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Nonlinear regularized long-wave models with a new integral transformation applied to the fractional derivative with power and Mittag-Leffler kernel

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Abstract

This paper presents a fundamental solution method for nonlinear fractional regularized long-wave (RLW) models. Since analytical methods cannot be applied easily to solve such models, numerical or semianalytical methods have been extensively considered in the literature. In this paper, we suggest a solution method that is coupled with a kind of integral transformation, namely Elzaki transform (ET), and apply it to two different nonlinear regularized long wave equations. They play an important role to describe the propagation of unilateral weakly nonlinear and weakly distributer liquid waves. Therefore, these equations have been noticed by scientists who study waves their movements. Particularly, they have been used to model a large class of physical and engineering phenomena. In this context, this paper takes into consideration an up-to-date method and fractional operators, and aims to obtain satisfactory approximate solutions to nonlinear problems. We present this achievement, firstly, by defining the Elzaki transforms of Atangana–Baleanu fractional derivative (ABFD) and Caputo fractional derivative (CFD) and then applying them to the RLW equations. Finally, numerical outcomes giving us better approximations after only a few iterations can be easily obtained.

Keywords: Atangana–Baleanu fractional derivative; Caputo fractional derivative; Approximate-analytical solution; Nonlinear regularized long wave model; Elzaki transform

1 Introduction

In recent decades, many studies have been performed on modeling with noninteger order calculus. These illustrative studies and developments in applied sciences have found out that fractional calculus has a great importance in mathematical modeling due to the memory effect. Hence, fractional calculus theory and its informative applications are attracting attention all over the world every day. New fractional operators that have different features have been defined and used extensively to model real-life problems. The emergence of the new operators in the literature can be considered as a result of the reproduction of new problems that model different types of real-life events. For this reason,

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approaching real-life problems in terms of their fractional order versions has facilitated modeling and solving them with a proper method. Therefore, this approach has been applied to a very wide area of science, for example, to physical and chemical problems [1–5], in engineering sciences [6–10], to financial instruments [11–14], in geosciences [15], to epidemic models [16–24], in the analysis of biological models [25, 26], etc. However, in recent years some novel fractional derivative operators without singular kernel have been investigated by using the exponential function [27], Mittag-Leffler function [28, 29], generalized Mittag-Leffler [30, 31] function, and normalized sinc function [32]. Especially these fractional operators have been preferred by the researchers who want to model and solve a real-life problem. Since these operators include a non-singular kernel, a problem coupled with them can be resolved easily and accurately. Furthermore, in terms of the integral transforms of these operators, numerical computations can be easily performed. Many researchers have studied these fractional operators, see, e.g., [33–60]. In the literature, there are some integral transform methods that can be applied to the solution of fractional differential equations. In this context, Elzaki transform (ET) is one of the integral transforms [61]. Some important solution methods related to the real-life problems and their numerical simulations obtained via the new integral transformation have been investigated by several researchers [62–66].

In this paper, two different fractional homogeneous nonlinear RLW equations are considered. In the literature there are several special versions of the RLW. Some scientists obtained that the RLW equations are better models than the classical Korteweg–de Vries (KdV) equation [67]. We apply the Elzaki transform coupled with the classical Caputo and ABC operators to two special RLW equations. Then we obtain their approximate solutions and analyze the numerical simulations of the solutions. The nonlinear RLW equations are given by [68, 69]

$${}_0^*D_t^q \phi(x, t) - \phi_{xxt}(x, t) + \left(\frac{\phi^2(x, t)}{2} \right)_x = 0 \quad (1)$$

with the initial condition

$$\phi(x, 0) = x \quad (2)$$

and

$${}_0^*D_t^q \phi(x, t) + \phi_x(x, t) - \phi_{xx}(x, t) + \phi^2(x, t)\phi_x(x, t) + \frac{1}{6} \left[e^{(-2x+4t)} \phi_{xt}(x, t) \right]_x = 0 \quad (3)$$

with the initial condition

$$\phi(x, 0) = \exp(-x), \quad (4)$$

where $(x, t) \in R \times [0, T]$, $0 < q \leq 1$, and ${}_0^*D_t^q$ represents the classical Caputo or Atangana–Baleanu operator of order q .

The reason for dealing with fractional-order systems is the memory and hereditary properties which are complex behavioral patterns of physical systems giving us a more realistic way to model nonlinear regularized long-wave models. In the fractional-order models, the memory property allows for the integration of more information from the

past which predicts and translates into the models more accurately. Also, the hereditary property describes the genetic profile along with age and status of the immune system. Because of such properties, fractional-order calculus have found wide applications in modeling dynamics processes in many well-known fields. On the other hand, the physical structures and illustrative applications of such problems have been extensively considered in the literature. The nonlinear RLW equation plays an important role in the study of nonlinear dispersive waves on account of its description to a wide range of important physical phenomena such as shallow water waves and ion-acoustic plasma. Especially fractional versions of these models have been studied by many researchers [6, 70]. For more details on the physical importance of the RLW equation, see Stoker and Waves [71].

The rest of the paper is organized as follows: In Sect. 2, some preliminary results required for the formulation of the problem are provided. Sect. 3 provides the description of the method via a new integral transformation. Main results, numerical simulations, and graphical representations are presented in Sect. 4. Finally, Sect. 5 concludes all the major findings of the present research study.

2 Preliminaries

In this section, we present some fundamental concepts of fractional derivatives with and without a singular kernel, their Elzaki transform, and fractional integrals.

Definition 1 The Caputo fractional derivative (CFD) is given as [72]:

$${}_0^C \mathcal{D}_t^q(u(t)) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{u^{(m)}(\eta)}{(t-\eta)^{q+1-m}} d\eta, & m-1 < q < m, \\ \frac{d^m}{dt^m} u(t), & q = m. \end{cases} \quad (5)$$

Definition 2 The Atangana–Baleanu fractional derivative in the Caputo mean (ABC) is given as [28]:

$${}_m^{ABC} \mathcal{D}_t^q(u(t)) = \frac{N(q)}{1-q} \int_m^t u'(\eta) E_q \left[-\frac{q(t-\eta)^q}{1-q} \right] d\eta, \quad (6)$$

where $u \in H^1(\alpha, \beta)$, $\beta > \alpha$, $q \in [0, 1]$. In Eq. (6), $N(q)$ represents a normalization function that equals to 1 when $q = 0$ and $q = 1$.

Definition 3 The fractional integral of the ABC operator (Atangana–Baleanu fractional integral) is presented by [28]

$${}_m^{AB} \mathcal{I}_t^q(u(t)) = \frac{1-q}{N(q)} u(t) + \frac{q}{\Gamma(q)N(q)} \int_m^t u(\eta)(t-\eta)^{q-1} d\eta. \quad (7)$$

Definition 4 The Elzaki transform defined for the exponential function is given in the set \mathcal{A} [61, 73] as

$$\mathcal{A} = \{u(t) : \exists Z, p_1, p_2 > 0, |u(t)| < Ze^{\frac{|t|}{p_i}}, \text{ if } t \in (-1)^i \times [0, \infty)\}. \quad (8)$$

For a selected function in the set, Z is a finite number, but p_1, p_2 can be finite or infinite.

Definition 5 The Elzaki transform of a given function $u(t)$ is defined as [73]

$$\mathcal{E}\{u(t)\}(\omega) = \tilde{U}(\omega) = \omega \int_0^\infty e^{-\frac{t}{\omega}} u(t) dt, \quad (9)$$

where $t \geq 0, p_1 \leq \omega \leq p_2$.

Theorem 1 (Convolution theorem for the Elzaki transformation, [74]) *The following equality holds:*

$$\mathcal{E}\{u * v\} = \frac{1}{\omega} \mathcal{E}\{u\} \mathcal{E}\{v\}, \quad (10)$$

where $\mathcal{E}\{\cdot\}$ is the Elzaki transform.

Definition 6 The Elzaki transform of the CFD operator ${}_0^C \mathcal{D}_t^q(u(t))$ is given by [75]

$$\mathcal{E}\left\{{}_0^C \mathcal{D}_t^q(u(t))\right\}(\omega) = \omega^{-q} \tilde{U}(\omega) - \sum_{k=0}^{m-1} \omega^{2-q+k} u^{(k)}(0), \quad (11)$$

where $m-1 < q < m$.

Theorem 2 *The Elzaki transform of the ABC fractional derivative ${}_0^{ABC} \mathcal{D}_t^q(u(t))$ is given by*

$$\mathcal{E}\left\{{}_0^{ABC} \mathcal{D}_t^q(u(t))\right\}(\omega) = \frac{N(q)\omega}{q\omega^q + 1 - q} \left(\frac{\tilde{U}(\omega)}{\omega} - \omega u(0) \right), \quad (12)$$

where $\mathcal{E}\{u(t)\}(\omega) = \tilde{U}(\omega)$.

Proof By Definition 2, we have the following:

$$\mathcal{E}\left\{{}_0^{ABC} \mathcal{D}_t^q(u(t))\right\}(\omega) = \mathcal{E}\left\{\frac{N(q)}{1-q} \int_0^t u'(\eta) E_q\left[-\frac{q(t-\eta)^q}{1-q}\right] d\eta\right\}(\omega). \quad (13)$$

Then, considering the definition of Elzaki transform and its convolution, we have

$$\begin{aligned} \mathcal{E}\left\{{}_0^{ABC} \mathcal{D}_t^q(u(t))\right\}(\omega) &= \mathcal{E}\left\{\frac{N(q)}{1-q} \int_0^t u'(\eta) E_q\left[-\frac{q(t-\eta)^q}{1-q}\right] d\eta\right\} \\ &= \frac{N(q)}{1-q} \frac{1}{\omega} \mathcal{E}\{u'(t)\} \mathcal{E}\left\{E_q\left[-\frac{qt^q}{1-q}\right]\right\} \\ &= \frac{N(q)}{1-q} \left[\frac{\tilde{U}(\omega)}{\omega} - \omega u(0) \right] \left[\int_0^\infty e^{-\frac{t}{\omega}} E_q\left[-\frac{qt^q}{1-q}\right] dt \right] \\ &= \frac{N(q)\omega}{q\omega^q + 1 - q} \left[\frac{\tilde{U}(\omega)}{\omega} - \omega u(0) \right]. \end{aligned} \quad (14)$$

□

Lemma 1 ([69]) *The solution of the specially-defined homogeneous generalized RLW problem (1) with the initial condition (2) is given by $\phi(x, t) = \frac{x}{1+t}$.*

Lemma 2 ([68]) *The solution of the specially-defined homogeneous generalized RLW problem (3) with the initial condition (4) is given by $\phi(x, t) = \exp(-x + 2t)$.*

3 Description of the method via a new integral transformation

In this part of the study, we will present the fundamental methodology which has been used in this study. To investigate this methodology, we take into account the following general form of a fractional nonlinear PDE:

$${}_0^*D_t^q \phi(x, t) + L\langle \phi(x, t) \rangle + N\langle \phi(x, t) \rangle = \theta(x, t), \quad (15)$$

$$(x, t) \in [0, 1] \times [0, T], \quad \kappa - 1 < q \leq \kappa,$$

with initial condition

$$\frac{\partial^z \phi}{\partial t^z}(x, 0) = \mu_z(x), \quad z = 0, 1, \dots, \kappa - 1, \quad (16)$$

and the boundary conditions

$$\phi(0, t) = \gamma_0(t), \quad \phi(1, t) = \gamma_1(t), \quad t \geq 0, \quad (17)$$

where μ_z , θ , γ_0 , and γ_1 are known functions. In Eq. (15), we represent the linear part of the equation with $L\langle \cdot \rangle$, the nonlinear part with $N\langle \cdot \rangle$, and ${}_0^*D_t^q$ denotes the ABC or Caputo fractional derivatives. We characterize the recursive steps for solving the suggested problems (1)–(2) and (3)–(4). Using the Elzaki transform of the CFD in Eq. (11) and ABC in Eq. (12), we consider $\mathcal{E}\{\phi(x, t)\}(\omega) = \tilde{\xi}(x, \omega)$ for Eq. (15). Then we can obtain the transformed functions for the Caputo fractional derivative

$$\tilde{\xi}(x, \omega) = \omega^q (\tilde{\theta}(x, \omega) - \mathcal{E}\{L\langle \phi(x, t) \rangle + N\langle \phi(x, t) \rangle\}) + \omega^2 \phi(x, 0). \quad (18)$$

In addition, we get the transformed functions for the ABC derivative as

$$\tilde{\xi}(x, \omega) = \left(\frac{q\omega^q + 1 - q}{N(q)} \right) [\tilde{\theta}(x, \omega) - \mathcal{E}\{L\langle \phi(x, t) \rangle + N\langle \phi(x, t) \rangle\}] + \omega^2 \phi(x, 0), \quad (19)$$

where $\mathcal{E}\{\theta(x, t)\} = \tilde{\theta}(x, \omega)$. Also considering the Elzaki transforms of the boundary conditions, we get

$$\mathcal{E}\{\gamma_0(t)\} = \tilde{\xi}(0, \omega), \quad \mathcal{E}\{\gamma_1(t)\} = \tilde{\xi}(1, \omega), \quad \omega \geq 0. \quad (20)$$

Then, applying the perturbation method, we achieve the solution of Eqs. (15)–(17) as

$$\tilde{\xi}(x, \omega) = \sum_{\varepsilon=0}^{\infty} \chi^\varepsilon \tilde{\xi}_\varepsilon(x, \omega), \quad \varepsilon = 0, 1, 2, \dots \quad (21)$$

The nonlinear part in Eq. (15) can be computed from

$$N\langle \phi(x, t) \rangle = \sum_{\varepsilon=0}^{\infty} \chi^\varepsilon \Phi_\varepsilon(x, t), \quad (22)$$

and the components $\Phi_\varepsilon(x, t)$ are given in [42] as

$$\Phi_\varepsilon(\phi_0, \phi_1, \dots, \phi_\varepsilon) = \frac{1}{\varepsilon!} \frac{\partial^\varepsilon}{\partial \varpi^\varepsilon} \left[N \left(\sum_{i=0}^{\infty} \varpi^i \phi_i \right) \right]_{\lambda=0}, \quad \varepsilon = 0, 1, 2, \dots \quad (23)$$

Substituting Eqs. (21) and (22) into Eq. (18), we get the solution components for the Caputo operator:

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) = & -\chi \omega^q \left(\mathcal{E} \left\{ L \left\langle \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) \right\rangle + \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} \right) \\ & + \omega^q (\tilde{\theta}(x, \omega)) + \omega^2 \phi(x, 0), \end{aligned} \quad (24)$$

and substituting Eqs. (21) and (22) into Eq. (19), we get the recursive relation which gives the solution for the Atangana–Baleanu operator:

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) = & -\chi \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \left[\mathcal{E} \left\{ L \left\langle \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) \right\rangle + \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} \right] \\ & + \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \tilde{\theta}(x, \omega) + \omega^2 \phi(x, 0). \end{aligned} \quad (25)$$

Then, by solving Eqs. (24) and (25) with respect to χ , we identify the following Caputo homotopies:

$$\begin{aligned} \chi^0 : \tilde{\xi}_0(x, \omega) &= \omega^q (\tilde{\theta}(x, \omega)) + \omega^2 \phi(x, 0), \\ \chi^1 : \tilde{\xi}_1(x, \omega) &= -\omega^q \mathcal{E} \{ L \langle \phi_0(x, t) \rangle + \Phi_0(x, t) \}, \\ \chi^2 : \tilde{\xi}_2(x, \omega) &= -\omega^q \mathcal{E} \{ L \langle \phi_1(x, t) \rangle + \Phi_1(x, t) \}, \\ &\vdots \\ \chi^{n+1} : \tilde{\xi}_{n+1}(x, \omega) &= -\omega^q \mathcal{E} \{ L \langle \phi_n(x, t) \rangle + \Phi_n(x, t) \}. \end{aligned} \quad (26)$$

Moreover, we define the following ABC homotopies:

$$\begin{aligned} \chi^0 : \tilde{\xi}_0(x, \omega) &= \omega^2 \phi(x, 0) + \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \tilde{\theta}(x, \omega), \\ \chi^1 : \tilde{\xi}_1(x, \omega) &= -\left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ L \langle \phi_0(x, t) \rangle + \Phi_0(x, t) \}, \\ \chi^2 : \tilde{\xi}_2(x, \omega) &= -\left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ L \langle \phi_1(x, t) \rangle + \Phi_1(x, t) \}, \\ &\vdots \\ \chi^{n+1} : \tilde{\xi}_{n+1}(x, \omega) &= -\left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ L \langle \phi_n(x, t) \rangle + \Phi_n(x, t) \}, \end{aligned} \quad (27)$$

when $\chi \rightarrow 1$, we obtain that Eqs. (26) and (27) show the approximate solution for problems (24) and (25), thus the solution is given by

$$\Delta_n(x, \omega) = \sum_{\sigma=0}^n \tilde{\xi}_{\sigma}(x, \omega). \quad (28)$$

Applying the inverse ET to Eq. (28), we obtain the approximate solution of Eq. (15),

$$\phi(x, t) \cong \phi_n(x, t) = \mathcal{E}^{-1} \{ \Delta_n(x, \omega) \}. \quad (29)$$

4 Main results and numerical simulations

In this section, we examine the Elzaki transform by considering the problems given in Eqs. (1)–(4). First, we solve problem (1) with the initial condition (2) by using the Elzaki transform method coupled with the Caputo derivative operator. We get by applying the Elzaki transform

$$\tilde{\xi}(x, \omega) = \omega^q \mathcal{E} \left\{ \phi_{xxt}(x, t) - \left(\frac{\phi^2(x, t)}{2} \right)_x \right\} + \omega^2 \phi(x, 0). \quad (30)$$

At this step, we apply the Elzaki perturbation transform method to Eq. (30) and get

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) &= \chi \omega^q \mathcal{E} \left\{ \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right)_{xxt} \right\} \\ &\quad - \chi \omega^q \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} + \omega^2 \phi(x, 0). \end{aligned} \quad (31)$$

Now if we apply the inverse Elzaki transform to Eq. (31), we have

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) &= \chi \mathcal{E}^{-1} \left\{ \omega^q \mathcal{E} \left\{ L \left\langle \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) \right\rangle \right\} \right\} \\ &\quad - \rho \mathcal{E}^{-1} \left\{ \omega^q \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} \right\} + \mathcal{E}^{-1} \{ \omega^2 \phi(x, 0) \}. \end{aligned} \quad (32)$$

In Eq. (32), the $\Phi_{\varepsilon}(\cdot)$ values are functions that show the nonlinear terms given in Eq. (23) and they are examined by this way:

$$\begin{aligned} \Phi_0(\phi) &= \phi_0(\phi_0)_x, \\ \Phi_1(\phi) &= \phi_0(\phi_1)_x + \phi_1(\phi_0)_x, \\ \Phi_2(\phi) &= \phi_0(\phi_2)_x + \phi_1(\phi_1)_x + \phi_2(\phi_0)_x, \\ &\vdots \end{aligned} \quad (33)$$

Then, we have the solution steps for the Caputo operator by considering the corresponding powers of χ :

$$\begin{aligned} \chi^0: \phi_0(x, t) &= \mathcal{E}^{-1} \{ \omega^2 x \} = x, \\ \chi^1: \phi_1(x, t) &= -\mathcal{E}^{-1} \{ \omega^q \mathcal{E} \{ \Phi_0(x, t) \} \} = -\frac{xt^q}{\Gamma(q+1)}, \\ \chi^2: \phi_2(x, t) &= -\mathcal{E}^{-1} \{ \omega^q \mathcal{E} \{ \Phi_1(x, t) \} \} = -\frac{2xt^{2q}}{\Gamma(2q+1)}, \\ \chi^3: \phi_3(x, t) &= -\mathcal{E}^{-1} \{ \omega^q \mathcal{E} \{ \Phi_2(x, t) \} \} = -\frac{6xt^{3q}}{\Gamma(3q+1)}, \end{aligned} \quad (34)$$

$$\vdots$$

$$\chi^{n+1} : \phi_{n+1}(x, t) = -\mathcal{E}^{-1} \left\{ \omega^q \mathcal{E} \left\{ \Phi_n(x, t) \right\} \right\} = (-1)^{n+1} \frac{(n+1)! x t^{(n+1)q}}{\Gamma((n+1)q+1)}.$$

Therefore, the approximate solution of the problem is given by

$$\begin{aligned} \phi(x, t) = x & \left(1 - \frac{t^q}{\Gamma(q+1)} + \frac{2t^{2q}}{\Gamma(2q+1)} - \frac{6t^{3q}}{\Gamma(3q+1)} + \cdots \right. \\ & \left. + (-1)^{n+1} \frac{(n+1)! t^{(n+1)q}}{\Gamma((n+1)q+1)} \right), \end{aligned} \quad (35)$$

giving the integer-order ($q = 1$) solution of the problem, $\phi(x, t) = \frac{x}{1+t}$.

On the other hand, we consider the problem by using the Elzaki transform coupled with the Atangana–Baleanu operator. First of all, we apply the Elzaki transform to the problem:

$$\tilde{\xi}(x, \omega) = \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ \phi_{xxt}(x, t) - \left(\frac{\phi^2(x, t)}{2} \right)_x \right\} + \omega^2 \phi(x, 0). \quad (36)$$

We apply the Elzaki perturbation transform method to Eq. (36) and get

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) = \chi & \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right)_{xxt} \right\} \\ & - \chi \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} + \omega^2 \phi(x, 0). \end{aligned} \quad (37)$$

Now if we take the inverse ET of Eq. (37) and have

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) = \chi \mathcal{E}^{-1} & \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ L \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) \right) \right\} \right\} \\ & - \chi \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} \right\} \\ & + \mathcal{E}^{-1} \{ \omega^2 \phi(x, 0) \}. \end{aligned} \quad (38)$$

In Eq. (38), the $\Phi_{\varepsilon}(\cdot)$ terms are nonlinear polynomials that have been mentioned in Eq. (23). Following the same steps to obtain nonlinear polynomials, we get the following:

$$\begin{aligned} \chi^0 : \phi_0(x, t) &= \mathcal{E}^{-1} \{ \omega^2 x \} = x, \\ \chi^1 : \phi_1(x, t) &= -\mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ \Phi_0(x, t) \} \right\} \\ &= -\frac{x}{N(q)} \left(\frac{qt^q}{\Gamma(q+1)} + 1 - q \right) \\ \chi^2 : \phi_2(x, t) &= -\mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ \Phi_1(x, t) \} \right\} \\ &= \frac{2x}{(N(q))^2} \left(\frac{(qt^q)^2}{\Gamma(2q+1)} + \frac{2q(1-q)t^q}{\Gamma(q+1)} + (1-q)^2 \right), \end{aligned}$$

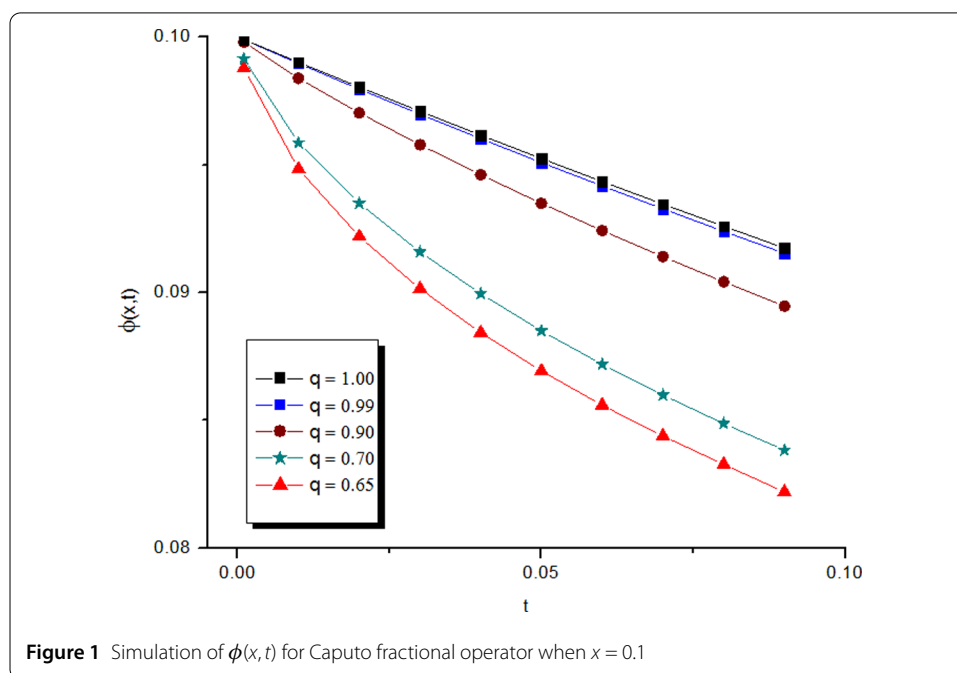
$$\begin{aligned}
\chi^3 : \phi_3(x, t) &= -\mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ \phi_2(x, t) \} \right\} \\
&= -\frac{x}{(N(q))^3} \frac{t^{3q} q^3 (4\Gamma^2(q+1) + \Gamma(2q+1))}{\Gamma^2(q+1)\Gamma(3q+1)} \\
&\quad - \frac{x}{(N(q))^3} \left(\frac{t^{2q} q^2 (1-q)(14\Gamma^2(q+1) + \Gamma(2q+1))}{\Gamma^2(q+1)\Gamma(2q+1)} \right. \\
&\quad \left. + \frac{10q(1-q)^2 t^q}{\Gamma(q+1)} + 5(1-q)^2 \right) \\
&\vdots
\end{aligned}$$

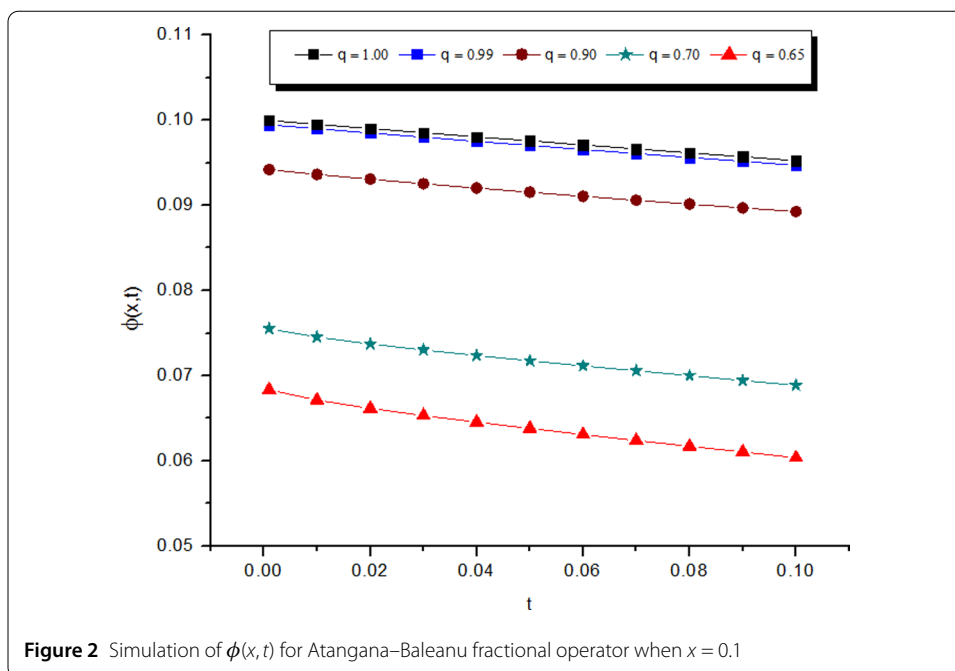
Therefore, the approximate solution depending on the ABC operator is the following:

$$\begin{aligned}
\phi(x, t) &= \sum_{\sigma=0}^n \phi_{\sigma}(x, t) \\
&= x \left(1 - \frac{1}{N(q)} \left(\frac{qt^q}{\Gamma(q+1)} + 1 - q \right) + \frac{2}{(N(q))^2} \left(\frac{(qt^q)^2}{\Gamma(2q+1)} + \frac{2q(1-q)t^q}{\Gamma(q+1)} \right. \right. \\
&\quad \left. \left. + (1-q)^2 \right) \right) - \frac{x}{(N(q))^3} \left(\frac{t^{3q} q^3 (4\Gamma^2(q+1) + \Gamma(2q+1))}{\Gamma^2(q+1)\Gamma(3q+1)} + \frac{10q(1-q)^2 t^q}{\Gamma(q+1)} \right. \\
&\quad \left. + 5(1-q)^2 \right) - \frac{x}{(N(q))^3} \left(\frac{t^{2q} q^2 (1-q)(14\Gamma^2(q+1) + \Gamma(2q+1))}{\Gamma^2(q+1)\Gamma(2q+1)} \right) + \dots,
\end{aligned}$$

giving the integer-order ($q = 1$) solution of the problem, $\phi(x, t) = \frac{x}{1+t}$.

The following Figs. 1 and 2 show the behavior of the solutions for different values of fractional order for Caputo and Atangana–Baleanu operator, respectively. For both Caputo and AB operators, the wave damping has been observed over time. In addition, it is





observed that for the AB operator the wave damping is slower than for the Caputo operator.

Secondly, we obtain the solution of the problem in Eqs. (3)–(4) by using the Elzaki transform method coupled with the Caputo and ABC derivative operators. We apply at first the Elzaki transform of the Caputo derivative to Eqs. (3)–(4):

$$\begin{aligned} \tilde{\xi}(x, \omega) = \omega^q \mathcal{E} \left\{ \phi_{xx}(x, t) - \phi_x(x, t) - \frac{1}{6} [e^{(-2x+4t)} \phi_{xt}(x, t)]_x - \phi^2(x, t) \phi_x(x, t) \right\} \\ + \omega^2 \phi(x, 0). \end{aligned} \quad (39)$$

At this step, we apply the Elzaki perturbation transform method to Eq. (39) and get

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) = \chi \omega^q \mathcal{E} \left\{ \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right)_{xx} - \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right)_x \right\} \\ - \chi \omega^q \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} + \omega^2 \phi(x, 0). \end{aligned} \quad (40)$$

Applying the inverse Elzaki transform to Eq. (40), we have

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) = \chi \mathcal{E}^{-1} \left\{ \omega^q \mathcal{E} \left\{ L \left\langle \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) \right\rangle \right\} \right\} \\ - \chi \mathcal{E}^{-1} \left\{ \omega^q \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} \right\} + \mathcal{E}^{-1} \{ \omega^2 \phi(x, 0) \}. \end{aligned} \quad (41)$$

In Eq. (41), the $\Phi_\varepsilon(\cdot)$ values are functions that show the nonlinear terms given in Eq. (23) and they are examined by the following way for the problem (3):

$$\begin{aligned}\Phi_0(\phi) &= \phi_0^2(\phi_0)_x, \\ \Phi_1(\phi) &= \phi_0^2(\phi_1)_x + 2\phi_0\phi_1(\phi_0)_x, \\ \Phi_2(\phi) &= \phi_0^2(\phi_2)_x + 2\phi_0\phi_1(\phi_1)_x + (\phi_1^2 + 2\phi_0\phi_2)(\phi_0)_x, \\ &\vdots\end{aligned}\quad (42)$$

We obtain the recursive relation for the Caputo operator by considering the corresponding powers of χ :

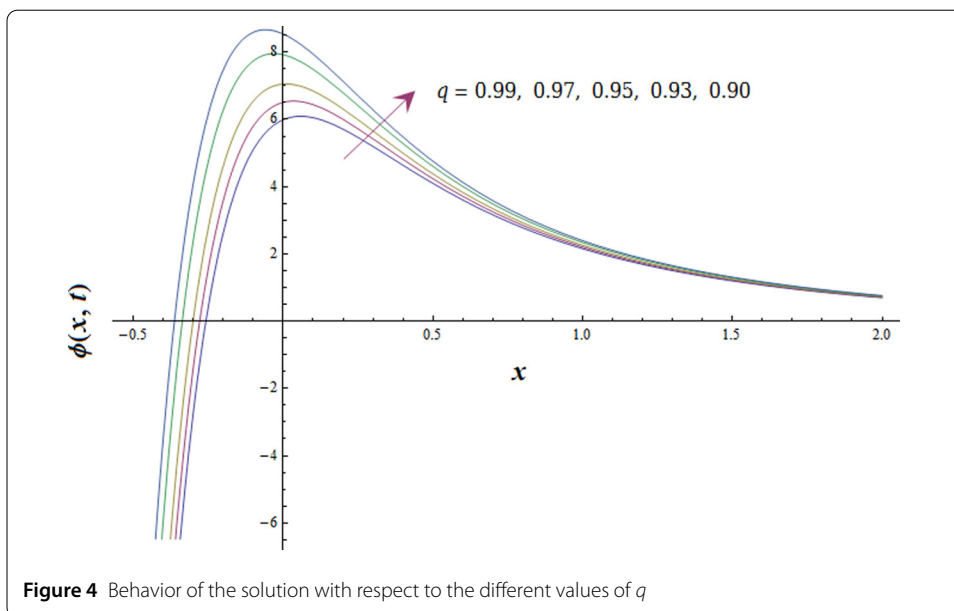
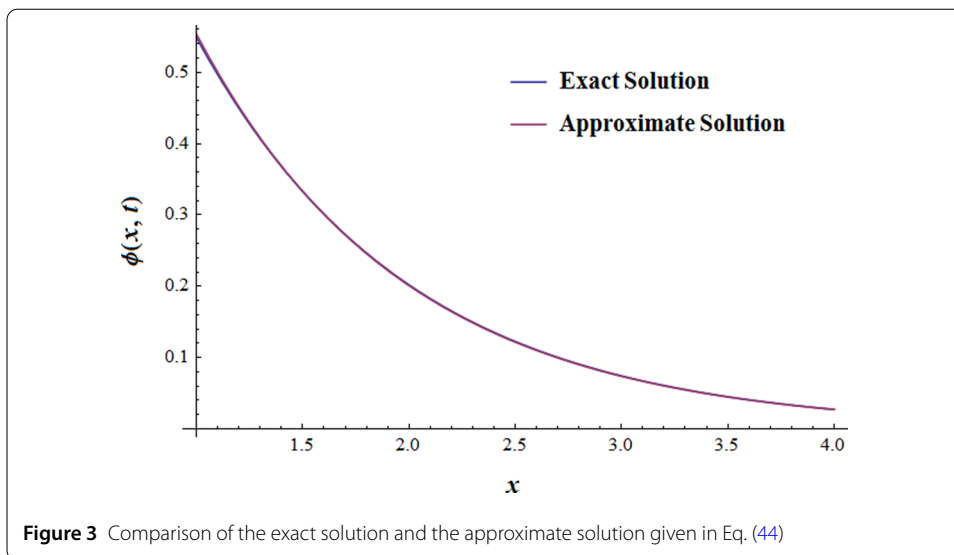
$$\begin{aligned}\chi^0: \phi_0(x, t) &= \mathcal{E}^{-1}\{\omega^2 e^{-x}\} = e^{-x}, \\ \chi^1: \phi_1(x, t) &= \mathcal{E}^{-1}\{\omega^q \mathcal{E}\{L(\phi_0(x, t))\}\} - \mathcal{E}^{-1}\{\omega^q \mathcal{E}\{\Phi_0(x, t)\}\} \\ &= \mathcal{E}^{-1}\{\omega^q \mathcal{E}\{2e^{-x}\}\} - \mathcal{E}^{-1}\{\omega^q \mathcal{E}\{-e^{-3x}\}\} \\ &= (2e^{-x} + e^{-3x}) \frac{t^q}{\Gamma(q+1)}, \\ \chi^2: \phi_2(x, t) &= \mathcal{E}^{-1}\{\omega^q \mathcal{E}\{L(\phi_1(x, t))\}\} - \mathcal{E}^{-1}\{\omega^q \mathcal{E}\{\Phi_1(x, t)\}\} \\ &= \mathcal{E}^{-1}\left\{\omega^q \mathcal{E}\left\{(4e^{-x} + 18e^{-3x} + 5e^{-5x}) \frac{t^q}{\Gamma(q+1)}\right\}\right\} \\ &\quad - \mathcal{E}^{-1}\left\{\omega^q \mathcal{E}\left\{\left(e^{-3x+4t} + \frac{5}{2}e^{-5x+4t}\right) \frac{qt^{q-1}}{\Gamma(q+1)}\right\}\right\} \\ &= (4e^{-x} + 18e^{-3x} + 5e^{-5x}) \frac{t^{2q}}{\Gamma(2q+1)} - \left(e^{-3x} + \frac{5}{2}e^{-5x}\right) \\ &\quad \times \frac{2^{1-2q} e^{2t} \sqrt{\pi} \left(-\frac{1}{t}\right)^{\frac{1}{2}-q} J_{-\frac{1}{2}+q}(-2t)}{\Gamma(q)}, \\ &\vdots\end{aligned}\quad (43)$$

Thus, the approximate solution of the problem is given by

$$\begin{aligned}\phi(x, t) &= e^{-x} + (2e^{-x} + e^{-3x}) \frac{t^q}{\Gamma(q+1)} + (4e^{-x} + 18e^{-3x} + 5e^{-5x}) \frac{t^{2q}}{\Gamma(2q+1)} \\ &\quad - \left(e^{-3x} + \frac{5}{2}e^{-5x}\right) \frac{2^{1-2q} e^{2t} \sqrt{\pi} \left(-\frac{1}{t}\right)^{\frac{1}{2}-q} J_{-\frac{1}{2}+q}(-2t)}{\Gamma(q)} + \dots,\end{aligned}\quad (44)$$

where $J_\alpha(x)$ is the Bessel function of the first kind and Eq. (44) gives the integer-order ($q = 1$) solution of the problem $\phi(x, t) = \exp(-x + 2t)$.

In Fig. 3, it can be observed that only a few components of the series obtained by Elzaki transform method are needed to get close to the exact solution. It has been observed in Fig. 4 that, as the value of the fractional parameter decreases, the wavelength increases.



On the other hand, we consider the problem by using the Elzaki transform coupled with the Atangana–Baleanu operator. First of all, we apply the Elzaki transform to the problem:

$$\begin{aligned}
 \tilde{\xi}(x, \omega) &= \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ \phi_{xx}(x, t) - \phi_x(x, t) \} \\
 &\quad \times \mathcal{E} \left\{ -\frac{1}{6} [e^{(-2x+4t)} \phi_{xt}(x, t)]_x \right\} \\
 &\quad \times \mathcal{E} \{ -\phi^2(x, t) \phi_x(x, t) \} + \omega^2 \phi(x, 0).
 \end{aligned} \tag{45}$$

Then, we apply the Elzaki perturbation transform method to Eq. (45) and get

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) &= \chi \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \\ &\times \left(\mathcal{E} \left\{ \left[\left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right)_{xx} - \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right)_x \right]_x \right\} \right. \\ &\left. - \mathcal{E} \left\{ \left[-\frac{e^{(-2x+4t)}}{6} \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \tilde{\xi}_{\varepsilon}(x, \omega) \right) \right]_x \right\} \right) \\ &- \chi \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} + \omega^2 \phi(x, 0). \end{aligned} \quad (46)$$

If we take the inverse Elzaki transform of the last equation, we have

$$\begin{aligned} \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) &= \chi \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ L \left(\sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \phi_{\varepsilon}(x, t) \right) \right\} \right\} \\ &- \chi \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ \sum_{\varepsilon=0}^{\infty} \chi^{\varepsilon} \Phi_{\varepsilon}(x, t) \right\} \right\} \\ &+ \mathcal{E}^{-1} \{ \omega^2 \phi(x, 0) \}. \end{aligned} \quad (47)$$

In Eq. (47), the $\Phi_{\varepsilon}(\cdot)$ terms are the nonlinear polynomials that have been mentioned in Eq. (23). Following the same steps to obtain nonlinear polynomials, we get

$$\begin{aligned} \chi^0 : \phi_0(x, t) &= \mathcal{E}^{-1} \{ \omega^2 e^{-x} \} = e^{-x}, \\ \chi^1 : \phi_1(x, t) &= \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ (\phi_0)_{xx} - (\phi_0)_x - \frac{1}{6} [e^{(-2x+4t)} (\phi_0)_{xt}]_x \right\} \right\} \\ &- \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ \Phi_0(x, t) \} \right\} \\ &= \frac{e^{-3x} + 2e^{-x}}{N(q)} \left(\frac{t^q}{\Gamma(q)} + 1 - q \right), \\ \chi^2 : \phi_2(x, t) &= \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \left\{ (\phi_1)_{xx} - (\phi_1)_x - \frac{1}{6} [e^{(-2x+4t)} (\phi_1)_{xt}]_x \right\} \right\} \\ &- \mathcal{E}^{-1} \left\{ \left(\frac{q\omega^q + 1 - q}{N(q)} \right) \mathcal{E} \{ \Phi_1(x, t) \} \right\}, \\ &= \frac{4e^{-x} + 5e^{-5x} + 18e^{-3x}}{N^2(q)} \left(\frac{q^2 t^{2q}}{\Gamma(2q+1)} + \frac{(1-q)2qt^q}{\Gamma(q+1)} + (1-q)^2 \right) \\ &- \frac{(15e^{-5x} + 6e^{-3x})}{6\Gamma(q-11)N^2(q)} \times \left(\Gamma(q)(1-q)e^{4t}t^{q-11} \right. \\ &\left. + q \left(2^{1-2q} e^{2t} \sqrt{\pi} \left(-\frac{1}{t} \right)^{\frac{1}{2}-q} J_{-\frac{1}{2}+q}(-2t) \right) \right), \\ &\vdots \end{aligned}$$

Therefore, the approximate solution depending on the ABC operator is the following:

$$\begin{aligned} \phi(x, t) = & e^{-x} + \frac{e^{-3x} + 2e^{-x}}{N(q)} \left(\frac{t^q}{\Gamma(q)} + 1 - q \right) + \frac{4e^{-x} + 5e^{-5x} + 18e^{-3x}}{N^2(q)} \\ & \times \left(\frac{q^2 t^{2q}}{\Gamma(2q+1)} + \frac{(1-q)2qt^q}{\Gamma(q+1)} + (1-q)^2 \right) - \frac{(15e^{-5x} + 6e^{-3x})}{6\Gamma(q-1)N^2(q)} \\ & \times \left(\Gamma(q)(1-q)e^{4t}t^{q-1} + q \left(2^{1-2q}e^{2t}\sqrt{\pi} \left(-\frac{1}{t} \right)^{\frac{1}{2}-q} J_{-\frac{1}{2}+q}(-2t) \right) \right) + \cdots, \quad (48) \end{aligned}$$

giving the integer-order ($q = 1$) solution of the problem, $\phi(x, t) = \exp(2t - x)$.

In Fig. 5, we have given the comparison of the exact solution and the approximate solution given in Eq. (48) and in Fig. 6, it can be seen the solutions to the problem which is given by Eq. (3) with respect to the different values of fractional parameter in the sense of Atangana–Baleanu operator. By taking account of the findings of the paper, we can observe that only a few components of the series obtained by the perturbation method coupled with the Elzaki transform provide almost the exact solution. Moreover, this study differs from the others on the nonlinear RLW equation in that it has pointed out the difference in the behaviors of two fractional derivative operators and it has employed the Elzaki transform of the AB operator. On the other hand, the scheme that has been defined in the second section identifies the components of the series solution. It is possible to calculate more components in the scheme to increase the approximation accuracy. Numerical results show how a high degree of accuracy, and in most cases the n -term approximation $\phi(x, t)$ is accurate already for pretty small values of n . In this context, we have used only the first three components $\phi(x, t) = \phi_0(x, t) + \phi_1(x, t) + \phi_2(x, t)$ to approximate the exact solution and to generate the surfaces shown in all figures in this paper. It can be regarded as a major advantage of the solution method to obtain the solutions even with fewer terms. Another advantage of the method is in simplifying the calculations by avoiding the difficulties and massive computational work compared with traditional numerical methods, because the homotopy Elzaki transform method (HETM) appears to be very promising

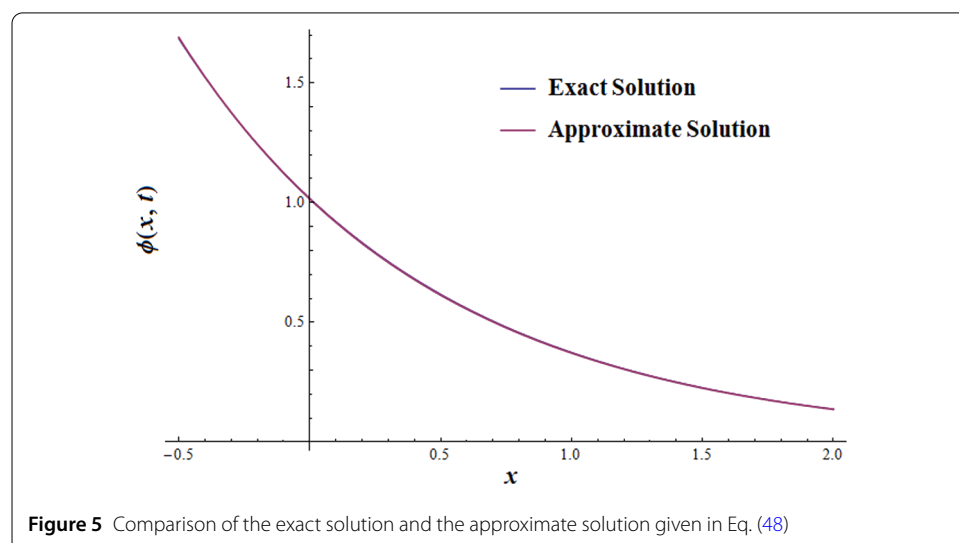
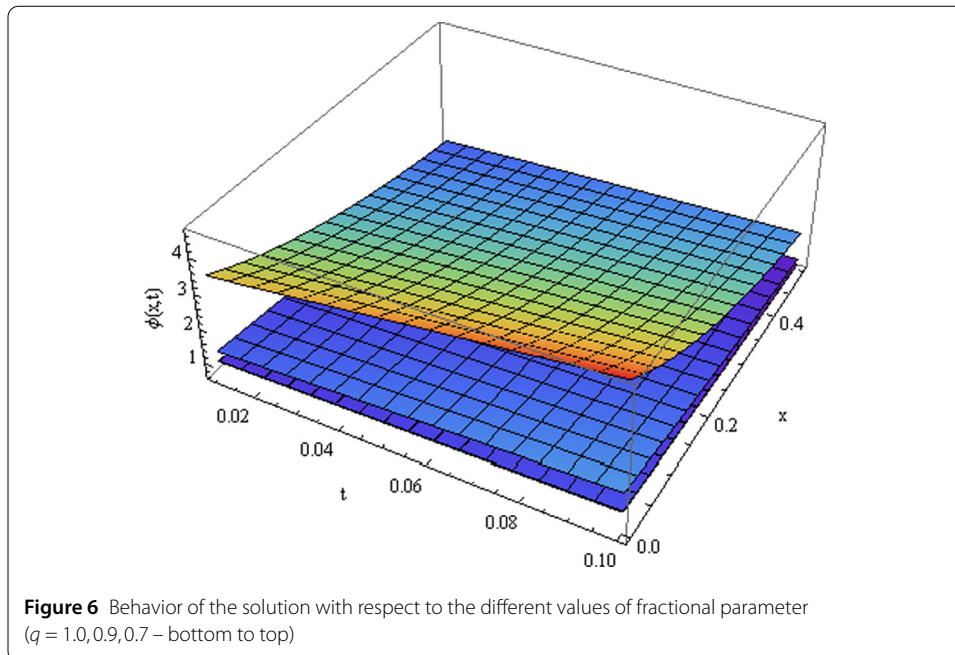


Figure 5 Comparison of the exact solution and the approximate solution given in Eq. (48)



for solving nonlinear partial differential equations without linearization, perturbation, or discretization.

5 Conclusion

The present work computes the approximate solutions of some special regularized long-wave equations of fractional order by utilizing a new integral transform technique, namely Elzaki transformation. Firstly, we have defined the Elzaki transformation of the Atangana–Baleanu fractional operator and have applied it to the suggested problems. The reliability and effectiveness of the employed scheme lie in the fact that it has a strong ability to provide a suitable convergence region for the solution. The high accuracy of results and simple solution procedure establish the dominance of this computational scheme over other existing numerical techniques. In addition, we have demonstrated the differences between the Caputo and Atangana–Baleanu fractional operator in finding the approximate solutions of the mentioned illustrative problems. The numerical outcomes reveal that the fractional derivative operators used in this study are very useful for modeling real-life problems and they have great advantages when considering their Elzaki transform to interpret some illustrative physical problems. Especially, the Atangana–Baleanu fractional operator has some additional advantages due to its nonsingular and nonlocal construction. It is pointed out by some researchers that Mittag-Leffler function is more effective in modeling the physical and engineering problems than the power function, and, since the AB operator has a nonlocal kernel, it provides better explanation of the memory within structure and media with distinct scales. As a different point, it can be stated that the utilized scheme for approximate solution is highly efficient and useful to handle many nonlinear equations describing real systems.

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Authors' contributions

All authors contributed equally to this article. All authors read and approved the final manuscript.

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