac{1}{4}\left[1 + e^{-j\omega} + e^{-j2\omega}\right]\Big|_{\omega = \frac{2\pi k}{N}}$

$$= \frac{1}{4} e^{-j\omega}\,[1 + 2\cos\omega] \quad \text{where} \quad \omega = \frac{2\pi k}{N} \text{ and } N = 3$$

$$= \frac{1}{4} e^{-j2\pi k/3}\left[1 + 2\cos\left(\frac{2\pi k}{3}\right)\right]$$

Hence, $X(k) = \frac{1}{4} e^{-j2\pi k/3}\left[1 + 2\cos\left(\frac{2\pi k}{3}\right)\right]$ where $k = 0, 1, \ldots, N-1$.

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$$x(n) = [\cos(0), \cos(\pi/4), \cos(\pi/2), \cos(3\pi/4)]$$
$$= \{1, 0.707, 0, -0.707\}$$

The N-point DFT of the sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, k = 0, 1, \ldots N-1.$$

The DFT is

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi nk/4}, k = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\pi nk/2}, k = 0, 1, 2, 3$$

For $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) = 1$$

For $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\pi(1)n/2}$$
$$= 1 + 0.707\, e^{-j\pi/2} + 0 + (-0.707) e^{-j3\pi/2}$$
$$= 1 + (0.707)(-j) + 0 - (0.707)(j)$$
$$= 1 - j\,1.414$$

For $k = 2$

$$X(2) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi(2)n/2} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi n}$$
$$= 1 + (0.707)\, e^{-j\pi} + 0 + (-0.707)\, e^{-j3\pi}$$
$$= 1 + (0.707)(-1) + 0 + (-0.707)(-1) = 1$$

For $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j\pi(3)n/2}$$
$$= 1 + (0.707)\, e^{-j3\pi/2} + 0 + (-0.707)\, e^{-j9\pi/2}$$
$$= 1 + (0.707)(j) + 0 + (-0.707)(-j) = 1 + j\,1.414$$
$$X(k) = \{1, 1 - j\,1.414, 1, 1 + j\,1.414\}$$

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If $x(n)$ is a complex-valued sequence, then the N-point DFT given in Eq. 6.10 can be expressed as

$$X(k) = X_R(k) + j\,X_I(k) = \sum_{n=0}^{N-1} [x_R(n) + j\,x_I(n)]\left[\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}\right]$$

Equating the real and imaginary parts of the above equation, we have

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi nk}{N} + x_I(n)\sin\frac{2\pi nk}{N}\right] \tag{6.14}$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi nk}{N} - x_I(n)\cos\frac{2\pi nk}{N}\right] \tag{6.15}$$

The direct computation of the DFT requires $2N^2$ evaluations of trigonometric functions, $4N^2$ real multiplications and $4N(N-1)$ real additions. Also, this is primarily inefficient as it does not exploit the symmetry and periodicity properties of the phase factor W_N, which are given below.

Symmetry Property $\quad W_N^{k+N/2} = -W_N^k$

Periodicity Property $\quad W_N^{k+N} = W_N^k \qquad (6.16)$

An efficient algorithm for DFT computation is the fast Fourier transform algorithm because FFT algorithms exploit the above two properties.

6.4.1 Radix-2 FFT

By adopting a divide and conquer approach, a computationally efficient algorithm for the DFT can be developed. This approach depends on the decomposition of an N-point DFT into successively smaller size DFTs. If N is factored as $N = r_1\, r_2\, r_3 \ldots r_L$ where $r_1 = r_2 = \ldots = r_L = r$, then $N = r^L$. Hence, the DFT will be of size 'r' , where this number 'r' is called the ***radix*** of the FFT algorithm. In this section, the most widely used radix-2 FFT algorithms are described. FFT algorithms take advantage of the periodicity and symmetry of the complex number $W_N^{nk}\left(= e^{-j\left(\frac{2\pi}{N}\right)nk}\right)$.

6.4.2 Decimation-in-Time (DIT) Algorithm

In this case, let us assume that $x(n)$ represents a sequence of N values, where N is an integer power of 2, that is, $N = 2^L$. The given sequence is decimated (broken) into two $\frac{N}{2}$ point sequences consisting of the even numbered values of $x(n)$ and the odd numbered values of $x(n)$.

The N-point DFT of sequence $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk},\ \ 0 \le k \le N-1 \tag{6.17}$$

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Similarly, from Eq. 6.23, we get

$$H(0) = C(0) + W_N^0\, D(0)$$

$$H(1) = C(1) + W_N^2\, D(1)$$

$$H(2) = C(0) + W_N^4\, D(0)$$

$$H(3) = C(1) + W_N^6\, D(1)$$

The above process, of reducing an L-point DFT (L is a power of 2) to an $L/2$-point DFTs, can be continued until we are left with 2-point DFTs, or there are L ($= \log_2 N$) stages, to be evaluated.

A 2-point DFT, $F(k)$, $k = 0, 1$ may be evaluated as,

$$F(0) = f(0) + f(1) W_8^0$$

$$F(1) = f(0) + f(1)\, W_8^4$$

$f(n)$, $n = 0, 1$ is a 2-point sequence.

$W_8^0 = 1$, $W_8^4 = -1$

Therefore, $F(0) = f(0) + W_2^0 f(1) = f(0) + f(1)$

$$F(1) = f(0) + W_2^1 f(1) = f(0) - f(1)$$

The corresponding flow graph of a 2-point DFT is shown in Fig. 6.7.

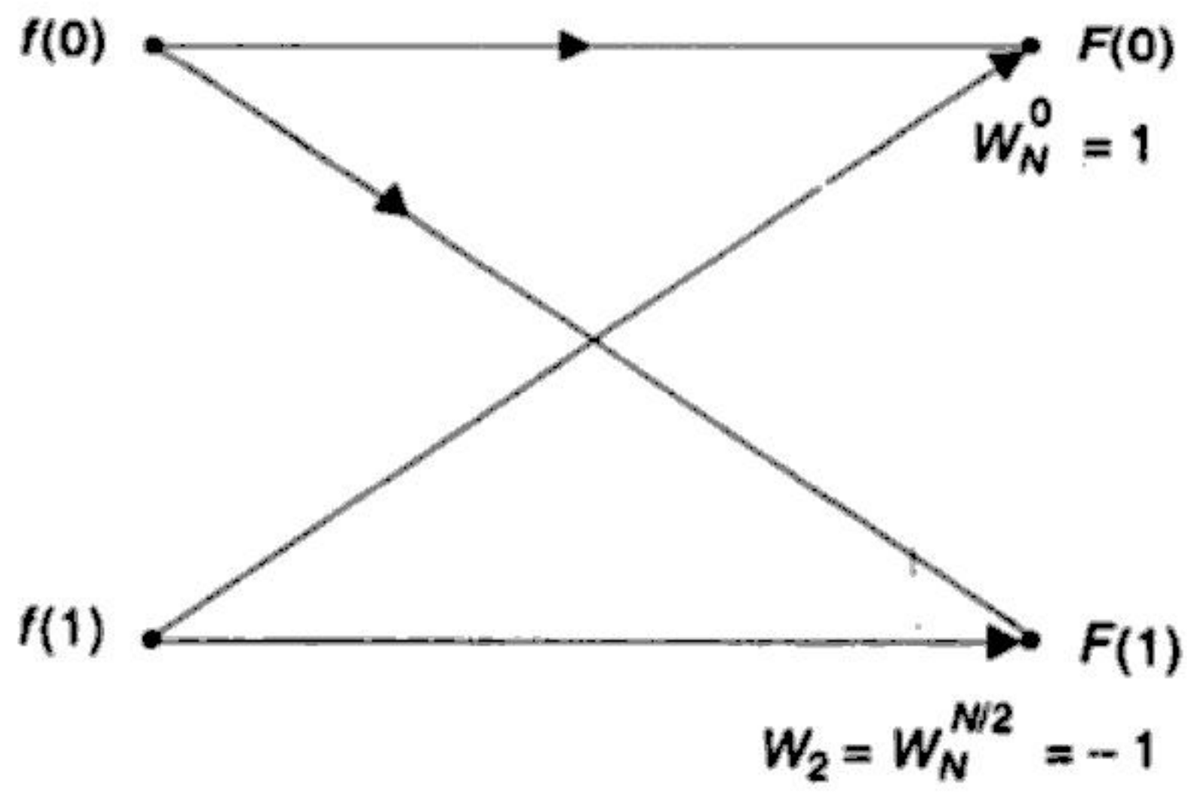

Fig. 6.7 *Flow Graph of a 2-Point DFT*

The number of complex multiplications and additions required was $N + 2(N/2)^2$ or $N + N^2/2$ for the original N-point transform when decomposed into two $(N/2)$-point transforms. When the $(N/2)$-point transforms are decomposed into $N/4$-point transforms, the factor of $(N/2)^2$ is replaced by $\frac{N}{2} + 2(N/4)^2$. Hence, the overall computation requires $N + N + 4(N/4)^2$ complex multiplications and additions.

Figure 6.8 shows the complete flow graph of the decimation-in-time FFT algorithm for $N = 8$, which consists of three stages. The first stage computes the four 2-point DFTs, the second stage computes the two 4-point DFTs and finally the third stage computes the desired 8-point DFT.

From the flow diagram, it is observed that the input data have been shuffled, that is, appear in "bit reversed" order, depicted in Table 6.1 for $N = 8$.

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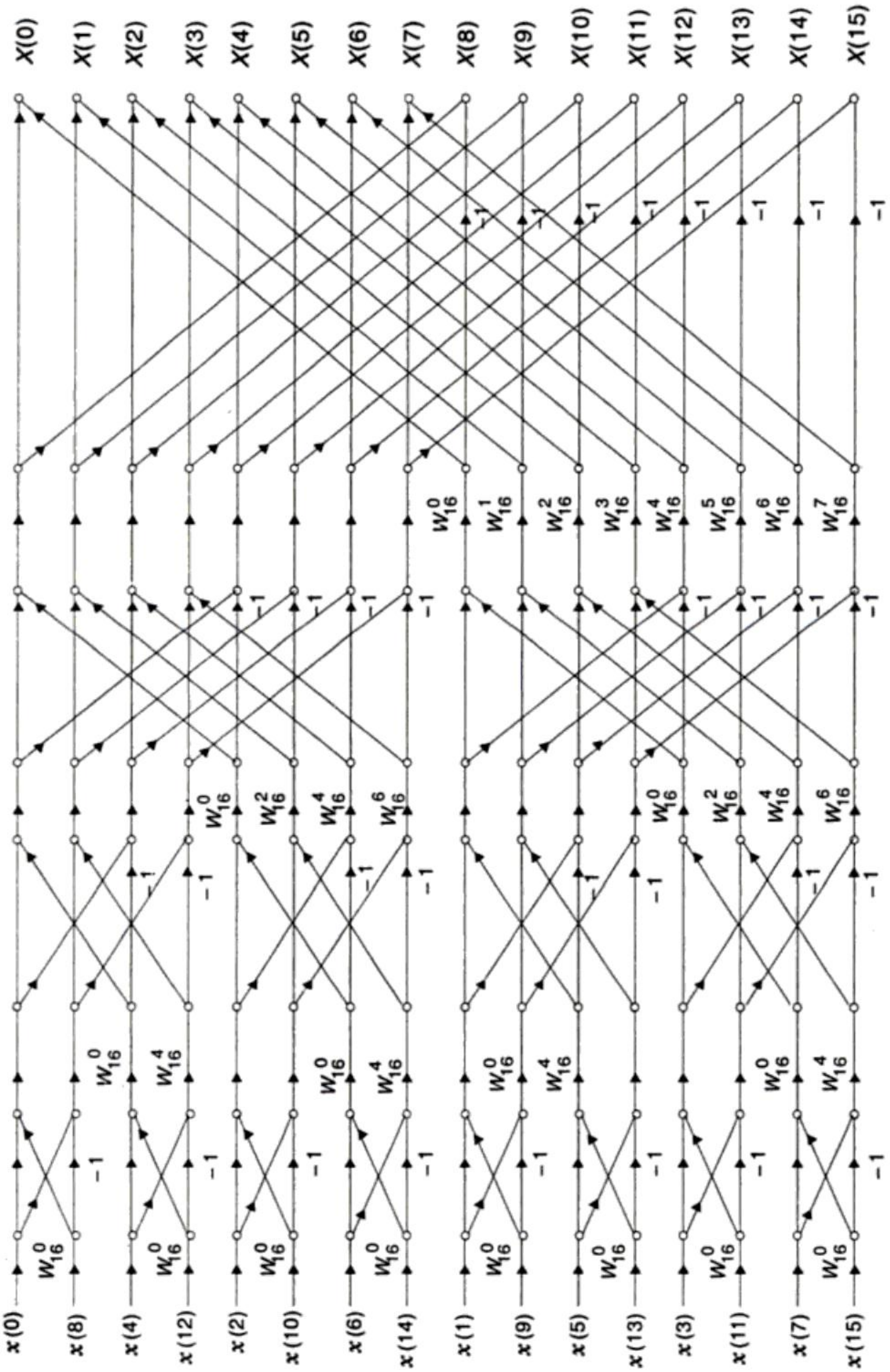

Fig. 6.11 *Flow-Graph of the 16-point DIT FFT Algorithm*

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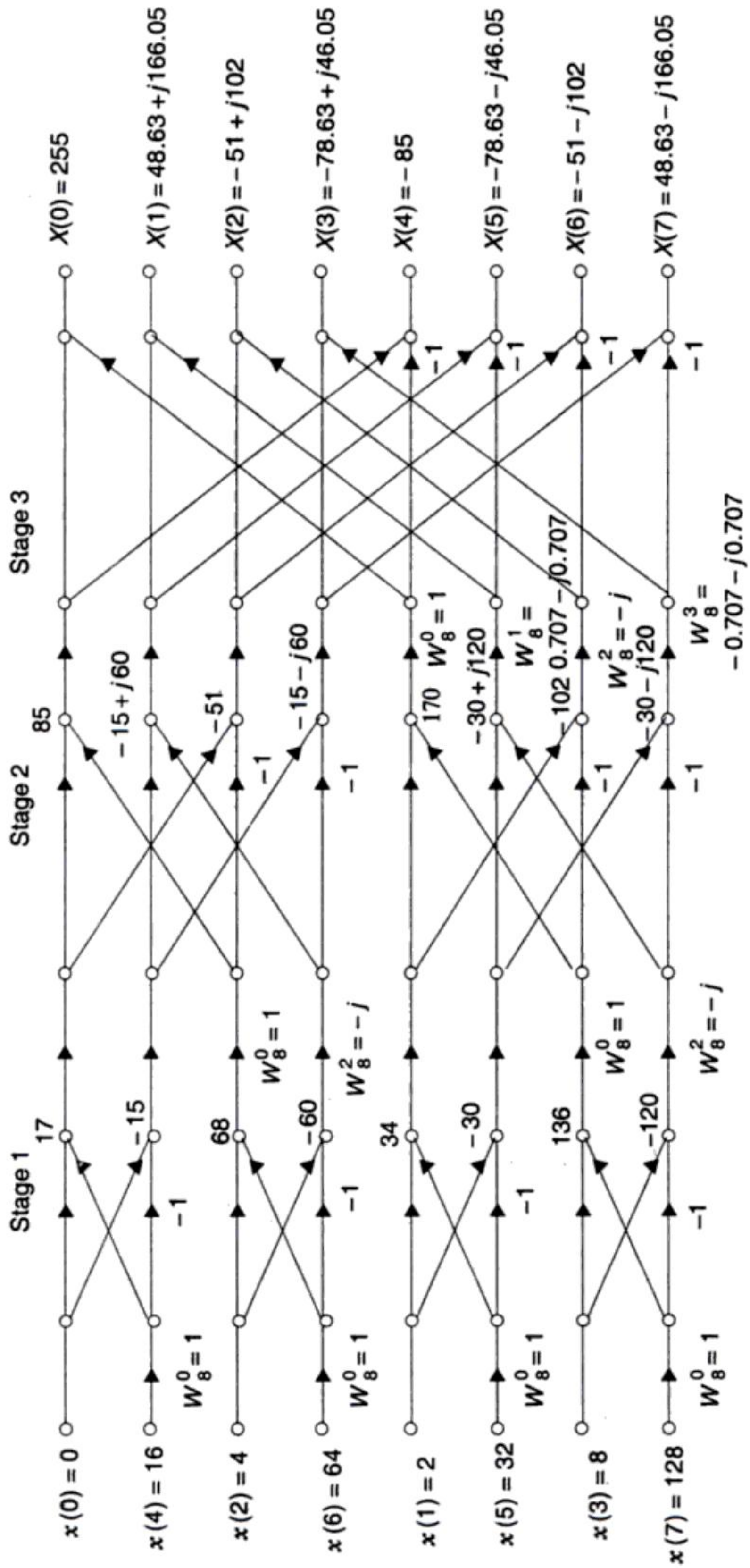

Fig. E6.17

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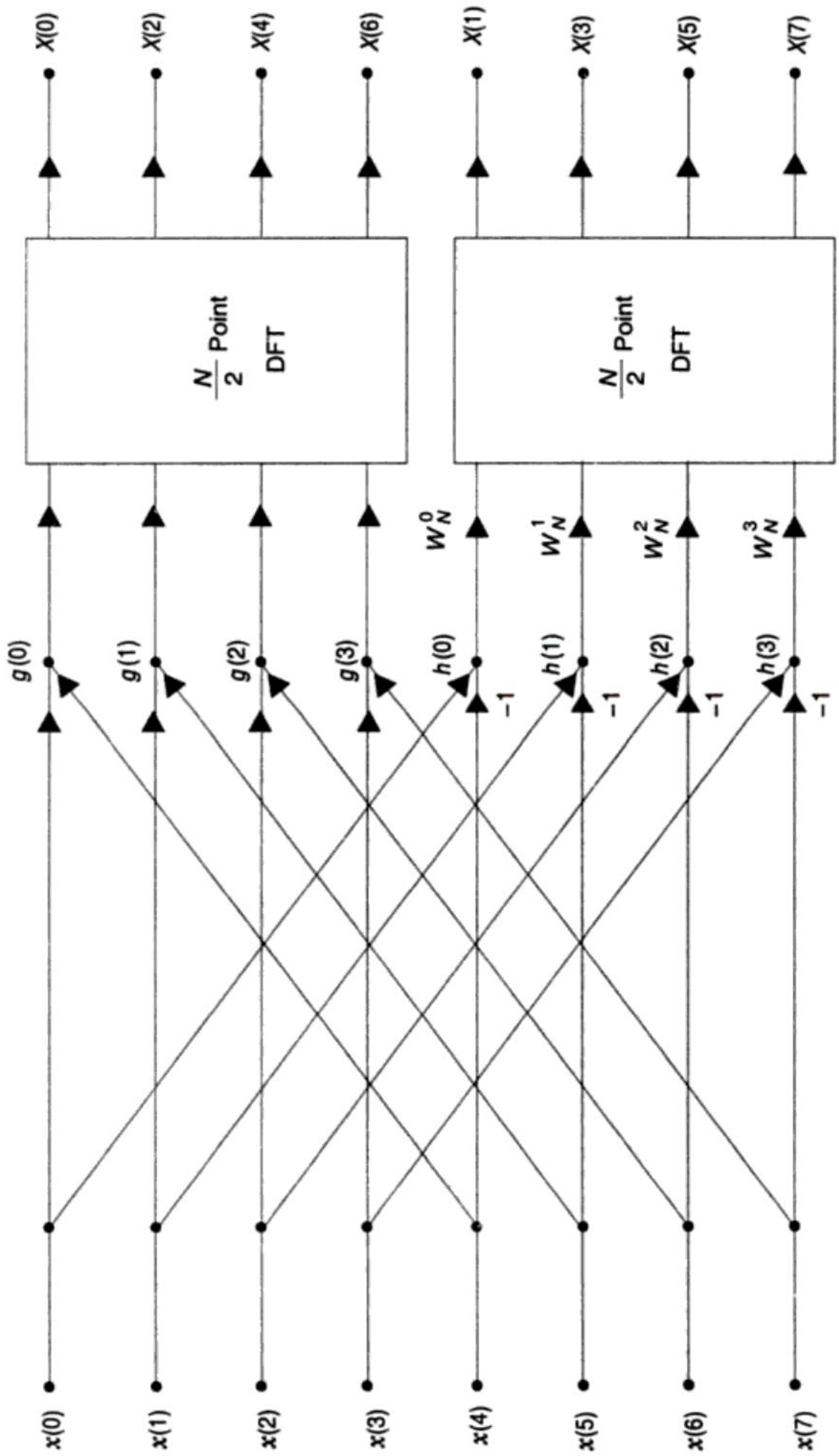

Fig. 6.12 *Flow Graph of the First Stage of Decimation-In-Frequency FFT for N = 8*

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It is observed from Fig. 6.15 that in the DIF FFT algorithm the input sequence $x(n)$ appears in natural order while the output $X(k)$ appears in the bit-reversed order. The algorithm has in place calculations given below with the butterfly structure shown in Fig. 6.14.

$$A = a + b$$
$$B \triangleq (a - b)\, W_N^r \tag{6.33}$$

Example 6.19 Given $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$, find $X(k)$ using DIF FFT algorithm.

Solution Given $N = 8$.

We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$.

Hence, $W_8^0 = 1$, $W_8^1 = 0.707 - j\,0.707$

$W_8^2 = -j$, $W_8^3 = -0.707 - j\,0.707$

Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. E 6.19.

Hence, $X(k) = \{20, -5.828 - j\,2.414, 0, -0.172 - j\,0.414, 0, -0.172 + j\,0.414, 0, -5.828 + j\,2.414\}$

Example 6.20 Given $x(n) = 2^n$ and $N = 8$, find $X(k)$ using DIF FFT algorithm.

Solution $x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$

Given $N = 8$.

We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$

Hence, $W_8^0 = 1$

$W_8^1 = 0.707 - j\,0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j\,0.707$

Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. E 6.20.

Hence, $X(k) = \{255, 48.63 + j166.05, -51 + j102, -78.63 + j46.05, -85, -78.63 - j46.05, -51 - j102, 48.63 - j166.05\}$

Example 6.21 Given $x(n) = n + 1$ and $N = 8$, find $X(k)$ using DIF FFT algorithm.

Solution $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Given $N = 8$.

We know that $W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$.

Hence, $W_8^0 = 1$

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$$W_8^1 = 0.707 - j\,0.707$$
$$W_8^2 = -j$$
$$W_8^3 = -0.707 - j\,0.707$$

Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. E 6.21.

Hence, $X(k) = \{36, -4 + j\,9.656, -4 + j\,4, -4 + j\,1.656, -4, -4 - j1.656, -4 - j4, -4 - j\,9.656\}$

Example 6.22 Compute the DFTs of the sequence $x(n) = \cos \dfrac{n\pi}{2}$, where $N = 4$, using DIF FFT algorithm.

Solution Given $N = 4$ and $x(n) = \{1, 0, -1, 0\}$

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$
$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

Using DIF FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig. E6.22.

Therefore, $X(k) = \{0, 2, 0, 2\}$

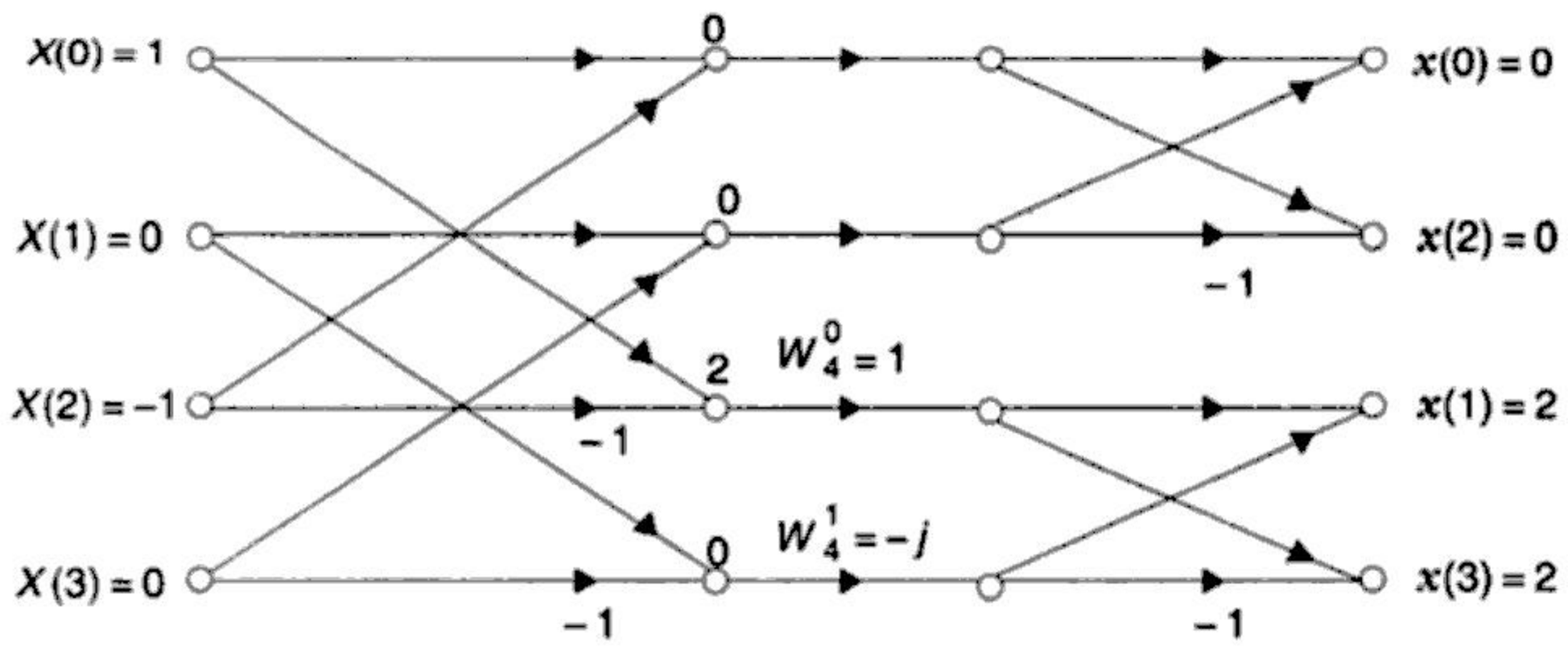

Fig. E6.22

6.5 COMPUTING AN INVERSE DFT BY DOING A DIRECT DFT

An FFT algorithm for calculating the DFT samples can also be used to evaluate efficiently the inverse DFT (IDFT). The inverse DFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}, \quad n = 0, 1, \ldots, N-1. \tag{6.34}$$

Taking complex conjugate of the above expression, we obtain

$$Nx^*(n) = \sum_{k=0}^{N-1} X^*(k)\, W^{nk} \tag{6.35}$$

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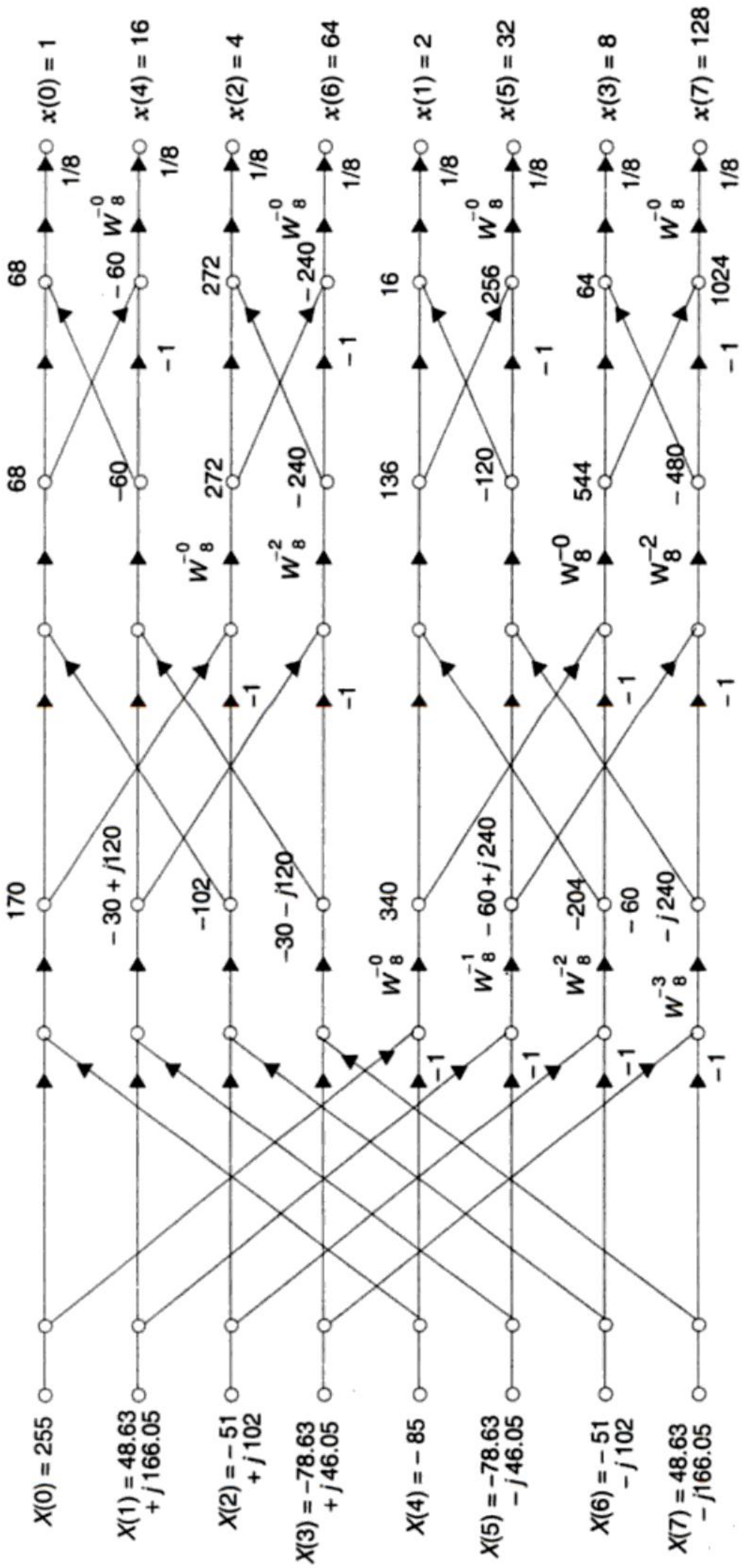

Fig. E6.25

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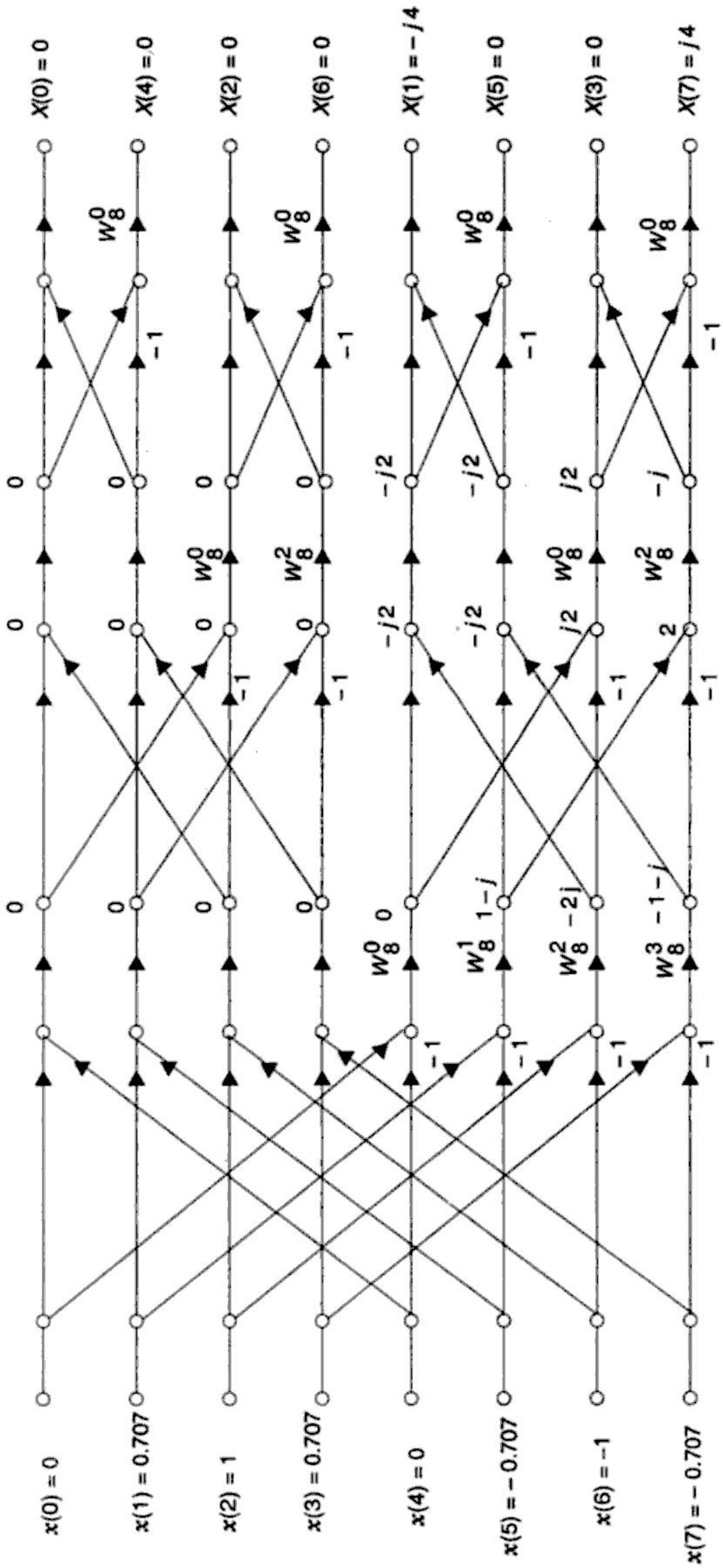

Fig. E6.27(b)

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$$= X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k)$$

where, $$X_1(k) = \sum_{n=0}^{1} x(3n) W_6^{3nk} = x(0) + x(3) W_6^{3k}$$

$$X_2(k) = \sum_{n=0}^{1} x(3n+1) W_6^{3nk} = x(1) + x(4) W_6^{3k}$$

$$X_3(k) = \sum_{n=0}^{1} x(3n+2) W_6^{3nk} = x(2) + x(5) W_6^{3k}$$

Also, $X_i(k+2) = X_i(k)$

$$X_1(0) = x(0) + x(3) W_6^0$$

$$X_1(1) = x(0) + x(3) W_6^3$$

$$X_2(0) = x(1) + x(4) W_6^0$$

$$X_2(1) = x(1) + x(4) W_6^3$$

$$X_3(0) = x(2) + x(5) W_6^0$$

$$X_3(1) = x(2) + x(5) W_6^3$$

$$X(k) = X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k)$$

$$X(0) = X_1(0) + W_6^0 X_2(0) + W_6^0 X_3(0)$$

$$X(1) = X_1(1) + W_6^1 X_2(1) + W_6^2 X_3(1)$$

$$X(2) = X_1(2) + W_6^2 X_2(2) + W_6^4 X_3(2)$$

$$= X_1(0) + W_6^2 X_2(0) + W_6^4 X_3(0)$$

$$X(3) = X_1(3) + W_6^3 X_2(3) + W_6^6 X_3(3)$$

$$= X_1(1) + W_6^3 X_2(1) + W_6^6 X_3(1)$$

$$X(4) = X_1(4) + W_6^4 X_2(4) + W_6^8 X_3(4)$$

$$= X_1(0) + W_6^4 X_2(0) + W_6^2 X_3(0)$$

$$X(5) = X_1(5) + W_6^5 X_2(5) + W_6^{10} X_3(5)$$

$$= X_1(1) + W_6^5 X_2(1) + W_6^4 X_3(1)$$

Figure E6.29(b) shows the decimation-in-time FFT flow diagram for $N = 6 = 3 \,.\, 2$.

(c) To evaluate the DFT values for $x(n) = \{1, 2, 3, 4, 5, 6\}$

$$x(n) = \{1, 2, 3, 4, 5, 6\}$$

$$W_6^0 = e^{\frac{-j2\pi}{6} \cdot 0} = 1$$

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The decimation-in-time FFT flow diagram for $N = 12$ is shown in Fig. E6.30.

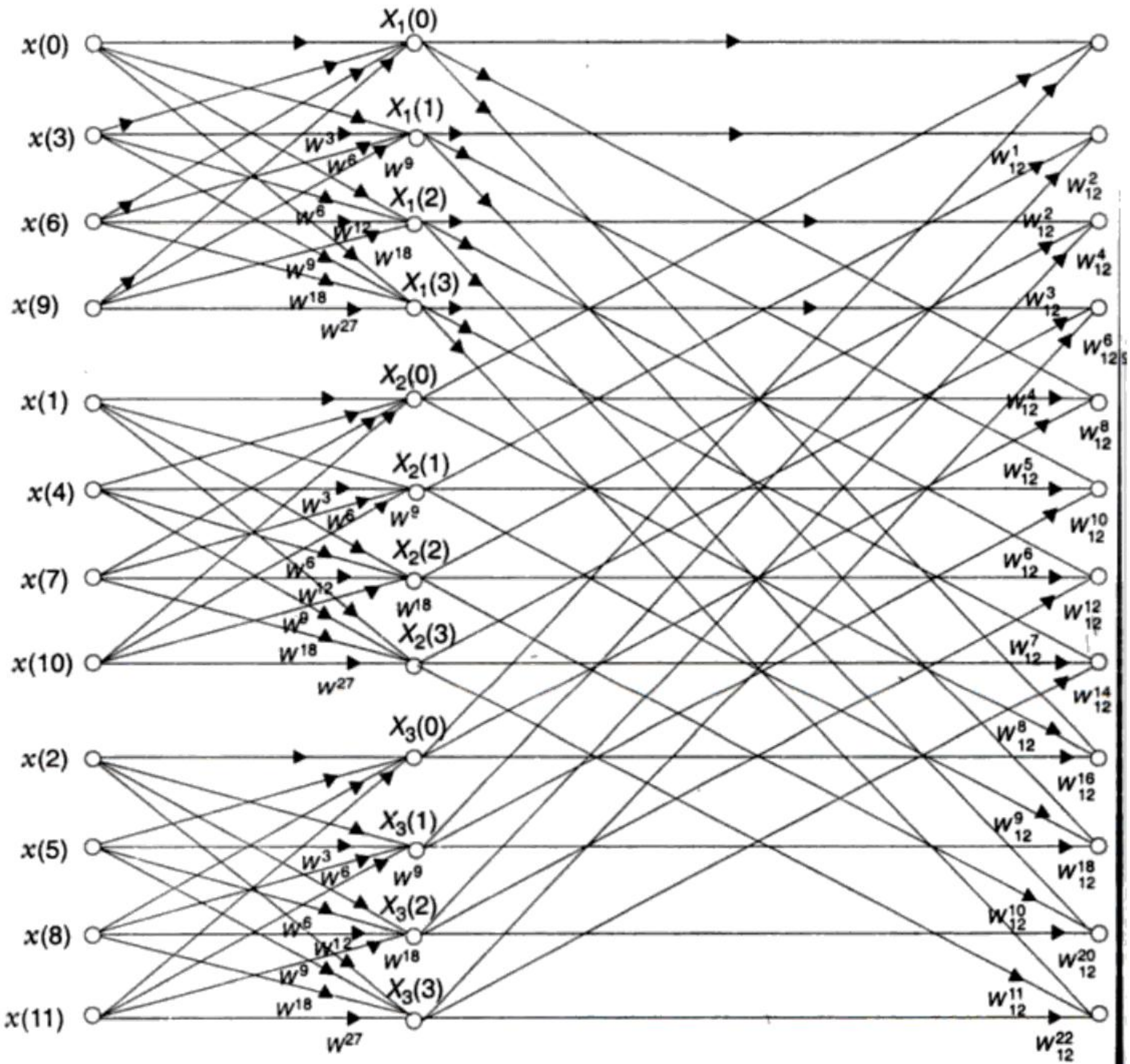

Fig. E6.30 *Flow Diagram of 12-Point Composite-Radix FFT*

Example 6.31 Develop a radix-4 DIT FFT algorithm for evaluating the DFT for $N = 16$ and hence determine the 16-point DFT of the sequence

$$x(n) = \{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1\}$$

Solution For $N = 16 = 4 \,.\, 4$, the Eq. 6.40 becomes

$$X(k) = \sum_{n=0}^{3} x(4n+2)\, W_{16}^{4nk} + \sum_{n=0}^{3} x(4n+1)\, W_{16}^{(4n+1)k}$$

$$+ \sum_{n=0}^{3} x(4n+2)\, W_{16}^{(4n+2)k} + \sum_{n=0}^{3} x(4n+3)\, W_{N}^{(4n+3)k}$$

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$$X(9) = X_1(1) + W_{16}^{9}X_2(1) + W_{16}^{18}X_3(1) + W_{16}^{27}X_4(1)$$
$$= 0 + (-j2)(-0.923 + j0.3826) + 0 + (-j2)(-0.382 + j0.923)$$
$$= -j2\{-0.923 + j0.3826 - 0.382 + j0.923\}$$
$$= -j2\{-1.3056 + j1.3056\}$$
$$= 2.6112 + j2.6112$$

$$X(10) = X_1(2) + W_{16}^{10}X_2(2) + W_{16}^{20}X_3(2) + W_{16}^{30}X_4(2)$$
$$= 0 + 0 + 0 + 0 = 0$$

$$X(11) = X_1(3) + W_{16}^{11}X_2(3) + W_{16}^{22}X_3(3) + W_{16}^{33}X_4(3)$$
$$= 0 + 0 + 0 + 0 = 0$$

$$X(12) = X_1(0) + W_{16}^{12}X_2(0) + W_{16}^{24}X_3(0) + W_{16}^{36}X_4(0)$$
$$= 0 + 4(-j) + 0 + j4 = 0$$

$$X(13) = X_1(1) + W_{16}^{13}X_2(1) + W_{16}^{26}X_3(1) + W_{16}^{39}X_4(1)$$
$$= 0 + (-j2)(0.3826 + j0.923) + 0 + (-j2)(-0.9238 - j0.382)$$
$$= -j0.7652 + 1.85 + j1.85 - 0.7652$$
$$= 1.0848 + j1.0848$$

$$X(14) = X_1(2) + W_{16}^{14}X_2(2) + W_{16}^{28}X_3(2) + W_{16}^{42}X_4(2)$$
$$= 0 + 0 + 0 + 0 = 0$$

$$X(15) = X_1(3) + W_{16}^{15}X_2(3) + W_{16}^{30}X_3(3) + W_{16}^{45}X_4(3)$$
$$= 0 + 0 + 0 + 0 = 0$$

The computed DFT values of the given sequence are also shown in Fig. E 6.31.

Example 6.32 Develop a DIF FFT algorithm for decomposing the DFT for $N = 6$ and draw the flow diagrams for (a) $N = 3.2$ and (b) $N = 2.3$.

Solution

(a) To develop DIF FFT algorithm for $N = 3.2$

$$X(k) = \sum_{n=0}^{5} x(n)W_6^{nk} = \sum_{n=0}^{2} x(n)W_6^{nk} + \sum_{n=3}^{5} x(n)W_6^{nk}$$

$$X(k) = \sum_{n=0}^{2} x(n)W_6^{nk} + \sum_{n=0}^{2} x(n+3)W_6^{(n+3)k}$$

$$= \sum_{n=0}^{2}\left[x(n) + x(n+3)W_6^{3k}\right]W_6^{nk}$$

$$X(2k) = \sum_{n=0}^{2}\left[x(n) + x(n+3)W_6^{6k}\right]W_6^{2nk} = \sum_{n=0}^{2}[x(n) + x(n+3)]W_6^{2nk}$$

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The group delay is defined as the delayed response of the filter as a function of ω to a signal.

Linear phase filters are those filters in which the phase delay and group delay are constants, i.e. independent of frequency. Linear phase filters are also called constant time delay filters. Let us obtain the conditions FIR filters must satisfy in order to have constant phase and group delays and hence obtain the conditions for having a linear phase. For the phase response to be linear

$$\frac{\Phi(\omega)}{\omega} = -\tau, \quad -\pi \le \omega \le +\pi$$

Therefore,

$$\Phi(\omega) = -\omega\tau$$

where τ is a constant phase delay expressed in number of samples. Using Eq. 7.2,

$$\Phi(\omega) = \tan^{-1} \frac{\text{Im}\, H(e^{j\omega})}{\text{Re}\, H(e^{j\omega})} = -\omega\tau$$

or

$$\omega\tau = \tan^{-1} \frac{\sum_{n=0}^{M-1} h(n) \sin \omega n}{\sum_{n=0}^{M-1} h(n) \cos \omega n}$$

or

$$\tan \omega\tau = \frac{\sum_{n=0}^{M-1} h(n) \sin \omega n}{\sum_{n=0}^{M-1} h(n) \cos \omega n}$$

Simplifying, we get

$$\sum_{n=0}^{M-1} h(n) \sin(\omega\tau - \omega n) = 0 \tag{7.3}$$

and a solution to Eq.7.3 is given by

$$\tau = \frac{(M-1)}{2} \tag{7.4}$$

and

$$h(n) = h(M-1-n) \text{ for } 0 < n < M-1 \tag{7.5}$$

If Eqs 7.4 and 7.5 are satisfied, then the FIR filter will have constant phase and group delays and thus the phase of the filter will be linear. The phase and group delays of the linear phase FIR filter are equal and

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where

$$H_r(f) = \sum_{n=-\infty}^{\infty} h(n) \cos 2\pi n f T \tag{7.11}$$

and

$$H_i(f) = \sum_{n=-\infty}^{\infty} h(n) \sin 2\pi n f T \tag{7.12}$$

From Eq. 7.11 and 7.12 we infer that $H_r(f)$ is an even function and $H_i(f)$ is an odd function of frequency. If $h(nT)$ is an even sequence, the imaginary part of the transfer function, $H_i(f)$, will be zero and if $h(nT)$ is an odd sequence, the real part of the transfer function, $H_r(f)$, will be zero. Thus an even unit impulse response yields a real transfer function and an odd unit impulse response yields an imaginary transfer function. A real transfer function has 0 or $\pm\ \pi$ radians phase shift, while an imaginary transfer function has $\pm\ \pi/2$ radians phase shift. Therefore, by making the unit impulse response either even or odd, one can generate a transfer function that is either real or imaginary.

In the design of digital filters two interesting situations are often sought after.

(i) For filtering applications, the main interest is in the amplitude response of the filter, where some portion of the input signal spectrum is to be attenuated and some portion is to be passed to the output with no attenuation. This should be accomplished without phase distortion. Thus the amplitude response is realised by using only a real transfer function. That is

$$H(e^{j\omega}) = H_r(f)$$

and

$$H_i(f) \equiv 0$$

(ii) For filtering plus quadrature phase shift. the applications include integrators, differentiators and Hilbert transform devices. For all these applications the desired transfer function is imaginary. Thus, the required amplitude response is realised by using only $H_i(f)$. That is

$$H(e^{j\omega}) = j\, H_i(f)$$

and

$$H_r(f) \equiv 0$$

Design Equations

The term $H(e^{j\omega})$ is periodic in the sampling frequency and hence both $H_r(f)$ and $H_i(f)$ are also periodic in the sampling frequency. Both $H_r(f)$ and $H_i(f)$ can be expanded in a Fourier series. Since the real part of the transfer function, $H_r(f)$, is an even function of frequency, its Fourier series will be of the form

$$H_r(f) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f T) \tag{7.13}$$

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type-I design and type-II design. In the Type-I design, the set of frequency samples includes the sample at frequency $\omega = 0$. In some cases, it may be desirable to omit the sample at $\omega = 0$ and use some other set of samples. Such a design procedure is referred to as the Type-II design.

Type-I Design

The samples are taken at the frequency

$$\omega_k = \frac{2\pi k}{M}, \ = 0, 1, \ldots, M-1 \tag{7.22}$$

The samples of the desired frequency response at these frequencies are given by

$$\begin{aligned}\tilde{H}(k) &= H_d\left(e^{j\omega}\right)\Big|_{\omega=\omega_k}, \ k = 0, 1, \ldots, M-1 \\ &= H_d(e^{j2\pi k/M}), \quad k = 0, 1, \ldots, M-1\end{aligned} \tag{7.23}$$

This set of points can be considered as DFT samples, then the filter coefficients $h(n)$ can be computed using the IDFT,

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} \tilde{H}(k)\, e^{j2\pi nk/M}, n = 0, 1, \ldots, M-1 \tag{7.24}$$

If these numbers are all real, then these can be considered as the impulse response coefficients of an FIR filter. This can happen when all the complex terms appear in complex conjugate pairs, and then all the terms can be matched by comparing the exponentials. The term $\tilde{H}(k)$ $e^{j2\pi nk/M}$ should be matched with the term that has the exponential $e^{-j2\pi nk/M}$ as a factor. The matching terms are then $\tilde{H}(k)\, e^{j2\pi nk/M}$ and $\tilde{H}(M-k)\, e^{j2\pi n(M-k)/M}$ since $2\pi n\,(M-k)/M = 2\pi n - (2\pi nk/M)$. These terms are complex conjugates if $\tilde{H}(0)$ is real and

(i) For M odd:

$$\tilde{H}(M-k) = \tilde{H}^*(k), \qquad k = 1, 2, \ldots, (M-1)/2 \tag{7.25}$$

(ii) For M even:

$$\begin{aligned}\tilde{H}(M-k) &= \tilde{H}^*(k), \qquad k = 1, 2, \ldots, M/2-1 \\ \tilde{H}(M/2) &= 0\end{aligned} \tag{7.26}$$

The desired frequency response $H_d(e^{j\omega})$ is chosen such that it satisfies the Eqs 7.25 and 7.26 for M odd or even, respectively. The filter coefficients can then be written as

$$h(n) = \frac{1}{M}\left[\tilde{H}(0) + 2\sum_{k=1}^{(M-1)/2} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/M}\right]\right], M \text{ odd} \tag{7.27}$$

and $$h(n) = \frac{1}{M}\left[\tilde{H}(0) + 2\sum_{k=1}^{M/2-1} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/M}\right]\right], M \text{ even} \tag{7.28}$$

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7.4.3.3 Hanning Window Function

The window function of a causal Hanning window is given by

$$w_{Hann}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.44}$$

The window function of a non-causal Hanning window is expressed by

$$w_{Hann}(n) = \begin{cases} 0.5 + 0.5\cos\dfrac{2\pi n}{M-1}, & 0 < |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

The width of the main lobe is approximately $8\pi/M$ and the peak of the first side lobe is at –32dB.

7.4.3.4 Blackman Window Function

The window function of a causal Blackman window is expressed by

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

The window function of a non-causal Blackman window is given by

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & \text{for } |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{7.45}$$

The width of the main lobe is approximately $12\pi/M$ and the peak of the first side-lobe is at –58dB.

7.4.3.5 Bartlett Window Function

The window function of a non-causal Bartlett window is expressed by

$$w_{Bart}(n) = \begin{cases} 1+n, & -\dfrac{M-1}{2} < n < 1 \\ 1-n, & 1 < n < \dfrac{M-1}{2} \end{cases}$$

Table 7.1 gives the important frequency-domain characteristics of some window functions.

Table 7.1 *Frequency-Domain Characteristics of Some Window Functions*

Type of Window	*Approximate Transition Width of Main Lobe*	*Minimum Stopband Attenuation (dB)*	*Peak of first Sidelobe (dB)*
Rectangular	$4\pi / M$	–21	– 13
Bartlett	$8\pi / M$	–25	– 27
Hanning	$8\pi / M$	–44	– 32
Hamming	$8\pi / M$	–53	– 43
Blackman	$12\pi / M$	–74	–58

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Solution The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-3\pi/4}^{3\pi/4} e^{-j3\omega}\, e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{\sin 3\pi(n-3)/4}{\pi(n-3)}, \quad n \neq 3 \text{ and } h_d(3) = \frac{3}{4}$$

The filter coefficients are,

$h_d(0) = 0.0750$, $h_d(1) = -0.1592$, $h_d(2) = 0.2251$, $h_d(3) = 0.75$
$h_d(4) = 0.2251$, $h_d(5) = -0.1592$, $h_d(6) = 0.0750$

The Hamming window function is,

$$w(n) = \begin{cases} 0.54 - 0.46\cos\dfrac{2\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, with $M = 7$,

$w(0) = 0.08$, $w(1) = 0.31$, $w(2) = 0.77$, $w(3) = 1$, $w(4) = 0.77$,

$w(5) = 0.31$, $w(6) = 0.08$.

The filter coefficients of the resultant filter are then,

$$h(n) = h_d(n).w(n) \quad n = 0, 1, 2, 3, 4, 5, 6.$$

Therefore,

$h(0) = 0.006$, $h(1) = -0.0494$, $h(2) = 0.1733$, $h(3) = 0.75$,
$h(4) = 0.1733$, $h(5) = -0.0494$ and $h(6) = 0.006$.

The frequency response is given by

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{6} h(n)\, e^{-j\omega n} \\ &= e^{-j3\omega}\,[\,h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega] \\ &= e^{-j3\omega}\,[\,0.75 + 0.3466\cos\omega - 0.0988\cos 2\omega + 0.012\cos\omega] \end{aligned}$$

Example 7.9 Design an FIR digital filter to approximate an ideal low-pass filter with passband gain of unity, cut-off frequency of 850 Hz and working at a sampling frequency of $f_s = 5000$ Hz. The length of the impulse response should be 5. Use a rectangular window.

Solution The desired response of the ideal low-pass filter is given by

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le f \le 850 \text{ Hz} \\ 0, & f > 850 \text{ Hz} \end{cases}$$

The above response can be equivalently specified in terms of the normalised ω_c. The normalised $\omega_c = 2\pi f_c / f_s = 2\pi(850)/(5000) = 1.068$ rad/sec. Hence, the desired response is

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High-Pass FIR Filter

$$h_d(n) = \begin{cases} -\left(\dfrac{2f_c}{F}\right)\dfrac{\sin 2\pi n f_c / F}{2\pi n f_c / F}, & \text{for } n > 0 \\ 1 - \dfrac{2f_c}{F}, & \text{for } n = 0 \end{cases} \tag{7.65}$$

where

$$f_c = 0.5\,(f_p + f_s) \quad \text{and} \quad \Delta F = f_p - f_s \tag{7.66}$$

Bandpass FIR Filter

$$h_d(n) = \begin{cases} \dfrac{1}{n\pi}\left[\sin\left(2\pi n f_{c2} / F\right) - \sin\left(2\pi n f_{c1} / F\right)\right], & \text{for } n > 0 \\ \dfrac{2}{F}\left(f_{c2} - f_{c1}\right), & \text{for } n = 0 \end{cases} \tag{7.67}$$

where

$$\begin{aligned} f_{c1} &= f_{p1} - \frac{\Delta F}{2} & f_{c2} &= f_{p2} + \frac{\Delta F}{2} \\ \Delta F_l &= f_{p1} - f_{s1} & \Delta F_h &= f_{s2} - f_{p2} \\ \Delta F &= \min\left[\Delta F_l, \Delta F_h\right] \end{aligned} \tag{7.68}$$

Bandstop FIR Filter

$$h_d(n) = \begin{cases} \dfrac{1}{n\pi}\left[\sin\left(2\pi n f_{c1} / F\right) - \sin\left(2\pi n f_{c2} / F\right)\right], & \text{for } n > 0 \\ \dfrac{2}{F}\left(f_{c1} - f_{c2}\right) + 1, & \text{for } n = 0 \end{cases} \tag{7.69}$$

where

$$\begin{aligned} f_{c1} &= f_{p1} + \frac{\Delta F}{2} & f_{c2} &= f_{p2} - \frac{\Delta F}{2} \\ \Delta F_l &= f_{s1} - f_{p1} & \Delta F_h &= f_{p2} - f_{s2} \\ \Delta F &= \min\left[\Delta F_l, \Delta F_h\right] \end{aligned} \tag{7.70}$$

Example 7.10 Design a low-pass digital FIR filter using Kaiser window satisfying the specifications given below.

Passband cut-off frequency, f_p = 150 Hz, stopband cut-off frequency, f_s = 250 Hz, passband ripple, A_p = 0.1 dB, stopband attenuation, A_s = 40 dB and sampling frequency, F = 1000 Hz.

Solution A computer program can be written for the design of Kaiser window digital filter using the functions given in Appendix. The computer output is given below.

From Eq. 7.53, δ = 0.005756.

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There are four different cases that result in a linear phase FIR filter, viz., (i) symmetric unit impulse response and the length of the filter, M odd, (ii) symmetric unit impulse response and M even, (iii) antisymmetric unit impulse response and M odd, and (iv) antisymmetric unit impulse response and M even. The first case is discussed below in detail and other cases are listed in Table 7.2.

In the symmetric unit impulse response case, $h(n) = H(M-1-n)$. The real-valued frequency response characteristics $|H(e^{j\omega})| = |H_r(e^{j\omega})|$, given in Eq. 7.14, is

$$|H(e^{j\omega})| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{(M-3)}{2}} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \tag{7.74}$$

Let $k = (M-1)/2 - n$. Then Eq. 7.74 can be written as

$$|H(e^{j\omega})| = \sum_{k=0}^{\frac{(M-1)}{2}} a(k)\cos\omega k \tag{7.75}$$

where

$$a(0) = h\left(\frac{M-1}{2}\right) \tag{7.76}$$

$$a(k) = 2h\left(\frac{M-1}{2} - k\right) \text{ for } 1 \le k \le \frac{M-1}{2}$$

The magnitude response for the other cases are similarly converted to a compact form as given in Table 7.2.

Table 7.2 *Magnitude Response Functions for Linear Phase FIR Filters*

Filter Type	$Q(\omega)$	$P(\omega)$
Case (i) - Symmetric and M odd $h(n) = h(M-1-n)$	1	$\sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$
Case (ii) - Symmetric and M even $h(n) = h(M-1-n)$	$\cos\frac{\omega}{2}$	$\sum_{k=0}^{(M/2)-1} b(k)\cos\omega k$
Case (iii) - Antisymmetric and M odd $h(n) = -h(M-1-n)$	$\sin\omega$	$\sum_{k=0}^{(M-3)/2} c(k)\cos\omega k$
Case (iv) - Antisymmetric and M even $h(n) = -h(M-1-n)$	$\sin\frac{\omega}{2}$	$\sum_{k=0}^{(M/2)-1} d(k)\cos\omega k$

From Table 7.2, it can be seen that the magnitude response function can be written as given in Eq. 7.77, for the four different cases.

$$|H(e^{j\omega})| = Q(\omega)\,P(\omega) \tag{7.77}$$

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7.28 What is an FIR half-band digital filter ? Explain with a suitable illustration.
7.29 What is an optimal linear phase FIR filter ? What parameters are optimised in these filters ?
7.30 State and explain the alternation theorem.
7.31 What are extra ripple filters ?
7.32 What are maximal ripple filters ?
7.33. Explain the Remez exchange algorithm used in the design of optimal filters.

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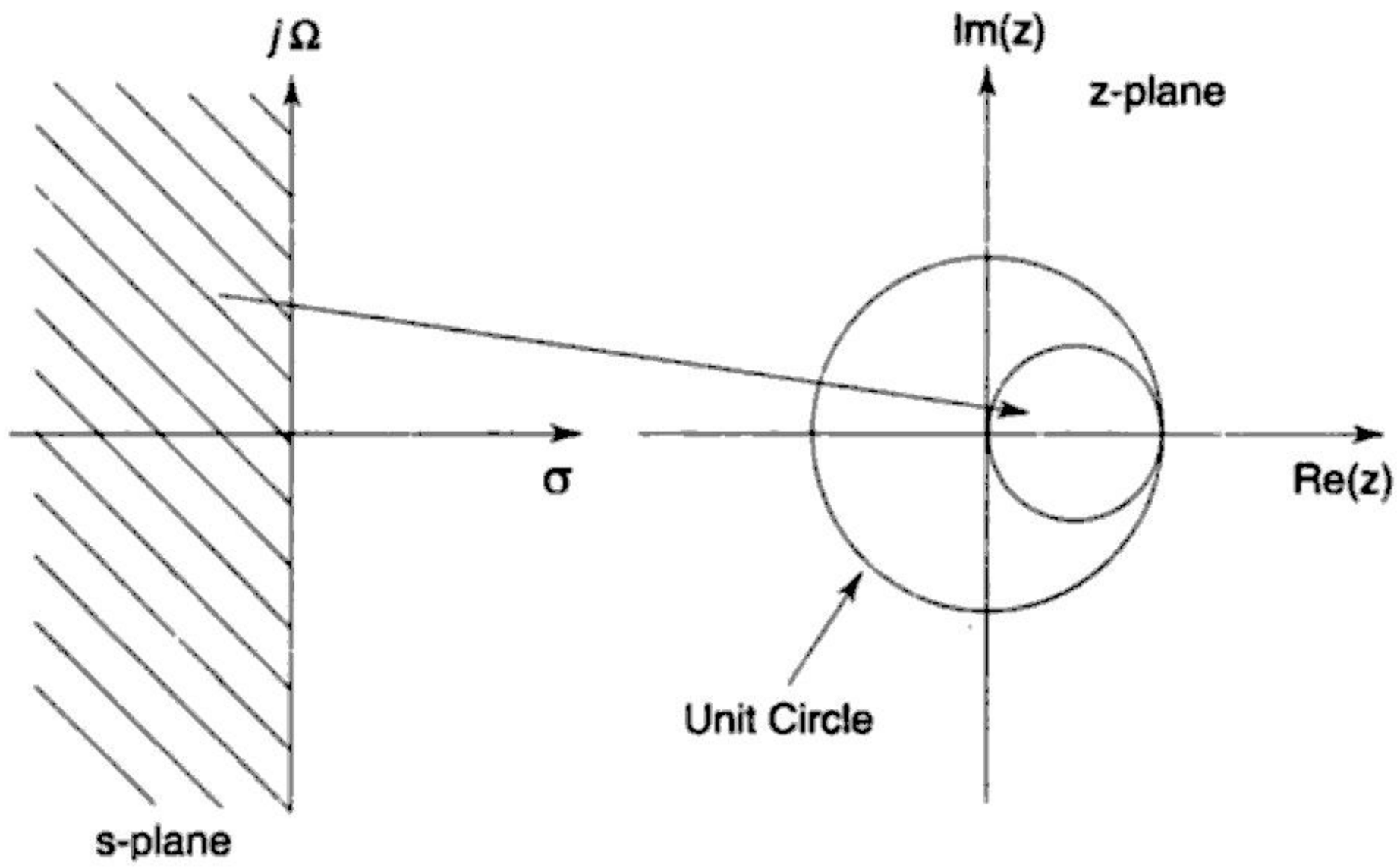

Fig. 8.2 *The Mapping of Eq. 8.5 into the z-plane*

It can be seen that the mapping of Eq. 8.5 takes the left-half plane of s-domain into the corresponding points inside the circle of radius 0.5 and centre at $z = 0.5$, and the right-half of the s-plane is mapped outside the unit circle. As a result, this mapping results in a stable analog filter transformed into a stable digital filter; however, as the locations of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass filters and bandpass filters having smaller resonant frequencies. Neither a high-pass filter nor a band reject filter can be realised using this technique.

The forward difference can be substituted for the derivative instead of the backward difference. This gives,

$$\frac{dy(t)}{dt} = \frac{y(nT+T) - y(nT)}{T}$$

$$= \frac{y(n+1) - y(n)}{T} \tag{8.12}$$

The transformation formula will be

$$s = \frac{z-1}{T} \tag{8.13}$$

or,

$$z = 1 + sT \tag{8.14}$$

The mapping of Eq. 8.14 is shown in Fig. 8.3. This results in a worse situation than the backward difference substitution for the derivative. When $s = j\,\Omega$, the mapping of these points in the s-domain results in a straight line in the z-domain with coordinates $(z_{real}, z_{imag}) = (1, \Omega T)$. Consequently, stable analog filters do not always map into stable digital filters.

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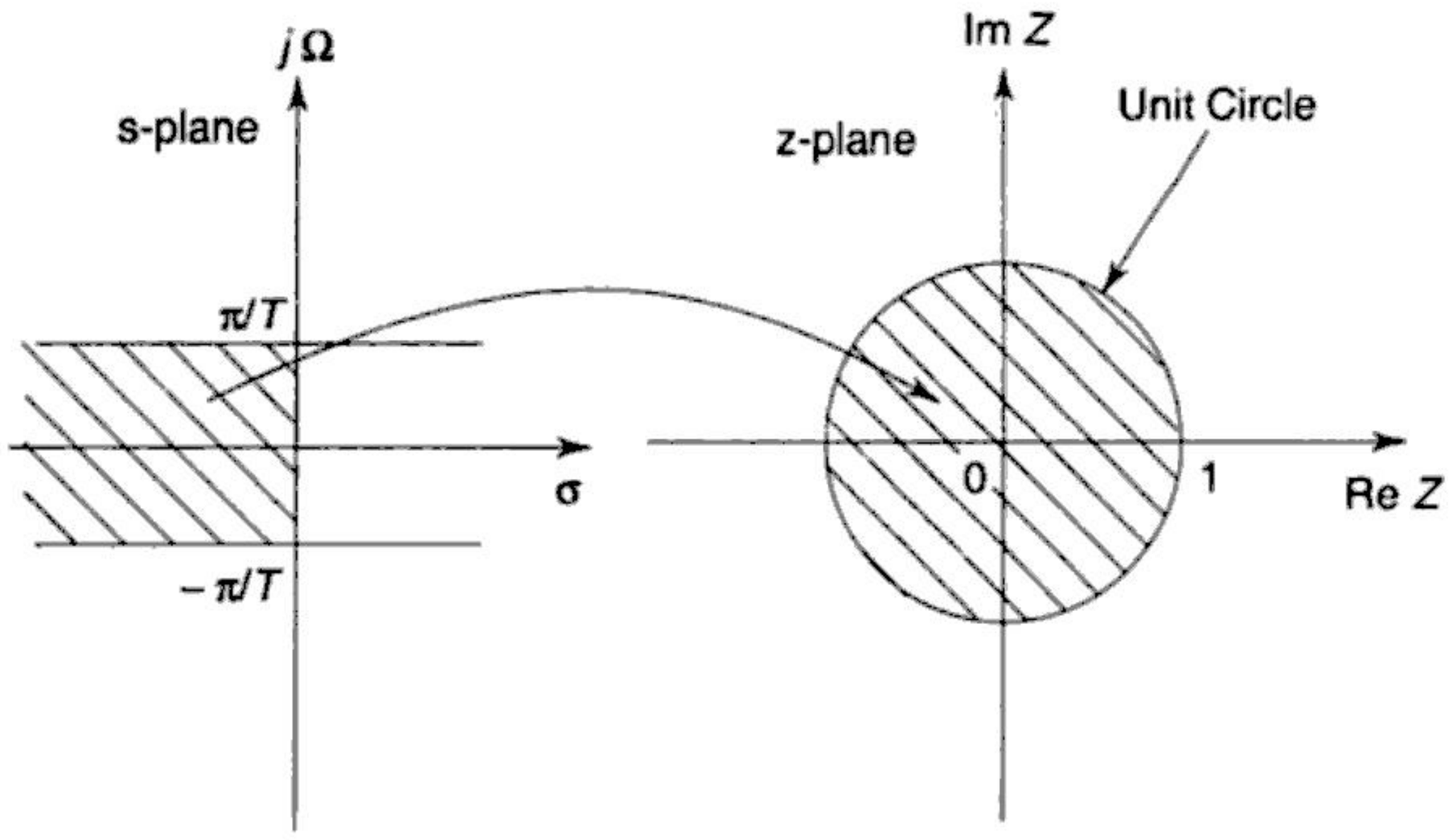

Fig. 8.4 *The Mapping of* $z = e^{sT}$

Example 8.4 Convert the analog filter into a digital filter whose system function is

$$H(s) = \frac{s + 0.2}{(s + 0.2)^2 + 9}$$

Use the impulse invariant technique. Assume $T = 1$s.

Solution The system response of the analog filter is of the standard form

$$H(s) = \frac{s + a}{(s + a)^2 + b^2}$$

where $a = 0.2$ and $b = 3$. The system response of the digital filter can be obtained using Eq. 8.27.

$$H(z) = \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT}\, z^{-2}}$$

$$= \frac{1 - e^{-0.2T}(\cos 3T)\, z^{-1}}{1 - 2e^{-0.2T}(\cos 3T)\, z^{-1} + e^{-0.4T}\, z^{-2}}$$

Taking $T = 1$s,

$$H(z) = \frac{1 - (0.8187)(-0.99)\, z^{-1}}{1 - 2(0.8187)(-0.99)\, z^{-1} + 0.6703\, z^{-2}}$$

That is,

$$H(z) = \frac{1 + (0.8105)\, z^{-1}}{1 + 1.6210\, z^{-1} + 0.6703\, z^{-2}}$$

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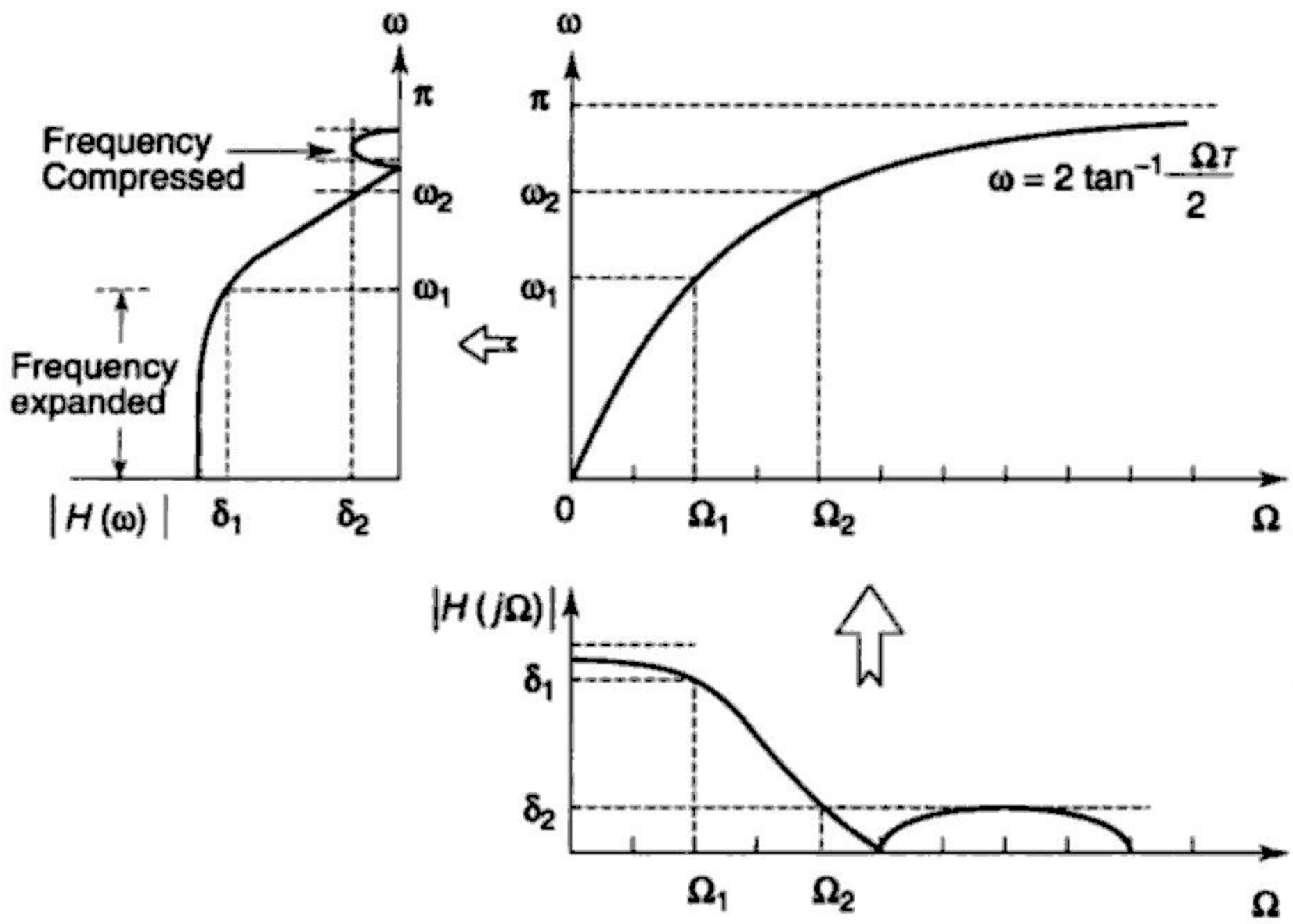

Fig. 8.5 *Relationship Between ω and Ω as Given in Eq. 8.40*

The sampling period is obtained from the above equation using

$$T = \frac{2}{\Omega_c} \tan \frac{\omega_r}{2} = \frac{2}{3} \tan \frac{\pi}{8} = 0.276 \text{ s}$$

Using bilinear transformation,

$$H(z) = H(s)\big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\dfrac{2}{T}\dfrac{(z-1)}{(z+1)} + 0.1}{\left[\dfrac{2}{T}\dfrac{(z-1)}{(z+1)} + 0.1\right]^2 + 9}$$

$$= \frac{(2/T)(z-1)(z+1) + 0 \cdot 1(z+1)^2}{[(2/T)(z-1) + 0.1(z+1)]^2 + 9(z+1)^2}$$

Substituting $T = 0.276$ s,

$$H(z) = \frac{1 + 0.027\, z^{-1} - 0.973\, z^{-2}}{8.572 - 11.84\, z^{-1} + 8.177\, z^{-2}}$$

Example 8.8 Apply bilinear transformation to

$$H(s) = \frac{2}{(s+1)(s+3)}$$

with $T = 0.1$s.

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Example 8.14 A prototype low-pass filter has the system response $H(s) = \dfrac{1}{s^2 + 2s + 1}$. Obtain a bandpass filter with $\Omega_0 = 2$ rad/s and $Q = 10$. $\Omega_0^2 = \Omega_1 \cdot \Omega_2$ and $Q = \dfrac{\Omega_0}{\Omega_2 - \Omega_1}$

Solution From Table 8.1, the required transformation is

$$s \to \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}, \text{ i.e.}$$

$$s = \Omega_c \frac{s^2 + \Omega_0^2}{s(\Omega_0/Q)} = \Omega_c \frac{s^2 + 2^2}{s(2/10)} = 5\Omega_c\left(\frac{s^2 + 4}{s}\right)$$

Therefore,

$$H(s) = H(s)\big|_{s = 5\Omega_c\left(\frac{s^2+4}{s}\right)}$$

$$H(s) = \frac{0.04\, s^2}{\Omega_c^2 s^4 + 0.4\Omega_c s^3 + (8\Omega_c^2 + 0.01)s^2 + 1.6\Omega_c s + 16\Omega_c^2}$$

Example 8.15 Transform the prototype low-pass filter with system function

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

into a high-pass filter with cutoff frequency Ω_c^*.

Solution From the Table 8.1, the desired transformation is,

$$s \to \frac{\Omega_c \Omega_c^*}{s}$$

Thus we have,

$$H_{hpf}(s) = \frac{\Omega_c}{\left(\dfrac{\Omega_c \Omega_c^*}{s}\right) + \Omega_c}$$

$$= \frac{s}{s + \Omega_c^*}$$

8.9.2 DIGITAL FREQUENCY TRANSFORMATION

As in the analog domain, frequency transformation is possible in the digital domain also. The frequency transformation is done in the digital domain by replacing the variable z^{-1} by a function of z^{-1}, i.e. $f(z^{-1})$. This mapping must take into account the stability criterion. All the poles lying within the unit circle must map onto itself and the unit circle must

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Chapter 9

Realisation of Digital Linear Systems

9.1 INTRODUCTION

For the design of digital filters, the system function $H(z)$ or the impulse response $h(n)$ must be specified. Then the digital filter structure can be implemented or synthesised in hardware/software form by its difference equation obtained directly from $H(z)$ or $h(n)$. Each difference equation or computational algorithm can be implemented by using a digital computer or special purpose digital hardware or special programmable integrated circuit.

In order to implement the specified difference equation of the system, the required basic operations are addition, delay and multiplication by a constant. A chosen structure determines a computational algorithm. Generally, different structures give different results. The most common methods for realising digital linear systems are direct, cascade and parallel forms, and state variable realisations.

In this chapter, direct-forms, cascade and parallel forms, ladder form and state variable realisations of IIR filters, direct-form and cascade realisations of FIR filters, and realisations of linear phase FIR systems are discussed.

9.2 BASIC REALISATION BLOCK DIAGRAM AND THE SIGNAL-FLOW GRAPH

The output of a finite order linear time invariant system at time n can be expressed as a linear combination of the inputs and outputs ,

$$y(n) = \sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \tag{9.1}$$

where a_k and b_k are constants with $a_0 \neq 0$ and $M \leq N$.

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Example 9.3 Determine direct Forms I and II for the second-order filter given by

$$y(n) = 2b \cos \omega_0 y(n-1) - b^2 y(n-2) + x(n) - b \cos \omega_0 x(n-1)$$

Solution Taking z–transform for the given function, we get

$$Y(z) = 2b \cos \omega_0 z^{-1} Y(z) - b^2 z^{-2} Y(z) + X(z) - b \cos \omega_0 z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - b \cos \omega_0 z^{-1}}{1 - 2b \cos \omega_0 z^{-1} + b^2 z^{-2}}$$

Direct form I

$$H(z) = H_1(z) H_2(z)$$

Therefore, $Y(z) = H_1(z) H_2(z) X(z)$

In this form, the intermediate sequence $w(n)$ is introduced between $H_1(n)$ and $H_2(n)$

Let $\quad H_1(z) = 1 - b \cos \omega_0 z^{-1} = \dfrac{W(z)}{X(z)}$

Therefore, $X(z)(1 - b \cos \omega_0 z^{-1}) = W(z)$

$$x(n) - b \cos \omega_0 x(n-1) = w(n)$$

and

$$H_2(z) = (1 - 2b \cos \omega_0 z^{-1} + b_2 z^{-2})^{-1} = \frac{Y(z)}{W(z)}$$

$$Y(z) = \frac{W(z)}{(1 - 2b \cos \omega_0 z^{-1} + b^2 z^{-2})}$$

$$Y(z)\left[1 - 2b \cos \omega_0 z^{-1} + b^2 z^{-2}\right] = W(z)$$

The inverse z–transform of this function is

$$y(n) - 2b \cos \omega_0 y(n-1) + b^2 y(n-2) = w(n)$$

The direct Form I realisation structure of the above function is shown in Fig. E 9.3(a).

Direct Form II

$$Y(z) = H_2(z) H_1(z) X(z)$$

$$H_2(z) = (1 - 2b \cos \omega_0 z^{-1} + b^2 z^{-2})^{-1} = \frac{U(z)}{X(z)}$$

Let $\quad H_1(z) = 1 - b \cos \omega_0 z^{-1} = \dfrac{Y(z)}{U(z)}$

$$X(z) = U(z)\{1 - 2b \cos \omega_0 z^{-1} + b^2 z^{-2}\}$$

Hence, $\quad x(n) = u(n) - 2b \cos \omega_0 u(n-1) + b^2 u(n-2)$

$$Y(z) = U(z)\{1 - b \cos \omega_0 z^{-1}\}$$

Hence, $\quad y(n) = u(n) - b \cos \omega_0 u(n-1)$

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$$= \frac{27}{5}\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)} - \frac{2}{5}\frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}$$

The parallel realisation of this transfer function is shown in Fig. E9.5(b).

9.3.4 Transposed Forms

If two digital filter structures have the same transfer function, then they are called **equivalent structures**. A simple way to generate an equivalent structure from a given realisation structure is via the **transpose operation**. The transposed form is obtained by (i) reversing the paths, (ii) replacing pick-off nodes by adders, and vice-versa, and (iii) interchanging the input and output nodes. For a single input-output system, the transposed structure has the same transfer function as the original realisation structure.

While using infinite precision arithmetic, any realisation structure will behave identically to any other equivalent structure. But, due to the finite word length limitations, the behaviour of a particular realisation structure is totally different from its equivalent structures. Therefore, it is important to select a structure which has the minimum quantization effects from the finite word length implementation.

Figure 9.10 shows the transposed direct form I realisation network of Fig 9.4. Figure 9.11 also illustrates the direct form I network of Fig. 9.10 which is redrawn to have the input on the left and output on the right conventionally. Figure 9.12 shows the transposed direct form II structure.

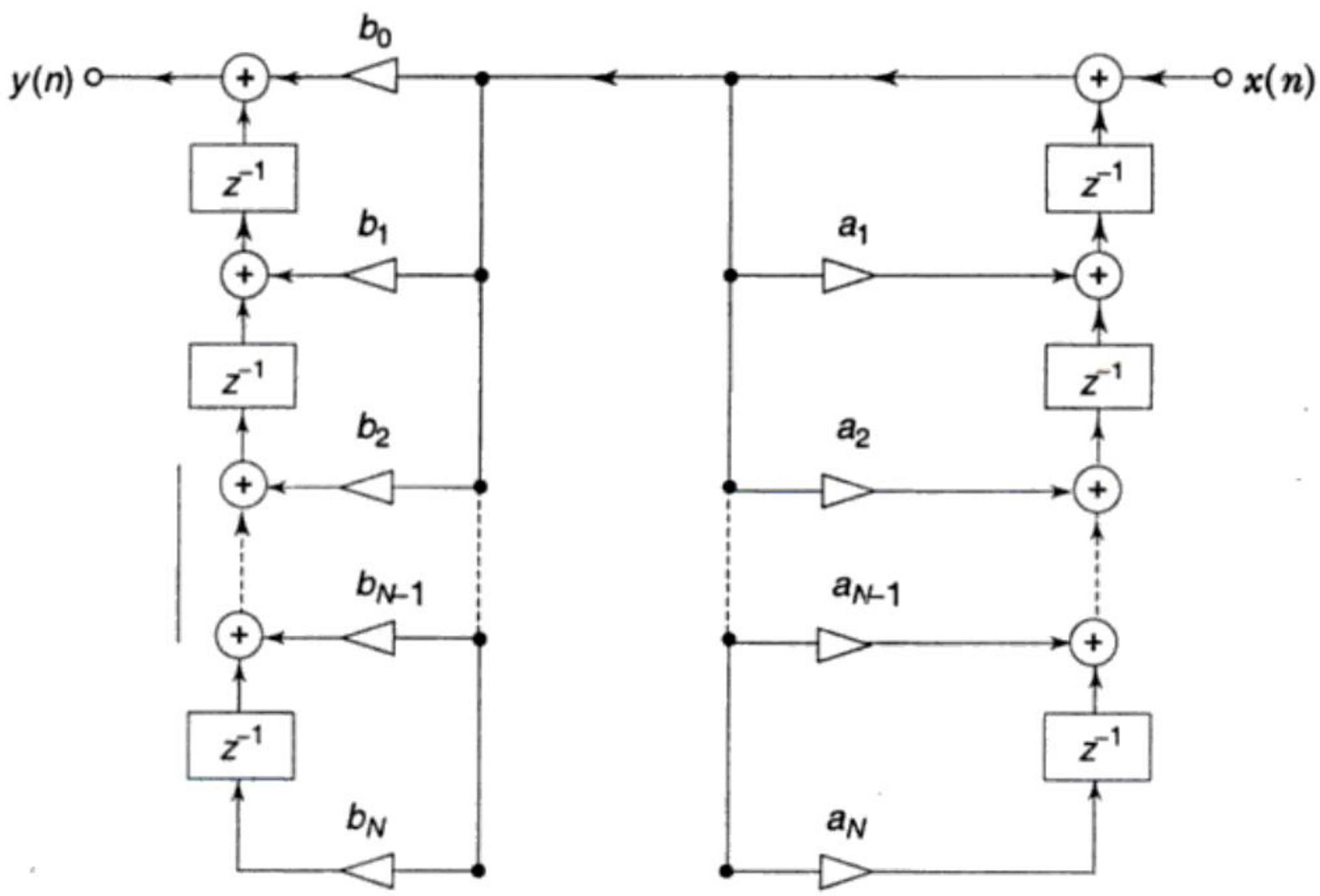

Fig. 9.10 *Transposed Direct Form I Realisation Structure of Fig. 9.4*

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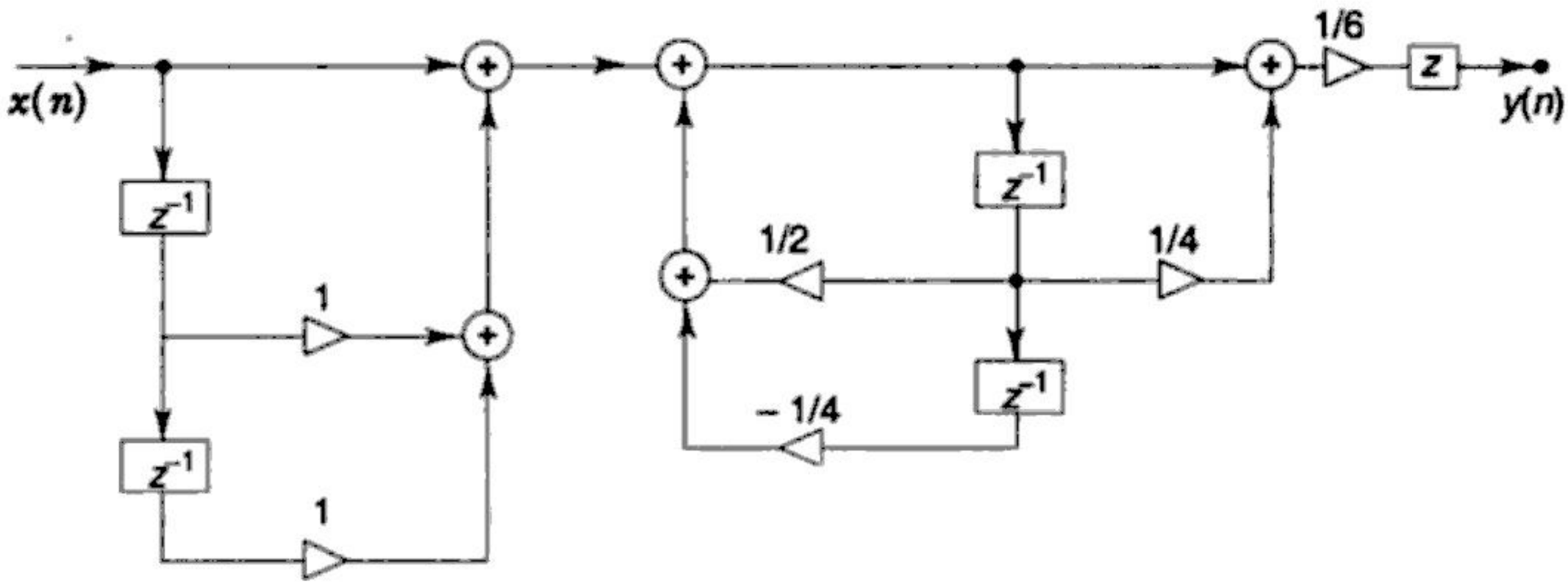

Fig. E9.9(a)

Parallel Realisation

$$H(z) = \frac{\frac{z}{6} + \frac{5}{24} + \frac{5}{24}z^{-1} + \frac{1}{24}z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} = \frac{\frac{z}{6}\left(1 + \frac{5}{4}z^{-1} + \frac{5}{4}z^{-2} + \frac{1}{4}z^{-3}\right)}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$$

Dividing the denominator by the numerator, we obtain

$$\begin{array}{r|l}
 & z^{-1} + 7 \\
\hline
\frac{1}{4}z^{-2} - \frac{1}{2}z^{-1} + 1 & \frac{1}{4}z^{-3} + \frac{5}{4}z^{-2} + \frac{5}{4}z^{-1} + 1 \\
 & \frac{1}{4}z^{-3} - \frac{1}{2}z^{-2} + z^{-1} \\
\hline
 & \frac{7}{4}z^{-2} + \frac{1}{4}z^{-1} + 1 \\
 & \frac{7}{4}z^{-2} + \frac{7}{2}z^{-1} + 7 \\
\hline
 & \frac{15}{4}z^{-1} - 6 \\
\hline
\end{array}$$

Therefore, $$H(z) = \frac{z}{6}\left[7 + z^{-1} + \frac{-6 + \frac{15}{4}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}\right]$$

$$= z\left[\frac{7}{6} + \frac{1}{6}z^{-1} + \frac{-1 + \frac{5}{8}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}\right]$$

$$H(z) = z\,[H_1(z) + H_2(z)]$$

where $H_1(z) = \frac{7}{6} + \frac{1}{6}z^{-1}$ and $H_2(z) = \dfrac{-1 + \frac{5}{8}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$

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Cascade realisation

$$H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$
$$= H_1(z) \,.\, H_2(z)$$

This function is realised in FIR cascade form as shown in Fig. E9.13(b)

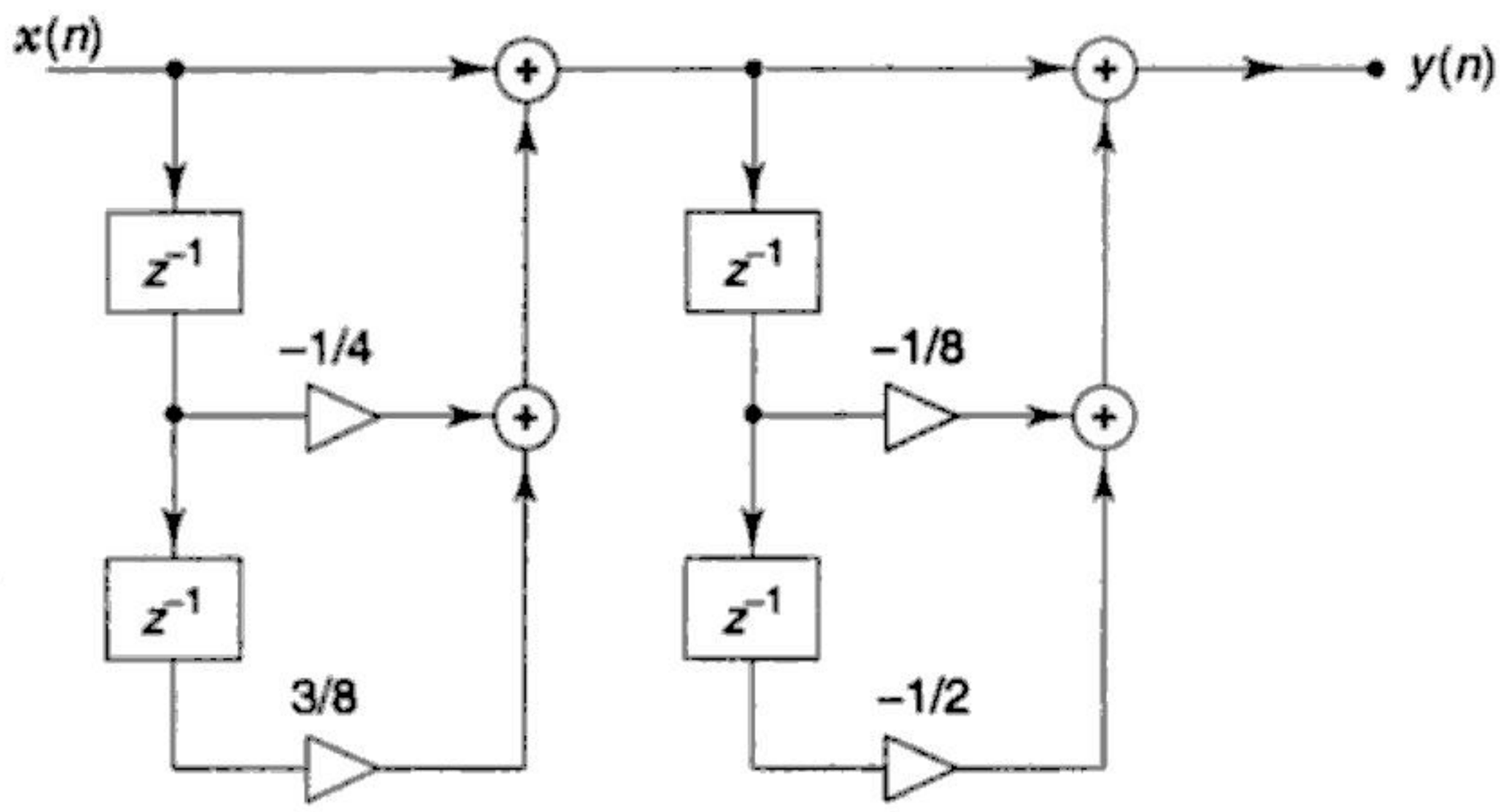

Fig. E9.13(b)

9.4.3 Realisation of Linear Phase FIR Systems

If the impulse response is symmetric about its origin, linear phase results. If all zero filter has a linear phase response, a special non-recursive structure that reduces the number of multiplications by approximately one-half can be implemented. The impulse response for a causal filter begins at zero and ends at $M - 1$.

As discussed in Section 7.2, a linear phase FIR filter of length M is characterised by

$$h(n) = h\,(M - 1 - n) \tag{9.10}$$

The symmetry property of a linear phase FIR filter is used to reduce the multipliers required in these realisations. Using this condition, the z-transform of the impulse response can be expressed as

$$H(z) = Z[h(n)] = \sum_{n=0}^{M-1} h(n)\, z^{-n} \tag{9.11}$$

For M even

$$H(z) = \sum_{n=0}^{M/2-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] \tag{9.12}$$

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Table 10.1 *Fixed point binary representation*

Number	*Sign magnitude*	*Two's complement*
7	0 1 1 1	0 1 1 1
6	0 1 1 0	0 1 1 0
5	0 1 0 1	0 1 0 1
4	0 1 0 0	0 1 0 0
3	0 0 1 1	0 0 1 1
2	0 0 1 0	0 0 1 0
1	0 0 0 1	0 0 0 1
0	0 0 0 0	0 0 0 0
−0	1 0 0 0	0 0 0 0
−1	1 0 0 1	1 1 1 1
−2	1 0 1 0	1 1 1 0
−3	1 0 1 1	1 1 0 1
−4	1 1 0 0	1 1 0 0
−5	1 1 0 1	1 0 1 1
−6	1 1 1 0	1 0 1 0
−7	1 1 1 1	1 0 0 1

Consider a number x, whose original length is 'L' bits. Let this number be quantised (truncated or rounded) to 'B' bits, as shown below. This quantised number is represented by $Q(x)$. Both x and $Q(x)$ are shown below. Note that $B < L$.

	Sign bit	0	1	2		L-2	L-1
$x =$	X	X	X	X	$\cdots$	X	X

and

	Sign bit	0	1	2		B-2	B-1
$Q(x) =$	X	X	X	X	$\cdots$	X	X

A truncation error, ε_T, is introduced in the input signal and thus the quantised signal is

$$Q_T(x) = x + \varepsilon_T \tag{10.1}$$

The range of values of the error due to truncation of the signal is analysed here for both sign magnitude and two's complement representations.

(i) *Truncation error for sign magnitude representation* When the input number x is positive, truncation results in reducing the magnitude of the number. Thus, the truncation error is negative and the range is given by

$$-(2^{-B} - 2^{-L}) \le \varepsilon_T \le 0 \tag{10.2}$$

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(iii) Finite arithmetic effects are less
(iv) Filter order required in multirate application are low and
(v) Sensitivity to filter coefficient lengths are less

While designing multirate systems, effects of aliasing for decimation and pseudoimages for interpolators should be avoided.

11.2 SAMPLING

Let $x(t)$ be a continuous time varying signal. The signal $x(t)$ is sampled at regular interval of time with sampling period T as shown in Fig. 11.1. The sampled signal $x(nT)$ is given by

$$x(nT) = x(t)|_{t=nT} = x(nT), -\infty < n < \infty \tag{11.1}$$

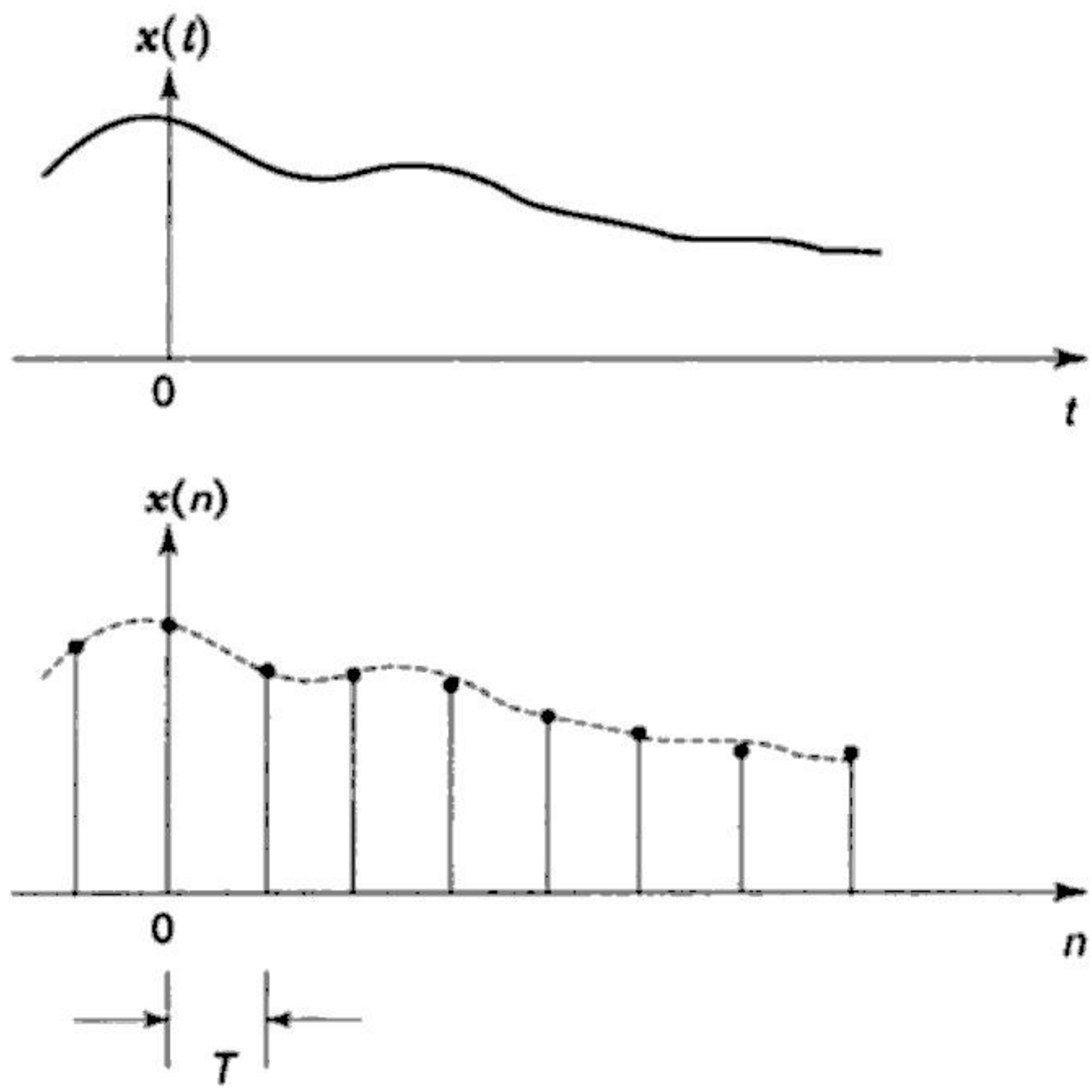

Fig. 11.1 *Sampling of a Continuous Time Signal*

A sampling process can also be interpreted as a modulation or multiplication process, as shown in Fig.11.2.

The continuous time signal $x(t)$ is multiplied by the sampling function $s(t)$ which is a series of impulses (periodic impulse train); the resultant signal is a discrete time signal $x(n)$.

$$x(n) = x(t)\, s(t)|_{t=nT}, -\infty < n < \infty \tag{11.2}$$

Sampling Theorem

According to the sampling theorem, a band–limited signal $x(t)$ having finite energy, which has no frequency components higher than f_h hertz, can be completely reconstructed from its samples taken at the rate of $2f_h$ samples per second.

The sampling rate of $2\,f_h$ samples per second is the Nyquist rate and its reciprocal $1/2\,f_h$ is the Nyquist period.

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$$w(m) = \begin{cases} x(m/L), & m = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases} \tag{11.15}$$

The z–transform of the signal $w(m)$ is given by,

$$\begin{aligned} W(z) &= \sum_{m=-\infty}^{\infty} w(m)\, z^{-m} \\ &= \sum_{m=-\infty}^{\infty} x(m)\, z^{-mL} \\ &= X(z^L) \end{aligned} \tag{11.16}$$

When considered over the unit circle, $z = e^{j\omega'}$,

$$W\left(e^{j\omega'}\right) = X\left(e^{j\omega' L}\right) \tag{11.17}$$

where $\omega' = 2\pi f T'$. The spectra of the signal $w(m)$ contains the images of baseband placed at the harmonics of the sampling frequency $\pm 2\pi/L$, $\pm 4\pi/L$. To remove the images, an anti-imaging (low-pass) filter is used. The ideal characteristic of the low-pass filter is given by,

$$H\left(e^{j\omega'}\right) = \begin{cases} G, & |\omega'| \le 2\pi f T'/2 = \pi / L \\ 0, & \text{otherwise} \end{cases} \tag{11.18}$$

where G is the gain of the filter and it should be L in the passband. The frequency response of the output signal is given by,

$$\begin{aligned} Y\left(e^{j\omega'}\right) &= H\left(e^{j\omega'}\right) X\left(e^{j\omega' L}\right) \\ &= \begin{cases} GX\left(e^{j\omega' L}\right), & |\omega'| \le \pi / L \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{11.19}$$

The output signal $y(m)$ is given by

$$\begin{aligned} y(m) &= \sum_{k-\infty}^{\infty} h(m-k)\, w(k) \\ &= \sum_{k=-\infty}^{\infty} h(m-k)\, x(k/L), \quad k/L \text{ an integer} \end{aligned} \tag{11.20}$$

Fig. 11.7 shows the entire process of interpolation by a factor L. The interpolation process is also known as up sampler.

Example 11.2 Obtain the two-fold expanded signal $y(n)$ of the input signal $x(n)$.

$$x(n) = \begin{cases} n, & n > 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution The output signal $y(n)$ is given by

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$$Y(e^{j\omega'}) = \begin{cases} \dfrac{L}{M} X\left(e^{j\omega' L/M}\right), & \text{for } |\omega'| \le \min\left[\pi, \dfrac{\pi M}{L}\right] \\ 0, & \text{otherwise} \end{cases}$$

11.4 SIGNAL FLOW GRAPHS

A signal flow graph comprises of branches and nodes, where branches represents the signal operations and nodes represents the connection points.

Figure 11.9 shows the various branch operations in the signal flow graphs. When signals enter a branch, they are associated with the input

(a) Unit Delay

$x(n)$ $\quad Z^{-1} \quad$ $y(n) = x(n-1)$

$X(e^{j\omega})$ $\quad$ $Y(e^{j\omega}) = e^{-j\omega} X(e^{j\omega})$

$X(z)$ $\quad$ $Y(z) = z^{-1} X(z)$

(b) M-sample delay

$x(n)$ $\quad Z^{-M} \quad$ $y(n) = x(n-M)$

$X(e^{j\omega})$ $\quad$ $Y(e^{j\omega}) = e^{-jM\omega} X(e^{j\omega})$

$X(z)$ $\quad$ $Y(z) = z^{-M} X(z)$

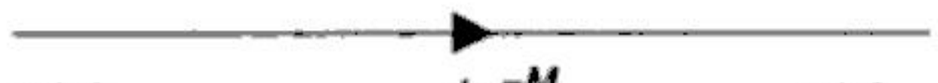
(c) GAIN (multiplication by a constant)

$x(n)$ $\quad k \quad$ $y(n) = kx(n)$

$X(e^{j\omega})$ $\quad$ $Y(e^{j\omega}) = kX(e^{j\omega})$

$X(z)$ $\quad$ $Y(z) = kX(z)$

(d) GAIN AND DELAY by a factor M

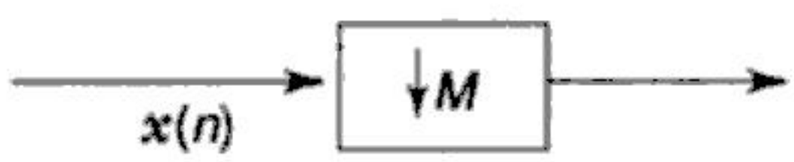


$x(n)$ $\quad kz^{-M} \quad$ $y(n) = kx(n-M)$

$X(e^{j\omega})$ $\quad$ $Y(e^{j\omega}) = ke^{-jM\omega} X(e^{j\omega})$

$X(z)$ $\quad$ $Y(z) = kz^{-M} X(z)$

(e) Sampling rate compressor / down sampler



$y(m) = x(Mm)$

$X(e^{j\omega})$

$X(z)$

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{l=0}^{M-1} X(e^{j(\omega - 2\pi l)/M})$$

$$Y(z) = \frac{1}{M} \sum_{l=0}^{M-1} X(z^{1/M} w)$$

$$w = e^{-j2\pi l/M}$$

Fig. 11.9 *(Contd.)*

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Combining systems three, four yields

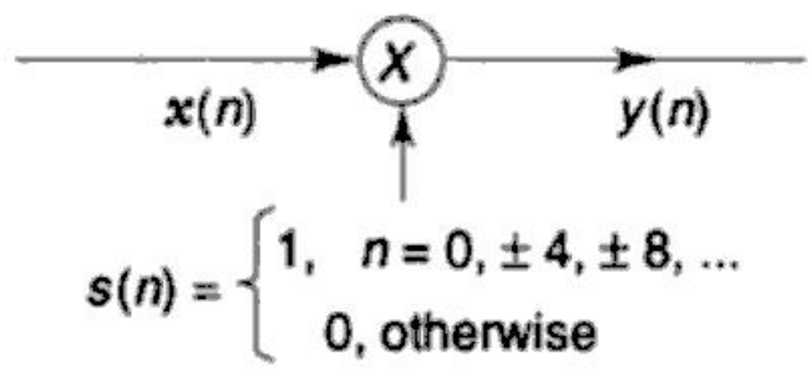

Fig. E 11.3 (d)

11.5 FILTER STRUCTURES

11.5.1 FIR Direct Form Structure

The general input-output relationship of an FIR filter is given by

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k) \tag{11.24}$$

The filter is represented by N-point impulse response $h(k)$, i.e., $h(k)$ takes values only in the range $k = 0, \ldots, N-1$, outside this interval $h(k)$ is zero. Fig. 11.12 shows the signal flow graph representation of an FIR filter.

FIR filters are normally designed with linear phase. Hence the impulse response is symmetric given by

$$h(k) = h(N-1-k)$$

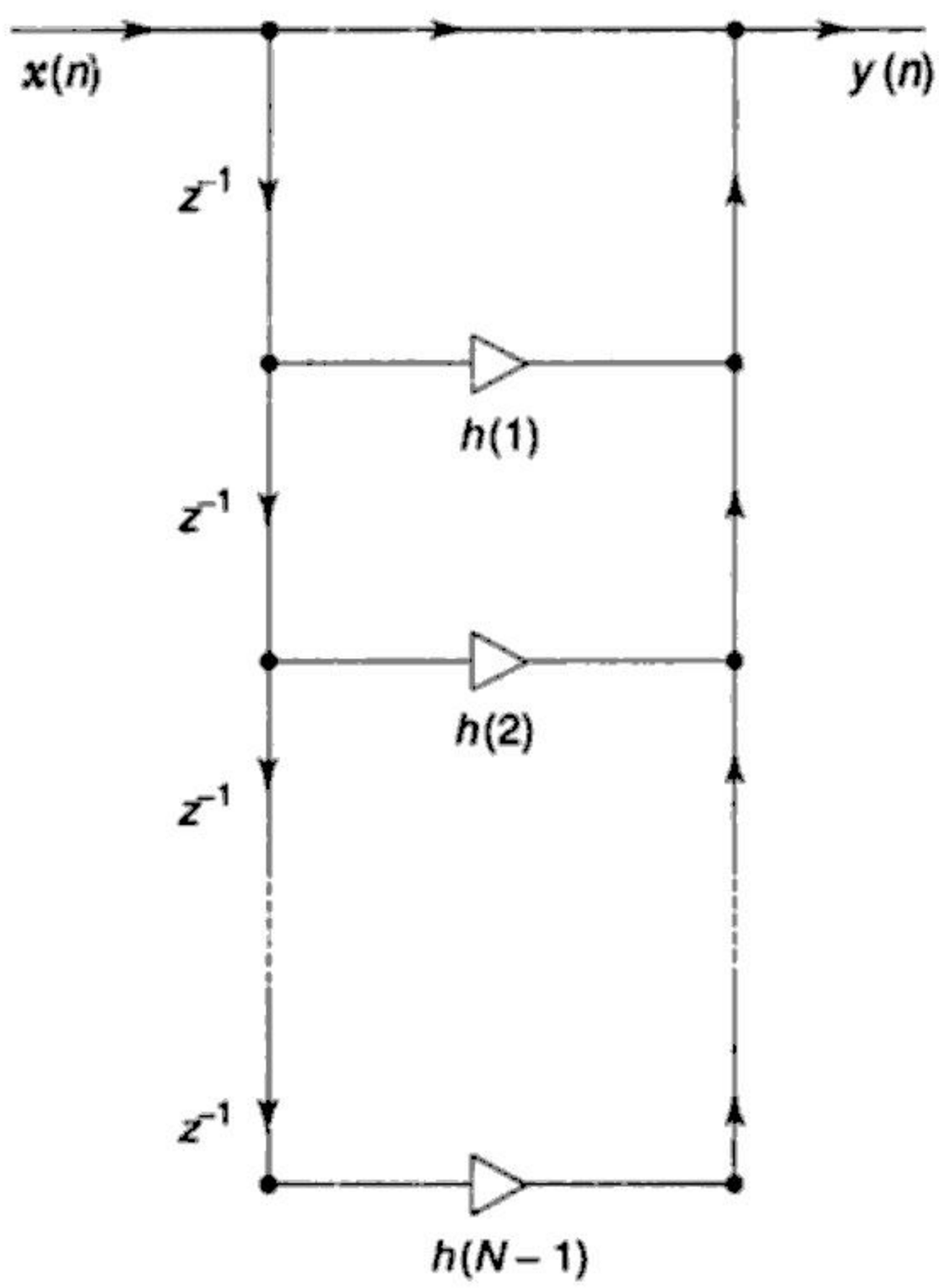

Fig. 11.12 *The Signal Flow Graph Representation of an FIR Filter*

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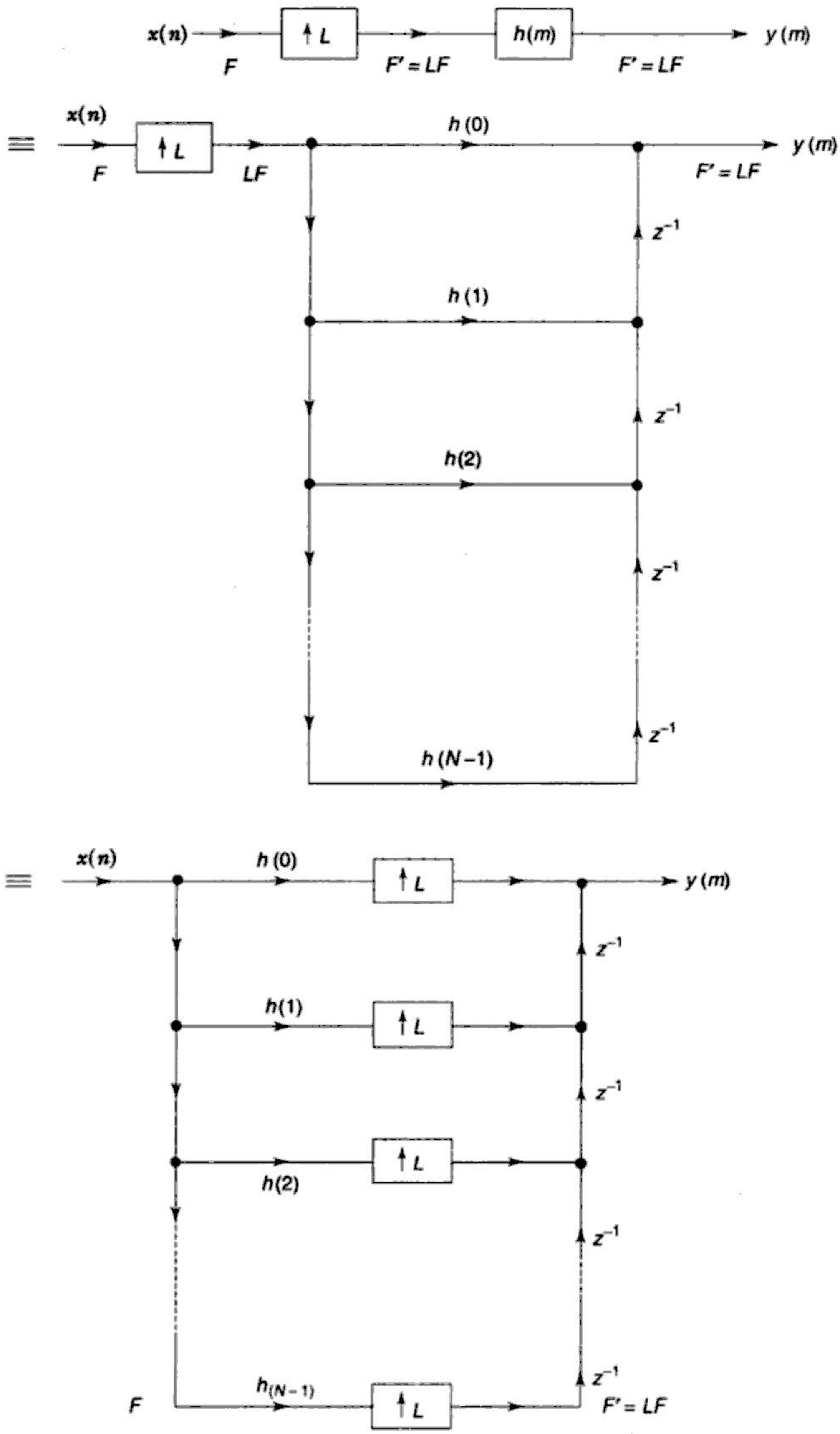

Fig. 11.15 *Direct form Representation of the 1 to L Interpolator*

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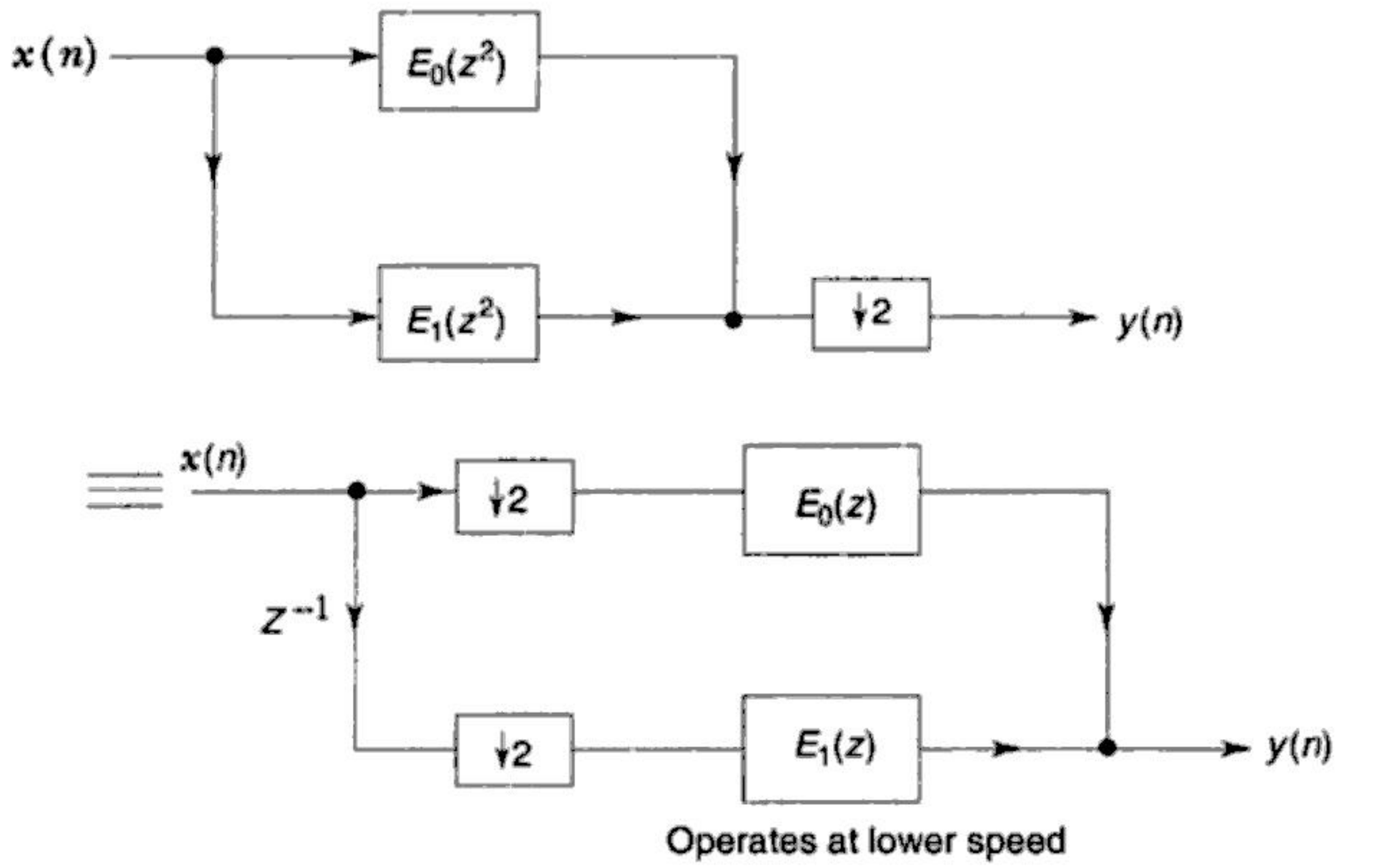

Fig. 11.18(c) *The Polyphase Implementation of Decimator by a Factor of two*

outputs swapped. Because of this, the interpolator structures derived in this section are of similar form to the decimator structures. The interpolator consisting of an interpolation (anti-imaging) filter $h(n)$ and an up sampler by a factor two is shown in Fig. 11.18(d).

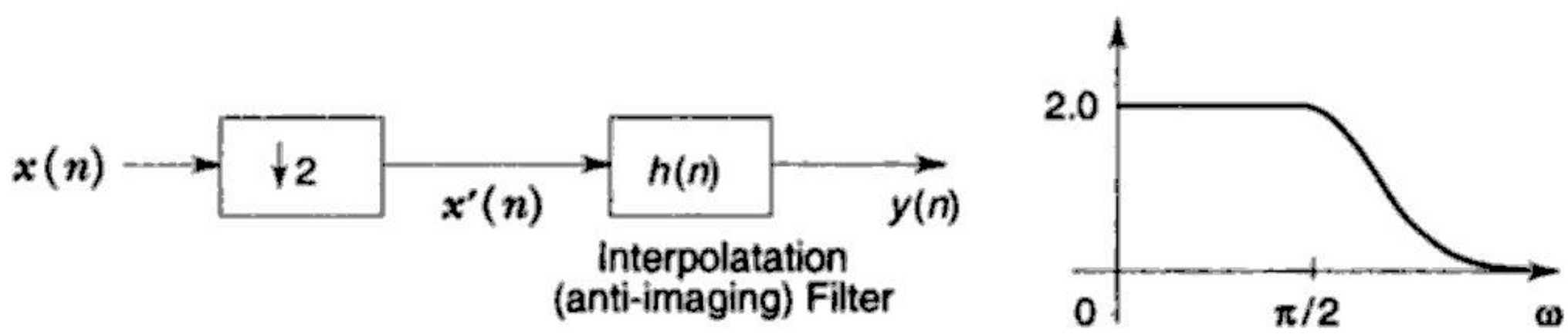

Fig. 11.18(d) *Interpolation by Factor Two and its frequency Response*

Assuming that the anti-imaging FIR filter has N coefficients, the filtering described by $y(n) = x'(n) * h(n) = \sum_{k=0}^{N-1} x'(k)\, h(n-k)$. Figure 11.18(e) represents the direct implementation of the FIR filter in transversal structure. The polyphase implementation structure with its interpolator identity is shown in Fig. 11.18(f).

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The frequency response of the low-pass filter is given by,

$$H(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega < \pi / M \\ 0, & \text{otherwise} \end{cases} \tag{11.38}$$

11.7.2 Filter Design for FIR Decimators and Interpolators

The various methods for FIR filter design are given below

(i) Window method
(ii) Optimal, equiripple linear phase method
(iii) Half-band designs
(iv) FIR interpolator design based on-time domain filter specifications
(v) Classical interpolation designs: linear and Lagrangian

Let us consider the equiripple FIR filter design. The equiripple response of the filter is shown in Fig. 11.24.

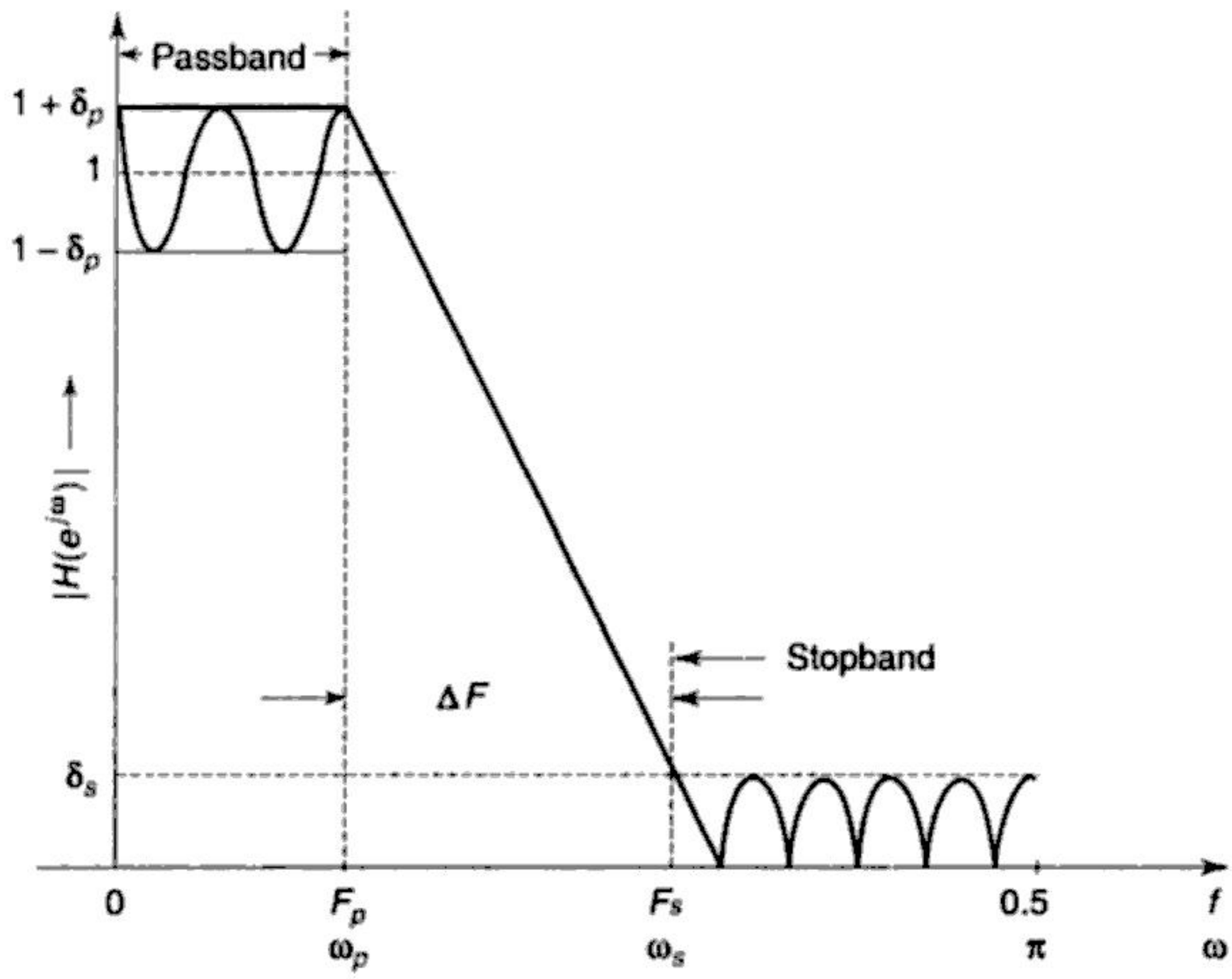

Fig. 11.24 *The Equiripple Response of the Filter*

The design equations for calculating the stopband and passband frequency are discussed below. Let the highest frequency of the decimated signal or the total bandwidth of the interpolated signal be $\omega_c \le \pi$, then the passband frequency is given by,

$$\omega_p \le \begin{cases} \omega_c / L, & 1 \text{ to } L \text{ interpolator} \\ \omega_c / M, & M \text{ to } 1 \text{ decimator} \\ \min(\omega_c / L, \omega_c / M), & \text{conversion by } L / M \end{cases} \tag{11.39}$$

The stopband frequency is given by

$$\omega_s = \begin{cases} \pi / L, & 1 \text{ to } L \text{ interpolator} \\ \pi / M, & M \text{ to } 1 \text{ decimator} \\ \min(\pi / L, \pi / M), & \text{conversion by } L / M \end{cases} \tag{11.40}$$

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Example 12.2 Consider the discrete time signal

$$x(n) = \cos(2\pi f_1 n) + \cos(2\pi f_2 n),\ n = 0, 1, \ldots, 7$$

Obtain the power spectrum for the data sequence length $L = 8, 16, 32$ for different values of f_1 and f_2, where $f_2 = f_1 + \Delta f$ and Δf is small deviation from f_1 or simply frequency separation.

Solution The power spectrum for the discrete time signal $x(n)$ is

$$P_{xx}\left(\frac{k}{L}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j\pi nk/L}\right|^2$$

The power spectrum is calculated for different values of L, by appending zeros to the original sequence.

The plot for power spectrum is shown in Fig. E12.2.

Let $f_1 = 0.6$ and $\Delta_f = 0.05$ which results $f_2 = 0.65$.

$$x(n) = \cos(2\pi(0.6)n) + \cos(2\pi(0.65)n)$$

The power spectrum of $x(n)$ for various values of L is shown in Fig. E12.2(a).

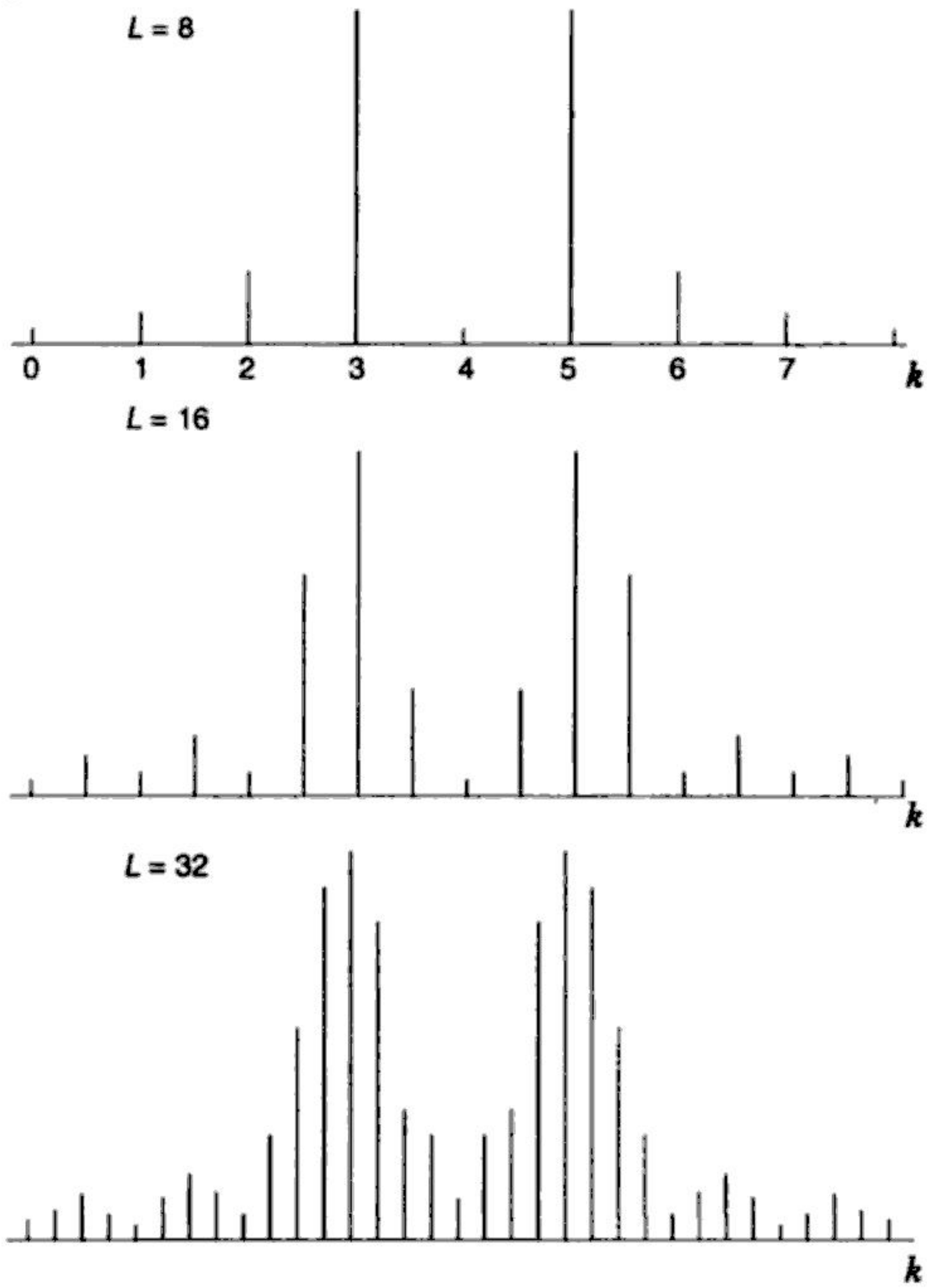

Fig. E12.2(a) *Power Spectrum* $\frac{1}{N}|X(k)|^2$ *Versus k for* $\omega_1 = 2\pi(0.6)$ *and* $\omega_2 = 2\pi(0.65)$

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12.5.4 Limitations of Non-Parametric Methods for Power Spectrum Estimation

(1) It requires long data sequences to obtain the necessary frequency resolution.
(2) Spectral leakage effects because of windowing.
(3) The assumption of the autocorrelation estimate $r_{xx}(m)$ to be zero for $m \geq N$. This assumption limits the frequency resolution and quality of the power spectrum estimate.
(4) Assumption that the data are periodic with period N.
These assumptions may not be realistic.

Example 12.3 Show that

$$\text{(a) } E\left[P_{xx}(f_1)P_{xx}(f_2)\right] = \sigma_x^4\left\{1+\left[\frac{\sin\pi(f_1+f_2)N}{N\sin\pi(f+f_2)}\right]^2+\left[\frac{\sin\pi(f_1-f_2)N}{N\sin\pi(f_1-f_2)}\right]^2\right\}$$

$$\text{and (b) var}\left[P_{xx}(f)\right] = \sigma_x^4\left\{1+\left(\frac{\sin 2\pi f N}{N\sin 2\pi f}\right)^2\right\}$$

Use the expression for the fourth joint moment for Gaussian random variable.

$$E[X_1X_2X_3X_4] = E[X_1X_2]\,E[X_3X_4] + E[X_1X_3]\,E[X_2X_4] + E[X_1X_4]\,E[X_2X_3]$$

Solution

(a) $E[P_{xx}(f_1)P_{xx}(f_2)] =$

$$E\left[\frac{1}{N^2}\left|\sum_{n_1=0}^{N-1}x(n_1)e^{-j2\pi f_1 n_1}\right|^2\left|\sum_{n_2=0}^{N-1}x(n_2)e^{-j2\pi f_2 n_2}\right|^2\right]$$

$$= \frac{1}{N^2}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sum_{n_3=0}^{N-1}\sum_{n_4=0}^{N-1}E\left[x(n_1)x(n_2)x(n_3)x(n_4)\right]e^{-j2\pi f_1(n_1-n_3)}e^{-j2\pi f_2(n_2-n_4)}$$

But, $E[x(n_1)x(n_2)x(n_3)x(n_4)] = E[x(n_1)x(n_2)]\,E[x(n_3)x(n_4)] + E[x(n_1)x(n_3)]\,E[x(n_2)x(n_4)] + E[x(n_1)x(n_4)]\,E[x(n_2)x(n_3)]$

$= \sigma_x^4$, $n_1 = n_2$ and $n_3 = n_4$; $n_1 = n_3$ and $n_2 = n_4$; $n_1 = n_4$ and $n_2 = n_3$

For $n_1 = n_4$ and $n_2 = n_3$,

$$e^{-j2\pi f_1(n_1-n_3)}e^{-j2\pi f_2(n_2-n_4)} = e^{-j2\pi f_1(n_1-n_3)}e^{-j2\pi f_2(n_3-n_1)}$$
$$= e^{-j2\pi f_1(n_1-n_3)}e^{-j2\pi f_2(n_1-n_3)}$$
$$= e^{-j2\pi(f_1-f_2)(n_1-n_3)}$$

For $n_1 = n_2$ and $n_3 = n_4$

$$e^{-j2\pi f_1(n_1-n_3)}e^{-j2\pi f_2(n_2-n_4)} = e^{-j2\pi f_1(n_1-n_3)}e^{-j2\pi f_2(n_1-n_3)}$$
$$= e^{-j2\pi(f_1-f_2)(n_1+n_3)}$$

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$$a_m = \begin{bmatrix} a_m(1) \\ a_m(2) \\ \cdot \\ \cdot \\ \cdot \\ a_m(m) \end{bmatrix} = \begin{bmatrix} a_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \begin{bmatrix} d_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ K_m \end{bmatrix}$$

where d_{m-1}, K_m are to be computed. Partition the $m \times m$ autocorrelation matrix as

$$R_m = \begin{bmatrix} R_{m-1} & \cdots & r_{m-1}^{b} \\ \cdots & & \\ \cdots & & \\ \cdots & & \\ r_{m-1}^{bt} & & r_{xx}(0) \end{bmatrix}$$

where $$r_{m-1}^{b} = \begin{bmatrix} r_{xx}(m-1) \\ r_{xx}(m-2) \\ \cdot \\ \cdot \\ \cdot \\ r_{xx}(1) \end{bmatrix}$$

$$r_{m-1} = \begin{bmatrix} r_{xx}(1) \\ r_{xx}(2) \\ \cdot \\ \cdot \\ \cdot \\ r_{xx}(m-1) \end{bmatrix}$$

r_{m-1}, r_{m-1}^{b} are vectors where the elements of one vector and reversed in the other vector.

Solution for $R_m a_m = -r_m$, becomes

$$R_m = \begin{bmatrix} R_{m-1} & \cdots & r_{m-1}^{b} \\ \cdots & & \\ \cdots & & \\ \cdots & & \\ r_{m-1}^{bt} & & r_{xx}(0) \end{bmatrix} \left\{ \begin{bmatrix} a_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \begin{bmatrix} d_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ K_m \end{bmatrix} \right\} = - \begin{bmatrix} r_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ r_{xx}(m) \end{bmatrix}$$

In equation form,

$$R_{m-1} a_{m-1} + R_{m-1} d_{m-1} + K_m r_{m-1}^{b} = -r_{m-1}$$

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$$= \sum_{k=0}^{\infty} a^k w(n-k)$$

Combining both the solutions,

$$x(n) = ca^n + \sum_{k=0}^{\infty} a^k w(n-k)$$

Assuming $x(0) = 0$,

$$c = -\sum_{k=0}^{\infty} a^k w(-k)$$

Hence,

$$x(n) = -a^n \sum_{k=0}^{\infty} a^k w(-k) + \sum_{k=0}^{\infty} a^k w(n-k)$$

$$= -\sum_{k=n}^{\infty} a^k w(n-k) + \sum_{k=0}^{\infty} a^k w(n-k)$$

$$= -\sum_{k=0}^{n-1} a^k w(n-k)$$

Mean

$$E[w(n)] = 0$$

$$E[x(n)] = E\left[\sum_{k=0}^{n-1} a^k w(n-k)\right]$$

$$= \sum_{k=0}^{n-1} a^k E[w(n-k)]$$

$$= 0 \text{ for all } n$$

Auto correlation

$$\gamma_{xx}(l) = E[x(n)x(n-l)]$$

$$= E\left[\sum_{k=0}^{n-1} a^k w(n-k) \sum_{k=0}^{n-1} a^i w(n-l-i)\right]$$

$$= E\left[\sum_{k=0}^{n-1} \sum_{i=0}^{n-1} a^{k+i} w(n-k) w(n-l-i)\right]$$

$$= \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} a^{k+i} E[w(n-k)] w(n-l-i)$$

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13.2.2 Adaptive Equalisation

Consider a digital communication system shown in Fig. 13.3. The bandwidth of the channel has to be used efficiently. The requirement is to design a reliable system with data to be transmitted at a higher rate. The factors that affect the data in the channel are inter-symbol interference and thermal noise. The output of the receiving filter is

$$s(t) = \sum_k a_k \, p(t - kT_s) \tag{13.4}$$

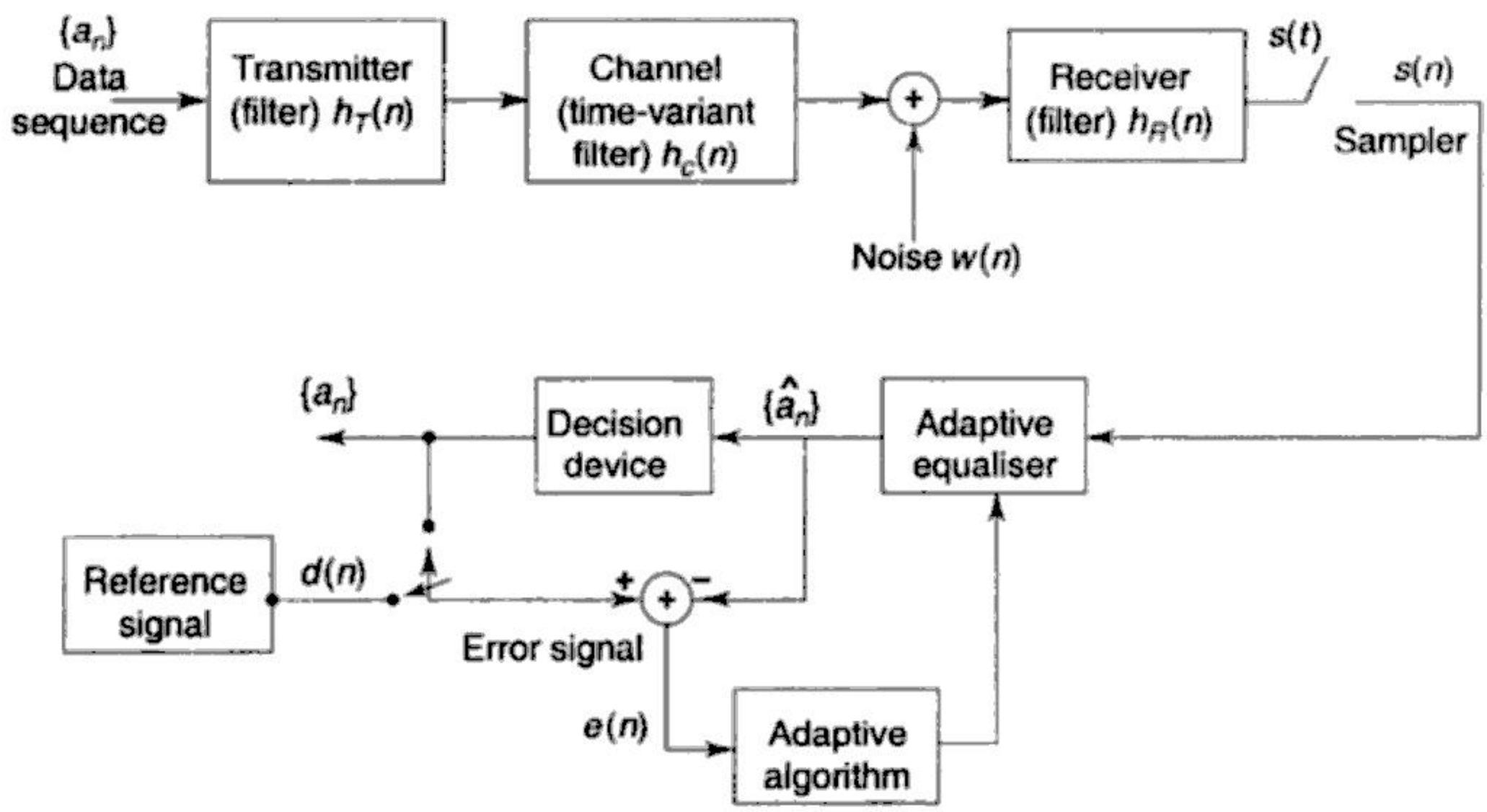

Fig. 13.3 *Adaptive Equalisation*

where T_s is the duration of the signalling interval and $p(t)$ is the impulse response of the cascade connection of the transmitting filter, the channel and the receiving filter.

The sampled version of $s(t)$ is

$$s(k) = \sum_n a_n \, p(k - n)$$

$$s(k) = a_k \, p(0) + \sum_{\substack{n \\ n \neq k}} p(k - n) \tag{13.5}$$

The first term corresponds to the desired symbol and the remaining terms represent inter-symbol interference. To avoid the ISI problem, the transmitter and receiver filters should be properly designed considering the channel characteristics. Since the channel has random characteristics, filters are designed based on the average channel characteristics. But this may not reduce ISI to a larger extent. Hence adaptive equalisers can be used. The purpose of using adaptive equaliser is for compensating the channel distortion so that the detected signal will be reliable. The adaptive equalisation process is done in two steps, i.e. (i) training mode and (ii) tracking mode. In the training mode, a known test signal, normally a PN sequence is transmitted. Using a

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$X_M(n)\, h_M(n-1)$ is the observed filter output at time n based on the filter coefficients at time $n-1$.

$$X_M^t(n)\, h_M(n-1) = \hat{d}\,(n, n-1) = \hat{d}\,(n) \tag{13.53}$$

$$e_M(n, n-1) = d\,(n) - \hat{d}\,(n, n-1) \tag{13.54}$$

Hence, $h_M(n) = e_M(n)$

$$h_M(n) = h_M(n-1) + K_M(n)\, e_M(n) \tag{13.55}$$

where

$$K_M(n) = P_M(n)\, X_M^*(n)$$

When n is small, $R_M(n)$ may not have an inverse. Hence add initially $\delta\, I_M$ to $R_M(n)$, where δ is a small positive constant.

Steps in the RLS Algorithm

Initial conditions

$$h_M(-1) = 0$$

$$P_M(-1) = \delta\, I_M$$

Steps in calculation of filter coefficients:

(i) Filter output

$$\hat{d}\,(n) = X_M^t(n)\, h_M(n-1)$$

(ii) Error

$$e_M(n) = d\,(n) - \hat{d}\,(n)$$

(iii) Kalman gain vector

$$K_M(n) = \frac{P_M(n-1)\, X_M^*(n)}{w + X_M^t(n)\, P_M(n-1)\, X_M^*(n)}$$

(iv) Inverse of the correlation matrix

$$P_M(n) = \frac{1}{w}\,[P_M(n-1) - K_M(n)\, X_M^t(n)\, P_M(n-1)]$$

(v) Filter coefficient vector

$$h_M(n) = h_M(n-1) + K_M(n)\, e_M(n)$$

With optimization, the mean square error is

$$\varepsilon\, M_{\min} = \sum_{l=0}^{n} w^{n-l}\, |d\,(l)|^2 - h_M^t(n)\, D_M^*(n)$$

13.6. THE FORWARD–BACKWARD LATTICE METHOD

The mth stage of the Mth order lattice filter is shown in Fig. 13.9 where $m = 1, 2, \ldots, M$. The equations can be written as

$$f_m(i) = f_{m-1}(i) + \gamma_m^{(f)}\, b_{m-1}(i-1)$$

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where

$\mu_m(n)$ is the time varying step size parameter

Let

$$\mu_m(n) = \frac{1}{E_{m-1}(n)} \tag{13.78}$$

and $E_{m-1}(n) = w\,E_{m-1}(n-1) + f^2_{m-1}(n) + b^2_{m-1}(n-1)$ (13.79)

This procedure is referred to as the gradient adaptive lattice(GAL) algorithm.

REVIEW QUESTIONS

13.1 What is the need for adaptivity?

13.2 Give some examples of applications where adaptive filtering is done.

13.3 Explain how a dynamic system can be modelled.

13.4 What is the need for adaptive equalisation in a digital communication system? Explain.

13.5 Explain how a narrow band signal can be extracted when it is embedded in a broad band noise.

13.6 Explain how noise introduced in the system can be minimised.

13.7 How is the effect of echo minimised in a telephone communication?

13.8 Using MMSE criterion, derive the expression for optimal filter coefficients.

13.9 Explain in detail the LMS algorithm for direct FIR filters.

13.10 Derive the mean square error in RLS algorithm with optimisation.

13.11 Explain the forward–backward lattice algorithm.

13.12 Discuss the gradient adaptive lattice algorithm.

13.13 Consider the adaptive FIR filter shown in Fig. Q13.13. The system S (z) is characterised by

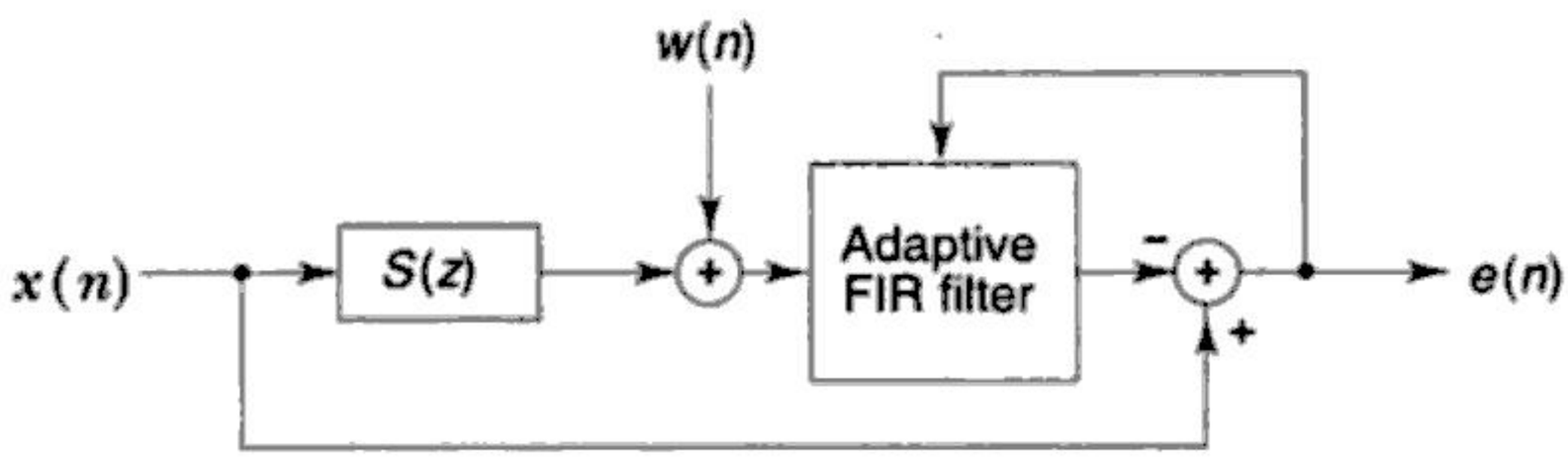

Fig. Q13.13

$$S(z) = \frac{1}{1 - 0.9\,z^{-1}}$$

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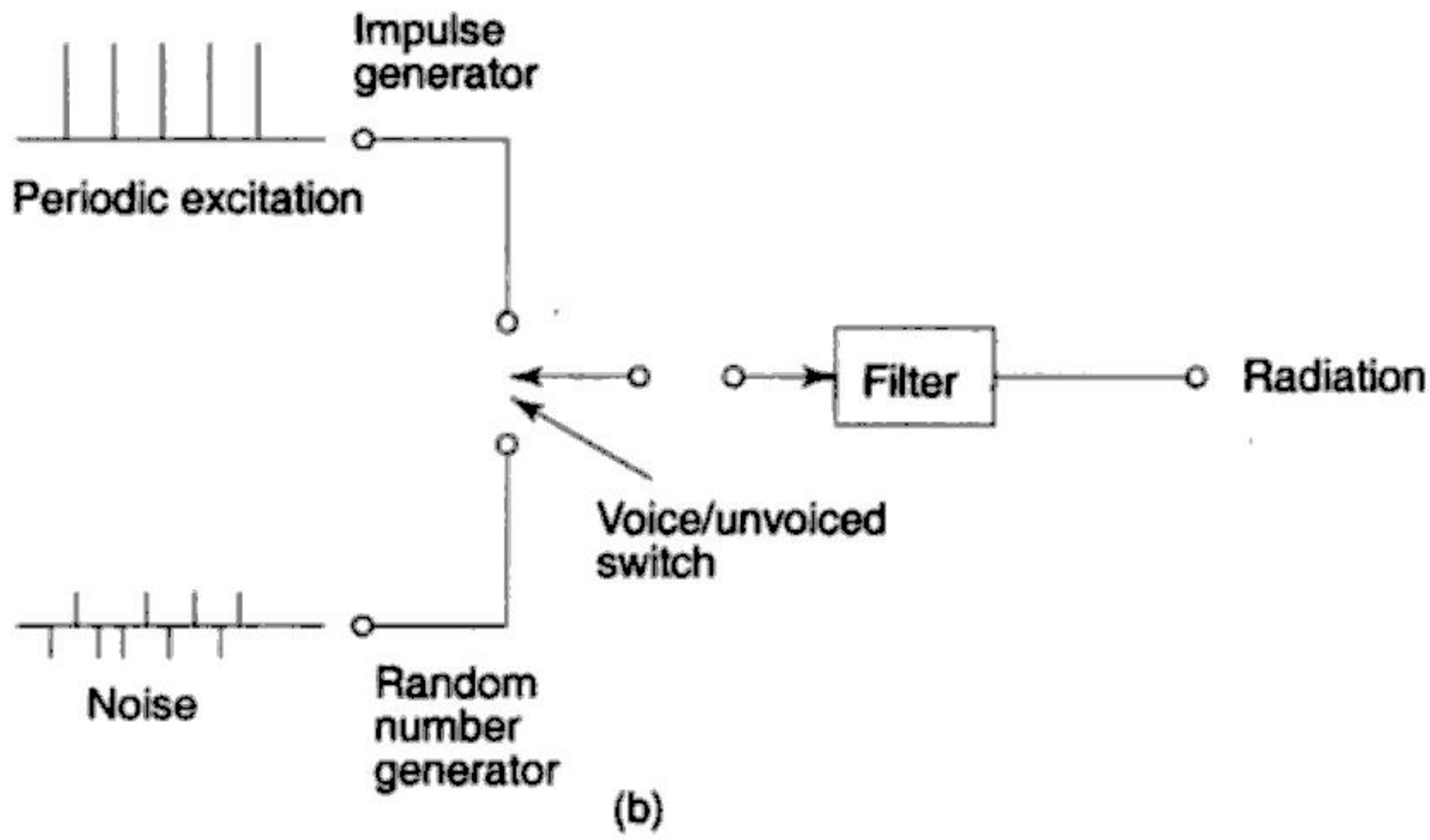

Fig. 14.2 *Models of Vocal Organs*

$$f_1 = \frac{1}{4}\frac{c}{l} \text{ Hz, where } c \text{ is the velocity of light in m/sec and } l \text{ is in cm,}$$

$$= \frac{1}{4}\frac{3 \times 10^8}{17} \approx 500 \text{ Hz}$$

The normal resonant modes of vibration of the vocal tract are called ***formants***.

In many speech processing systems, the source of excitation and the vocal tract system are assumed to be independent. A simple speech prediction model is shown in Fig. 14.2(b).

The excitation sources are an impulse generator and a random number noise generator. The rate at which the impulse generator generates pulse is equal to the rate of oscillation of the vocal cords. The filter represents the vocal tract system. For representing quasi-random turbulence and pressure build-up, a random number generator is used. The sources are given as input to a linear, time varying digital filter. The filter represents the vocal tract system whose coefficients are varied in short intervals of time.

14.2.2 Analysis of Speech Signals

Frequency domain analysis can be done by using FFT. It requires a data segment with suitable length such that the harmonics can be resolved and the data within that length should be approximately stationary. Windowing can be done to extract the required length of the data segment. The pitch of the human voice ranges from a minimum of 80 Hz(males) to a maximum of 400 Hz(females). If three periods of 80 Hz are considered, then the selected data length is for 37.5 ms.

14.2.3 Short Time Spectrum Analysis

Consider the discrete time signal, $x(nT)$, $-\infty < n < \infty$. The Fourier transform of $x(nT)$ is

$$\mathcal{F}\{x(nT)\} = X(e^{j\omega T})$$

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The signal $x(n)$ is split into many narrow band signals which occupy contiguous frequency bands using analysis filter bank. These signals are down-sampled, yielding sub-band signals, which are then compressed using encoders. The compressed signals are multiplexed and transmitted. On the receiving side, reverse operations are carried out. The received signal is demultiplexed, decoded, up-sampled and then passed through a synthesis filter bank. The outputs of the synthesis filter bank are combined to yield the signal $y(n)$. By using sub-band coding, better compression ratio is achieved, because each sub-band signal can be represented using a different number of bits. This is possible because of the signal's spectral characteristics.

A 7 KHz wide band audio signal is coded using sub-band coding based on a five band QMF bank (based on the investigation results of Richards and Jayant). The frequency ranges are 0 – 875 Hz, 875 – 1750 Hz, 1750 – 3500 Hz, 3500 – 5250 Hz and 5250 – 7000 Hz. The realisation of the structure is shown in Fig. 14.9. Adaptive differential PCM is used for encoding the sub-band signals.

Table 14.1

Frequency Channels (in Hz),	*Bit allocation*
0 – 875	5 bits/sample
875 – 1750	5 bits/sample
1750 – 3500	4 bits/sample
3500 – 5250	4 bits/sample
5250 – 7000	3 bits/sample

Hence the total bit rate of the encoder is 56 kbps.

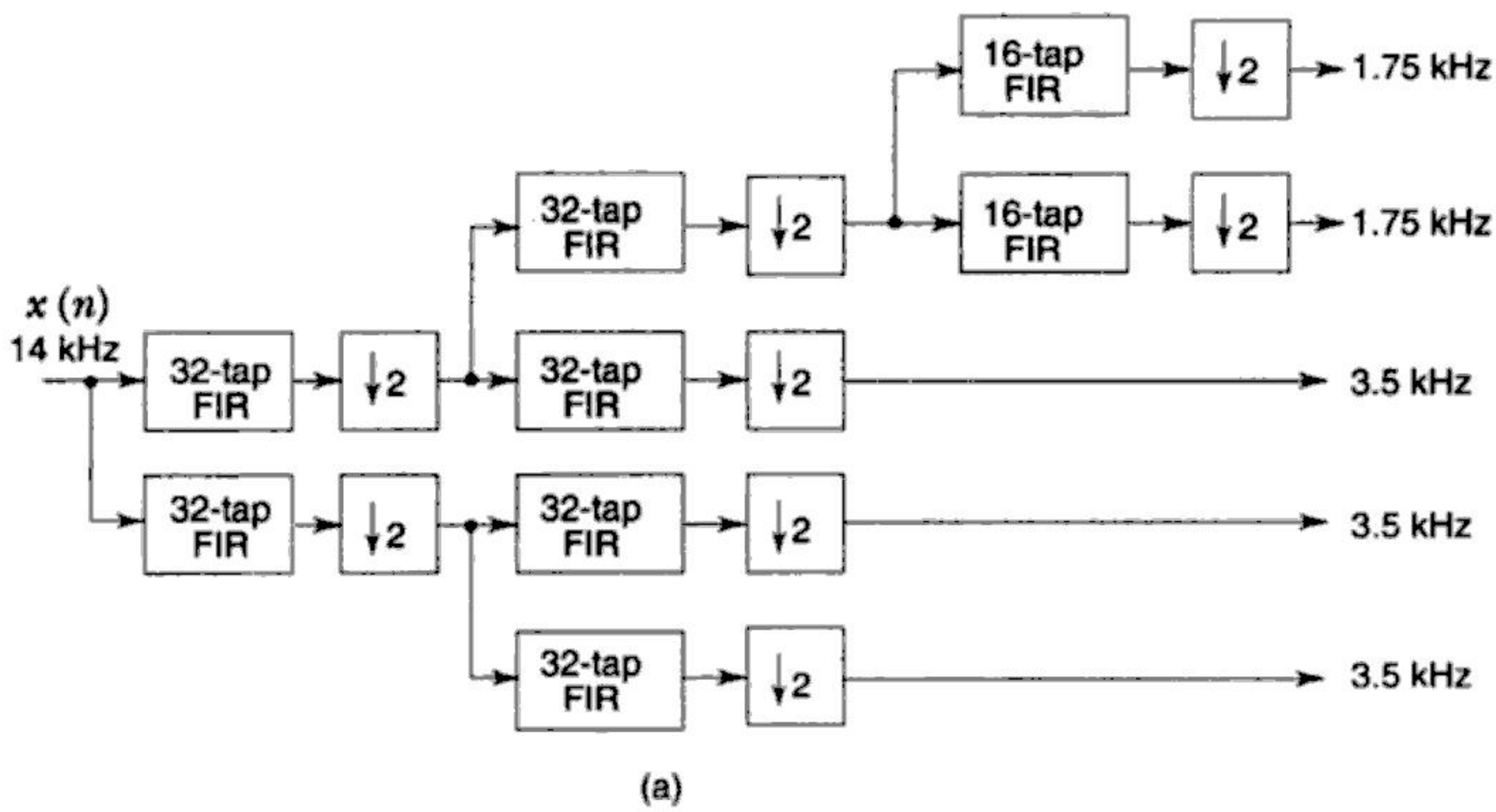

Fig. 14.9 *(Contd.)*

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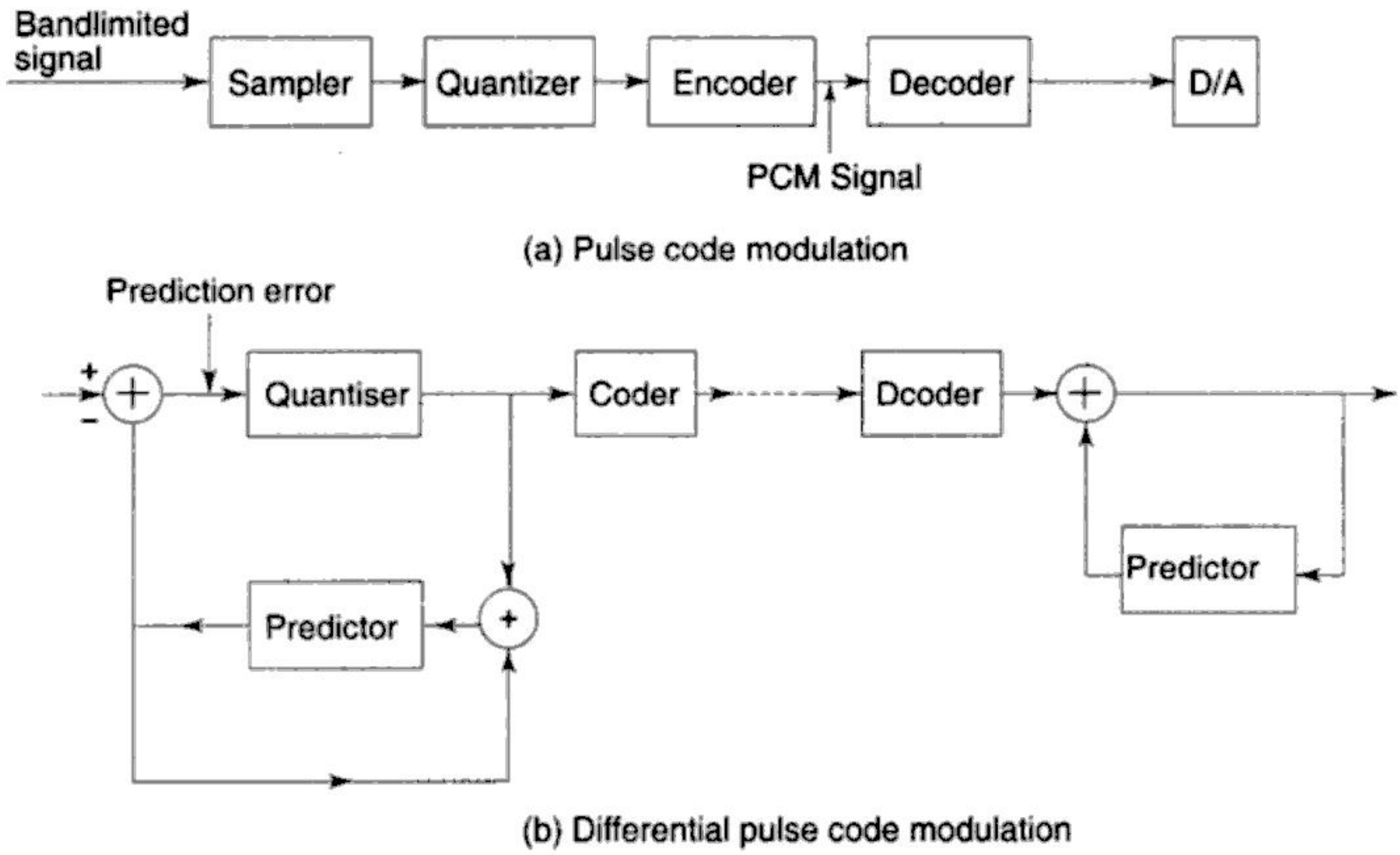

Fig. 14.19 *Image Compression Techniques*

14.5 INTRODUCTION TO WAVELETS

A wavelet is a "small wave", which has its energy concentrated in time. It gives a tool for the analysis of transient, nonstationary, or time-varying phenomena. It not only has an oscillating wavelike characteristic but also has the ability to allow simultaneous time and frequency analysis with a flexible mathematical foundation.

Traditional signal processing techniques such as Fourier transform and short time Fourier transform (STFT) are poorly suited for analysing signals which have abrupt transitions superimposed on lower frequency backgrounds such as speech, music and bio-electric signals. On the otherhand, the wavelet transform, a new tool, provides a novel approach to the analysis of such signals. The wavelet transform has a multi-resolution capability. The multi-resolution signal processing used in computer vision, sub-band coding developed for speech and image compression, and wavelet series expansions developed in applied mathematics are recognised as different views of signal theory. The wavelet theory provides a unified frame work for a number of techniques which had been developed independently for various signal processing applications.

Time-Frequency Representations

Stationary signals, whose spectral characteristics do not change with time, are represented as a function of time or frequency. Non-stationary signals, which involve both time and frequency, especially the auditory and visual perceptions require time-frequency analysis. The time-

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then the lower levels of decomposition would correspond to higher frequency sub-bands and the nth level would correspond to the lowest frequency sub-bands. Generally, most of the energy in an image is concentrated in the low frequency region. Thus, as one moves from higher levels, i.e. region of approximate co-efficients to lower levels, i.e. region of detail co-efficients of sub-band decomposition, there will be a decrease in the energy content of the sub-band. In otherwords, most of wavelet co-efficients in the detail region have very low values and these can be quantised to zero without affecting the perceived quality of the image significantly. In addition, there are spatial self-similarities across the sub-bands, i.e. the similarities occupy the same spatial position in each sub-band, but at different resolutions. The embedded zero-tree wavelet (EZW) coding algorithm exploits these properties to give excellent compression results.

(ii) Video Compression Today's main stream MPEG (Moving Picture Expert Group) standard is geared towards transmission of sequences of images. These algorithms use a type of compression known as discrete cosine transform (DCT). MPEG compression solutions are more expensive, Houston Advanced Research Centre (Houston, TX), claims HARC-C can achieve still-image compression ratios of about 300 to 1 and video compression ratios upto 480 to 1.

(iii) Audio Compression Wavelet based audio compression can be achieved by using wavelet based digital filters in sub-band decomposition. In this, the wavelet packet decomposition of audio frames is obtained and then quantised to keep the distortion below the masking threshold for the frame. The masking threshold of a frame is the minimum sound pressure level (SPL) at which the tone becomes audible. This technique depends on choosing the most appropriate wavelet from a library of wavelets adaptively.

(b) Sound Synthesis

Sound synthesis is a very important application of WT. Victor Wickerhauser has suggested that wavelet packets could be useful in sound synthesis. His idea is that wavelet packet generator could replace a large number of oscillators. Through experimentation, a musician could determine combinations of wave packets that produce especially interesting sounds. Wickerhauser feels that sound synthesis is a natural use of wavelets. A sample of the notes produced by the instrument could be decomposed into its wavelet packet coefficients. Reproducing the note would then require reloading those coefficients into a wave packet generator and playing back the result.

(c) Computer and human vision

Marr's theory was that image processing in the human visual system has a complicated hierarchical structure that involves several layers of

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Chapter 15

MATLAB Programs

15.1 INTRODUCTION

MATLAB stands for MATrix LABoratory. It is a technical computing environment for high performance numeric computation and visualisation. It integrates numerical analysis, matrix computation, signal processing and graphics in an easy-to-use environment, where problems and solutions are expressed just as they are written mathematically, without traditional programming. MATLAB allows us to express the entire algorithm in a few dozen lines, to compute the solution with great accuracy in a few minutes on a computer, and to readily manipulate a three-dimensional display of the result in colour.

MATLAB is an interactive system whose basic data element is a matrix that does not require dimensioning. It enables us to solve many numerical problems in a fraction of the time that it would take to write a program and execute in a language such as FORTRAN, BASIC, or C. It also features a family of application specific solutions, called ***toolboxes***. Areas in which toolboxes are available include signal processing, image processing, control systems design, dynamic systems simulation, systems identification, neural networks, wavelength communication and others. It can handle linear, non-linear, continuous-time, discrete-time, multivariable and multirate systems. This chapter gives simple programs to solve specific problems that are included in the previous chapters. All these MATLAB programs have been tested under version 5.3 of MATLAB and version 4.2 of the signal processing toolbox.

15.2 REPRESENTATION OF BASIC SIGNALS

MATLAB Programs for the generation of unit impulse, unit step, ramp, exponential, sinusoidal and cosine sequences are as follows.

% Program for the generation of unit impulse signal

```
clc;clear all;close all;
t=-2:1:2;
```

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15.15 UPSAMPLING A SINUSOIDAL SIGNAL

%Program for upsampling a sinusoidal signal by factor L

```
N = input('Input length of the sinusoidal sequence=');
L = input('Up Samping factor=');
fi = input('Input signal frequency =');
% Generate the sinusoidal sequence for the specified length N
n = 0: N-1;
x = sin (2*pi*fi*n);
% Generate the upsampled signal
y = zeros(1,L*length(x));
y([1:L:length(y)])=x;
%Plot the input sequence
subplot (2,1,1);
stem(n,x);
title('Input Sequence');
xlabel('Time n');
ylabel('Amplitude');
%plot the output sequence
subplot (2,1,2);
stem (n,y(1:length(x)));
title (['output sequence, upsampling factor =', num2str(L)]);
xlabel ('Time n');
ylable ('Amplitude');
```

15.16 DOWN SAMPLING A SINUSOIDAL SEQUENCE

%Program for down sampling a sinusoidal sequence by a factor M

```
N=input('Input length of the sinusoidal signal =');
M=input('Down sampling factor =');
fi=input('Input signal frequency =');
%Generate the sinusoidal sequence
n=0 : N-1;
m=0 : N*M-1;
x=sin(2*pi*fi*m);
%Generate the down sampled signal
y=x([1 : M : length(x)]);
%Plot the input sequence
subplot (2,1,1);
stem(n,x(1:N));
title ('Input sequence'):
xlabel ('Time n');
ylabel ('Amplitude');
%Plot the down sampled signal sequence
subplot(2,1,2);
stem(n,y);
title (['Output sequence down sampling factor', num2str(M)];
```

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```
  end
     s=s/length(e1);
     if(y4==rs)
     break
  end
     ni=ni+1;
     w=w1;
  end
figure(1);subplot(2,2,1);plot(z);title('near end signal');
subplot(2,2,2);plot(rs);title('echo produced in the hybrid');
subplot(2,2,3);plot(fs);title('desired signal ');
subplot(2,2,4);plot(fs1);title('echo added with desired
signal');
figure(2);subplot(2,1,1);plot(y4);title('estimated echo
signal using LMS algorithm');
subplot(2,1,2);plot(fs1-y4);title('echo cancelled signal');
```

15.39 CANCELLATION OF ECHO PRODUCED ON THE TELEPHONE—PASS BAND CHANNEL

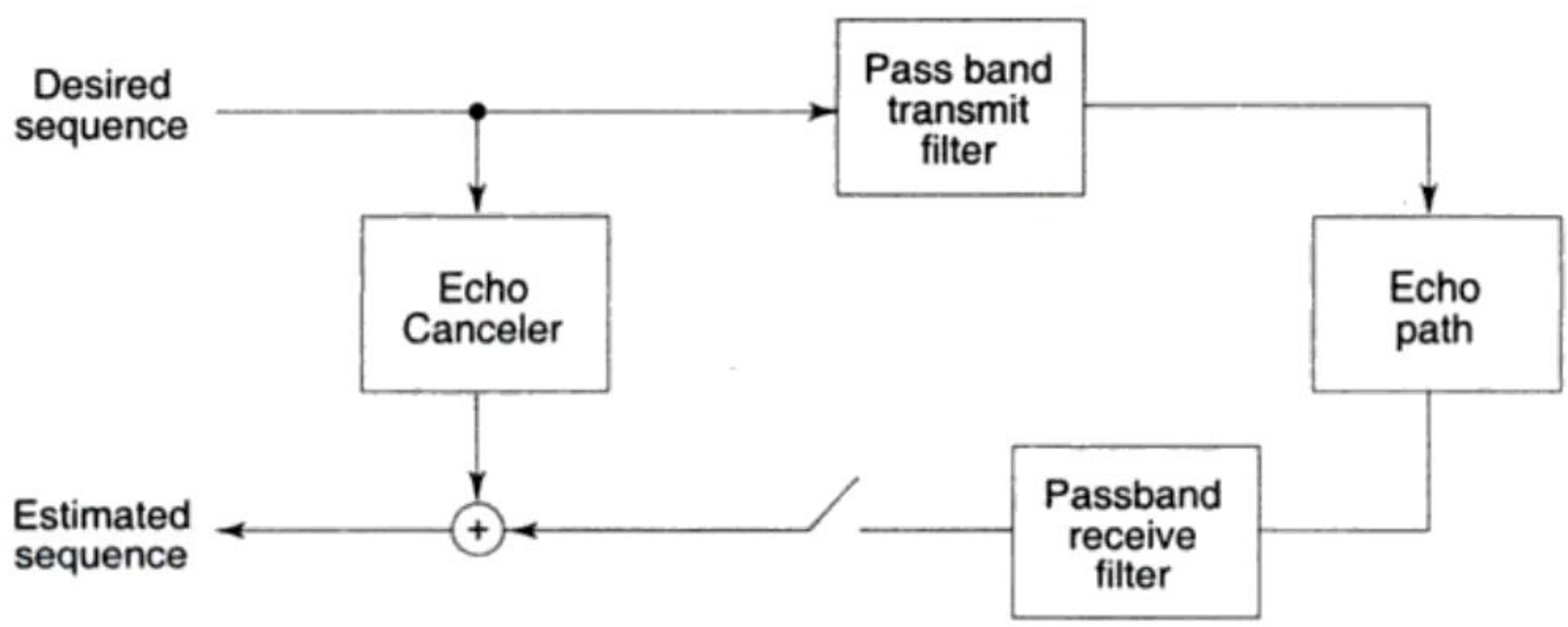

Fig. E15.39 *Pass Band Channel Echo Canceler*

```
%Simulation program for passband echo cancellation shown in
Fig. E15.39 using LMS algorithm
clc;close all;clear all;
format long
fd=8000;fs=16000;fc=8000;
f=4000;t=0:.01:1;%d=sin(2*pi*f*t/fd);
%Near end signal
ns=[5 2 5 2 5 5 2 2 2 5 5 2 2 2 2 2 5 5 2 5 2 5 5 5 5 5 5 5 5 5 5 5 5];
%Near end input signal is digitally modulated and plotted
y=dmod(ns,fc,fd,fs,'psk');
sbplot(2,2,1);plot(y);title('input signal');
xlabel('Time ——');ylabel('Amplitude ——;');
%Echo is generated due to mismatch of hybrid impedances
h=5*randn(1,length(ns));
```

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15.6 (i) *Determine the range of values of the parameter 'a' for which the linear time invariant system with impulse response*

$$h(n) = \begin{cases} a^n, & n \geq 0, n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

is stable.

(ii) *Determine the response of the system with impulse response* $h(n) = a^n\, u(n)$ *to the input signal* $x(n) = u(n) - u(n-10)$

15.7 Consider the system described by the difference equation $y(n) = a\,(y+1) + bx\,(n)$. *Determine 'b' in terms of a so that* $\Sigma h(n) = 1$.

(a) *Compute the zero-state step response s(n) of the system and choose 'b' so that* $s\,(\infty) = 1$.

(b) *Compare the values of 'b' obtained in parts (a) and (b). What did you observe?*

15.8 Compute and sketch the convolution $y(n)$ *and correlation* $r_{xh}\,(n)$ *sequences for the following pair of signals and comment on the results obtained.*

$$x_1(n) = \left\{\begin{matrix} 1, 2, 4 \\ \uparrow \end{matrix}\right\} \qquad h_1(n) = \left\{\begin{matrix} 1, 1, 1, 1, 1 \\ \uparrow \end{matrix}\right\}$$

$$x_2(n) = \left\{\begin{matrix} 0, 1, -2, 3, -4 \\ \uparrow \end{matrix}\right\} \qquad h_2(n) = \left\{\begin{matrix} \tfrac{1}{2}, 1, 2, 1, \tfrac{1}{2} \\ \uparrow \end{matrix}\right\}$$

$$x_3(n) = \left\{\begin{matrix} 1, 2, 3, 4 \\ \uparrow \end{matrix}\right\} \qquad h_3(n) = \left\{\begin{matrix} 4, 3, 2, 1 \\ \uparrow \end{matrix}\right\}$$

$$x_4(n) = \left\{\begin{matrix} 1, 2, 3, 4 \\ \uparrow \end{matrix}\right\} \qquad h_4(n) = \left\{\begin{matrix} 1, 2, 3, 4 \\ \uparrow \end{matrix}\right\}$$

15.9 Consider the recursive discrete-time system described by the difference equation.

$$y(n) = a_1 y(n-1) - a_2(n-2) + b_0\, x(n)$$

where $a_1 = -0.8,\ a_2 = 0.64,\ b_0 = 0.866$

(a) *Write a program to compute and plot the impulse response h(n) of the system for* $0 \leq n \leq 49$.

(b) *Write a program to compute and plot the zero-state step response s(n) of the system for* $0 \leq n \leq 100$.

(c) *Define an FIR system with impulse response* $h_{FIR}\,(n)$ *given by*

$$h_{FIR}(n) = \begin{cases} h(n), & 0 \leq n \leq 19 \\ 0, & \text{elsewhere} \end{cases}$$

where h(n) is the impulse response computed in part (a). Write a program to compute and plot its step response.

(d) *Compare the results obtained in part (b) and part (c) and explain their similarities and differences.*

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43. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$

44. $\int \frac{x\,dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2} + C$

45. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right) + C$

46. $\int x\sqrt{x^2 - a^2}\,dx = \frac{\sqrt{(x^2 - a^2)^3}}{3} + C$

47. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

48. $\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - a\cos^{-1}\left(\frac{a}{x}\right) + C$

49. $\int \sqrt{\frac{a + x}{b + x}}\,dx = \sqrt{(a + x)(b + x)} + (a - b)\ln\left(\sqrt{a + x} + \sqrt{b + x}\right) + C$

50. $\int \sqrt{\frac{a + x}{b - x}}\,dx = -\sqrt{(a + x)(b - x)} - (a + b)\sin^{-1}\sqrt{\frac{b - x}{a + b}} + C$

51. $\int \frac{dx}{\sqrt{(x - a)(b - x)}} = 2\sin^{-1}\sqrt{\frac{x - a}{b - a}} + C$

52. $\int f(x)\left[\frac{dg(x)}{dx}\right]dx = f(x)\,g(x) - \int g(x)\left[\frac{df(x)}{dx}\right]dx$

Definite Integrals

1. $\int_0^\infty \left[\frac{a}{a^2 + x^2}\right]dx = \frac{\pi}{2},\ a > 0$

2. $\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}},\ a > 0$, and n is a positive integer

3. $\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}},\ a > 0$

4. $\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a},\ a > 0$

5. $\int_0^\infty \sin^2(nx)\,dx = \int_0^\pi \cos^2(nx)\,dx = \pi/2$, n is an integer

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