Microsoft

Programming assignment 6
Quantum Fourier Transform

Mariia Mykhailova
Senior Software Engineer
Microsoft Quantum Systems

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Task 1.1: Prepare periodic state

$$\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{k}{2^n}} |k\rangle$$

Hint: which basis state can be mapped to this state using QFT?

Look at the QFT representation for a basis state $|j\rangle$:

$$|j\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{jk}{2^n}} |k\rangle$$

Comparing this to our given state, we can recognize that $j = 1$ (in little-endian it is 10…0, the first qubit stores least-significant bit)

```
X(qs[0]);
QFTLE(LittleEndian(qs));
```

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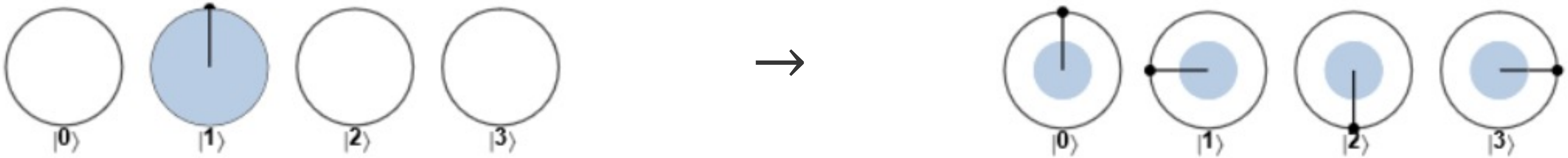
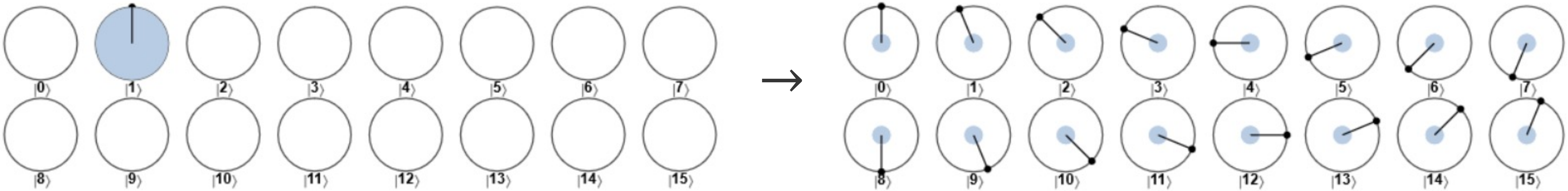
Task 1.1: Prepare periodic state

$$|1\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{k}{2^n}} |k\rangle$$





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Task 1.2: Prepare superposition of odd states

$$\frac{1}{\sqrt{2^{n-1}}}(|1\rangle + |3\rangle + \cdots + |2^n - 1\rangle)$$

Hint: Which superposition of two basis states can be mapped to this state?

An equal superposition of all states is

$$QFT|0\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{0k}{2^n}} |k\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^0 |k\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |k\rangle$$

We need to add a term that will cancel even terms and amplify odd ones:

$$\frac{1}{\sqrt{2^n}}(-|0\rangle + |1\rangle - |2\rangle + |3\rangle + \cdots + |2^n - 1\rangle) =$$

$$= -\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} (-1)^{k \bmod 2} |k\rangle = -\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{k}{2}} |k\rangle = -QFT|2^{n-1}\rangle$$

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Task 1.2: Prepare superposition of odd states

$$\frac{1}{\sqrt{2^{n-1}}}(|1\rangle + |3\rangle + \cdots + |2^n - 1\rangle)$$

The desired state is the QFT transform of the following state:

$$\frac{1}{\sqrt{2^{n-1}}}(|0\rangle - |2^{n-1}\rangle)$$

In Q# implementation, these little-endian states are $|0 \ldots 00\rangle - |0 \ldots 01\rangle$

```
// prepare |0...00⟩ - |0..01⟩
H(Tail(qs));
Z(Tail(qs));
// apply Fourier transform
QFTLE(LittleEndian(qs));
```

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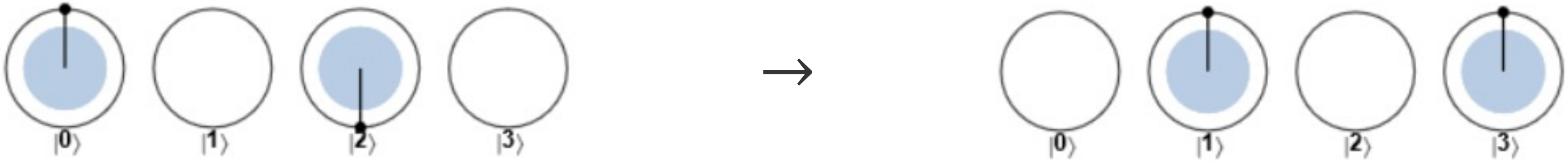
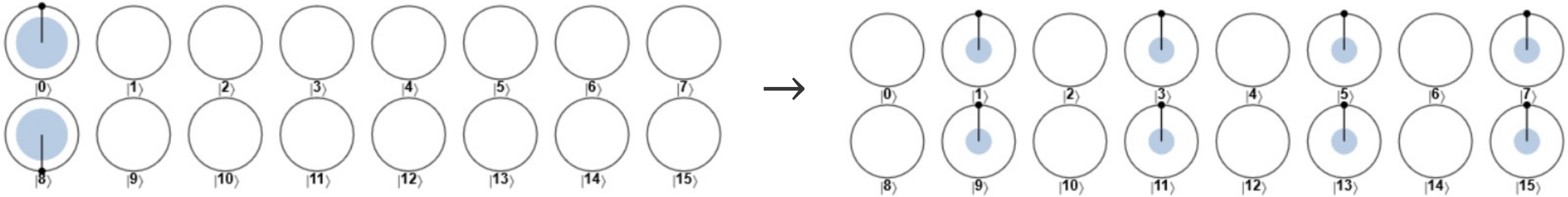
Task 1.2: Prepare superposition of odd states

$$\frac{1}{\sqrt{2^{n-1}}}(|0\rangle - |2^{n-1}\rangle) \rightarrow \frac{1}{\sqrt{2^{n-1}}}(|1\rangle + |3\rangle + \cdots + |2^n - 1\rangle)$$





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Task 1.3: Prepare superposition of cosines

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{k=0}^{2^n-1}\cos\frac{2\pi k}{2^n}|k\rangle$$

Hint: use superposition of two basis states

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{k=0}^{2^n-1}\cos\frac{2\pi k}{2^n}|k\rangle = \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}\frac{1}{\sqrt{2}}(e^{2\pi i\frac{k}{2^n}} + e^{-2\pi i\frac{k}{2^n}})|k\rangle =$$

$$= QFT\frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle) = QFT\frac{1}{\sqrt{2}}(|10\ldots0\rangle + |11\ldots1\rangle)$$

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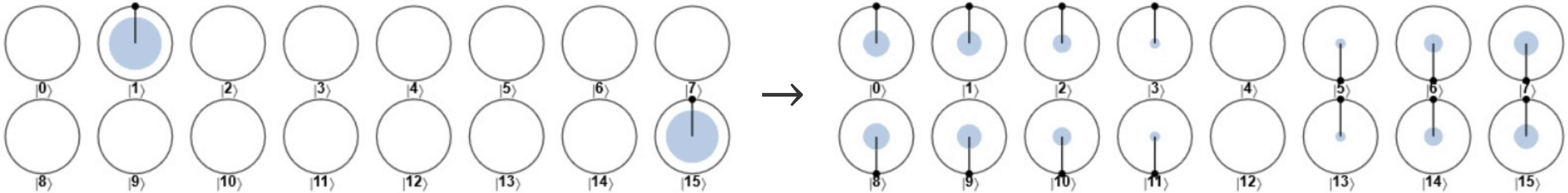
Task 1.3: Prepare superposition of cosines

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{k=0}^{2^n-1}\cos\frac{2\pi k}{2^n}|k\rangle = QFT\frac{1}{\sqrt{2}}(|10\ldots0\rangle + |11\ldots1\rangle)$$




```
X(qs[0]);
H(qs[1]);
for (i in 2..Length(qs)-1) { CNOT(qs[1], qs[i]); }
QFTLE(LittleEndian(qs));
```

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Task 1.4: Prepare superposition of sines

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{k=0}^{2^n-1}\sin\frac{2\pi k}{2^n}|k\rangle$$

Hint: prepare this state up to a global phase

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{k=0}^{2^n-1}\sin\frac{2\pi k}{2^n}|k\rangle = \frac{1}{i}\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}\frac{1}{\sqrt{2}}\left(e^{2\pi i\frac{k}{2^n}} - e^{-2\pi i\frac{k}{2^n}}\right)|k\rangle =$$

$$= \frac{1}{i}QFT\frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle) = \frac{1}{i}QFT\frac{1}{\sqrt{2}}(|10\ldots0\rangle - |11\ldots1\rangle)$$

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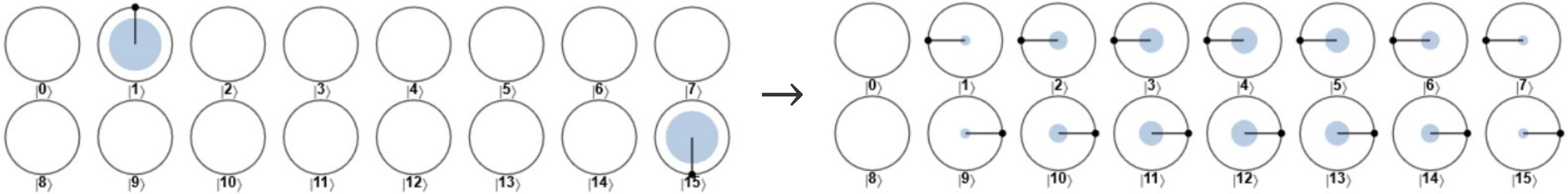
Task 1.4: Prepare superposition of sines

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{k=0}^{2^n-1}\sin\frac{2\pi k}{2^n}|k\rangle = \frac{1}{i}QFT\frac{1}{\sqrt{2}}(|10\ldots0\rangle - |11\ldots1\rangle)$$




```
X(qs[0]);
H(qs[1]);
for (i in 2..Length(qs)-1) { CNOT(qs[1], qs[i]); }
Z(qs[1]);
QFTLE(LittleEndian(qs));
```

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Task 2.1: Distinguish two periodic states

Given $\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{kF}{2^n}} |k\rangle$, figure out F.

Hint: this state looks like applying QFT to what state?

$$|F\rangle \to \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i \frac{Fk}{2^n}} |k\rangle$$

Using Adjoint QFT allows to restore basis state F from the given one:

```
Adjoint QFTLE(LittleEndian(qs));
// now the register qs contains the frequency F
let F = MeasureInteger(LittleEndian(qs));
return F == 2^f0 ? 0 | 1;
```

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