3) (10 pts) ANL (Summations and Recurrence Relations)

Let a, b, c, and d, be positive integer constants with $a < b$. ***Without using the arithmetic sum formula***, prove that

$$\sum_{i=a}^{b}(ci+d) = \frac{(c(a+b)+2d)(b-a+1)}{2}$$

$$\sum_{i=a}^{b}(ci+d) = \sum_{i=1}^{b} ci - \sum_{i=1}^{a-1} ci + \sum_{i=a}^{b} d$$

$$= \frac{cb(b+1)}{2} - \frac{c(a-1)a}{2} + (b-a+1)d$$

$$= \frac{c(b^2+b-(a^2-a))}{2} + \frac{2d(b-a+1)}{2}$$

$$= \frac{c(b^2+b-a^2+a)}{2} + \frac{2d(b-a+1)}{2}$$

$$= \frac{c(b^2-a^2+b+a)}{2} + \frac{2d(b-a+1)}{2}$$

$$= \frac{c((b-a)(b+a)+(b+a))}{2} + \frac{2d(b-a+1)}{2}$$

$$= \frac{c(b+a)(b-a+1)}{2} + \frac{2d(b-a+1)}{2}$$

$$= \frac{(c(a+b)+2d)(b-a+1)}{2}$$

The key to this proof is setting up the usual summation formulas and then recognizing that $b^2 - a^2$ factors. Once this term is factorized, it can be recognized that (a+b) is a factor in the first term that can be pulled out. This, in turn, reveals (b - a + 1) to be a factor between the two larger terms, something that could have been anticipated since (b - a + 1) is a factor in the result on the RHS and also an immediate factor in the last term after the initial algebra.

Grading: 2 pts to split up summations, 2 pts for evaluating sums to i, 1 pt for evaluating all constant sums, 5 pts for simplification to the RHS. Note - the grader must be very careful to give partial credit on the last five points. Indeed, a bulk of the work of this problem is in this algebra that can be done in many different ways. Give partial based on how accurate and what percentage of the total algebra was carried out. Any correct answer gets full credit. Any response with a single isolated error should just get 1 point off. An answer that is about halfway there (factors out c, tries to do something with the squares) should get 2 or 3 of the 5 points based on discretion.