Noise Analysis of Phase-Locked Loops

Amit Mehrotra

***Abstract*—This work addresses the problem of noise analysis of phase-locked loops (PLLs). The problem is formulated as a stochastic differential equation and is solved in the presence of circuit white noise sources yielding the spectrum of the PLL output. Specifically, the effect of loop filter characteristics, phase-frequency detector, and phase noise of the open-loop voltage-controlled oscillator (VCO) on the PLL output spectrum is quantified. These results are derived using a full nonlinear analysis of the VCO in the feedback loop and cannot be predicted using traditional linear analyses or the phase noise analysis of open-loop oscillators. The computed spectrum matches well with measured results; specifically, the shape of the output spectrum matches very well with measured PLL output spectra reported in the literature for different kinds of loop filters and phase detectors. The PLL output spectrum computation only requires the phase noise of the VCO, loop filter and phase detector noise, phase detector gain, and loop filter transfer function and does not require the transient simulation of the entire PLL which can be very expensive. The noise analysis technique is illustrated with some examples.**

***Index Terms*—Noise analysis, phase-locked loops, stochastic differential equations.**

I. Introduction

PHASE-LOCKED loops (PLLs) and delay-locked loops (DLLs) are extensively used in microprocessors and digital signal processors for clock generation and as frequency synthesizers in RF communication systems for clock extraction and generation of a low-phase-noise local oscillator signal from an on-chip voltage-controlled oscillator (VCO) which might have a higher open-loop noise performance. The basic block diagram of a PLL is shown in Fig. 1. The phase of a local VCO signal is compared with the phase of a (hopefully) low-noise reference signal and the difference of the two phases is low-pass filtered and applied to the controlling node of the VCO. If the input signal frequency is within the VCO tuning range, the VCO output is also "locked" to the same frequency as the input signal and the phase difference between the two signals is very small. In RF communication systems, frequency synthesizer noise directly degrades the overall noise performance of the system. Similarly, timing jitter in PLLS of high-performance processors degrades the timing margins of the overall design. Hence, the accurate prediction of PLL noise performance is critical for the design of these systems.

Noise generation mechanisms for PLLs and DLLs are very different. In a DLL, the voltage noise from each of the delay stages accumulates for one period of the input reference signal and then the output phase is aligned with the input signal phase. On the other hand, in a PLL, a VCO is present in the feedback loop and the difference in the phase of the reference and the VCO signal is filtered and used as the control signal of the VCO. Therefore, one of the noise sources of the VCO is the difference of the phase noise of the reference signal and the VCO output along with the noise in the loop filter, phase detector, and frequency dividers. This paper addresses the problem of noise analysis of PLLs.

Manuscript received April 30, 2001; revised May 3, 2002. This paper was recommended by Associate Editor M. Gilli.

The author is with the Computer and Systems Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: amehrotr@uiuc.edu).

Publisher Item Identifier 10.1109/TCSI.2002.802347.

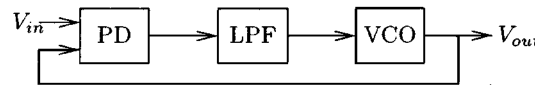


Fig. 1. PLL block diagram.

The starting point of this work is [1]–[3] where noise analysis of open-loop oscillators based on a novel perturbation analysis of an oscillatory system of equations was presented. However, the PLL is a phase feedback system and special techniques are required for solving the associated system of equations. In this work, a system of stochastic differential equations governing the behavior of the PLL VCO phase are developed. The PLL is assumed to be locked to a reference periodic signal which is assumed to have Brownian motion phase deviation. It is shown in Section III that the PLL output phase, in locked condition, is a sum of two stochastic processes: the Brownian motion phase deviation of the reference signal and one component of an appropriate multidimensional Ornstein–Uhlenbeck process. Similar to [2] and [3], it is shown that the PLL output is asymptotically wide-sense stationary. Using the statistics of the phase deviation process, a general expression for the power spectral density (PSD) of the PLL output is obtained. This expression is used to derive the PSD of the PLL output for some specific loop filter configurations in Section IV. From the output spectrum, it can be observed that that the PLL output PSD closely follows the reference signal spectrum for very small offset frequencies and follows the open-loop VCO output spectrum for high offset frequencies. This fact has been experimentally observed and widely reported in the literature. Finally, experimental results on an example circuit are presented in Section V.

II. Previous Work

Noise analysis of PLLs is probably the least understood topic in RF noise analysis. Some existing works present an intuitive explanation of how various noise sources affect the overall noise of a PLL [4]–[8]. Techniques borrowed from linear noise analysis are used to predict the output phase noise spectrum and the relative importance of VCO phase noise, reference signal phase noise, and noise in the phase detector and loop filter as a function of the loop bandwidth [6], [8]–[10]. Output characteristics of sampling PLLs has also been analyzed in the presence of white noise at the PLL input [11], [12]. However, the PLL circuit noise and reference signal phase noise have not been considered. In all these approaches, a precise mathematical characterization of the noisy PLL output and quantification of how much each noise sources contribute to the PLL output noise is not present. Moreover, it has been shown [1]–[3] that noise generation in an oscillator is inherently a nonlinear phenomenon and linear noise analysis techniques for circuits containing VCOs are not rigorously justified. Also, the VCO output itself is a stochastic process and VCO phase noise cannot be viewed as an additive noise source. Similarly, translation of random phase deviations in the VCO phase to the VCO output PSD is a nonlinear phenomenon and the use of linear analysis based techniques for this purpose is also not justified. Behavioral level noise analysis techniques for PLLs have also been reported [13], [14] which can also include power supply noise [15], [16]. However, numerical integration involved in such methods can be expensive. Also, the oscillator phase noise models used in these works [15] is not rigorously justified. The VCO modeled in these works is a ring oscillator and it is not clear how to extend this approach to PLLs with other kinds of VCOs such as one based on an LC tank.

III. PLL Noise Analysis

Let the input reference signal of the PLL in Fig. 1 be periodic with period T, i.e., of angular frequency $\omega_0 = 2\pi/T$. Since this reference is also generated by a real oscillator, it also has Brownian motion phase error $\alpha_{\text{in}}(t)$[1], i.e., the reference signal is of the form $x_{\text{in}}(t+\alpha_{\text{in}}(t))$ where $\alpha_{\text{in}}(t)=\sqrt{c_{\text{in}}}B_{\text{in}}$. $B_{\text{in}}(t)$ is a one-dimensional (1-D) Brownian motion process. Note that, throughout this discussion, "phase" has units of time. Phase in radians can be recovered by multiplying $\alpha(t)$ by the appropriate angular frequency ω. The advantage of using this formulation is that the process of zero-delay frequency division of either the reference signal or the VCO output *does not* affect the analysis presented below (except for adding more noise).

PLL noise analysis presented in this paper utilizes the results presented in [1]–[3] for noise in open-loop VCOs which are summarized below.

- The phase deviation of the open-loop VCO output is governed by the following stochastic differential equation:

$$\frac{d\alpha_{\text{open loop vco}}}{dt} = v^T(t+\alpha_{\text{open loop vco}}(t))\xi_p(t) \quad (1)$$

  where $\xi_p(t) \in \mathbb{R}^p$ is a vector of p uncorrelated VCO white noise sources and $v(\cdot) \in \mathbb{R}^p = [v_1(\cdot) \quad v_2(\cdot) \quad \dots \quad v_p(\cdot)]$ is a periodic function which depends on the noise source intensities and the response of the linearized oscillator circuit [1].
- It was shown that asymptotically $\alpha_{\text{open loop vco}}(t)$ becomes a Brownian motion process whose variance increases linearly with time at a rate c, i.e., $\alpha_{\text{open loop vco}} = \sqrt{c}B(t)$, where c is the time average of the inner product of the vector $v(\cdot)$, i.e.

$$c = \frac{1}{T}\int_0^T v^T(t)v(t)dt.$$

- The noisy oscillator output was shown to be of the form $x_s(t+\alpha_{\text{open loop vco}}(t))$ where $x_s(t)$ is the noiseless periodic steady-state response of the oscillator.

In a PLL, the difference of the reference and the VCO phase is filtered and applied to the VCO control node. Hence, (1) is modified as follows:

$$\frac{d\alpha_{\text{vco}}}{dt} = v^T(t+\alpha_{\text{vco}}(t))\xi_p(t) + v_{\text{control}}(t+\alpha_{\text{vco}}(t))\gamma(t). \quad (2)$$

Here $\gamma(t)$ is the VCO input and $v_{\text{control}}(\cdot) \in \mathbb{R}$ is the component of $v(\cdot)$ which corresponds to a unit noise source present at the control node of the VCO. The form of (2) is valid only if the variance of $\gamma(t)$ is bounded for all t. If this is not the case, perturbation analysis on which (1) and (2) are based becomes invalid. Note that the input in (2) is a stochastic process and therefore this equation is a stochastic differential equation and techniques from stochastic calculus need to be used to solve this equation [17], [18].

Now the important assumption that the PLL is in lock with the reference signal is introduced. By this, it is implied that the VCO output is *locked* to the same frequency as the reference signal.[1] Define $\beta(t)$ as follows:

$$\beta(t) = \alpha_{\text{vco}}(t) - \alpha_{\text{in}}(t). \quad (3)$$

Further, it is also assumed that $\beta(t)$ has *bounded* variance for all t. This assumption implies that the uncertainty in the phase of the PLL output grows only as fast as the uncertainty in the reference signal phase. If this is not the case, the PLL goes out of lock once the phase difference exceeds a certain critical value. Noise analysis of PLLs seems to be of little use when the VCO is not locked to the reference.

It should be mentioned that there exists a nonzero probability with which the PLL goes *out* of lock, even in the presence of small noise [19], [20]. This probability and the associated *first exit time* out of the *basin of attraction* (i.e., locked state) can also be calculated for these systems using large deviation techniques [20]. However, this probability is very small and this case will not be considered here.

Since $\gamma(t)$ is a filtered version of $\beta(t)$, $\beta(t)$ and $\gamma(t)$ are related by the following differential equation:

$$G\frac{dx}{dt} = Ex + F\xi_q(t) \quad (4)$$

where $x = [\beta(t) \quad \gamma(t) \quad \dots]^T$ is a vector of state variables, $x \in \mathbb{R}^n$, $n = 1 + o_{lpf}$, $o_{lpf} > 0$ is the order of the low-pass filter,[2] $G, E \in \mathbb{R}^{n-1\times n}$, and $F \in \mathbb{R}^{n-1\times q}$, q is the number of noise sources in the low-pass filter and the phase detector. Note that the coefficient matrices G, E, and F are *independent* of time. This assumes that the reference signal frequency is not drifting with time and the VCO remains locked to the reference. This implies that the variations around the VCO control voltage are very small. This also follows from the assumption that in the locked state $\gamma(t)$ has bounded variance. If the filter transfer function is stable, the bounded variance of $\gamma(t)$ also implied a bounded variance of $\beta(t)$.

PLL noise analysis now proceeds as follows.

1) Equations (2)–(4) are solved using stochastic differential equation techniques and an expression of $\beta(t)$ is obtained.

[1]Or a multiple thereof if frequency dividers are used.

[2]The formulation in (4) may not be valid when the loop filter is not present. This case is discussed separately in Section IV.

$\gamma(t)$ and other components of x are not required for the output spectrum calculation and need not be computed separately.

2) Since $\alpha_{\rm vco}(t)$ is a stochastic process, PLL VCO output $x_s(t+\alpha_{\rm vco}(t))$ is also a stochastic process. Using the expression of $\beta(t)$ obtained in step 1), the following autocorrelation can be computed:

$$R_{x_s,x_s}(t,\tau) = \mathbb{E}\left[x_s(t+\alpha_{\rm vco}(t))x_s^*(t+\tau+\alpha_{\rm vco}(t+\tau))\right]$$

where $\mathbb{E}[\cdot]$ represents the expectation operator.

3) It can be shown that the asymptotically $R_{x_s,x_s}(t,\tau)$ is independent of t, i.e., the PLL VCO output is a wide-sense stationary stochastic process. The PSD of this output is computed using the stationary autocorrelation function computed in step 2).

A. Solution of the PLL Phase Equation

Using the fact that $\alpha_{\rm in}(t)$ is a scaled Brownian motion process, (2) can be rewritten as

$$\frac{d\beta}{dt} = v^T(t+\alpha_{\rm in}(t)+\beta(t))\xi_p(t) + v_{\rm control}(t+\alpha_{\rm in}(t)+\beta(t))\gamma(t) - \sqrt{c_{\rm in}}\xi_{\rm in}(t) \quad (5)$$

where $\xi_{\rm in}(t)$ is the white noise process which is the time derivative of $B_{\rm in}(t)$. If only the asymptotic behavior of $\beta(t)$ is of interest, (5) can be simplified using the averaging principle for stochastic differential equations [21]. According to this principle, since $\alpha_{\rm in}(t)$ is a scaled Brownian motion process, i.e., its variance grows unbounded with time, $v(\cdot)$ is periodic in its argument and $\beta(t)$ is assumed to have finite variance for all t, the asymptotic behavior of $\beta(t)$ is governed by the following differential equation:[3]

$$\frac{d\beta}{dt} = C_{\rm vco}^T\xi_p(t) + \sqrt{c_{\rm control}}\gamma(t) - \sqrt{c_{\rm in}}\xi_{\rm in}(t) \quad (6)$$

where

$$C_{\rm vco}^T = \begin{bmatrix}\sqrt{c_1} & \sqrt{c_2} & \dots & \sqrt{c_p}\end{bmatrix}^T$$
$$c_i = \frac{1}{T}\int_0^T v_i^2(t)dt$$

and

$$c_{\rm control} = \frac{1}{T}\int_0^T v_{\rm control}^2(t)dt.$$

Equations (4) and (6) can be combined to obtain a linear differential equation of the form

$$\dot{x} = -Ax + D\xi_{p+q+1}$$

for appropriate $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{n\times(p+q+1)}$ where $\xi_{p+q+1}(t) = \begin{bmatrix}\xi_p(t) & \xi_q(t) & \xi_{\rm in}(t)\end{bmatrix}^T$ is a vector of $(p+q+1)$ uncorrelated white noise processes. This equation can be written in stochastic differential equation form as

$$dx = -Axdt + DdB_{p+q+1}(t) \quad (7)$$

where $B_{p+q+1}(t)$ is a $(p+q+1)$-dimensional Brownian motion process.

[3]That is, the trajectory of $\beta(t)$ in (5) converges to the solution of (6) with probability 1.

Equation (7) is linear in x with constant coefficients. Hence traditional linear noise analysis techniques can be used to find the spectrum of $\beta(t)$. However, what is actually needed are the second-order statistics of $\beta(t)$ which can be used to compute the autocorrelation function of the VCO output.

Equation (7) is known as an n-dimensional Ornstein–Uhlenbeck process [18] and its variance is bounded if the real parts of the eigenvalues of A are positive. Similar to the ordinary differential equation case, the solution of (7) can be written as [17]

$$x(t) = DB_{p+q+1}(t) - \int_0^t A\exp(A(s-t))DdB_{p+q+1}(s).$$

It can be shown that [18]

$$\mathbb{E}\left[x(t_1)x^T(t_2)\right] = \int_0^{\min(t_1,t_2)} \exp(A(s-t_1))DD^T \times \exp\left[A^T(s-t_2)\right]ds.$$

It therefore follows that

$$\mathbb{E}\left[\beta(t_1)\beta(t_2)\right] = \int_0^{\min(t_1,t_2)} e\exp(A(s-t_1))DD^T \times \exp\left[A^T(s-t_2)\right]ds\, e^T \quad (8)$$

where $e = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$.

Similarly, it can be shown that[4]

$$\mathbb{E}\left[\beta(t_1)\alpha_{\rm in}(t_2)\right] = e\sqrt{c_{\rm in}}A^{-1}\exp(A\min(0,t_2-t_1))Df \quad (9)$$

where $f = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}^T$. Let A be diagonalized as

$$A = W\Lambda W^{-1}$$

where $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ is a diagonal matrix of eigenvalues of A and W is a matrix of the corresponding eigenvectors. Then

$$\mathbb{E}\left[\beta(t_1)\alpha_{\rm in}(t_2)\right] = \sum_{i=1}^n \mu_i\exp(-\lambda_i\min(0,t_2-t_1)) \quad (10)$$

for appropriate value of μ_i (see the Appendix).

Also, asymptotically, we have

$$\mathbb{E}\left[\beta(t_1)\beta(t_2)\right] = \sum_{i=1}^n \nu_i\exp(-\lambda_i|t_1-t_2|) \quad (11)$$

for appropriate values of ν_i (see the Appendix).

B. PLL Output Spectrum

The expectations in Section III-A can be used to obtain the autocorrelation function and the PSD of the PLL output. Recall that the VCO output in the presence of phase deviation $\alpha_{\rm vco}(t)$ is given by $x_s(t+\alpha_{\rm vco}(t))$ where $x_s(\cdot)$ is the T-periodic noiseless output of the VCO which is locked to the reference signal. Since $\alpha_{\rm vco}(t)$ is a stochastic process, $x_s(t+\alpha_{\rm vco}(t))$ is also a stochastic process. Since $x_s(t)$ is T-periodic, it can be expanded in a Fourier series as

$$x_s(t) = \sum_{i=-\infty}^{\infty} X_i\exp(\jmath i\omega_0 t).$$

[4]There is also an $\exp(-At_1)$ term in this expression which vanishes asymptotically if A has eigenvalues with positive real parts.

The autocorrelation function of the VCO output can now be computed as follows:

$$R_{x_s,x_s}(t,\tau) = \sum_{i,k=-\infty}^{\infty} X_i X_k^* \exp(\jmath(i-k)\omega_0 t)\exp(-\jmath k\omega_0\tau) \times \mathbb{E}\left[\exp\big(\jmath\omega_0(i\alpha_{\text{vco}}(t) - k\alpha_{\text{vco}}(t+\tau))\big)\right].$$

Here, X_k^* is the complex conjugate of X_k. It can be shown that $\alpha_{\text{vco}}(t)$ is asymptotically a zero mean Gaussian process and therefore

$$\mathbb{E}\left[\exp\left(\jmath\omega_0\left(i\alpha_{\text{vco}}(t) - k\alpha_{\text{vco}}(t+\tau)\right)\right)\right] = \exp\left(-\frac{1}{2}\omega_0^2\sigma^2(t,\tau)\right)$$

where $\sigma^2(t,\tau) = \mathbb{E}\left[[i\alpha_{\text{vco}}(t) - k\alpha_{\text{vco}}(t+\tau)]^2\right]$. Using (10) and (11), $\sigma^2(t,\tau)$ can be evaluated as

$$\begin{aligned}\sigma^2(t,\tau) =& (i-k)^2 c_{\text{in}} t + k^2 c_{\text{in}}\tau - 2ikc_{\text{in}}\min(0,\tau) \\ & - 2ik\sum_{l=1}^{n}\mu_l - 2ik\sum_{l=1}^{n}(\mu_l+\nu_l)\exp(-\lambda_l|\tau|) \\ & + (i^2+k^2)\sum_{l=1}^{n}(\nu_l + 2\mu_l).\end{aligned}$$

Substituting the above expression of $\sigma^2(t,\tau)$ in the autocorrelation expression, note that $R_{x_s,x_s}(t,\tau)$ vanishes asymptotically for $i \neq k$, since $\exp(-0.5(i-k)^2\omega_0^2 c_{\text{in}} t)$ drops to zero asymptotically. Hence, only terms corresponding to $i = k$ survive. Therefore,

$$\begin{aligned}R_{x_s,x_s}(t,\tau) = & \sum_{i=-\infty}^{\infty} X_i X_i^* \exp(-\jmath i\omega_0\tau) \\ & \times \exp\left[-\frac{1}{2}\omega_0^2 i^2\left[c_{\text{in}}|\tau| + 2\sum_{l=1}^{n}(\nu_l+\mu_l) \times \left[1-\exp(-\lambda_l|\tau|)\right]\right]\right].\end{aligned} \quad (12)$$

Note that the autocorrelation function of the output is asymptotically independent of t, i.e., the PLL output is wide-sense stationary. A similar observation was made in [1]–[3] for open-loop oscillators. The Fourier transform of (12), which is the PSD of the output, is given by

$$\begin{aligned}S_{x_s,x_s}(\omega) =& \sum_{i=-\infty}^{\infty}\sum_{k_1,\ldots,k_n=0}^{\infty} 2X_i X_i^* \exp\left[-\omega_0^2 i^2\sum_{l=1}^{n}(\mu_l+\nu_l)\right] \\ & \times \frac{\left[\prod_{l=1}^{n}\left[i^2\omega_0^2(\mu_l+\nu_l)\right]^{k_l}\right]\left(\frac{1}{2}\omega_0^2 i^2 c_{\text{in}} + \sum_{l=1}^{n}k_l\lambda_l\right)}{\left[\prod_{l=1}^{n}k_l!\right]\left[\left(\frac{1}{2}\omega_0^2 i^2 c_{\text{in}} + \sum_{l=1}^{n}k_l\lambda_l\right)^2 + (\omega + i\omega_0)^2\right]}.\end{aligned} \quad (13)$$

In practice, one is usually interested in the the PSD around the first harmonic which is defined (in dBc/Hz) as

$$10\log_{10}\left(\frac{S_{x_s,x_s}(\omega-\omega_o)}{|X_1|^2}\right).$$

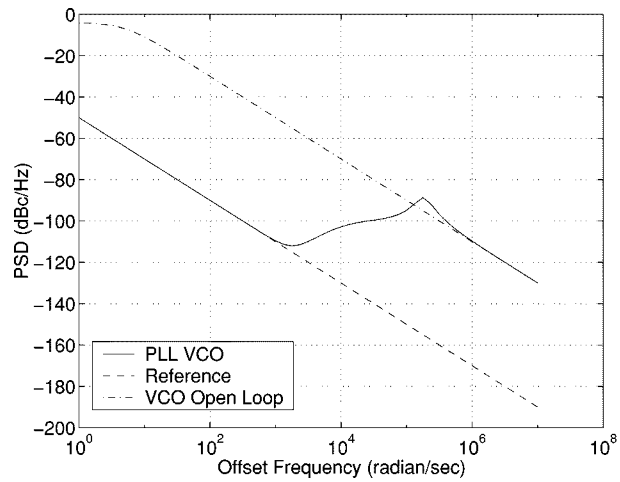


Fig. 2. PSD of the PLL output with no loop filter.

IV. PLL Examples

While (13) is valid for any loop filter transfer function, it offers little insight into the actual nature of the output spectrum. In this section, four specific examples of loop filters will be presented and their corresponding PLL output spectrum will be computed. Even for these simple examples, the computed PLL output PSD is remarkably similar in shape to measured results. For simplicity, the phase detector and loop filter will be assumed to be noiseless. A circuit-level example will be presented in Section V.

A. PLL Without Loop Filter

First consider the simplest of PLLs, one without a loop filter. In this case, $\gamma(t) = -k_{pd}\beta(t)$ where k_{pd} is the phase detector gain. Hence, (6) becomes

$$\frac{d\beta}{dt} = -\sqrt{c_{\text{pll}}}\beta + C_{\text{vco}}^T\xi_p(t)$$

where $\sqrt{c_{\text{pll}}} = k_{pd}\sqrt{c_{\text{control}}}$. Also let $c_{\text{vco}} = \sum_i c_i$. Therefore, $\lambda_1 = \sqrt{c_{\text{pll}}}$. Also $D = [C_{\text{vco}}^T \quad -\sqrt{c_{\text{in}}}]$. For this example, it can be shown that $\nu_1 = DD^T/\sqrt{c_{\text{pll}}} = (c_{\text{in}} + c_{\text{vco}})/\sqrt{c_{\text{pll}}}$ and $\mu_1 = -(c_{\text{in}}/\sqrt{c_{\text{pll}}})$. The resulting output spectrum around the first harmonic is shown in Fig. 2 for $\omega_0 = 10^{10}$ rad/s, $c_{\text{in}} = 10^{-25}$ sec, $c_{\text{vco}} = 10^{-19}$ s, and $c_{\text{pll}} = 10^{11}$ 1/s^2. This corresponds to a phase noise performance of −130 dBc/Hz at 10^4 rad/s offset for the reference signal, −70 dBc/Hz dBc/Hz for the open-loop VCO and −97 dBc/Hz for the PLL. Also shown in the figure are PSDs of the reference input signal and the open-loop VCO output. Note that the PLL output spectrum follows that reference input signal spectrum for low offset frequencies and open-loop VCO spectrum for large offset frequencies. In between, the output spectrum is almost constant. Note that the offset frequency beyond which the PLL output spectrum follows the open-loop VCO spectrum is approximately $\sqrt{c_{\text{pll}}}$, i.e., the bandwidth of the PLL. Also note that, at high offset frequencies, the PLL output PSD is slightly higher than the open-loop VCO spectral density. This is because there is no loop filter present in the circuit to remove the high frequency noise component of the phase noise of the reference signal. In the next few examples, where a loop filter is included, the PLL

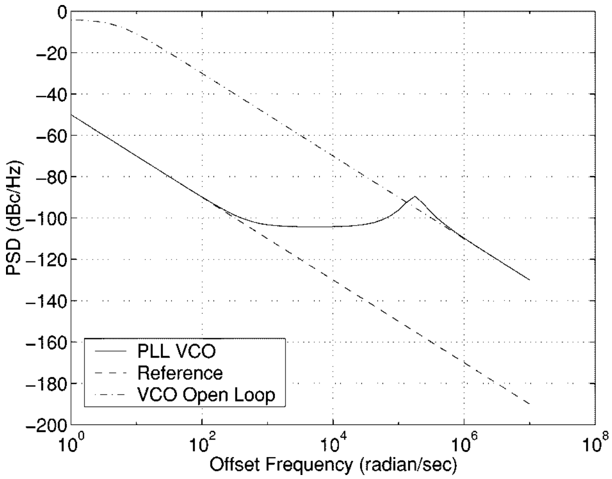


Fig. 3. PSD of a PLL output with a first-order filter.

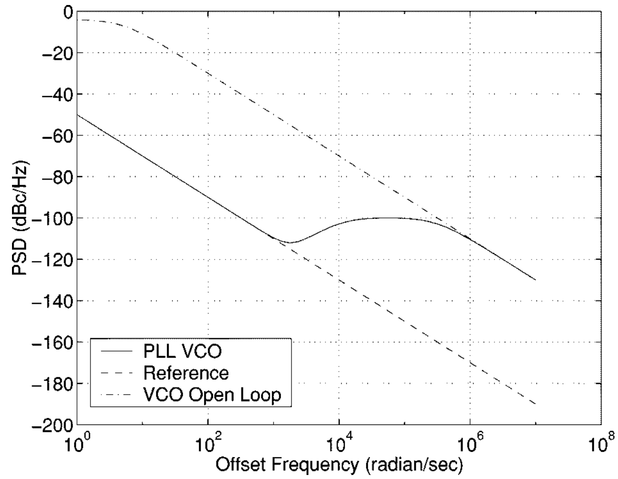


Fig. 4. CPPLL spectrum.

output coincides with the open-loop VCO output for high offset frequencies.

B. PLL With a First-Order Filter

For this case, $\beta(t)$ and $\gamma(t)$ are related by the following equation:

$$\frac{1}{\omega_{lpf}}\frac{d\gamma}{dt}+\gamma(t)=-k_{pd}\beta(t)$$

where ω_{lpf} is the corner frequency of the low-pass filter. Equation (6) can therefore be written as

$$\frac{d}{dt}\begin{bmatrix}\beta\\ \overline{\gamma}\end{bmatrix}=-\begin{bmatrix}0 & -\sqrt{c_{\rm pll}}\\ \omega_{lpf} & \omega_{lpf}\end{bmatrix}\begin{bmatrix}\beta\\ \overline{\gamma}\end{bmatrix}+\begin{bmatrix}C_{\rm vco}^T & -\sqrt{c_{\rm in}}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi_p(t)\\ \xi_{\rm in}(t)\end{bmatrix}$$

where $\overline{\gamma}(t)=\gamma(t)/k_{pd}$ and,[5] as before, $\sqrt{c_{\rm pll}}=k_{pd}\sqrt{c_{\rm control}}$. The eigenvalues of the A matrix are given by

$$\lambda_{1,2}=\frac{\omega_{lpf}\pm\sqrt{\omega_{lpf}^2-4\omega_{lpf}\sqrt{c_{\rm pll}}}}{2}.$$

For this PLL, it can be shown that

$$\begin{aligned}\mu_1=&\frac{c_{\rm in}\lambda_2}{(\lambda_1-\lambda_2)\lambda_1}\\ \mu_2=&\frac{c_{\rm in}\lambda_1}{(\lambda_2-\lambda_1)\lambda_2}\\ \nu_1=&\frac{c_{\rm in}+c_{\rm vco}}{(\lambda_1-\lambda_2)^2}\left(\frac{\lambda_2^2}{2\lambda_1}-\frac{\lambda_1\lambda_2}{2(\lambda_1+\lambda_2)}\right)\\ \nu_2=&\frac{c_{\rm in}+c_{\rm vco}}{(\lambda_1-\lambda_2)^2}\left(\frac{\lambda_1^2}{2\lambda_2}-\frac{\lambda_1\lambda_2}{2(\lambda_1+\lambda_2)}\right).\end{aligned}$$

The resulting output spectrum around the first harmonic is shown in Fig. 3. The loop filter corner frequency is chosen to be 10^5 rad/s. All other parameters are the same as in Section IV-A. Note that the addition of the loop filter introduces a bump in the flat portion of the spectrum. This bump becomes more pronounced as the bandwidth of the loop filter is decreased. Also the PSD is lower than in Fig. 2 for the flat portion of the spectrum. The phase noise performance at 10^4 rad/s offset is -104 dBc/Hz.

C. Charge Pump PLL (CPPLL)

The phase detectors described in Sections IV-A and IV-B suffer from the limitation that the phase difference between the input and the VCO output is not zero in steady state. Zero phase error can be accomplished by using an integrator after the linear phase detector (also known as the charge pump phase detector). However, this degrades the stability of the loop. This stability is recovered by introducing an additional zero in the charge pump transfer function. The filter is realized in practice by using the series combination of a capacitor and a resistor. The charge pump can be modeled by a linear transfer function of the form $k_{pd}(s+\omega_1/s)$ where ω_1 is the zero frequency. After some rearranging, (6) can be written as

$$\frac{d}{dt}\begin{bmatrix}\beta\\ \overline{\gamma}\end{bmatrix}=-\begin{bmatrix}0 & -\sqrt{c_{\rm pll}}\\ \omega_1 & \sqrt{c_{\rm pll}}\end{bmatrix}\begin{bmatrix}\beta\\ \overline{\gamma}\end{bmatrix}+\begin{bmatrix}C_{\rm vco}^T & -\sqrt{c_{\rm in}}\\ -C_{\rm vco}^T & \sqrt{c_{\rm in}}\end{bmatrix}\begin{bmatrix}\xi_p(t)\\ \xi_{\rm in}(t)\end{bmatrix}$$

where $\overline{\gamma}(t)$ and $\sqrt{c_{\rm pll}}$ are defined as before.

The resulting output spectrum around the first harmonic is shown in Fig. 4 using the same parameters as in Section IV-B. Note that, as the offset frequency is reduced, the output PSD initially follows the VCO spectrum, flattens out at a certain level, drops and then starts following the reference signal spectrum. At 10^4 rad/s offset frequency, the PSD is -103 dBc/Hz.

The above charge pump suffers from a critical effect. Since the charge pump drives the series combination of a resistor and a capacitor, each time a current is injected into the filter, the control voltage experiences a large jump which is detrimental for the transient behavior of the VCO [22]. Therefore, a second capacitor is usually placed in parallel to the series combination of the resistor and capacitor to suppress the initial step. The overall charge pump can be modeled by a linear transfer function of the

[5]This scaling also helps the numerical stability of this computation.

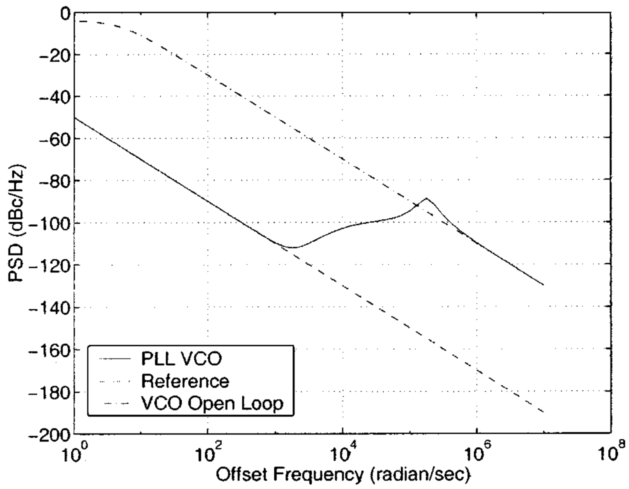


Fig. 5. CPPLL spectrum (second-order loop filter).

form $k_{pd}(\omega_1+s/s(1+s/\omega_2))$. Note that the loop is now second order. After some rearranging, (6) can be written as

$$\frac{d}{dt}\begin{bmatrix}\beta\\ \gamma\\ \delta\end{bmatrix} = -\begin{bmatrix}0 & -\sqrt{c_{\rm pll}} & 0\\ 0 & 0 & -1\\ \omega_1\omega_2 & \omega_2\sqrt{c_{\rm pll}} & \omega_2\end{bmatrix}\begin{bmatrix}\beta\\ \gamma\\ \delta\end{bmatrix} + \begin{bmatrix}C_{\rm vco}^T & -\sqrt{c_{\rm in}}\\ 0 & 0\\ -\omega_2 C_{\rm vco}^T & \omega_2\sqrt{c_{\rm in}}\end{bmatrix}\begin{bmatrix}\xi_p(t)\\ \xi_{\rm in}(t)\end{bmatrix}.$$

The resulting output spectrum around the first harmonic is shown in Fig. 5 using the same parameters as the previous CPPLL and $\omega_2 = 20\omega_1$. Note that the second capacitor again introduces a bump in the output PSD of the CPPLL. As ω_2 is reduced, the bump becomes more pronounced. At 10^4 rad/s offset frequency, the PSD is −103 dBc/Hz.

V. Experimental Results

The algorithm for computing the PLL output spectrum is implemented in MATLAB. From (6), it follows that the noise analysis of the VCO need not be a part of the PLL noise analysis. The VCO parameters required for PLL noise analysis are $c_{\rm vco} = \sum_i c_i$ and $c_{\rm control}$, and these can be computed separately for the open-loop VCO using techniques presented elsewhere [1]. Also note that the size of the A matrix in (7) is very small (typical values of n are 4–5). Even if an active loop filter is used [8], a separate transfer function analysis of the loop filter can be carried out to determine G, E, and F matrices in (4). Therefore, the spectrum calculation involves the diagonalization of a very small matrix and this process is very efficient. In practice, the infinite summations in (13) are truncated to some finite integer. Unlike noise analysis of many other periodic circuits, PLL noise analysis *does not* require a transient analysis of the entire circuit. Transient analysis of a PLL is very expensive because of widely separated time constants present in the circuit and CPU times of the order of a few hours are common.[6] Assuming that the PLL locks to the reference frequency, the VCO control voltage can be computed such that the VCO output is also at the appropriate frequency and its noise analysis can then be performed. The charge pump phase frequency detector (PFD) consists of digital circuits as well (flip-flops). Therefore, the overall characteristics of a PDF is very nonlinear. However, this nonlinearity manifests itself only when the phase difference between the VCO output and the reference signal is large and affects the settling and acquisition behavior of the PLL but it *does not* affect the noise analysis. PLL noise analysis can therefore be viewed as being performed at the *system level* without necessarily requiring the transistor level description of the entire circuit. This is possible due to the unique nature of the PLL where the loop filter path has very slowly varying signals when the PLL is locked and the VCO noise can be characterized completely using very few parameters [1].

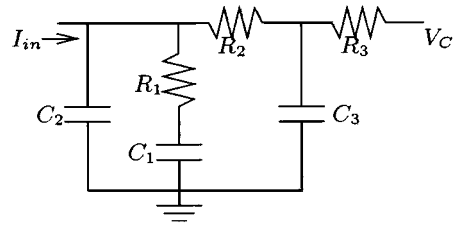


Fig. 6. Loop filter reported in [23].

Measured PLL output spectra are widely reported in the literature [8], [23]–[25]. However, only Parker and Ray [23] were considerate enough to report the details of the open-loop VCO spectrum, loop filter implementation, and the charge pump; therefore, their circuit will be used as an example. The phase noise of the 1.6-GHz open-loop oscillator was measured at −99 dBc/Hz at a 100-kHz offset. This corresponds to $c_{\rm vco} = 4.9177 \times 10^{-19}$ s.[7] The loop filter used in that work is shown in Fig. 6 which creates a three-pole one-zero network. Therefore, the phase detector/loop filter transfer function is given by

$$\frac{V_C(s)}{\phi_e(s)} = \frac{I_p R_1 C_1}{2\pi(C_1+C_2+C_3)} \times \frac{s+\frac{1}{R_1C_1}}{1+s\frac{R_1C_1(C_2+C_3)+R_2C_3(C_1+C_2)}{C_1+C_2+C_3}+s^2\frac{R_1R_2C_1C_2C_3}{C_1+C_2+C_3}}.$$

For this filter, $I_P = 25\ \mu$A, $C_1 = 50$ pF, $C_2 = C_3 \approx 3.5$ pF, and $R_1 = 50$ kΩ, and the reference signal frequency is 1/26th of the VCO frequency. The choice of R_2 is such that the bandwidth of the overall filter is not affected. In this analysis, noise due to the resistors present in the loop filter and the transistors present in the charge pump are also considered. Fig. 7 shows the PLL output spectrum as a function of offset frequency. The simulated spectrum compares very well with the measured spectrum reported by Parker and Ray [23] which is reproduced in Fig. 8. They reported a 9-dB peaking of the PLL output above the open-loop oscillator noise at 200-kHz offset. The simulated spectrum displays about 8 dB peaking at this offset frequency. This difference can be attributed to the simplified model of transistor noise used in the current work. Also shown in the figure is the PLL spectrum considering VCO noise only. It is evident that, at small offset frequencies, noise contributions of the loop filter resistors and the phase detector is nonnegligible.

[6]Transient analysis may be required anyway for predicting the transient behavior of the PLL.

[7]Oscillator phase noise in dBc/Hz at large $\omega_{\rm offset}$ is given by $10\log_{10} c_{\rm vco}(\omega_0/\omega_{\rm offset})^2$ [1]–[3].

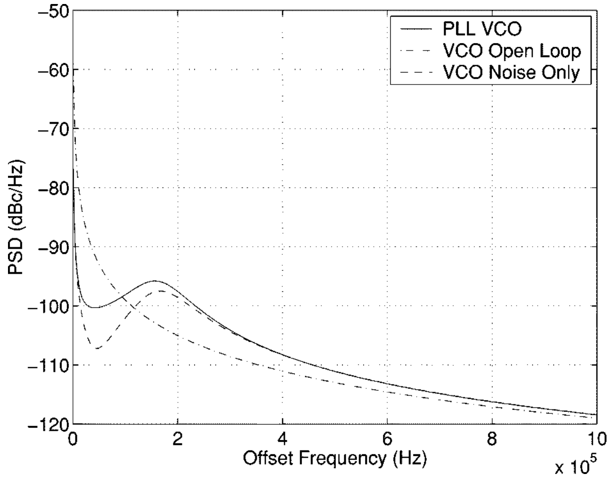


Fig. 7. Plot of the open and closed-loop VCO spectra.

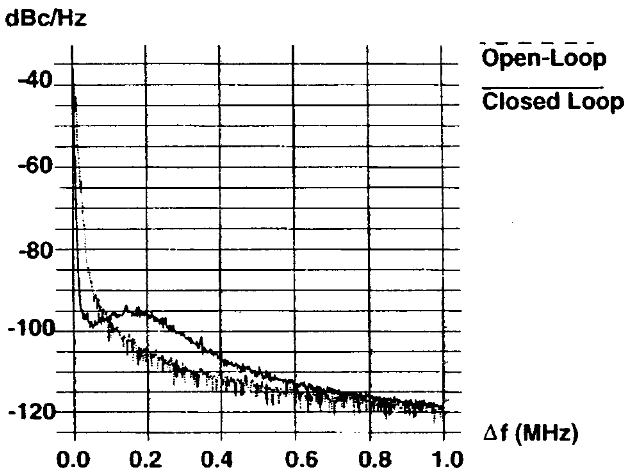


Fig. 8. Plot of measured open and closed-loop VCO spectra [23].

VI. Conclusion

A noise analysis technique for PLLs in the presence of circuit white noise sources and Brownian motion phase deviation in the reference signal is presented. The problem is formulated as a stochastic differential equation and techniques for obtaining the asymptotic solution of this equation are discussed. It is shown that, in the locked state, the PLL output phase can be expressed as a sum of the reference signal phase (Brownian motion process) and one component of a multidimensional Ornstein–Uhlenbeck process which has asymptotically bounded variance. The PLL output is shown to be asymptotically wide-sense stationary and an expression of its spectrum is obtained. This technique is used to compute the output spectra of a few popular PLL configurations. A circuit level example is also presented and it is demonstrated that phase detector and loop filter noise also contribute to the PLL output noise for small offset frequencies.

The examples presented in Sections IV and V assumed that the reference signal has less noise compared to the VCO. This analysis is also valid for the case when the reference signal is more *noisy* than the VCO signal. Such applications include clock recovery circuits in RF communication systems and disk-drive read channels. For the case of disk-drive read channels, $c_{\text{in}} > c_{\text{vco}}$, since the timing signal is generated by the rotation of a mechanical motor. For RF communication systems, the received signal phase picks up additional *bounded variance* noise components over and above the (potentially small) Brownian motion phase deviation of the oscillator which generates these signals. The spectrum calculation techniques presented here can be used to show that a zero is necessary in the low-pass filter for acceptable noise performance of such PLLs.

This technique assumes that frequency dividers, if present, add negligible delay to the signal. Also, the phase detector is modeled as a linear continuous-time approximation of the actual digital implementation. The effect of relaxing these assumptions on the PLL noise performance is currently under investigation. Additionally, this technique can only handle white noise sources. For noise with long-term correlations, i.e., flicker noise, the steps outlines above are not rigorously justified. The authors of [26] used the modulated stationary noise model to analyze flicker noise. However, the asymptotic arguments in this formulation need to be carefully examined before these results can be carried over to the flicker noise case as well.

Appendix

Calculation of μ_i and ν_i

From (10), we have

$$\mathbb{E}[\beta(t_1)\alpha_{\text{in}}(t)] = w_1\sqrt{c_{\text{in}}}\,\text{diag}\left[\frac{\exp(\lambda_1\min(0,t_2-t_1))}{\lambda_1},\ldots,\right.$$
$$\left.\frac{\exp(\lambda_2\min(0,t_2-t_1))}{\lambda_2}\right] d_{p+q+1}$$

where w_1 is the first row of W and d_{p+q+1} is the last column of $W^{-1}D$. Therefore,

$$\mu_i = \sqrt{c_{\text{in}}}\frac{w_{1_i}d_{(p+q+1)_i}}{\lambda_i}$$

where w_{1_i} and $d_{(p+q+1)_i}$ are the ith components of w_1 and d_{p+q+1}, respectively.

Now consider

$$e\exp(A(s-t_1))DD^T\exp\left[A^T(s-t_2)\right]e^T = eW$$
$$\times\exp(\Lambda(s-t_2))W^{-1}DD^TW^{-T}\exp(\Lambda(s-t_2))W^Te^T.$$

Let $X = W^{-1}DD^TW^{-T}$. Note that X is symmetric. Therefore,

$$e\exp(A(s-t_1))DD^T\exp\left[A^T(s-t_2)\right]e^T$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}w_{1_i}w_{1_j}x_{ij}\exp(\lambda_i(s-t_1)+\lambda_j(s-t_2)).$$

Further, it can also be shown that

$$\int_0^{\min(t_1,t_2)}\Big[\exp(\lambda_i(s-t_1)+\lambda_j(s-t_2))$$
$$+\exp(\lambda_j(s-t_1)+\lambda_i(s-t_2))\Big]ds$$
$$=\frac{\exp(-\lambda_i|t_1-t_2|)+\exp(-\lambda_j|t_1-t_2|)}{\lambda_i+\lambda_j}$$
$$-\frac{\exp(-\lambda_i t_1-\lambda_j t_2)+\exp(-\lambda_j t_1-\lambda_i t_2)}{\lambda_i+\lambda_j}.$$

Therefore,

$$\nu_i = \frac{w_{1_i}^2 x_{ii}}{2\lambda_i} + \sum_{j=1, j\neq i}^{n} \frac{w_{1_i} w_{1_j} x_{ij}}{2(\lambda_i + \lambda_j)}.$$

ACKNOWLEDGMENT

The author would like to express his gratitude to P. Grubel for useful feedback on the work and an anonymous reviewer for his patient review and in-depth comments on the manuscript.

REFERENCES

[1] A. Mehrotra, "Simulation and modeling techniques for noise in radio frequency integrated circuits," Ph.D. dissertation, Univ. California, Berkeley, 1999.

[2] A. Demir, A. Mehrotra, and J. Roychowdhury, "Phase noise in oscillators: A unifying theory and numerical methods for characterization," in *Proc. 1998 Design Automation Conf.*, 1998, pp. 26–31.

[3] ——, "Phase noise in oscillators: A unifying theory and numerical methods for characterization," *IEEE Trans. Circuits Syst. I*, vol. 47, pp. 655–674, May 2000.

[4] V. F. Kroupa and L. Šojdr, "Phase-lock loops of higher orders," in *Proc. 2nd Int. Conf. Frequency Control and Synthesis*, 1989, pp. 65–68.

[5] B. Kim, T. C. Weigandt, and P. R. Gray, "PLL/DLL system noise analysis for low jitter clock synthesizer design," in *Proc. 1994 IEEE Int. Symp. Circuits and Systems*, vol. 4, 1994, pp. 31–34.

[6] U. L. Rohde, *Microwave and Wireless Synthesizers: Theory and Design*. New York: Wiley, 1997.

[7] J. C. Nallatamby, M. Prigent, J. C. Sarkissian, R. Quere, and J. Obregon, "A new approach to nonlinear analysis of noise behavior of synchronized oscillators and analog-frequency dividers," *IEEE Trans. Microwave Theory Tech.*, vol. 46, pp. 1168–1171, Aug. 1998.

[8] L. Lin, L. Tee, and P. R. Gray, "A 1.4 GHz differential low-noise CMOS frequency synthesizer using a wideband PLL architecture," in *Dig. Tech. Papers, IEEE Int. Solid-State Circuits Conf.*, 2000, pp. 204–205.

[9] K. Lim, S. Choi, and B. Kim, "Optimal loop bandwidth design for low noise PLL applications," in *Proc. Asia and South Pacific Design Automation Conf. 1997*, pp. 425–428.

[10] K. Lim, C.-H. Park, and B. Kim, "Low noise clock synthesizer design using optimal bandwidth," in *Proc. 1998 IEEE Int. Symp. Circuits and Systems*, vol. 1, 1998, pp. 163–166.

[11] G. Kolumbán, "Frequency domain analysis of sampling phase-locked loops," in *Proc. 1988 IEEE Int. Symp. Circuits and Systems*, vol. 1, 1988, pp. 611–614.

[12] D. Asta and D. N. Green, "Analysis of a hybrid analog/switched-capacitor phase-locked loop," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 183–197, Feb. 1990.

[13] A. Demir, "Analysis and simulation of noise in nonlinear electronic circuits and systems," Ph.D. dissertation, Univ. California, Berkeley, 1997.

[14] W. E. Thain Jr and J. A. Connelly, "Simulating phase noise in phase-locked loops with a circuit simulator," in *Proc. 1995 IEEE Int. Symp. Circuits and Systems*, vol. 3, 1995, pp. 1760–1763.

[15] L. Wu, H. Jin, and W. C. Black Jr, "Nonlinear behavioral modeling and simulation of phase-locked and delay-locked systems," in *Proc. IEEE 2000 Custom Integrated Circuits Conf.*, 2000, pp. 447–450.

[16] P. Heydani and M. Pedram, "Analysis of jitter due to power-supply noise in phase-locked loops," in *Proc. IEEE 2000 Custom Integrated Circuits Conf.*, 2000, pp. 443–446.

[17] B. K. Øksendal, *Stochastic Differential Equations: An Introduction With Applications*. Berlin, Germany: Springer-Verlag, 1998.

[18] C. W Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*, 2nd ed, ser. Springer Series in Synergetics. Berlin, Germany: Springer-Verlag, 1983, vol. 13.

[19] P. Dupuis and H. J. Kushner, "Stochastic systems with small noise, analysis and simulation; A phase locked loop example," *SIAM J. Appl. Math.*, vol. 47, no. 3, pp. 643–661, June 1987.

[20] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*. Boston, MA: Jones & Bartlett, 1993.

[21] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*. Berlin, Germany: Springer-Verlag, 1984.

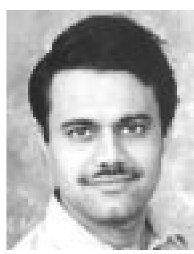
[22] B. Razavi, *Design of Analog CMOS Integrated Circuits*. New York: McGraw-Hil, 2000.

[23] J. F. Parker and D. Ray, "A 1.6-GHz CMOS PLL with on-chip loop filter," *IEEE J. Solid-State Circuits*, vol. 33, pp. 337–343, Mar. 1998.

[24] A. Ali and J. L. Tham, "A 900 MHz frequency symthesizer with integrated LC voltage-controlled oscillator," in *Dig. Tech. Papers, IEEE Int. Solid-State Circuits Conf.*, 1996, pp. 390–391.

[25] J. Craninckx and M. Steyaert, "A fully integrated CMOS DCS-1800 frequency synthesizer," in *Dig. Tech. Papers, IEEE Int. Solid-State Circuits Conf.*, 1998, pp. 372–373.

[26] A. Demir, E. Liu, and A. Sangiovanni-Vincentelli, "Time-domain non Monte-Carlo noise simulation for nonlinear dynamic circuits with arbitrary excitations," *IEEE Trans. Computer-Aided Design*, vol. 15, pp. 493–505, May 1996.



Amit Mehrotra received the B. Tech. degree in electrical engineering from the Indian Institute of Technology, Kanpur, India, in 1994, and the M.S. and Ph.D. degrees from the University of California at Berkeley in 1996 and 1999, respectively.

In August 1999, he joined the University of Illinois at Urbana-Champaign where he is currently an Assistant Professor with the Department of Electrical and Computer Engineering and a Research Assistant Professor, the Illinois Center for Integrated Micro-Systems group at the Coordinated Science Laboratory. His research interests include RF, analog, and mixed-signal circuit design, for mobile communication systems, simulation techniques for RF and mixed-signal circuits and systems, interconnect performance and modeling issues in VLSI and novel circuits and physical design issues for high-performance VLSI designs, and model-order reduction of linear and nonlinear circuits.