3) (10 pts) ANL (Summations and Recurrence Relations)

Using the iteration technique, find a tight Big-Oh bound for the recurrence relation defined below:

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2, \text{ for } n > 1$$
$$T(1) = 1$$

Hint: You may use the fact that $\sum_{i=0}^{\infty}(\frac{3}{4})^i = 4$ and that $3^{log_2 n} = n^{log_2 3}$, and that $log_2 3 < 2$.

Iterate the given recurrence two more times:

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2$$
$$T(n) = 3(3T\left(\frac{n}{4}\right) + (\frac{n}{2})^2) + n^2$$
$$T(n) = 9T\left(\frac{n}{4}\right) + \frac{3n^2}{4} + n^2$$
$$T(n) = 9T\left(\frac{n}{4}\right) + n^2(1 + \frac{3}{4})$$
$$T(n) = 9(3T\left(\frac{n}{8}\right) + (\frac{n}{4})^2) + n^2(1 + \frac{3}{4})$$
$$T(n) = 27T\left(\frac{n}{8}\right) + \frac{9n^2}{16} + n^2(1 + \frac{3}{4})$$
$$T(n) = 27T\left(\frac{n}{8}\right) + n^2(1 + \frac{3}{4} + \frac{9}{16})$$

In general, after the k^{th} iteration, we get the recurrence

$$T(n) = 3^k T\left(\frac{n}{2^k}\right) + n^2(\sum_{i=0}^{k-1}(\frac{3}{4})^i)$$

To solve the recurrence, find k such that $\frac{n}{2^k} = 1$. This occurs when $n = 2^k$ and $k = log_2 n$. Plug into the equation above for this value of k to get:

$$T(n) = 3^{logn}T(1) + n^2\left(\sum_{i=0}^{k-1}\left(\frac{3}{4}\right)^i\right) \le 3^{logn} + n^2\left(\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i\right) = n^{log_2 3} + 4n^2 = O(n^2)$$

Grading: 1 pt for copying recurrence, 1 pt for getting 2^{nd} iteration, 2 pts for getting third iteration (in any form), 3 pts for kth iteration, 1 pt for what to plug in form, 2 pts to complete the problem. (Be somewhat generous as this is probably the hardest problem on the exam.)