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Q1 a) Yes

Reflexive : If $(a,a) \in R$ & $(a,a) \in S$

then $(a,a) \in R \cap S$

Symmetric : Let $(a,b) \in R \cap S$

then $(a,b) \in R$ & $(a,b) \in S$

$\Rightarrow (b,a) \in R$ & $(b,a) \in S$

$\Rightarrow (b,a) \in R \cap S$

Transitive : Let (a,b) & $(b,c) \in R \cap S$

$\therefore$ They both belong R&S

$\Rightarrow (a,c) \in R$ and $(a,c) \in S$

$\therefore (a,c) \in R \cap S$

b) No

If $X = \{a,b,c\}$

$R = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$

$S = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$

Since R & S are equivalence relⁿ

But $(a,c) \notin R \cup S$

while (a,b) & $(b,c) \in R \cup S$

$\therefore$ Not transitive

Q2 Given: θ is a congruence on a lattice L

To Prove: $a \equiv b \pmod{\theta}$

$a \equiv b \pmod{\theta}$ & $c \equiv c \pmod{\theta}$

$\Rightarrow a \vee c \equiv b \vee c \pmod{\theta}$ & $a \wedge c \equiv b \wedge c \pmod{\theta}$

Now:

given: $\forall a, b, c \in L$, $a \vee c \equiv b \vee c \pmod{\theta}$
and $a \wedge c \equiv b \wedge c \pmod{\theta}$ — ①

consider $a, b, c, d \in L$ s.t
$a \equiv b \pmod{\theta}$ & $c \equiv d \pmod{\theta}$

then, $a \vee c \equiv b \vee c \pmod{\theta}$ (From ①) — ②

& $c \vee b \equiv d \vee b \pmod{\theta}$ (From ①)

$b \vee c = c \vee b$

$\therefore b \vee c \equiv d \vee b \pmod{\theta}$ — ③

From ② & ③

$a \vee c \equiv b \vee d \pmod{\theta}$

Similarly by duality

$a \wedge c \equiv b \wedge d \pmod{\theta}$

Hence proved.

Q3 Part I g is well defined

$[a]_\theta = [b]$

$\Leftrightarrow a \equiv b \pmod{\theta}$

$\Leftrightarrow f(a) = f(b)$

$\Leftrightarrow g([a]) = g([b])$

$\therefore$ g is well-defined & one one

Part 2 g is isomorphism

g is onto

To show $\forall k \in K$, $\exists [l] \in L/\theta$ s.t. $g([l]) = k$

$\because f$ is onto $\therefore \exists l \in L$ s.t

$f(l) = k \Rightarrow g([l]) = f(l) = k$

g is homomorphism:

$g([a] \vee [b]) = g([a \vee b])$

$= f(a \vee b) = f(a) \vee f(b)$

$= g([a]) \vee g([b])$

and using duality principle, we can show the same for meet

Thus g is one-one & onto & homomorphism

$\therefore$ It is isomorphism

Part III $\ker q = \sigma$

$(a, b) \in \sigma$

$\Leftrightarrow [a]_\sigma = [b]_\sigma$

$\Leftrightarrow q(a) = q(b) \Leftrightarrow (a, b) \in \ker q$

Part IV $g \cdot p = f$

$$g(q(l)) = g([l]_\sigma) = f(l)$$

Q4. Let $X = \{x \in L \mid x \le f(x)\}$

and $a = \sup(X)$

To prove : $f(a) = a$

Now, $0 \in X$ since L is complete

$\sup(X)$ exists because L is complete

$\forall x \in X, \ x \le a$

$\therefore \ \forall x \in X \quad f(x) \le f(a)$ (Isotone)

$x \le f(x) \le f(a)$

$\therefore f(a)$ is upper bound

$\therefore \ a \le f(a)$ (since least upper bound is a) — ①

$\Rightarrow f(a) \le f(f(a))$ (Isotone)

$\therefore f(a) \in X$

$\Rightarrow \ f(a) \le a$ — ②

From ① & ②

$\therefore \exists \ a \in L$ such that $f(a) = a$