3) (10 pts) ANL (Recurrence Relations)

Use the iteration technique to solve the following recurrence relation in terms of n:

$$T(n) = 2T(n/2) + 1, for\ all\ integers\ n > 1$$
$$T(1) = 1$$

Find a tight Big-Oh answer.

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$
$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + 1$$
$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + 1\right) + 1$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 2 + 1$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 3$$
$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + 1$$
$$T(n) = 4\left(2T(\frac{n}{8}\right) + 1) + 3$$
$$T(n) = 8T(\frac{n}{8}) + 4 + 3$$
$$T(n) = 8T(\frac{n}{8}) + 7$$

Based on these three iterations, we see that after k iterations, the recurrence is

$$T(n) = 2^k T(\frac{n}{2^k}) + (2^k - 1)$$

Plug in the value of k such that $\frac{n}{2^k} = 1$ to this recurrence. This means that $2^k = n$. Substituting, we get:

$$T(n) = nT(1) + (n - 1)$$
$$T(n) = n + (n - 1)$$
$$T(n) = 2n - 1$$

It follows that T(n) = O(n).

Grading: 2 pts for iteration with T(n/4), 2 pts for T(n/8). 2 pts for general expression after k iterations, 1 pt for the value to plug in for k. 3 pts to finish the problem.