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Lecture Notes on

# Algorithmic Algebra

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Draft—September 3, 2015 and forever

# Preface

This lecture notes are produced as a part of the course *CS6842: Algorithmic Algebra* which was a course offered during August to November semester in 2013 and 2015 at the CSE Department of IIT Madras.

## Acknowledgements

Thanks to Alexander Shrestov and Markus Blaser for creating nice templates for lecture notes which are being combined and in this document.

## Scribe status

Each lecture has a field called **status**. It tells which stage of the edit pipeline is the document currently. Each scribe will be set to choose to either Monday-Tuesday (MT) or Wednesday-Friday (WF) pair of lectures. There are four versions.

1. Alpha ( $\alpha$ ) (deadline  $X + 2$ ): Rough draft. The MT Scribe is expected to turn-in the rough notes by  $X+2$  (Thursday night 10pm) and WF scribe is expected to turn this in by Sunday night 10pm.
2. Beta ( $\beta$ ) (deadline  $X + 4$ ) : Final draft. The MT scribe is expected to turn this in by Saturday night 10pm and WF scribe is expected to turn this in by Tuesday night 10pm.
3. Gamma ( $\gamma$ ) (deadline  $X + 5$ ) : Corrected draft. This is done by the TAs and given to the instructor.
4. Delta ( $\delta$ ) (deadline  $X + 6$ ) : Final notes. The instructor puts out this final version.

Even after these edits, it is possible that there are still errors in the draft, which may not get noticed. If you find errors still, please report at the git hub itself, or send emails to the instructor or the TAs.

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CS6842 – Algorithmic Algebra

*Instructor:* Jayalal Sarma

*Scribe:* K Dinesh

*Date:* Aug 3, 2015

**Status:**  $\delta$

Lecture

**1**

## Introduction, Motivation and the Language

### 1.1 Overview of the course. Administrative, Academic policies

## 1.2 Introduction and Motivation

Main theme of this course is to use algebra to solve computational problems. Let us consider the following two problems :

**Plagiarism check** Given two  $C$  programs  $P_1$  and  $P_2$ , check if they are the same under renaming of variables.

**Molecule detection** Given two chemical molecules check if they have the same structure.

Consider the following simpler variant of plagiarism checking where the program submitted has just its variables renamed and all other portions of the program are the same. In this case, given the two versions of the program, we need to check if there is a way to rename the variables of one program to get the other one.

In the second case one could view the given molecules as graphs. The problem is again similar problem, where we want to see if there is way to rename the vertex labels of the graph so that under the relabelling both the graphs are the same.

Our aim in both cases is to check whether the two structures are same under renaming. We are interested in solving this problem on graphs.

**Definition 1.1** (Graph Isomorphism). *Two graphs  $X_1(V_1, E_1)$ ,  $X_2(V_2, E_2)$  are said to be isomorphic if there is a bijective map  $\sigma : V_1 \rightarrow V_2$  such that  $\forall (u, v) \in V_1 \times V_1$ ,*

$$(u, v) \in E_1 \iff (\sigma(u), \sigma(v)) \in E_2$$

**Problem 1.2.** *The graph isomorphism problem is the decision problem of checking if given two graphs  $X_1, X_2$  are isomorphic.*

We are also interested in the following special case of the above problem called graph automorphism problem.

**Definition 1.3** (Graph Automorphism). *For a graph  $X(V, E)$ , an automorphism of  $X$  is a renaming of the vertices of  $X$  given by a bijective map  $\sigma : V \rightarrow V$  such that  $\forall (u, v) \in V \times V$ ,*

$$(u, v) \in E \iff (\sigma(u), \sigma(v)) \in E$$

We are interested in the set of all bijections such that they are automorphisms of  $X$ . We denote this by  $Aut(X)$ .

**Definition 1.4.** *For a graph  $X$  on  $n$  vertices,  $Aut(X) = \{\sigma \mid \sigma : [n] \rightarrow [n], \sigma \text{ is an automorphism of } X\}$*

Note that an identity map which takes a vertex to itself always belongs to  $Aut(X)$  for all graphs  $X$ . Hence the question is : are there any bijections other than the identity map as automorphism of  $X$ .

**Problem 1.5** (Graph Automorphism Problem). *Given a graph  $X$  does  $Aut(X)$  has any element other than the identity element.*



One way to see bijections is via permutations. This is because, every bijection define a permutation and vice versa.

Let  $X$  be an  $n$  vertex graph. Denote  $S_n$  to be the set of all permutations on  $n$  elements. Hence  $Aut(X)$  can be defined as  $\{\sigma \mid \sigma \in S_n \text{ and } \sigma \text{ is an automorphism of } G\}$ .

We now show that the set  $Aut(X)$  has some nice properties. Given  $\sigma_1, \sigma_2 \in Aut(X)$ , we can compose two permutations as  $\sigma_1 \circ \sigma_2 = (\sigma_1(\sigma_2(1)), \sigma_1(\sigma_2(2)), \dots, \sigma_1(\sigma_2(n)))$ . This is same as applying  $\sigma_2$  on identity permutation and then applying  $\sigma_1$  to the result. We show that  $Aut(X)$  along with the composition operation  $\circ$  gives us many nice properties.

- If  $\sigma_1, \sigma_2 \in Aut(X)$ , then  $\sigma_1 \circ \sigma_2$  is also an automorphism of  $X$ . The reason is that for any  $(u, v) \in X \times X$ ,  $(u, v) \in E \iff (\sigma_2(u), \sigma_2(v)) \in E$  Now applying  $\sigma_1$  on the previous tuple, we get that  $(\sigma_2(u), \sigma_2(v)) \iff (\sigma_1 \circ \sigma_2(u), \sigma_1 \circ \sigma_2(v)) \in E$ . Hence  $(u, v) \in E \iff (\sigma_1 \circ \sigma_2(u), \sigma_1 \circ \sigma_2(v)) \in E$ . This tells that  $Aut(X)$  is *closed* under  $\circ$ .
- The composition operation is also *associative*.
- *Identity* permutation belongs to  $Aut(X)$  as observed before.
- Since we are considering bijections, it is natural to consider *inverse* for a permutation  $\sigma$  denoted  $\sigma^{-1}$  with the property that  $\sigma \circ \sigma^{-1}$  is identity permutation.

We gave a definition of inverse for an arbitrary permutation. But then a natural question is : given  $\sigma \in Aut(X)$ , is it true that  $\sigma^{-1}$  also belongs to  $Aut(X)$ . It turns out that it is true.

**Claim 1.6.** For the graph  $X(V, E)$   $\sigma \in Aut(X) \iff \sigma^{-1} \in Aut(X)$

*Proof.* Recall that  $\sigma \in Aut(G)$  iff  $\forall (u, v) \in V \times V$ ,  $(u, v) \in E \iff (\sigma(u), \sigma(v)) \in E$ . In particular, this must be true for  $(\sigma^{-1}(u), \sigma^{-1}(v))$  also. This means,

$$(\sigma^{-1}(u), \sigma^{-1}(v)) \in E \iff (\sigma(\sigma^{-1}(u)), \sigma(\sigma^{-1}(v))) \in E$$

By definition of  $\sigma^{-1}$  we get that  $(\sigma^{-1}(u), \sigma^{-1}(v)) \in E \iff (u, v) \in E$ . This shows that  $\sigma^{-1} \in Aut(X)$  □

Objects which satisfy these kind of properties are called groups.

### 1.3 Overview of the course

There are two major themes.

- Algorithms for permutation groups.
- Algorithms for polynomials.

## Algebraic Approach to Primality Testing

In this lecture, we will see an algebraic approach to solving a fundamental problem in Number Theory.

### 2.1 Application to Number Theory

Following is an algorithmic question that we are interested.

**Problem 2.1.** *Given a number  $n$  in its binary representation, check if it is a prime or not in time  $O(\text{poly}(\log N))$ .*

**Note 2.2.** *The trivial algorithms that we can think of will depend on  $n$  and hence takes time exponential in its input representation.*

Consider the following property about prime number proved by Fermat which is of interest in this context.

**Theorem 2.3** (Fermat's Little Theorem). *If  $N$  is a prime, then  $\forall a, 1 \leq a \leq N - 1$ ,*

$$a^N = a \pmod{N}$$

*Proof.* Fix an  $a \in \{1, 2, \dots, N - 1\}$ . Now consider the sequence  $a, 2a, \dots, (N - 1)a$ . The question we ask is : can any two of the numbers in this sequence be the same modulo  $N$ . We claim that this cannot happen. We give a proof by contradiction : suppose that there are two distinct  $r, s$  with  $1 \leq r < s \leq N - 1$  and  $sa = ra \pmod{N}$ . Then clearly  $N|(s - r)a$  which means  $N|a$  or  $N|(s - r)$ . But both cannot happen as  $a, s - r$  are strictly smaller than  $N$ .

This gives that all the  $N$  numbers in our list modulo  $N$  are distinct. Hence all numbers from  $1, 2, \dots, N - 1$  appear in the list when we go modulo  $N$ . Taking product of the list and the list modulo  $N$ , we get

$$(N - 1)!a^{N-1} = (N - 1)! \pmod{N}$$

By cancelling  $(N - 1)!$ , we get that  $a^{N-1} = 1 \pmod{N}$ . □

This tells that the above condition is necessary for a number to be prime. If this test is also sufficient (i.e, if the converse of the above theorem is true), we have a test for checking primality of

a number. But it turns out that this is not true due to the existence of Carmichael numbers which are not prime numbers but satisfy the above test.

So one wants a necessary and sufficient condition which can be used for primality testing. For this, we need the notion of polynomials.

**Definition 2.4.** A polynomial  $p(x) = \sum_{i=0}^d a_i x^i$  with  $a_d \neq 0$  denotes a polynomial in one variable  $x$  of degree  $d$ . Here  $a_i$ s are called coefficients and each term excluding the coefficient is called a monomial. A polynomial is said to be identically zero, if all the coefficients are zero.

A polynomial time algorithm for this problem has been found in 2002 by Manindra Agarwal, Neeraj Kayal and Nitin Saxena (called as the AKS algorithm). In their result, they used the following polynomial characterisation for a prime number.

**Theorem 2.5** (Polynomial formulation (Agarwal-Biswas 1999)). Let  $N \geq 1$  be an integer. Define a polynomial  $p_N(z) = (1+z)^N - 1 - z^N$ . Then

$$p_N(z) \equiv 0 \pmod{N} \iff N \text{ is prime}$$

Hence checking if  $N$  is prime or not boils down to checking if  $p_N(z)$  is identically 0 or not except for the fact that the underlying operations are done modulo  $N$ .

Proof of the theorem is as follows.

*Proof.* Note that  $p_N(z) = \sum_{i=1}^{N-1} \binom{N}{i} z^i$  and  $\binom{N}{i} = \frac{N(N-1)\dots(N-i+1)}{1 \cdot 2 \dots i}$  with  $1 \leq i \leq N-1$ .

If  $N$  is prime, then  $\binom{N}{i} = N \times k_i$  for some integer  $k_i$  as none of  $1, 2, \dots, i$  divides  $N$ . Hence  $\binom{N}{i} \pmod{N} = 0$  for every  $i$  and  $p_N(z) \equiv 0 \pmod{N}$  since the polynomial has all coefficients as zero.

If  $N$  is composite, we need to show that  $p_N(z) \not\equiv 0 \pmod{N}$  which means that there is at least one non-zero coefficient  $\binom{N}{i} \pmod{N}$ . Since  $N$  is composite, there exists a prime  $p$  such that  $p \mid N$ . Let  $p^k \mid N$  where  $k \geq 1$  is the largest exponent of  $p$  in the prime factorisation of  $n$ . Hence  $p^{k+1} \nmid N$ .

We first show that  $p_N(z)$  is a non zero polynomial by showing that the coefficient of  $z^i$  for  $i = p$ , which is  $\binom{N}{p}$ , is non zero modulo  $p^k$  showing<sup>2</sup>  $p_N(z)$  is not a zero polynomial modulo  $N$ . Let  $N = g \times p^k$ . In  $\binom{N}{p}$ ,  $p$  of the denominator divides  $N$ . Note that  $p$  cannot divide  $g$ , for if it does, then  $p^{k+1} \mid N$  which is not possible by choice of  $p$  and  $k$ .

$$\binom{N}{p} = \frac{g \times p^k (N-1) \dots (N-p+1)}{1 \cdot 2 \dots p} = \frac{g \times p^{k-1} (N-1) \dots (N-p+1)}{1 \cdot 2 \dots p-1} \quad (2.1)$$

Hence it must be that  $p^{k-1}$  divides  $\binom{N}{p}$ . This is because, there is no  $p$  left now in the denominator to divide  $N$ . Now  $p^k \nmid \binom{N}{p}$  because, none of the terms  $(N-1), \dots, (N-p+1)$  can have  $p$  as a factor since all these terms are obtained by subtracting at most  $p-1$  times from  $N$ . Hence  $\binom{N}{p} \not\equiv 0 \pmod{p^k}$ . This completes the proof.  $\square$

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<sup>1</sup>In class we asked the following question of finding an  $a \in \{1, 2, \dots, N-1\}$  such that  $p_N(a) \not\equiv 0 \pmod{N}$ . This has a counter example. Consider the case where  $N$  is a Carmichael number. By definition, Carmichael numbers are composite numbers that satisfy Fermat's Little theorem. Hence if  $N$  is Carmichael,  $\forall a \in \{1, 2, \dots, N-1\}$ , we get  $a^N = a \pmod{N}$ . This gives that  $\forall a \in \{1, 2, \dots, N-1\}$

$$p_N(a) = ((a+1)^N - a^N - 1) \pmod{N} = ((a+1) - a - 1) \pmod{N}$$

which is zero modulo  $N$ .

<sup>2</sup>Note that if  $N \mid \binom{N}{i}$  then  $p^k \mid \binom{N}{i}$ . Taking contrapositive, we get that  $\binom{N}{i} \not\equiv 0 \pmod{p^k}$  implies  $\binom{N}{i} \not\equiv 0 \pmod{N}$

Note that Fermat's Little Theorem is a special case of Agarwal-Biswas formulation of primality testing.

**Claim 2.6.** *Fermat's Little Theorem is a special case of Agarwal-Biswas theorem.*

*Proof.* To prove Fermat's Little theorem, we need one direction of implication of Agarwal-Biswas theorem :

$$\text{"If } N \text{ is prime, then } (1 + z)^N = (1 + z^N) \pmod{N} \text{"}$$

Note that Fermat's Little theorem asks about  $a^N \pmod{N}$  for  $a \in \{1, 2, \dots, N-1\}$ . Since the implication talks about  $z^N \pmod{N}$  and  $(1 + z)^N \pmod{N}$ , this suggests an induction strategy on  $a$ .

Let  $N$  be prime. We check the base case : for  $a = 1$ , the  $1^N = 1 \pmod{N}$ . Hence the base case is true. By induction, suppose that  $a^N = a \pmod{N}$  for  $a \in \{1, 2, \dots, N-1\}$ . Now,

$$\begin{aligned} (a+1)^N &= (a^N + 1) \pmod{N} && \text{[By Agarwal-Biswas as } N \text{ is prime]} \\ &= (a+1) \pmod{N} && \text{[By inductive hypothesis]} \end{aligned}$$

This completes the inductive case. □

This shows that checking if the polynomial  $p_N(z)$  is a zero polynomial is an if and only if check for primality of  $N$ . So checking primality of a number now boils down to checking if a polynomial is identically zero or not. This is a fundamental problem of polynomial identity testing. We will be discussing about polynomial identity testing and AKS algorithm in our next theme.

**Remark 2.7.** *One could consider two possible definitions of a polynomial being identically zero and they are not equivalent. Indeed, if all coefficients of a polynomial are zeros then it evaluates to zero on all substitutions of the variables. However, the converse need not be true in all underlying algebraic structures. For example, consider the polynomials  $x^p - x$  in a field  $\mathbb{Z}_p$  (operations are  $+$  and  $\times$  modulo  $p$ ). This is indeed a non-zero polynomial but all evaluations modulo  $p$  are zero.*

## Algebraic Approach to finding Perfect Matchings in Graphs

### 3.1 Application to Graph algorithms

Consider the following problem of finding perfect matching.

**Definition 3.1** (Finding Perfect Matching). *Given a bipartite graph  $G(V_1, V_2, E)$ , we need to come up with an  $E' \subseteq E$  such that  $\forall u \in V_1 \cup V_2$ , there is exactly one edge incident to it in  $E'$ .*

We shall give a polynomial formulation for the problem. Given  $G(V_1, V_2, E)$  with vertex sets  $|V_1| = |V_2| = n$ . Define an  $n \times n$  matrix  $A$  where  $A(i, j) = 1$  if  $(i, j) \in E$  and is 0 otherwise for all  $(i, j) \in V_1 \times V_2$ . Recall the determinant of  $A$  given by

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n A_{i, \sigma(i)}$$

where

$$\text{sign}(\sigma) = \begin{cases} -1 & \text{if } \text{inv}(\sigma) \text{ is even} \\ 1 & \text{if } \text{inv}(\sigma) \text{ is odd} \end{cases}$$

and  $\text{inv}(\sigma)$  is defined as  $|\{(i, j) \mid i < j \text{ and } \sigma(i) > \sigma(j), 1 \leq i < j \leq n\}|$ . We denote  $f(x) \equiv 0$  to denote that polynomial  $f(x)$  is the zero polynomial.

**Lemma 3.2.** *For the matrix  $A$  as defined as before,  $\det(A) \neq 0 \Rightarrow G$  has a perfect matching*

*Proof.* Let  $\det(A) \neq 0$ . Hence there exists a  $\sigma \in S_n$  such that  $\prod_{i=1}^n A_{i, \sigma(i)} \neq 0$ . Hence the edge set  $E' = \{(i, \sigma(i)) \mid 1 \leq i \leq n\}$  exists in  $G$  and since  $\sigma(i) = \sigma(j)$  iff  $i = j$  for every  $i, j$ ,  $E'$  form a perfect matching.

□

Note that converse of this statement is not true. For example, consider the bipartite graph whose  $A$  matrix is  $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . It can be verified that  $\det(A) = 0$  but the bipartite graph associated has a perfect matching.

Hence the next natural question, similar to our primality testing problem, is to ask for some kind of modification so that converse of previous lemma (lemma 3.2) is true. In the example considered, there were two perfect matchings in  $G$  having opposite sign due to which the determinant became 0. So the modification should ensure that perfect matchings of opposite signs does not cancel off in the determinant.

One way to achieve this is as follows. Define a matrix  $T$  as

$$T(i, j) = \begin{cases} x_{ij} & (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

where  $(i, j) \in V_1 \times V_2$ . This matrix is called as the Tutte matrix. Now, if we consider determinant of this matrix, we can see that the monomials corresponding a  $\sigma \in S_n$  can be a product of at most  $n$  variables. Hence  $\det(T)$  is a polynomial in  $n^2$  variables with degree at most  $n$ .

Also in the expansion of determinant, we can observe that each term picks exactly one entry from every row and column meaning each of the entries picked is from a distinct row and column. Hence each of the them is a bijection and hence is a permutation on  $n$  elements. Also corresponding to any permutation on  $n$  elements, we can get a term. This gives us the following observation.

**Observation 3.3.** *Set of monomials in  $\det(T)$  is in one-one correspondence with set of all permutations on  $n$ .*

We now give polynomial formulation for the problem of checking perfect matching in a bipartite graph.

**Claim 3.4.** *For the matrix  $T$  as defined before,  $\det(T) \neq 0 \iff G$  has a perfect matching*

*Proof.* By  $\det(T) \equiv 0$ , we mean that the polynomial  $\det(T)$  has all coefficients as zero. ( $\Rightarrow$ ) Since  $\det(T) \neq 0$ , there exists a  $\sigma \in S_n$  such that the monomial corresponding is non-zero and by definition of determinant it must be expressed as product of some  $n$  set of variables  $\prod_i T(i, \sigma(i))$ . The variable indices gives a matching and since there are  $n$  variables this is a perfect matching.

( $\Leftarrow$ ) Suppose  $G$  has a perfect matching given by a  $M$ . Let  $\tau$  denote the permutation corresponding to  $M$ . We now need to show that  $\det(T) \neq 0$ . To prove this, it suffices to show that there is a substitution to  $\det(T)$  which evaluates to a non-zero value.

Consider the following assignment,  $\forall i, j$

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in M \\ 0 & \text{otherwise} \end{cases}$$

From the formula of determinant,

$$\begin{aligned} \det(T) &= \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n T_{i, \sigma(i)} \\ &= \text{sign}(\tau) \prod_{i=1}^n A_{i, \tau(i)} + \sum_{\sigma \in S_n \setminus \{\tau\}} \text{sign}(\sigma) \prod_{i=1}^n T_{i, \sigma(i)} \end{aligned}$$

Now substituting  $x_{ij} = a_{ij}$ , we get that the first term evaluates to  $\text{sign}(\tau)$  since all the entries are 1. The second term evaluates to 0 since for all  $\sigma \neq \tau$ , there must be a  $j$  such that  $\sigma(j) \neq \tau(j)$ . Hence it must be that  $a_{j, \sigma(j)} = 0$  and hence the product corresponding to  $\sigma$  goes to 0.  $\square$

Hence to check if the bipartite graph  $G$  has a perfect matching or not, it suffices to check if the polynomial  $\det(T)$  is identically zero or not. Checking if a polynomial is identically zero or not is one of the fundamental question in this area.

Note that this problem becomes easy if the polynomial is given as sum of monomial form. In most of the cases, the polynomial will not be given this way. For example if we consider our problem, we are just given the  $T$  matrix and  $\det(T)$  is the required polynomial. Trying to expand  $\det(T)$  and simplifying will involve dealing with  $n!$  monomials which is not feasible.

Hence the computational question again boils down to checking if a polynomial is identically zero or not.

## Graphs, Groups and Generators

In this lecture we will pose three graph theoretic questions and find answers using approaches in algebra.

### 4.1 Graph Isomorphism, Automorphism and Rigidity

**Definition 4.1.** (*Graph Isomorphism.*) Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs. We say  $G_1 \cong G_2$  (read as  $G_1$  is isomorphic to  $G_2$ ) if there exists a bijection  $\sigma : V_1 \rightarrow V_2$  such that  $\forall (u, v) \in V_1 \times V_2$  we have

$$(u, v) \in E_1 \iff (\sigma(u), \sigma(v)) \in E_2$$

In other words, we say a graph  $G_1$  is isomorphic to  $G_2$  if there exists a relabeling of the vertices in  $G_1$  such that the adjacency and non-adjacency relationships in  $G_2$  is preserved.

**Observation 4.2.** If  $|V_1| \neq |V_2|$ , then  $G_1$  is not isomorphic to  $G_2$ .

The graph isomorphism problem is stated as follows.

**Problem 4.3 (Graph Isomorphism Problem).** Given two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , test if  $G_1 \cong G_2$  or not.

A natural question to ask in this setting is that if there is an isomorphism from a graph  $G$  to itself.

Let  $[n] = \{1, 2, \dots, n\}$ . Let  $S_n$  denote the set of all permutations from the set  $[n]$  to  $[n]$ . Let  $G = (V, E)$  be a graph. An automorphism of  $G$  is a bijection  $\sigma : V \rightarrow V$  such that  $\sigma(G) = G$ . Let

$$\text{Aut}(G) = \{\sigma \mid \sigma \in S_n \text{ and } \sigma(G) = G\}$$

be the set of all automorphisms of  $G$ . Ideally, we would like to compute the set of automorphism of a graph to itself.

**Problem 4.4 (Graph Automorphism Problem).** Given a graph  $G$ , list the elements of  $\text{Aut}(G)$ .



The above problem can be expected to be solved in polynomial time only if the output expected is polynomial in length. This brings up the size of the  $\text{Aut}(G)$  into question. Unfortunately, if  $G$  is the complete graph on  $n$  vertices. Then  $|\text{Aut}(G)| = n!$ . Hence the above question is not well-formulated.

Since identity permutation is trivially an automorphism for any graph, the  $\text{Aut}(G)$  is always a non-empty subset of  $S_n$  where  $n$  is the number of vertices. Hence, one can ask a natural decision variant of the above problem, namely the graph rigidity problem.

Formally, the graph rigidity problem is stated as follows.

**Problem 4.5 (Graph Rigidity Problem).** *Given a graph  $G$ , test if  $\text{Aut}(G)$  is trivial. That is, whether  $\text{Aut}(G)$  contains only the identity permutation or not.*

More than the size, in the first lecture of this course, we have seen that  $\text{Aut}(G)$  forms a *subgroup* of  $S_n$ . To utilize this structure, we first refresh the definition of an abstract group. From now on, we will use the letter  $X$  to denote a graph and  $G$  to denote a group.

## 4.2 Groups and Generators

**Definition 4.6.** (Groups.) *A set  $G$  together with a binary operation  $*$  is said to be a group if the following four conditions are met*

- **Closure** :  $a, b \in G$ , the element  $a * b \in G$ .
- **Associative** : For any  $a, b, c \in G$ , we have  $(a * b) * c = a * (b * c)$ .
- **Existence of Identity** : For any  $a \in G$  there exists a unique element  $e \in G$  such that  $a * e = e * a = a$ .
- **Existence of Inverse** : For any  $a \in G$  there exists a unique element  $b \in G$  (denoted by  $a^{-1}$ ) such that  $a * b = b * a = e$ .

**Example 4.7.** •  $S_n$  forms a group under composition.

- $(\mathbb{Z}_5, +)$  is a group.

From now on, we will use the letter  $X$  to denote a graph and  $G$  to denote a group.

**Remark 4.8.** Let  $(G, *)$  be a group. Let  $H \subseteq G$  such that  $(H, *)$  also forms a group. We say  $H$  is a subgroup of  $G$  and denote by  $H \leq G$ .

**Exercise 4.9.** For any graph  $X$ , the set  $\text{Aut}(X)$  forms a group under the composition operation. That is,  $\text{Aut}(G) \leq M$  where  $M = (S_n, \circ)$  is the permutation group.

Let  $(G, *)$  be a group. Let  $H \subseteq G$  such that  $(H, *)$  also forms a group. We say  $H$  is a *subgroup* of  $G$  and denote by  $H \leq G$ . We showed in the first lecture of the course that, for any graph  $X$ , the set  $\text{Aut}(X)$  forms a group under the composition operation. That is,  $\text{Aut}(G) \leq M$  where  $M = (S_n, \circ)$  is the symmetric group.

Let  $(G, +)$  be a finite group and  $g \in G$  be an element. Let  $g^2 = g * g, g^3 = g * g * g$ . Similarly  $g^k = \underbrace{g * g * \dots * g}_{k \text{ times}}$ . Now consider the set  $H = \{g, g^2, g^3, \dots\}$ . Since  $(G, *)$  is a finite group there must exist a  $k$  such that  $g^k = g$  in  $H$ .

**Lemma 4.10.** *Let  $(G, +)$  be a finite group and  $g \in G$  be an element. Let  $H = \{g, g^2, g^3, \dots\}$  be a set of elements. The unique identity  $e$  of  $G$  is in  $H$ .*

*Proof.* Since  $(G, *)$  is a finite group there must exist a  $k$  such that  $g^k = g$  in  $H$ . By definition,  $g^k = g^{k-1} * g = g$ . As  $(G, *)$  is a group,  $g^{-1}$  exists in  $G$ .

Therefore,

$$g^{k-1} * g * g^{-1} = g * g^{-1} = e$$

□

**Definition 4.11 (Generator).** *Let  $(G, +)$  be a finite group and  $g \in G$  be an element. We say an element  $g \in G$  is a generator of the set  $H$  if for every element  $h \in H$  there exists a  $m$  such that  $h = g^m$ . (denoted by  $H = \langle g \rangle$ ).*

**Observation 4.12.**  $H = \langle g \rangle$  is a subgroup of  $G$ . That is,  $H \leq G$ .

A quick example is that 1 is a generator for  $(\mathbb{Z}_5, +)$

**Definition 4.13.** *An group  $(G, *)$  that can be generated by a single element is called a cyclic group. For instance,  $(\mathbb{Z}_5, +)$ .*

Not every group is cyclic. For instance, one can verify that  $S_3$  (with 6 elements in it) is not cyclic.

**Definition 4.14 (Generating Set).** *Let  $S \subseteq G$  be the set  $\{u_1, \dots, u_k\}$ .  $S$  is said to be generating if  $\langle S \rangle = G$ .*

Having observed that  $Aut(X)$  could have potentially be of exponential size, a reasonable way to formulate the graph automorphism problem is in terms of the generating set of the group. For this, we establish first that  $Aut(X)$  have a generating set of size  $\text{poly}(n)$ ? In fact any group has !.

### 4.2.1 Lagrange's theorem

Let  $(G, *)$  be a group. Let  $H \leq G$ . For any  $g \in G$ , define the right coset of  $H$  in  $G$  to be

$$Hg = \{hg \mid h \in H\}$$

Let  $g_1, g_2 \in G$  and  $Hg_1, Hg_2$  be the corresponding right cosets. Are there elements that belong to more than one coset of  $H$  in  $G$ ? Is  $Hg_1 \cap Hg_2 \neq \phi$ ? If yes, then the set of right cosets of  $H$  in  $G$  form a partition of the ground set of  $G$ . In that case, how many such cosets are required to cover the entire set  $G$ ? Let us answer these two questions.

**Lemma 4.15.** *Let  $g_1, g_2 \in G$  and  $Hg_1 = \{hg_1 \mid h \in H\}, Hg_2 = \{hg_2 \mid h \in H\}$ . Then*

$$Hg_1 = Hg_2 \text{ or } Hg_1 \cap Hg_2 = \phi.$$

*Proof.* If  $g_1 = g_2$ , then by definition  $Hg_1 = Hg_2$ . Therefore let  $g_1 \neq g_2$ . We will prove : If  $Hg_1 \cap Hg_2 \neq \phi$  then  $Hg_1 = Hg_2$ . Let  $Hg_1 \cap Hg_2 \neq \phi, g \in Hg_1 \cap Hg_2$  we will show

(i)  $Hg_1 \subseteq Hg_2$  ; and

(ii)  $Hg_2 \subseteq Hg_1$ .

Since  $g \in Hg_1$  we know that there exists a  $h_1 \in H$  such that  $g = h_1g_1$ . Similarly  $g \in Hg_2$  suggests that there exists a  $h_2 \in H$  such that  $g = h_2g_2$ .

$$h_1g_1 = h_2g_2 = g$$

As  $(H, *)$  is a group by itself,  $h_1^{-1}$  and  $h_2^{-1}$  exists.

$$g_1 = h_1^{-1}h_2g_2 \quad (4.2)$$

$$g_2 = h_2^{-1}h_1g_1 \quad (4.3)$$

(i)  $Hg_1 \subseteq Hg_2$

Let  $g' \in Hg_1$ . This implies there exists a  $h' \in H$  such that  $g' = h'g_1$ . Therefore,

$$\begin{aligned} g' &= h'g_1 \\ g' &= h'(h_1^{-1}h_2g_2) \quad [\text{By equation (4.2)}] \end{aligned}$$

By closure property in  $(H, *)$ , we have  $h'' = h'h_1^{-1}h_2 \in H$ . Therefore  $g' = h''g_2$ ,  $g' \in Hg_2$ .

(ii)  $Hg_2 \subseteq Hg_1$

Let  $g' \in Hg_2$ . This implies there exists a  $h' \in H$  such that  $g' = h'g_2$ . Therefore,

$$\begin{aligned} g' &= h'g_2 \\ g' &= h'(h_2^{-1}h_1g_1) \quad [\text{By equation (4.3)}] \end{aligned}$$

By closure property in  $(H, *)$ , we have  $h'' = h'h_2^{-1}h_1 \in H$ . Therefore  $g' = h''g_1$ ,  $g' \in Hg_1$ .

□

**Lemma 4.16.** For every  $g \in G$ ,  $|Hg| = |H|$ .

*Proof.* By construction, for every element in  $H$  there exists an element in  $Hg$ . So  $|Hg| \leq |H|$ . We first argue that  $|H| \leq |Hg|$ . Suppose not. Let  $|Hg| < |H|$ . Then there exists  $h_1, h_2 \in H$ ,  $h_1 \neq h_2$  such that  $h_1g = h_2g$ . Since  $(G, *)$  is a group,  $g^{-1}$  exists. We have  $h_1gg^{-1} = h_2gg^{-1}$  which implies  $h_1 = h_2$ , a contradiction. □

**Theorem 4.17 (Lagrange's Theorem).** Let  $(G, *)$  be a group and  $H \leq G$ . Then  $|H|$  divides  $|G|$ .

*Proof.* Direct consequence of Lemmas 4.15 and 4.16 □

**Observation 4.18.** Let  $H = \langle g \rangle$  and  $H' = \langle H, g' \rangle$  where  $g' \in G \setminus H$ . Then  $H \leq H' \leq G$ . We have  $g \in H' \setminus H$ , therefore  $|H'| > |H|$ .  $H' \leq H$ . By Theorem 4.17,  $|H'| \geq 2|H|$ . This shows that every group has a generating set of size  $\log |G|$ .

**Remark 4.19.** We know that  $\text{Aut}(X) \leq S_n$  for any graph  $X = (V, E)$ . By Observation 4.18  $\text{Aut}(X)$  has a generating set of size  $\log |S_n| = \log(n!) \in \mathcal{O}(n \log n)$ .

## Orbit-Stabilizer Lemma

We first start with some notations and definitions. Order of a group  $G$  is number of elements in the group, that is  $|G|$ . Let  $H \leq G$  and  $g \in G$ . The right coset of  $H$  in  $G$  is defined as  $Hg = \{hg \mid h \in H\}$ . If the multiplication is on the left, we call it the left coset. That is,  $gH = \{gh \mid h \in H\}$ . In general, it is not necessary that the left and right cosets are the same. If  $H$  is a group such that left and right cosets are the same element-wise, (that is,  $\forall g \in G, Hg = gH$ ),  $H$  is said to be a normal subgroup of  $G$ .

Let  $H$  be a normal subgroup of  $G$ . Choose one element from each of these as a representative of the set (say  $[g]$  denote the coset representative of the coset  $Hg = gH$ ). These elements have a group structure among them. This requires a proof, which we will come to in the next few lectures.

### 5.1 Group Action and Orbits

Let  $G$  be a Subgroup of  $S_n$ . Let  $\alpha \in [n]$  and  $g \in G$ . We denote by  $\alpha^g$  is the image of  $\alpha$  under the permutation  $g$ . Orbit of an element  $\alpha$  in  $G$  is the set of elements it gets mapped to under permutations in  $G$ . More formally,

**Definition 5.1 (Orbit of  $\alpha$  in  $G$ ).** *The orbit of  $\alpha$  in  $G$  is defined as*

$$\alpha^G = \{\alpha^g \mid g \in G\}$$

This defines a natural relation among elements of  $[n]$ .

$$\alpha \sim \beta \leftrightarrow \exists g \in G, \alpha^g = \beta$$

We check that this is an equivalence relation.  $e \in G$ , where  $e$  is the identity element. Hence,  $\alpha \sim \alpha$ . Let  $\alpha \sim \beta$ . Thus  $\exists g \in G, \alpha^g = \beta$ . Hence,  $\alpha = \beta^{g^{-1}}$ . Thus,  $\beta \sim \alpha$ . The relation  $\sim$  is an equivalence relation. For transitivity, let  $\alpha \sim \beta, \beta \sim \gamma$ . By definition,  $\exists g_1, g_2 \alpha^{g_1} = \beta, \beta^{g_2} = \gamma$ . By composition of permutations,  $(\alpha^{g_1})^{g_2} = \gamma$ . Hence,  $\alpha \sim \gamma$ .

The stabilizer of  $\alpha$  in  $G$  is

$$G_\alpha = \{g \mid \alpha^g = \alpha\}$$

. This is the set of elements in  $G$  which sends  $\alpha$  to  $\alpha$  itself.

### 5.1.1 Orbit-Stabilizer Lemma

Is there any connection between the number of permutations in  $G$  that fixes  $\alpha$  and the number of elements to which  $\alpha$  can be taken to? The orbit stabilizer lemma, which is an easy consequence of Lagrange's theorem gives a neat answer.

**Lemma 5.2 (Orbit-Stabilizer Lemma).** *Let  $G \leq S_n$ . Then for any  $\alpha \in [n]$ ,*

$$|\alpha^G| * |G_\alpha| = |G|$$

*Proof.* We quickly observe that  $G_\alpha$  is a Subgroup of  $G$ . Indeed, identity belongs to  $G_\alpha$  trivially. If  $g$  and  $g'$  both fixes  $\alpha$ , then so does  $gg'$  and  $g'g$ . If  $g$  fixes  $\alpha$ , then so does  $g^{-1}$ . Since  $G_\alpha$  forms a Subgroup of  $G$ , by Lagrange's Theorem,

$$\frac{|G|}{|G_\alpha|} = \text{number of distinct right cosets of } G_\alpha \text{ in } G$$

To complete the proof, it suffices to argue that the number of distinct cosets of  $G_\alpha$  in  $G$  is the size of the orbit of  $\alpha$  under the action of  $G$ . We now show the bijection.

Let  $\beta \in \alpha^G$  via  $h \in G$ . That is,  $\alpha^h = \beta$ . Consider the map,

$$\Gamma : \beta \rightarrow \{g \in G \mid \alpha^g = \beta\}$$

We first show that this is well-defined map between the elements in the orbit of  $\alpha$  to the cosets. For that, we first show that  $\{g \in G \mid \alpha^g = \beta\}$  is indeed a right coset of  $G_\alpha$  in  $G$ . To argue this, it suffices to show that

$$\begin{aligned} \Gamma(\beta) &= \{g \in G \mid \beta = \alpha^g\} = \{g \in G \mid \alpha^h = \alpha^g\} = \{g \in G \mid \alpha^{gh^{-1}} = \alpha\} \\ &= \{g \in G \mid gh^{-1} \in G_\alpha\} = \{g \in G \mid g \in G_\alpha h = G_\alpha h \end{aligned}$$

Thus  $\Gamma$  is a function.

We now show that the function  $\Gamma$  is injective. Let  $\beta$  and  $\gamma$  be two different elements of the orbit of  $\alpha$  via the group elements  $h_\beta$  and  $h_\gamma$ . That is,

$$\Gamma(\beta) = \{g \in G \mid \alpha^g = \beta\}$$

$$\Gamma(\gamma) = \{g \in G \mid \alpha^g = \gamma\}$$

Indeed, these two sets cannot have an intersection since  $\beta \neq \gamma$ . Thus,  $\Gamma(\beta) \neq \Gamma(\gamma)$ .

We now argue that the function  $\Gamma$  is surjective. Consider any coset  $C = G_\alpha g$  of  $G_\alpha$  in  $G$ , where  $g \in G$ . We show that there is a  $\beta \in \alpha^G$  such that  $\Gamma(\beta) = C$ . Indeed, define  $\beta$  to be  $\alpha^g$ . Consider any  $h \in G$  such that  $\alpha^h = \beta$ .

1: JS says: Surjectivity proof to be completed

□

## 5.2 Graph Automorphism and Graph Isomorphism

We define the problems that we are interested in, technically.

**Problem 5.3** (GRAPH ISOMORPHISM PROBLEM (GI)). *Given a graph  $X_1 = (V_1, E_1)$  and  $X_2 = (V_2, E_2)$ , decide if  $X_1 \cong X_2$  or not.*

**Problem 5.4** (GRAPH AUTOMORPHISM PROBLEM (GA)). *Given a graph  $X = (V, E)$ , compute a Generating Set for  $\text{Aut}(X)$ .*

**Problem 5.5** (GRAPH RIGIDITY PROBLEM (GR)). *Given a graph  $X = (V, E)$ , decide if  $\text{Aut}(X)$  is trivial or not.*

**Problem 5.6** (COUNTING ISOMORPHISMS (#GI)). *Given a graph  $X_1 = (V_1, E_1), X_2 = (V_2, E_2)$ , output the number of Isomorphisms from  $X_1$  to  $X_2$ , in binary.*

**Problem 5.7** (COUNTING AUTOMORPHISMS (#GA)). *Given a graph  $X = (V, E)$ , compute the size of the automorphism group,  $|\text{Aut}(X)|$ , in binary.*

**Problem 5.8** (COMPUTING THE ISOMORPHISM (ISO)). *Given a graph  $X_1 = (V_1, E_1), X_2 = (V_2, E_2)$ , output an isomorphism map between  $V_1$  and  $V_2$  if it exists.*

We use the notion of polynomial time reducibility between two problems. Given two problems  $A, B$ , we say that  $A \leq B$  ( $A$  reduces to  $B$ ), if given a polynomial time (in terms of input size  $n$ ) algorithm for  $B$ , we can give out a polynomial time algorithm for  $A$ . We quickly observe some easy relationship among these problems. Indeed, if we can solve  $GA$ , we can solve  $GR$  as well. Given a graph  $X$ , to check if it is rigid, it is only a matter of checking if there is a nontrivial element in the generating set for the group  $\text{Aut}(X)$ . Hence we conclude that  $GR \leq GA$ . The same is the case for  $GR$  and  $\#GA$  as well. That is,  $GR \leq \#GA$ . Similarly, if we can compute the isomorphism, we can decide it as well. Trivially,  $GI \leq ISO$ . It is interesting to think about whether  $\#GA$  can be done using  $GA$ . That is, given the generating set of a permutation group (that is, subgroup of  $S_n$ ), can we compute the size of the generated group?

## Reduction Between Variants of GI

Our aim is to understand the relationship between various problems related to graph isomorphism that we listed in the last lecture. To quote the names, we talked about, graph isomorphism problem (GI), graph automorphism problem (GA), graph rigidity problem (GR), counting isomorphisms (#GI), counting automorphisms (GA), and computing the isomorphism (ISO). As a main tool towards understanding these, we now introduced a colored version of graph isomorphism.

### 6.1 Colored Graph Isomorphism Problem

The driving thought is the following. Consider the following problem. Given two graphs  $X_1$  and  $X_2$ , and consider a vertex  $u \in V_1$  and  $v \in V_2$ . Can we test if there is an isomorphism between  $X_1$  and  $X_2$  that maps  $u$  to  $v$  itself? To create a terminology, the scenario can also be described as : we will color vertex  $u$  and  $v$  with a color (say *blue*), the rest of the vertices in both  $X_1$  and  $X_2$  as red, and ask if there is a *color preserving isomorphism* between the graphs.

More formally, let  $X = (V, E)$  be a graph. Consider a coloring function  $\Psi : V(X) \rightarrow [c]$ , where  $i \in [c]$  denotes a color. Thus, for a vertex  $v$ ,  $\Psi(v)$  denotes its color. We call the set of vertices  $\Psi^{-1}(i)$  as the  $i^{\text{th}}$  color class, and we call the graph as  $c$ -colored.

**Problem 6.1** (COLORED GRAPH ISOMORPHISM (CGI)). *Given two  $c$ -colored graphs  $(X_1, \Psi_1)$  and  $(X_2, \Psi_2)$  decide if there exists  $\sigma : V(X_1) \rightarrow V(X_2)$  such that*

- *for all  $(u, v) \in V(X_1) \times V(X_2)$ ,  $(u, v) \in E(X_1)$  if and only if  $(\sigma(u), \sigma(v)) \in E(X_2)$*
- *for every  $u \in V(X_1)$ ,  $\Psi_1(u) = \Psi_2(\sigma(u))$  .*

How hard can colored graph isomorphism be? In general can there be an efficient algorithm which solves CGI for any  $c$ ? Indeed, an easy observation is that GI is a special case when  $c = 1$ . That is, give all vertices in the graphs the same color so that any isomorphism preserves the colors. Thus, CGI seems harder when number of vertices in a color class is allowed to be large. Indeed, later in the course, when the number of vertices in any color class is bounded by a constant, we will give a polynomial time algorithm for the problem. However, without any restrictions the problem is as hard as graph isomorphism. However, what about the cases when  $c > 1$ . at a first sight, they dont seem easier than GI. We start by showing that they are not harder for sure.

### 6.1.1 Gadget Trick : From Colored GI to Non-colored GI

We show that  $CGI \leq GI$ . That task is as follows, given  $(X_1, \Psi_1)$  and  $(X_2, \Psi_2)$  construct graphs  $X'_1$  and  $X'_2$  (uncolored graphs) such that

$$((X_1, \Psi_1), (X_2, \Psi_2)) \in CGI \iff (X'_1, X'_2) \in GI$$

The construction of  $X'_1$  and  $X'_2$  are as follows : For every vertex  $u \in V(X_1)$  such that  $u \in \Psi_1^{-1}(i)$ :

1. Add  $ni$  extra vertices to  $X_1$ .
2. Add edges from each of the extra vertices to  $u$ .

Do the same for  $(X_2, \Psi_2)$  to get  $X'_2$ . This completes the reduction.

We denote by  $Y_u$  the extra vertices that we added for the vertex  $u$ . Notice that the degree of all extra vertices added is 1 each, and the degree of all original vertices in color class  $i$  ( $i \geq 1$ ) is at least  $ni$ . The number of vertices in the new graph produced is at most  $n + \sum_{i=1}^c ni \leq O(n^2)$ . The reduction runs in polynomial time since we can construct the graphs  $X'_1$  and  $X'_2$  in polynomial time. Thus the only thing remaining is to prove the correctness of the reduction which we do by the following claim.

**Claim 6.2.**  $((X_1, \Psi_1), (X_2, \Psi_2)) \in CGI \iff (X'_1, X'_2) \in GI$

*Proof.* ( $\implies$ ) Let  $((X_1, \Psi_1), (X_2, \Psi_2)) \in CGI$ . We need to show that  $X'_1 \cong X'_2$ . From the assumption, we know that there exists  $\sigma : V(X_1) \rightarrow V(X_2)$  such that :

1.  $\forall (u, v) \in V(X_1) \times V(X_2), (u, v) \in E(X_1)$  if and only if  $(\sigma(u), \sigma(v)) \in E(X_2)$ .
2.  $\forall u \in V(X_1), \Psi_1(u) = \Psi_2(\sigma(u))$ .

To extend this to an isomorphism  $\sigma'$  between  $X'_1$  and  $X'_2$ , we need to define the images of the extra vertices that we added in the above construction. However, since the  $\sigma$  is color preserving, we know that for any  $u \in V(X_1)$ ,  $|Y_u| = |Y_{\sigma(u)}|$ . Extend  $\sigma$  to  $\sigma'$  by choosing any bijection between the vertices  $Y_u$  and  $Y_{\sigma(u)}$ .

We now argue that  $\sigma'$  is an isomorphism. Observe that, for any  $u \in V(X_1)$ , the only edges incident on  $Y_u$  are of the form  $(a, u)$  where  $a \in Y_u$ . Consider the pair  $(\sigma'(a), \sigma'(u))$  which is same as  $(a', \sigma(u))$  where  $a' \in Y_{\sigma(u)}$ . This is an edge in  $X'_2$  by construction. The converse of the above implications also hold, and hence  $X'_1 \cong X'_2$ .

( $\impliedby$ ) Suppose  $X'_1 \cong X'_2$ . We need to show that  $(X_1, \Psi_1) \cong (X_2, \Psi_2)$ . By assumption, there is a bijection  $\sigma : V(X'_1) \rightarrow V(X'_2)$  such that  $\forall (u, v) \in E(X'_1), (\sigma(u), \sigma(v)) \in E(X'_2)$ .

To begin with, we argue that  $\sigma$  must map elements of  $V(X_1)$  to  $V(X_2)$  itself. This is simply because the degrees of the vertices outside  $V(X_2)$  are all 1 and the degree of all vertices in  $V(X_1)$  in the graph  $X'_1$  are all at least  $n$ . Since  $\sigma$  is originally an isomorphism,  $\sigma$  restricted to  $V(X_1)$  immediately gives an isomorphism between  $X_1$  and  $X_2$ .

We now need to argue that the map  $\sigma$  preserves color. Suppose not. Let  $u \notin \Psi_2^{-1}(i)$ . Hence  $\sigma(u) \in \Psi_2^{-1}(j)$  and  $j \neq i$ . Note that  $ni + n > \deg(u) \geq ni$ . This implies  $n(i + 1) > \deg(u) \geq ni$ . And,  $\sigma(u) \in \Psi_2^{-1}(j) \Rightarrow n(j + 1) > \deg(u) \geq nj$ . Since both of these can not be true simultaneously, we have a contradiction. Hence,  $\sigma(u) \in \Psi_2^{-1}(i)$ . Thus,  $(X_1, \Psi_1) \cong (X_2, \Psi_2)$ .  $\square$



### 6.1.2 Computing $\text{ISO} \leq \text{GRAPHISO}$

We show that  $\text{ISO} \leq \text{CGI}$ . As  $\text{CGI} \leq \text{GI}$  we have  $\text{ISO} \leq \text{GI}$ .

Given  $X_1, X_2$  and oracle access to a blackbox which checks for isomorphism between any two given graphs, output a permutation that maps  $X_1$  to  $x_2$ .

The Reduction is as follows:

1. Check if  $X_1 \cong X_2$ . If NO, end.
2. For each vertex  $i_1 \in [n] = V_1$   
 Get  $X'_1$  by coloring  $i_1$  with color  $i_1$ .
  - Get  $X'_2$  by coloring  $i_2 \in V_2$  with color  $i_1$ .
  - Query CGI to check if  $X'_1 \cong X'_2$ .
  - Remove colors from  $X'_2$  and repeat the last two steps for  $i_2 \in V_2$  until we get a "yes" answer. For the  $i_2$  on which we get a "yes" answer, fix the color as  $i_1$  and do not reuse color  $i_1$  again in the outer loop.
3. Output the permutation.

To see the correctness : suppose the graphs are isomorphic. That is the algorithm reaches step (2), 3rd subpart is guaranteed to find a  $j$  for each vertex  $i$ . Observing that we will color a vertex on the left and the right with some color  $i_1$  on if we are sure that there is a color preserving isomorphism. Hence the isomorphism map, if exists, will be found. To see the running time, observe that we make atmost  $O(n^2)$  queries to  $\text{CGI}$  which in turn makes a single query each to  $\text{GI}$ . Thus we can solve  $\text{ISO}$  using at most  $O(n^2)$  queries to  $\text{GI}$ .

### 6.1.3 $\text{GRAPHISO} \leq \text{GRAPHAUTO}$

We need to construct a graph  $G$  such that the generating set  $S$  of  $\text{Aut}(G)$  enables to decide if  $X_1 \cong X_2$ .

Assume that the graphs are connected to begin with. The reduction is simply to take  $X = X_1 \cup X_2$ . Let  $S$  be the generating set of the reduction, check if there exists a  $\sigma \in S$  such that  $\sigma$  maps atleast one vertex in  $X_1$  to a vertex in  $X_2$ . We directly show the correctness of the reduction.

**Claim 6.3.**  $X_1 \cong X_2$  if and only if there exists a  $\sigma \in S$  and  $v \in V(X_1)$  such that  $\sigma(v) \in V(X_2)$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $X_1 \cong X_2$ . Then there exists a  $\tau$  which is an isomorphism from  $X_1$ , to  $X_2$ .  $\tau \in \text{Aut}(X)$ . Hence, there is a  $\sigma$  that maps a vertex in  $X_1$  to a vertex in  $X_2$ .

( $\Leftarrow$ ) Let there exist a  $\sigma \in S$  such that  $\sigma$  maps  $u \in X_1$  to a vertex in  $X_2$ . Let  $v \in X_1$  be such that  $\sigma(u) \in X_1$ . Since  $X_1$  is connected  $u, v$  are connected. But  $\sigma(u) \in X_2, \sigma(v) \in X_1$  are not connected which is a contradiction. Therefore for  $\sigma \in S$ ,  $\sigma$  maps all the vertices in  $X_1$  to  $X_2$ . Thus  $X_1 \cong X_2$ .  $\square$

What do we do if the graphs are not connected? We simply add an extra vertex each to both the graphs that is adjacent to all the vertices in the corresponding graphs. Since the new vertices have degree  $n$ , they can only be mapped to each other by any isomorphism (or an automorphism of  $X$ ). Thus  $\text{GI} \leq \text{GA}$ .

## Reduction of $\mathcal{GA}$ to $\mathcal{GI}$

### 7.1 Recap and Lecture overview

Let us denote by  $\text{Color} - \text{GI}$  the problem of finding whether there is a coloring preserving isomorphism. In the previous few lectures we showed that  $\text{Color} - \text{GI} \leq \mathcal{GI}$ . We also showed that  $\mathcal{GI} \leq \mathcal{GA}$ . We also showed how to compute an isomorphism map between two graphs ( $\text{COMPUTE-ISO}$ ) if we can check if two graphs are isomorphic. Recall that  $\text{Aut}(X)$  is the group of all automorphisms of a given graph  $X$  and  $\mathcal{GA}$  is the problem of obtaining a generator for  $\text{Aut}(X)$ . In this lecture, we show that  $\mathcal{GA} \leq \mathcal{GI}$ . That is, given a subroutine to solve  $\mathcal{GI}$ , we show how we can compute a generator set of  $\text{Aut}(X)$  for an input graph  $X$ . Along with  $\mathcal{GI} \leq \mathcal{GA}$  proved in the earlier class, this shows that the graph isomorphism and graph automorphism problems are equivalent in terms of hardness.

The main idea is that there is a unique way to express an element in  $G$  using Tower of subgroups of  $G$ .

### 7.2 Solving Graph Automorphism using Graph Isomorphism

#### 7.2.1 Tower of Subgroups of Group

**Definition 7.1** (Tower of Subgroups of  $G$ ). Let  $k \in \mathbb{N}$ . For a group  $G$ , the  $k$  subgroups of  $G$  namely  $G^{(1)}, G^{(2)}, \dots, G^{(k)} = \{id\}$  is said to form a tower of subgroups of  $G$  if

$$G = G^{(0)} \geq G^{(1)} \geq \dots, G^{(k)} = \{id\}$$

The above concept is defined for an arbitrary group, but we will use this specifically for understanding about Tower of subgroups of  $\text{Aut}(X)$ , for a given graph  $X$ .

Note that in the sequence,  $G^{(i)}$  is a subgroup of  $G^{(i-1)}$  for  $1 \leq i \leq k$ . Hence there is a coset structure that  $G^{(i)}$  generates (or induces) in  $G^{(i-1)}$  and we seek to exploit this structure to get a  $O(n \log n)$  sized generating set of  $\text{Aut}(X)$  efficiently, given a procedure for solving  $\mathcal{GI}$ .

**Definition 7.2.** For a group  $G$  and a subgroup  $H$  of  $G$  denote  $H : G$  to denote the set of cosets of  $H$  in  $G$  and the number of cosets in  $H : G$ , denoted,  $[H : G]$  is called as the index of  $H$  in  $G$ .



FIGURE 7.1: Tower of groups of  $G$  for two levels with cosets in the first level fixed as  $u_1, u_2, \dots, u_{\ell_1}$  and second level fixed as  $w_1, w_2, \dots, w_{\ell_2}$ .

For example, for  $1 \leq i \leq k$ ,  $[G^{(i+1)} : G^{(i)}]$  denotes the number of cosets of induced in  $G^{(i)}$  by  $G^{(i+1)}$ . Denote this number by  $\ell_i$ . Note that by Lagrange's theorem  $\ell_i = \frac{|G^{(i)}|}{|G^{(i+1)}|}$ . Hence  $\prod_i \ell_i = |G^{(0)}| = |G|$ .

Recall the notion of a *coset representative* of a coset, with respect a given group and a subgroup of the given group. For example, in the figure 7.1, consider the subgroup  $G^{(1)}$  of  $G$ . Here we choose  $u_1$  as the coset representative of the coset  $G^{(1)}u_1$ .

Note that any arbitrarily chosen element of that coset  $G^{(1)}u_1$  can be made its representative due to the following reason. For any two elements in the same coset, say  $g$  and  $g'$  we have  $G^{(1)}g = G^{(1)}g'$ , by the definition of the coset (since it generates the same coset)<sup>3</sup>. Hence, there is nothing special about  $u_1, g, g'$  and any elements of the coset can be chosen as its representative.

### 7.2.2 Unique representation of a group in terms of coset representatives

Given this setup, we are now ready to give a unique way of representing group elements once we fix a tower of subgroups for the group and a coset representatives at each level.

Consider an element  $g \in G^{(i-1)}$  for some  $i \geq 1$  and  $G^{(i+1)}$  be the subgroup in the next level in the tower of subgroups. Since the cosets of  $G^{(i+1)}$  partitions  $G^{(i)}$  it must be that  $g$  must lie in exactly one coset.

**Observation 7.3.** Consider the groups  $G^{(i)}$  and  $G^{(i-1)}$  for some  $i \geq 1$ . Let  $g \in G^{(i-1)}$  lie in the coset whose representative is  $u_1$ . Then there exists a unique  $h_i \in G^{(i)}$  such that

$$g = h_i u_1$$

<sup>3</sup>One way to show this is as follows : let  $g, g' \in G^{(1)}u_1$  where  $u_1$  is a coset representative. Hence  $g = h_1 u_1$ ,  $g' = h_2 u_1$  for  $h_1, h_2 \in G^{(1)}$ . For any  $w \in G^{(1)}g$ ,  $w = hg$  for some  $h \in G^{(1)}$ . Using the fact that  $g = h_1 u_1$ , we get  $w = h h_1 u_1$ . Hence  $w \in G^{(1)}u_1$ . Hence  $G^{(1)}g \subseteq G^{(1)}u_1$ . A similar argument shows that  $G^{(1)}u_1 \subseteq G^{(1)}g$ . This also shows that  $G^{(1)}u_1 = G^{(1)}g$ . Hence the cosets formed by  $g$  and  $g'$  are the same as the original coset.

*Proof.* By definition,  $g \in G^{(i)}u_1 \Rightarrow \exists h_i \in G^{(i)}$  such that  $g = h_i.u_1$ . Such a  $h_i$  is unique since if there is another  $h'_i \in G^{(i)}$  such that  $h_i u_1 = h'_i u_1$ , then  $h_i = g.u_1^{-1} = h'_i$ .  $\square$

This gives us a way to uniquely represent the elements of  $G$ .

**Theorem 7.4.** *Suppose we fix the tower of subgroups of  $G$  denoted  $G = G^{(0)} \geq G^{(1)} \geq \dots, G^{(k)} = \{id\}$  as well as the coset representatives for  $G^{(i+1)} : G^{(i)}$  for all  $0 \leq i \leq k-1$  in the tower of subgroups. Then, each element of the group  $G$  can be represented as a unique product of coset representatives.*

*Proof.* Let  $g \in G = G^{(0)}$ . Let  $\ell_1$  be the number of cosets in  $G^{(1)} : G^{(0)}$ . Let  $u_1, u_2, \dots, u_{\ell_1}$  be the coset representatives. By the above observation 7.3 (setting  $i = 1$ ), we know that there exists a unique  $h_1 \in G^{(1)}$  such that  $g = h_1 u_1$ . Now consider the cosets  $G^{(2)} : G^{(1)}$ . Since they partition  $G^{(1)}$ , we ask, in which coset does  $h_1$  belong to? By the application of the same claim to  $h_1$  and the pair of groups  $(G_1, G_2)$  we get a unique  $h_2$  such that  $h_1 = h_2 u_2$ , where  $u_2$  is the coset in which  $h_1$  belongs. Hence  $g$  can be expressed as  $h_2 u_2 u_1$ .

Continuing in this way, as we go down the tower of subgroups while fixing the coset representations, in each stage we get a unique  $h_i$  and  $u_i$  whose product gives  $h_{i-1}$  and we end up with a unique representation for the element  $g$ . Note that this representation is unique since at each stage the  $h_i$  were found in a unique way, and the coset representatives are fixed.  $\square$

Note that this also gives us a way to solve  $\#\mathcal{GA}$ . If we can count the number of coset representatives at each level (which is  $\ell_i$  by our notation), then  $\prod_i \ell_i = |G|$ .

### 7.2.3 Finding a generating set for $Aut(X)$

To find a generating set for  $Aut(X)$  it suffices to find a tower of sub-groups of  $Aut(X)$  and the coset representatives at each level.

Applying Theorem 7.4 for  $Aut(X)$ , we can represent each element of  $Aut(X)$  uniquely as a product (which in our case is composition operation) of a sequence of coset representatives once we define a suitable tower of subgroups for  $Aut(X)$  and compute the coset representatives efficiently. Hence it suffices to output these representatives to obtain a generating set.

We define the tower of subgroups in such a way that computing the coset representatives becomes efficient (using  $\mathcal{GI}$ ). Denote  $G^{(0)}$  as  $Aut(X)$ . The subgroups are defined as, for  $0 \leq i \leq n-1$ ,

$$G^{(i+1)} := \{g \in G^{(i)} \mid i^g = i\}$$

That is,  $G^{(i)}$  is the sub-group of automorphisms which maps all the nodes of the graph in the range  $\{1, \dots, i\}$  to themselves.

We can check that  $G^{(i)}$  is indeed a group since the composition of two permutations which maps all the nodes of the graph in the range  $\{1, \dots, i\}$  to themselves also does the same. Moreover, the identity permutation is the identity for this sub-group too, and the inverse permutation is defined in the standard way and satisfy the property of inverse element of a group. Hence, this is indeed a subgroup. By the definition of  $G^{(i)}$  it can be verified that the subgroups defined indeed forms a tower of subgroups of  $Aut(X)$ .

Now the task is reduced to finding the coset representatives of  $G^{(i+1)}$  in  $G^{(i)}$ .

**Claim 7.5.** *Let the number of cosets generated by  $G^{(i+1)}$  in  $G^{(i)}$  be  $\ell_i$ . Then*

$$\ell_i \leq n - i$$

*Proof.* Consider  $g \in G^{(i)}$ . Note that  $g$  maps the element  $i + 1$  to an element in the range  $\{i + 1, \dots, n\}$ . This is because the other elements are already fixed. Let  $k = (i + 1)^g$  be the element to which  $i + 1$  is mapped and  $r$  be the element such that  $r^g = i + 1$ . Then consider the permutation  $g'$  which retains other mappings from  $g$  but changes the above two mappings to  $(i + 1)^{g'} = i + 1$  and  $r^{g'} = k$ . Note that  $g'$  is also an automorphism since  $r$  being mapped to  $i + 1$  implies that if  $r$  is mapped to the image of  $i$  the automorphism would still be preserved (adjacency and non-adjacency is preserved). Also note that  $g'$  maps  $\{1, \dots, i + 1\}$  and hence is in  $G^{i+1}$ . This means that whatever is the number of automorphisms in  $G^i$  cannot exceed the number of automorphisms in  $G^{i+1}$  multiplied by  $n - i$  since the  $i + 1$  can potentially map to only  $n - i$  elements.  $\square$

This claim shows that at each level the number of representatives that we need to output is at most  $n - i$  and hence the size of generating set collected over all levels is at most  $\sum_{i=1}^n (n - i) = O(n^2)$ . The algorithm for finding the coset representatives is the following. We explain how to get the representatives for a level  $i$ .

Look for automorphisms in  $G^{(i)}$  which map all of  $\{1, \dots, i\}$  to themselves and map  $i + 1$  to each element in the set  $\{i + 1, \dots, n\}$ , one by one and ask the question: is there an automorphism which preserves these mappings. This gives us the cosets since the only freedom in the coset representatives is in where  $(i + 1)$  maps to.

We answer this using **Color – GI** problem. Since we already have a solution to  $\mathcal{GI}$ , as demonstrated in the previous lectures, we can use this to also solve **Color – GI**. Now, this can be done by using **Color – GI** in the following way.

1. Make two copies of the input graph  $X$  and call it  $X_1$  and  $X_2$ .
2. Colour the vertex  $v_k \in \{v_1, \dots, v_i\}$  by with the color  $k$  for both the graphs  $X_1$  and  $X_2$ .
3. Colour the vertex  $v_{i+1}$  in  $X_1$  and  $X_2$  with the same colour  $i + 1$  and use a new colour  $i + 2$  and colour the rest of the vertices in both the graphs with the colour  $i + 2$ .

In this way, we can cast the problem as a **Color – GI** and solve it using  $\mathcal{GI}$ . Each time we get an answer as yes, we then use the reduction  $\text{Color – GI} \leq \mathcal{GI}$  to find the actual permutation. This permutation is one of the coset representatives and hence in this manner we get each coset representative. Note that for each search for coset representative we make only polynomially many calls to  $\mathcal{GI}$  and the number of such representatives is upper bounded by  $n^2$  and hence our reduction is polynomial time and computes a generating set of  $\text{Aut}(X)$  in polynomial time, given that we can solve  $\mathcal{GI}$  in polynomial time.

Note that this procedure can be modified not only to compute an automorphism, but can also be used to compute  $|\text{Aut}(X)|$ . The reason is that the  $\ell_i$  which corresponds to number of coset representatives in  $G^{(i+1)} : G^{(i)}$  can be exactly computed in our setting. Our final task is to obtain a permutation in  $G^{(i)}$  that sends  $i + 1$  to  $\{i + 1, \dots, n\}$ . The set of permutations which satisfy this is nothing but the orbit of  $i + 1$  in  $G^{(i)}$ . Hence by Orbit stabilizer theorem,  $|G^{(i)}| = \left| (i + 1)^{G^{(i)}} \right| \dots |G^{(i+1)}|$ . This gives that  $\left| (i + 1)^{G^{(i)}} \right| = \frac{|G^{(i)}|}{|G^{(i+1)}|} = \ell_i$ . This tells that to obtain  $\ell_i$  it suffices to estimate the size of the orbit of  $i + 1$ .

The generating set that we obtained by fixing a tower of subgroups and coset representatives have many nice properties. We will see more about them in subsequent lectures.

**Definition 7.6** (Strong Generating Set). *A generating set for  $\text{Aut}(X)$  obtained in the manner above making use of coset representatives and tower of sub-groups is known as a strong generating set.*

## Some Group-theoretic problems in Permutation Groups

### 8.1 Recap

In the previous lecture we showed that  $\mathcal{GA} \leq \mathcal{GI}$ . We also defined certain generic group theoretic concepts like *tower of sub-groups*, *coset representatives* and the notion of *strong generating sets*. Also note that the reduction  $\# \mathcal{GA} \leq \mathcal{GI}$  can be done using the set of reductions  $\# \mathcal{GA} \leq \mathcal{GA} \leq \mathcal{GI}$  where the first reduction follows from the fact that the generating set of  $\text{Aut}(X)$  was in fact a strong generating set and hence each element of  $G$  could be obtained uniquely by composing elements of the generating set. This would imply that the size of the automorphism group is  $\prod_i \ell_i$  where  $\ell_i$  is

the number of cosets in the  $i^{\text{th}}$  level of the tower of subgroups of  $G$ .

Our aim is to cast the graph theoretic problems in group theoretic terms and solve the problem using the machinery of group theory. Towards this aim, we abstract the various problems and questions which were encountered while doing the previous reductions and give a set of four related group-theoretic problems.

### 8.2 Some Computational Questions in Group theory

Following are two natural question to ask.

**Problem 8.1 (Order Computation).** *Given a group  $G$  via its generating set, can we compute the order of the group, that is, the number of elements in the group.*

**Problem 8.2 (Membership Testing).** *Given an element  $g \in G$  and an  $H \leq G$  expressed via a generating set  $S$  (i.e.  $\langle S \rangle = H$ ), test if  $g \in H$  or not.*

Note that problem 2 can be solved using problem 1. This is because we could just add  $g$  to the generating set  $S$  of  $H$  and use problem 1 to obtain the sizes of the groups generated by  $S$  and  $S \cup \{g\}$ . If the sizes are unequal we conclude that  $g$  was not in  $H$ . This is because if  $g$  were in  $H$ , then the generating set  $S$  could have generated  $g$  as well and adding it to  $S$  could not have resulted in any new elements.

Here is a recursive strategy to solve Problem 1 (Order computation). The orbit stabilizer lemma that for any  $\alpha \in G$ ,

$$|\alpha^G| \cdot |G_\alpha| = |G|$$

. Suppose we can estimate  $|\alpha^G|$  which is the size of the orbit, then to compute  $|G|$ , we are left with the smaller recursive subproblem of estimating the size of  $|G_\alpha|$ .

To implement this strategy, we need to solve the two subproblems. (1) finding the size of the orbit and (2) finding the generating set of  $G_\alpha$ . Moreover, in last lecture, we also wanted to count the number of permutations in  $G^{(i)}$  which first  $i$  elements identically and maps the  $i + 1^{th}$  element to any of the  $n - i$  elements. We saw that this is actually a question of obtaining the size of orbit. This motivates the following question.

**Problem 8.3 (Orbit Computation).** *Given a group  $G \leq S_n$  (via its generating set  $S$ ), compute the orbits of the action of  $G$  on  $[n]$ .*

**Diversion:** The following is a slight diversion to address a question raised in class. So far we were interested in groups which are subgroups of  $S_n$ . But how do we answer the same questions for general abstract groups? We show that an abstract group can nevertheless be viewed as a subgroup of a permutation group.

**Claim 8.4.** *A group  $G$  acts on itself.*

*Proof.* Consider an element  $g \in G$ . Consider any arbitrary ordering of the elements in  $G$ . Then the multiplication of  $G$  by  $g$ ,  $G.g$  sends the elements of  $G$  to a permutation of themselves<sup>4</sup>. Hence, with each element  $g \in G$ , we can associate a permutation of the elements of  $G$  in the following way : if  $G = \{g_1, g_2, \dots, g_k\}$  then for a  $g_i \in G$ , corresponding  $\sigma_i \in S_k$  is defined as the one which satisfy  $(g_i g_1, g_i g_2, \dots, g_i g_k) = (g_{\sigma_i(1)}, g_{\sigma_i(2)}, \dots, g_{\sigma_i(k)})$

Collect all the permutations associated with the group elements and call it  $H$ . Note that the resulting set of permutations is forms a group since multiplication in original group translates to composition in the new group<sup>5</sup> and the identity element in the original group is associated with the identity permutation and so on.

Hence if the order of the group is  $k$ , then the resulting group of permutations is a sub-group of  $S_k$ . This defines the notion of a group acting on itself.  $\square$

**Note 8.5.** *Hence from now on, we will assume that  $G$  acts on an arbitrary set  $\Omega$  which we can think  $[n]$ .*

End of diversion.

## 8.3 Set Stabilizer Problem

We know that  $Aut(X)$  acts on  $[n]$ . This can also be visualized as  $Aut(X)$  acting on the set of potential edges in the graph  $X$ . What does it do to the actual edges in this action? Indeed, the

<sup>4</sup>The reason is that the resulting operation acts bijectively on the elements. That is if  $g_1 \neq g_2$  then  $gg_1 \neq gg_2$ . Also for any  $h = g_i x$ , there is a unique solution  $x = hg_i^{-1}$  in  $G$

<sup>5</sup> The statement amounts to showing that for any  $g_1, g_2 \in G$  with  $\sigma_1, \sigma_2 \in H$  as the corresponding elements defined by the map,  $g_1 g_2 \in G \iff \sigma_1 \circ \sigma_2 \in H$ . This is because,

$$\begin{aligned} g_1 g_2 (g_1, g_2, \dots, g_k) &= (g_1 g_{\sigma_2(1)}, g_1 g_{\sigma_2(2)}, \dots, g_1 g_{\sigma_2(k)}) \\ &= (g_{\sigma_1(\sigma_2(1))}, g_{\sigma_1(\sigma_2(2))}, \dots, g_{\sigma_1(\sigma_2(k))}) \end{aligned}$$



automorphisms map edges to edges and non-edges to non-edges. Hence, any edge of the graph gets mapped to another edge. To be more precise, we can also think of  $S_n$  as acting on the set of all potential edges given by  $K = \{\{i, j\} | i, j \in [n] \text{ and } i \leq j\}$ . In other words, the action of  $Aut(X)$  on the set  $E(X)$  does not change  $E(X)$  or it *fixes*  $E(X)$ . This leads us to the fourth problem called the *Set Stabilizer problem*.

**Definition 8.6** (Set Stabilizer of  $\Sigma$ ). *Given a  $G$  and a subset  $\Sigma \subseteq \Omega$ ,*

$$\text{SetStab}(\Sigma) = \{g \in G \mid \Sigma^g = \Sigma\}$$

$$\text{where } \Sigma^g = \{\alpha^g \mid \alpha \in \Sigma\}$$

Given a group  $G \leq S_n$  which acts on a set  $\Omega$ , we define the Set Stabilizer of  $\Sigma \subseteq \Omega$  as the set of all permutations in  $G$  which map elements of  $\Sigma$  to elements of  $\Sigma$  and all non elements of  $\Sigma$  to non-elements of  $\Sigma$ . Note that  $\text{SetStab}(\Sigma)$  is a sub-group of  $G$ . In the case of automorphism groups, the mapping is  $G = S_n$ ,  $\Omega = K$ ,  $\Sigma = E(X)$  and  $\text{SetStab}(\Sigma) = \text{Aut}(X)$ .

Having set up all the notation, let us state the computational question we seek to answer.

**Problem 8.7 (Set Stabilizer Problem).** *Given a group  $G$  via its generating set  $S$ , a set  $\Omega$  on which it acts and a subset  $\Sigma \subseteq \Omega$ , output a generating set for the group  $\text{SetStab}(\Sigma)$ .*

By our previous discussion on  $\text{Aut}(X)$  fixing the edges of  $X$ , we can conclude that  $\text{Aut}(X)$  is the stabilizer of  $E(X)$ . Hence the problem  $\mathcal{GA}$  reduces to the problem **SetStab**.

## 8.4 Orbit Computation

For the sake of completeness, let us restate the question.

**Orbit Computation** : Given a group  $G \leq S_n$  (via its generating set  $S$ ), compute the orbits of the action of  $G$  on  $[n]$ .

We can see that the orbits of the action of  $G$  on  $[n]$  partition the set into different subsets. Hence, we would want to compute the partitions. We cast this problem as a graph theoretic problem.

Let  $\Omega = [n]$ . We construct a directed graph  $X$  with  $V(X) = \Omega$  and  $E(X) = \{(\alpha, \beta) \mid \exists g \in G, \beta = \alpha^g\} \subseteq V(X) \times V(X)$ . We can have the edge  $(\alpha, \beta) \in E(X)$  labelled by  $g \in G$  if  $\alpha^g = \beta$ . To check if two elements  $\alpha, \beta$  lie in the same partition, it suffices to check if there is a directed path from  $\alpha$  to  $\beta$  (or from  $\beta$  to  $\alpha$ ). Hence the connected components of the graph corresponds to the orbits. In the next lecture, we will give another algorithm for solving this problem.

## Set Stabilizers and Point-wise Stabilisers

### 9.1 Recap and Exercise

In the previous class, we were looking at ways to solve  $\mathcal{AUT}(\mathcal{X})$ . We looked at 4 different problems in this context. Consider *Problem 3* of finding the orbit. We are interested in computing the size of the orbit of  $\alpha \in G$ . Since some of the students expressed difficulty in coming up with an algorithm, we decide to state the algorithm and leave the correctness proofs as an exercise.

#### 9.1.1 Orbit Computation

The following is an algorithm to calculate the orbit of  $\alpha$

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**Algorithm 1** Algorithm for Orbit Computation
 

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```

1: procedure ORBIT COMPUTATION( Input : Generating set  $S$  )
2:    $\Delta = \{\alpha\}$ 
3:   repeat
4:      $\Delta^1 = \Delta$ 
5:     for  $g \in S$  and  $\delta \in \Delta$  do
6:        $\Delta = \Delta^1 \cup \{\delta^g\}$ 
7:   until  $\Delta^1 \neq \Delta$ 

```

---

The algorithm was developed as step by step in the lecture which also developed its correctness. However, the formal proofs are left as an exercise. To be precise, it is left as an exercise to the reader to do the following : (1) Prove that Algorithm 1 eventually terminates. (2) Prove that Algorithm 1 terminates with  $\Delta = \text{Orbit}(\alpha)$  (3) Calculate the running time of Algorithm 1.

The problem can be viewed in terms of the graph. Consider the following definition of a directed graph  $X$ .  $V(X)$ , the vertex set of the graph  $G$  is equal to the set  $\Omega$ .  $E(X) = \{(\alpha, \beta) \mid \exists g \in S \text{ such that } \alpha^g = \beta\}$

The key observation is that, if there is a path in  $X$  from  $\alpha$  to  $\gamma$  then  $\gamma$  is in the orbit of  $\alpha$ . Finding the orbit of  $\alpha$  is equivalent to computing the transitive closure of the graph  $X$ . It is left as an exercise to the reader to complete the details of the above algorithm including rigorous proof of correctness.

## 9.2 Set Stabilisers Problem

We recall the problem that we defined in the last lecture. Given group  $G \leq S_n$  and a set  $\Sigma \subseteq \Omega$ , where  $G$  acts on  $\Omega$  and  $|\Omega| = n$ , we are interested in computing the generating set of the following group :

$$\mathcal{SETSTAB}(\Sigma) = \{g \in G \mid \Sigma^g = \Sigma\} \text{ where} \quad (9.4)$$

$$\Sigma^g = \{\alpha^g \mid \alpha \in \Sigma\} \quad (9.5)$$

As we saw in the last lecture, solving the set stabiliser problem gives an algorithm for obtaining  $\mathcal{AUT}(X)$ .

We now define a variant of the problem called the *Point Stabiliser Problem*

### Point-wise Stabiliser Problem

$$\mathcal{POINTSTAB}(\Sigma) = \{g \in G \mid \forall \alpha \in \Sigma, \alpha^g = \alpha\} \quad (9.6)$$

$\mathcal{POINTSTAB}$  is *easier* to solve than  $\mathcal{SETSTAB}$  The Point Stabilizer Problem is to obtain a generating set for the group  $\mathcal{POINTSTAB}(\Sigma)$ . It is easy to check that  $\mathcal{POINTSTAB}(\Sigma)$  is indeed a group.

In the reduction from  $\mathcal{GA}$  to  $\mathcal{GI}$ , we defined a tower of sub-groups where

$$G^{(i)} = \{g \in G \mid \forall 1 \leq j \leq i, j^g = j\} \quad (9.7)$$

Observe that each tower here is a point-wise stabiliser of  $\{1, 2, \dots, i\}$  For each  $G^{(i)}$ , we have  $\Omega = \{1, 2, \dots, n\}$  and  $\Sigma = \{1, 2, \dots, i\}$

Here is a general strategy for solving point-wise stabilizer problem. Without loss of generality, as above, assume that  $\Sigma$  is the first  $i$  elements of  $\Omega$ . The problem to solve is precisely the following, given the generators  $S$  of a group  $G$ , find the generators of  $G^{(i)}$ . A natural attack on the problem is stepwise, given the generating set of  $G$ , find that of  $G^{(1)}$ , and then using that to find the generating set of  $G^{(2)}$  and so on. In  $i$  levels of this recursion, we will be done.

The key problem that we identified for solving the order computation as well as point-stabilizer problem is the following. Given the generating set of  $G^{(i-1)}$ , find that of  $G^{(i)}$ . This is exactly what we will do in the next section.

### 9.2.1 Schreier's Lemma

Schrier gave a clever way of completing our task. If in addition to the generating set of  $G^{i-1}$ , we are also given the coset representatives of  $G^i$  as a subgroup in  $G^{(i-1)}$ .

In the following, we will call  $G$  to be the bigger group ( $G^{(i-1)}$ ) and  $H$  to be the subgroup  $G^{(i-1)}$ . Let  $R$  be set of right coset representatives of the subgroup  $H$  in  $G$ . The following lemma gives a direct way of writing the generating set for  $H$ .

**Lemma 9.1 (Schreier's Lemma).** *Let  $S$  be the generating set of  $G$  and  $R$  be the set of coset representatives of  $H$ .  $S' = \{r_1 g r_2^{-1} \mid r_1, r_2 \in R, g \in S\}$   $S' \cap H$  forms a generating set for  $H$ .*

Before we prove this, notice that  $R$  contains the identity of the groups. Hence the Set  $S \subseteq S'$ . By taking  $S' \cap H$ , we are extracting the elements in  $S'$  which are also in  $H$ . In order to do this, we do need a membership testing method for  $H$ . In our context,  $H$  is  $G^{(i)}$  and has an easy membership test given an element  $g \in G^{(i-1)}$ .

*Proof.* Define,

$$RS = \{rs \mid r \in R, s \in S\}$$

Since each element in  $RS$  can be written as  $(rsr_1^{-1})r_1$ , we immediately get that,

$$RS \subseteq S'R \tag{9.8}$$

Let  $\langle S' \rangle$  denotes the group generated by  $S'$ . By definition,  $S'R \subseteq \langle S' \rangle R$ . From 9.8, we have  $RS \subseteq \langle S' \rangle R$ . Therefore,  $RSS \subseteq \langle S' \rangle RS$ , and hence  $RSS \subseteq \langle S' \rangle RS$ . Repeating this process, we have  $\forall t \geq 0, RS^t \subseteq \langle S' \rangle R$

Since  $R$  contains the identity, and  $S$  generates  $G$ ,  $G \subseteq \bigcup_{i=1}^k RS^i$  for some fixed finite  $k$ . In fact, we will see later that  $k \leq |G|$ , but we do not need this bound here.

We argue that  $H \leq \langle S' \rangle$ . Since  $G \subseteq \langle S' \rangle R$ , if  $\langle S' \rangle$  does not contain  $H$ ,  $\langle S' \rangle R$  cannot cover  $G$ . Hence proved by contradiction. Therefore,  $S' \cap H$  generates  $H$ .

2: JS says: To be completed, why should it exactly generate  $H$ ? □

The following is a different view of the above proof. This was not done in the lecture, but we record this for completeness of this notes and for future use. We derive an equivalent statement to the above, by observing that for every  $r_1$  and  $a$ , there is a unique  $r_2$  such that  $r_1 a r_2^{-1} \in H$ . Call this  $r_2$  as  $\overline{r_1 a}$ . Hence the following is an equivalent restatement.

**Lemma 9.2 (Schreier's Lemma restated).** *Let  $H$  be a subgroup of  $G$ ,  $S$  be the generating set of  $G$  and  $R$  be the set of right coset representatives (containing identity), then,*

$$T = \{rs(\overline{rs})^{-1} \mid r \in R, s \in S\}$$

is a generating set for  $H$ , where  $\overline{g}$  denotes the coset representative corresponding to the element  $g$  (that is, the unique element in  $Hg \cap R$ ).

*Proof.* First observe that the elements of  $T$  are in  $H$  (that is the way we redefined it to begin with). Hence, it cannot generate a larger group. It is enough to show that  $T \cup T^{-1}$  generates all the elements of  $H$ .

Consider an arbitrary element  $h \in H$ . Since  $H \leq G$ , it can be written as the product of elements from  $S \cup S^{-1}$ . Let  $h = s_1 s_2 \dots s_k$ . We will incrementally keep modifying this product representations from left to right to obtain a product representation of  $h$  using elements of  $T \cup T^{-1}$ . We call this modification sequence to be  $h_1, h_2 \dots h_k$  where each  $h_i = h$ . In general,

$$h_i = t_1 t_2 t_i r_{i+1} s_{i+1} s_{i+2} \dots s_k$$

where  $t_i \in T \cup T^{-1}$  and  $r_{i+1} \in R$ .

The initial element  $h_0 = \prod_{i=1}^k s_i$  with  $r_1 = 1$ , which is exactly the original representation, and the final element  $h_k = (\prod_{i=1}^k t_i) r_{k+1}$ . Since  $h_k = h \in H$ , it must be that  $r_{k+1}$  is the identity. This gives the representation for  $h$  in terms of  $T \cup T^{-1}$ .

Now we complete the proof by defining  $h_{i+1}$  from  $h_i$ . We just need to define  $t_{i+1}$  and  $r_{i+2}$ . Take,  $t_{i+1} = r_{i+1} s_{i+1} \overline{r_{i+1} s_{i+1}}^{-1}$  and  $r_{i+2} = \overline{r_{i+1} s_{i+1}}$ . Clearly, the product  $h_{i+1}$  has the required form,  $t_{i+1} \in T \cup T^{-1}$  and  $r_{i+2} \in R$ . □

## Using Schreier's Lemma - Reduce Algorithm

Let  $G$  be a group generated by a set  $S$ . In the first part of this lecture, we went after a question from the previous lecture, about the value of  $k$  such that  $G \subseteq \cup_{t=0}^k S^t$ . This is an important question to be answered for the following variant of the membership problem. Given a group  $G \leq S_n$  via the generating set  $S$ , and a  $g \in S_n$ , we asked the problem of testing whether  $g \in G$ . In the previous versions, we expected a yes or no answer to this question. However, it is also useful to ask the computational version of this problem. That is, if  $g \in G$ , then give a representation of  $g$  as the product of elements in  $S$ . However, for this, the first question to ask is a bound on the length of any such representation.

### 10.1 Bounds on length of generating sequence

We start with the following easy bound.

**Proposition 10.1.**  $k \leq |G|$

*Proof.* We prove the statement by contradiction.

Let the smallest length of the generating sequence of some element  $g_1 \in G$  be  $|G| + m$ ,  $m > 0$ .

$$g_1 = \prod_{i=1}^{|G|+m} a_i \quad (10.9)$$

Consider partial products of the sequence above :

$$S_l = \prod_{i=1}^l a_i \quad (10.10)$$

There are  $|G| + m$  partial products. There are only  $|G|$  possible values for  $S_i$ . By Pigeon Hole Principle, we have the following :

$$\exists l_1, l_2, \quad l_1 < l_2 \quad S_{l_1} = S_{l_2} \quad (10.11)$$

This means, we can remove the elements of the generating sequence between  $l_1$  and  $l_2$ , which will generate the same element  $g_1$  (From Equation 10.11). Hence we get a shorter generating sequence for  $g_1$  which contradicts the original assumption. Thus,  $k \leq |G|$   $\square$

The above bound is too weak for our purpose because, in our context,  $|G|$  could be very large. Can we improve this bound in general? Unfortunately we end up proving that the bound is tight.

**Proposition 10.2.** *There is an  $n \in N$ , group  $G \leq S_n$  and a generating set  $S$  of  $G$ , and a  $g \in G$  such that the shortest product representation of sequence is of length  $|G|$ .*

*Proof.* Let  $p_1, p_2, \dots, p_m$  be the first  $m$  distinct prime numbers. Choose  $n = p_1 p_2 \dots p_m$ . We will define a  $G \leq S_n$ . The  $G$  that we define will be a cyclic group generated by the following element  $a$ .

$$a = (1, 2, \dots, p_1)(p_1 + 1, p_1 + 2 \dots p_2) \dots (\dots) \quad (10.12)$$

Let  $M$  be the order of the element  $a$ . That is,  $M$  is the smallest  $M$  such that  $a^M = id$ . First observation is that,  $M = \prod_{i=1}^m p_i$ , which is the lcm of the length of the above cycles. It is easy to see that the element  $a^{M-1}$  cannot have a shorter representation than as the product of  $a$ ,  $M - 1$  times. Notice that  $M \geq 2^m$ .

From the prime number theorem,  $m \geq \Omega(\frac{p_m}{\log p_m})$ . Also notice that  $n = \sum_{i=1}^m p_i \leq 1 + 2 + 3 \dots + p_m \leq p_m^2$ . Thus,  $p_m \geq \sqrt{n}$ .

Hence,  $m \geq \Omega(\frac{\sqrt{n}}{\log \sqrt{n}})$ . Thus,  $M \geq 2^{\frac{c\sqrt{n}}{\log n}} - 1$ .  $\square$

That is negative news. However, the above example does not rule out the existence of a polynomial length expression or computation for the element. For example,  $a^{M-1}$  can be expressed in terms of  $a$  through repeated squaring.

But then, how do we formulate our membership problem? This is where the strong generating strikes again. If we are allowed to change the generating set  $S$  to and  $S'$  (precomputation before seeing the  $g$ ) we can always compute a generating set such that we can output a short representation.

## 10.2 Need of a Reduction Step

We now return to the algorithm to compute the generators of a sub group. Using Schreier's lemma, we concluded that it's sufficient to compute the set  $S' \cap H$ .

Before getting into the algorithm for the same, let us first look at the size of the set that is generated.

By the definition of the set  $S'$ , we can only say

$$|S'| \leq |S||R|^2 \quad (10.13)$$

If we have to implement this recursively, we see that each level we are incurring a  $|R|^2$  term. Hence

$$|S'| \leq |S|l_1^2 \cdot l_2^2 \dots \quad (10.14)$$

We need to carefully control the size of the generating set  $S'$  at each level.

### 10.3 REDUCE algorithm

How do we reduce the generating set to control its size? Consider  $\pi, \psi \in S'$  such that  $1^\pi = 1^\psi = k$ . Do we need to keep both of them? Yes, indeed, it may be that  $2^\pi = k'$  and  $2^\psi = k''$ . We cannot afford to throw any of them away.

However, here is an observation. Consider replacing  $\psi$  with  $\pi^{-1}\psi$ .

We can observe the following :

1. We can still obtain all the elements  $\psi$  can be obtained  $\pi(\pi^{-1}\psi)$ . Observe that we also do not add any new elements in the process as well. Hence by this change, the subgroup that is generated remains the same.
2. Doing the above process of replacement for every pair of elements that map 1 to the same element, we end up with elements that all map 1 to different elements (except those that fix 1).
3.  $\pi^{-1}\psi$  fixes 1. Hence all the newly added elements fix 1.
4. We can repeat the same procedure for  $2, 3, \dots n$ .

The idea seems neat. At the end of the above procedure, for every pair  $(i, j)$ , we have exactly one  $\pi$  such that  $i^\pi = j$ . This immediately gives an upper bound of  $O(n^2)$  in the size of the sets.

However, there is a catch. When we repeat the same procedure for  $2, 3, \dots n$ , what is the guarantee that the property for 1 will not be violated?

The idea is that once we fix the elements which are mapping 1 to  $k$  where  $k \neq 1$ , all the extra elements which are mapping now 1 to 1 are pushed to the stage 2. In stage 2, the operations of the form  $\pi^{-1}\psi$  are done on elements which are stabilizing 1. Hence they will continue to stabilize 1. In the end, we may get several trivial elements, which we can simply discard.

To clarify this, we write this entire algorithm as a procedure.

---

**Algorithm 2** REDUCE Algorithm. Input : Generating set  $S$ , Output: Reduced Generating Set

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```

1:  $B_0 = B$ 
2:  $A[ ][ ]$ , an empty  $n \times n$  array
3: for  $i = 0$  to  $n - 1$  do
4:   for all  $\psi \in B_i$  do
5:      $j = i^\psi$ 
6:     if  $A[i][j]$  is empty then
7:       if  $j = i$  then
8:          $B_{i+1} = B_{i+1} \cup \{\psi\}$ 
9:       else
10:         $A[i][j] = \psi$ 
11:     else
12:        $\pi = A[i][j]$ 
13:        $B_{i+1} = B_{i+1} \cup \{\psi^{-1}\pi\}$ 
14: discard all trivial elements from  $\cup B_i$ 
15: return  $\cup B_i$ 

```

---

The correctness proof for the algorithm is left as an exercise.

## Pointwise Stabilizer, Membership Testing and Group Intersection

### 11.1 Recap

We first attacked the Membership Problem. We saw that this reduces to the problem of Order Computation. (See section 8.2)

To solve this problem of Order Computation, we use the Orbit-Stabiliser Lemma. This involved finding out the size orbit of an element  $\alpha$  (which we reduced to Graph Reachability), and then recursively finding the size of  $G_\alpha$  which is the stabiliser group of  $\alpha$ . Product of these two gives the order of the group.

Note that in the above approach,  $G$  is specified via a generating set. Hence before we recurse, we need to obtain a generating set for  $G_\alpha$ . For this purpose, we used Schreier's lemma to find a generating set. We also observed that size of the generating set can grow very fast. Using the reduction algorithm that Aditi very kindly explained in the previous lecture, we could ensure that the size of the generating set always stayed within  $O(n^2)$  (where  $n = |\Omega|$ ).

### 11.2 Finding Coset Representatives in the tower of subgroups

Now that we have obtained a way of obtaining a generating set for the subgroup, we are left with the problem of finding the orbit size. By Orbit-Stabiliser Lemma (Lemma 5.2), due to one-one correspondence with cosets, we get that to estimate orbit size, it suffices to estimate the number of cosets of the subgroup. In our setting this can be achieved by finding the coset representatives in the tower of subgroups.

Denote  $R$  as the set of right coset representatives of  $G^{(i+1)}$  in  $G^i$  for  $i \geq 0$ . Note that there are at most  $n - i$  cosets, since  $i + 1^{th}$  location can be taken to at most  $n - i$  different places. Recall that the set of locations where  $i + 1$  is taken to by  $G^{(i)}$  is precisely  $i + 1$ .

$$(i + 1)^{G^{(i)}} = \{k \in \{i + 1, \dots, n\} \mid \exists g \in G^{(i)}, (i + 1)^g = k\}$$

Let  $X_i \subseteq \{i + 1, \dots, n\}$  be the orbit of  $i + 1$  in  $G^{(i)}$  (We have seen an algorithm to do it in Section 9.1.1). For each  $k \in X_i$ , let  $g_{i+1,k}$  be an element in  $G^{(i)}$  that maps 1 to  $k$ . Note that such an element must exist in  $G^{(i)}$ . To find this group element, it suffices to obtain a path from  $i + 1$  to  $k$  in the graph of generating set (see Section 9.1.1 for definition) and take the product of edge labels on the path. The set  $R$  is now the collection of all  $g_{i+1,k}$  for each  $k \in X_i$ .



### 11.3 Algorithm for Pointwise Stabiliser Problem

We now piece together the above ideas to show that for any  $\Sigma \subseteq \Omega$ , the Generator set of  $\text{PointStab}(\Sigma) \leq S_n$  can be computed in  $\text{poly}(|\Omega|)$  time. Let  $i \geq 1$ . Recall that  $\text{PointStab}(\{1, 2, \dots, i\}) = G^{(i)}$ .

To compute a generating set for  $G^{(i)}$

1. Use the technique specified above to find coset representative for  $G^{(1)}$  in  $G$ . We consider  $G^{(0)} = S_n$  and use the generator set for  $S_n$  as  $\{(1)(2) \dots (n), (1\ 2), \dots, (n-1\ n)\}$  while constructing the graph for generating set.
2. Previous step gives us the set  $R$  which consists of the right coset representatives of  $G^{(1)}$  in  $G^{(0)}$ . Now we can apply Schreier's Lemma to obtain the generating set for  $G^{(1)}$ .
3. Apply the REDUCE algorithm (Algorithm 2) to obtain a smaller generator set.
4. Recurse by following the same procedure, and finding the generating set of  $G^{(2)}$  given  $G^{(1)}$ , until the desired  $G^{(i)}$  is found.

It can be seen the each step terminates in  $\text{poly}(|\Omega|)$  time and the number of recursive calls is at most  $i \leq |\Omega|$ . Hence overall runtime is polynomial.

### 11.4 Algorithm for Membership Testing

Note that we already saw how to check membership of an element in a group using an algorithm to compute order of the group. But this does not tell anything about how the element is expressed in terms of its generators. Hence we consider the following stronger version of Membership problem.

**Problem 11.1** (Stronger Membership Test). *Given a generating set  $A$  of  $G$  with,  $G \leq S_n$  and a  $g \in S_n$ , give a representation in terms of any desired generating set, and the generating set itself.*

---

#### Algorithm 3 MEMBERTEST : Algorithm for Membership Testing

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1: procedure MEMBERTEST(Input : Element  $g$ , Generating set  $A$  for  $G^{(i)}$ , Index  $i$ )
2:   if  $g = id$  then
3:     return true
4:    $X_i = (i+1)^{G^{(i)}}$  (Use Orbit Computation Algorithm 1)
5:   Compute  $R$ , the set of coset representatives of  $G^{(i+1)}$  in  $G^{(i)}$  (see Section 11.2)
6:    $k = (i+1)^g$  (Image of  $i+1$  on action of  $g$ )
7:   if  $k \notin X_i$  then
8:     return false
9:   else
10:     $B =$  Generating set of  $G^{(i+1)}$  (Use Schreier's Lemma)
11:    Apply Algorithm REDUCE to  $B$  and denote  $A'$  for the reduced set obtained
12:    Pick  $g_{ik}$  from  $R$  that maps  $i$  to  $k$ 
13:     $g' = g \circ g_{ik}^{-1}$ 
14:    return MEMBERTEST( $g', A', i+1$ )

```

---

Note that if insisting on giving the representation of  $g$  in terms of  $A$ , then the problem may not be solvable in polynomial time since there are groups which require large size to represent in terms of its generators (Proposition 10.2). Now we call **MEMBERTEST** on the input  $(g, A, 0)$  for checking if  $g \in \langle A \rangle$  or not. It is left as an exercise to show that the algorithm terminates in  $\text{poly}(|\Omega|)$  time.

## 11.5 Group Intersection Problem

We consider the following two problem :

**Problem 11.2** (Subgroup Problem). *Given groups  $G$  and  $H$  via their generating sets  $A$  and  $B$  respectively, test if  $H \leq G$*

**Problem 11.3** (Group Intersection Problem). *Given groups  $G$  and  $H$  via their generating sets  $A$  and  $B$  respectively, find the generating set of  $G \cap H$ <sup>6</sup>.*

Since we have already given an algorithm for membership checking, Subgroup problem has the following algorithm : for each  $b \in B$  check if  $b \in G$  using **MEMBERTEST** algorithm.

So how about Group Intersection problem ? We show that this problem is as hard as the Set Stabilizer Problem. We show this by first reducing **SetStab** to **GroupInter**.

**Claim 11.4.**  $\text{SetStab} \leq \text{GroupInter}$

*Proof.* Given  $\Sigma \subseteq \Omega$ , generating set  $K$  of  $G$  we need to produce groups  $J, H$  via generators  $A, B$  such that  $J \cap H = \text{SetStab}(\Sigma)$ . Consider the group,  $H = \text{Sym}(\Sigma) \times \text{Sym}(\Omega \setminus \Sigma)$  where  $\text{Sym}(T)$  consists of all permutations defined over  $T$  of length  $|T|$ . Hence  $\text{Sym}(T)$  fixes  $T$ . Note that  $H \leq S_n$  and consists of all permutations that have  $\Sigma$  mapped to  $\Sigma$ . Since  $\text{SetStab}(\Sigma)$  talks about permutations in  $G$  that fix  $\Sigma$ ,  $G \cap H$  is the  $\text{SetStab}(\Sigma)$ . Now it remains to give a generator for  $\text{Sym}(\Sigma)$ . It can be shown that if  $\Sigma = \{1, 2, \dots, n\}$ , then the generator set for  $\text{Sym}(\Sigma)$  is  $(1, \dots, n), (1\ 2), (2\ 3), \dots, (n-1\ n)$ .  $\square$

In the next lecture, we will show **GroupInter** reduces to **SetStab** thereby showing that they are equally hard.

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<sup>6</sup>Verify that if  $G$  and  $H$  are groups then  $G \cap H$  is also a group

## Group Intersection to Set Stabiliser and Jerrum's Filter

In last lecture, we completed the discussion on algorithms for Pointwise Stabilizer problem and Membership testing. We saw two new problems – Subgroup problem and Group Intersection problem and showed that Subgroup problem can be reduced to Membership testing and hence polytime solvable while Set Stabilizer problem reduces to Group Intersection problem. We continue with our discussion on Group Intersection and Set Stabilizer problem.

### 12.1 $\text{GroupInter} \leq \text{SetStab}$

We show that Group Intersection problem reduces to Set Stabilizer problem. Before we start, let us look at ways of combining groups.

**Definition 12.1** (Direct Product). *For groups  $G, H$ , define direct product of  $G$  and  $H$  denoted  $G \times H$  as the group*

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

*with group operation for  $(g_1, h_1), (g_2, h_2)$  defined as,*

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$$

*If  $G, H$  are subgroups of  $S_n$  acting on the set  $\Omega$ , then the group  $G \times H$  acts on  $\Omega \times \Omega$  with the action defined as  $(\alpha, \beta)^{(g, h)} = (\alpha^g, \beta^h)$  where  $\alpha, \beta \in \Omega$  and  $(g, h) \in G \times H$ .*

The reduction is as follows : given groups  $G$  and  $H$ , consider the group  $G \times H$  and  $\Sigma' = \{(i, i) \mid i \in \Omega\}$ . To establish correctness, we show that if we consider the action of  $G \times H$  on  $\Omega \times \Omega$ , then  $\text{SetStab}(\Sigma')$  is  $G \cap H$ . More precisely,

**Claim 12.2.**  $(G \cap H)_{\text{duplicate}} = \text{SetStab}(\Sigma')$  where  $(G \cap H)_{\text{duplicate}} = \{(g, g) \mid g \in G \cap H\}$  and Set Stabilizer computation is with respect to the action of  $G \times H$  on  $\Omega \times \Omega$ .

*Proof.* We show containment in both ways.

$[(G \cap H)_{\text{duplicate}} \subseteq \text{SetStab}(\Sigma')]$  Consider  $g \in G \cap H$ . Clearly  $(g, g) \in G \times H$ . For  $(i, i) \in \Sigma'$ ,  $(i, i)^{(g, g)} = (i^g, i^g) \in \Sigma'$  by definition of  $\Sigma'$ . Hence  $(g, g) \in \text{SetStab}(\Sigma')$ .

$[\text{SetStab}(\Sigma') \subseteq (G \cap H)_{\text{duplicate}}]$  Let  $(g, h) \in \text{SetStab}(\Sigma')$ . By definition,  $\forall (i, i) \in \Sigma'$  we have  $(i, i)^{(g, h)} = (i^g, i^h) = (j, j)$  for some  $j \in \Omega$ . In other words,  $i^g = i^h$  for all  $i$ . Hence it must be that  $g$  and  $h$  are the same giving  $g = h \in G \cap H$ . Hence  $(g, h) \in (G \cap H)_{\text{duplicate}}$ .  $\square$

Now that we know that **GroupInter** and **SetStab** problems are equivalent to each other and are both seemingly harder than the graph automorphism problem which we want to solve. Our next aim would be to solve special cases of these problems.

Before we do the special cases, we would like to do one interesting optimization for the **REDUCE** algorithm.

## 12.2 Jerrum's Filter: Obtaining generating set of size $n - 1$

3: JS says: All the arguments are in place after the edit. A little more cleanup is to be done, will do that at the end of the semester.

We know from Schreier's lemma how to obtain a strong generating set for  $G \leq S_n$  of size  $O(n^2)$ . We also know that for any subgroup of  $S_n$ , there exists a generating set of size  $O(n \log n)$ . A natural question is : can we improve this bound further ?

Mark Jerrum showed that one can give a generator of size  $n - 1$  instead. The process is algorithmic and is called as Jerrum's filter<sup>7</sup>.

### 12.2.1 Main ideas

Given a generating set  $S$  of  $G$  and  $G$  acts on a set  $\Omega$ , our aim is to obtain another generating set  $A$  for  $G$  of size  $\leq n - 1$ .

Given a set  $S$  of group elements, we define a undirected graph  $X_S(V, E)$  with  $V = \Omega$  and  $E \subseteq V \times V$  is defined as for each  $g \in S$ , let  $i_g$  denote the least index moved by  $g$  as  $i_g$  i.e.  $\forall i \leq i_g$   $i^g = i$ . Now we add the edge  $(i_g, i_g^g)$  to  $E$ . We also label this edge by  $g$ .

From the definition of  $X_S$ , the following can be observed.

**Observation 12.3.** *For a  $g \in S$   $g^{-1}$  maps  $i_g^g$  to  $i_g$ . Also both  $g, g^{-1}$  has the same least index.*

The main idea is the following : we maintain a set  $A$  of generators in such a way that  $X_A$  is acyclic. When we get a new  $g \in S$ , we add  $g$  to  $A$  and compute  $X_{A \cup \{g\}}$ . Now we make some operation on this graph  $X_{A \cup \{g\}}$  in such a way that the resultant graph has no cycles and the updated set  $A$  still generates  $G$ . This process is *online*, since we update the graph as and when  $g$  arrives and do not need the entire  $S$  a priori. Continuing in this manner, we get the  $X_A$  to be a forest and hence has  $n - 1$  edges.

Before explaining the details, we need the following measure.

**Definition 12.4** (Weight of  $T$ ). *Let  $T \subseteq S_n$ . Then*

$$wt(X_T) = \sum_{g \in T} i_g$$

Observe that for any  $T$ ,  $wt(T) \leq |T|n$ .

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<sup>7</sup>It turns out that this can be improved to  $\lfloor n/2 \rfloor$  (Neumann) and is tight. We will not be discussing it in this lecture.

### 12.2.2 The Algorithm

The main idea of the algorithm is to maintain a set  $A$  such that  $X_A$  is acyclic. In the construction, we ensure that we add edges corresponding to element only in  $A$  (as explained while we defined  $X_A$ ) and remove those edges not in  $A$ .

At the end of the process, we get an acyclic graph which is a forest with number of edges at most  $n - 1$ . Hence  $|A|$  is upper bounded by number of edges in the graph which is at most  $n - 1$ .

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**Algorithm 4** JERRUMFILTER : Computing a generating set of size at most  $n - 1$

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1: procedure JERRUMFILTER(Input : Generating set  $S$  for  $G$  given in online fashion)
2:   Let  $A$  be generating set maintained. Initially  $A = \emptyset$ 
3:   for a given  $g \in S$  do
4:     If  $g \notin A$ , then add  $g$  to  $A$  and add  $e_g = (i_g, i_g^g)$  to the graph  $X_A$ 
5:     if a cycle is created in  $X_A$  then
6:       Let  $C$  be the unique cycle created of length  $k$ 
7:       Let  $i_0$  be the least index in the cycle  $C$ ,  $g_0$  be an element corresponding to  $i_0$ 
8:       Let  $\{g_t\}_{t \in [k]}$  be the edge labels on the cycle  $C$  with  $g_0$  as label of edge incident to  $i_0$ 
9:       Denote  $g_0 g_1^{\epsilon_1}, \dots, g_k^{\epsilon_k}$  as  $h$  where  $\epsilon_i \in \{1, -1\}$  defines the direction
10:      Remove  $g_0$  and add  $h$  to the set  $A$  and update  $X_A$  to reflect the changes
11:      If  $h$  is identity, ignore by not adding it to  $A$ 

```

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The algorithm is self explanatory till step 5. Note that there is exactly one cycle formed by adding of  $e_g$ . This is because, so far we have maintained the invariant that  $X_A$  is acyclic. Let  $i$  be the least index in the cycle  $C$ . We claim that one of the edges incident to  $i_0$  labelled by  $g$  in  $C$  must have  $i = i_0^g$  as its least index. If not then the neighbours of  $i_0$  would not be the least moved element by the corresponding group element.

Since  $i_0$  is the least index in the cycle, it is fixed by all of  $g_0, g_1, \dots, g_k$  and therefore the least element moved by  $h$  is strictly more than  $i$ . Therefore, the weight of the graph increases by at least on 1 on performing this operation. Since the weight of  $S$  is upper bounded by  $\text{poly}(|S|, n)$ , the process terminates in in polynomially many steps.

On repeating this with all elements, we have a generating set  $A$  with an acyclic graph  $X_A$ , and therefore  $|A| \leq n - 1$ .

## Group Intersection Problem under Special Cases-I

### 13.1 Recap

In the previous lecture, we used *Jerrum's filter* to reduce the size of the generating set  $\langle S \rangle$  of a subgroup  $G$  of  $S_n$  to at most  $n - 1$ . Let us revisit the *Group Intersection Problem* :

**Problem 13.1 (Group Intersection).** *Given two groups  $G$  and  $H$  via their generating sets  $\langle A \rangle$  and  $\langle B \rangle$  respectively, with  $G \leq S_n$  and  $H \leq S_n$ , compute the generating set for the group  $G \cap H$ ?*

We also showed that the *Group Intersection Problem* can be reduced to the *Set Stabilizer Problem* and vice-versa. As  $GA \leq SETSTAB \equiv$  Group Intersection Problem both are harder than *Graph Automorphism Problem*. In this lecture we will be interested in solving Problem 13.1 in a special setting of *Normal Subgroups*. But before that let's review the basics of *Normal Subgroups*.

### 13.2 Normal Subgroups

We already know that if  $H$  and  $G$  are two groups and  $H \leq G$ , then the right cosets of  $H$  in  $G$  are given by  $Hg = \{hg \mid h \in H \text{ and } g \in G\}$ . Similarly the left cosets are given by  $gH = \{gh \mid h \in H \text{ and } g \in G\}$ . By *Lagrange's Theorem* we also know that any two right cosets of a subgroup  $H$  are either *disjoint* or *equal* i.e. for any  $g_1, g_2 \in G$  with  $g_1 \neq g_2$ , either  $Hg_1 \cap Hg_2 = \emptyset$  or  $Hg_1 = Hg_2$ . The same conditions hold true for the left cosets of  $H$  too.

For any  $H \leq G$  it is not necessary that  $Hg = gH$ . Normal subgroups achieve the above condition.

**Definition 13.2.** *Let  $H \leq G$  such that  $\forall g \in G, Hg = gH$ . Then  $H$  is called a Normal Subgroup of  $G$  and is denoted by  $H \triangleleft G$ .*

One special property of such a subgroup  $H$  follows directly from the definition: The set of all left cosets and right cosets remain the same. Consider the operation  $*$  defined as  $Hg_1 * Hg_2 = Hg_1g_2 \forall g_1, g_2 \in G$ .

**Observation 13.3.** *The set of all cosets form a group with respect to the operation  $*$  defined above.*

- **Closure** Since  $G$  has closure property and  $Hg_1 * Hg_2 = Hg_1g_2$ , hence this itself implies closure of the set of cosets under  $*$ .

- **Existence of Identity** The subgroup  $H$  serves as the identity element in the set as  $\forall g \in G, Hg * Hg^{-1} = Hgg^{-1} = H$ .
- **Existence of Inverse** Since  $\forall g \in G, Hg * Hg^{-1} = Hgg^{-1} = H$ , therefore inverse exists.
- **Associativity** Follows from the definition.

The set of all cosets of a normal subgroup  $H$  in  $G$  has a distinct name in the literature of Group theory.

**Definition 13.4.** Let  $H \triangleleft G$ . The set of all cosets  $\{Hg \mid g \in G, h \in H\}$  along with the binary operation  $*$  is called the Quotient Group of  $G$  by  $H$  (denoted by  $G/H$ ).

If  $g' \in Hg$ , then  $Hg' = Hg$  i.e.  $Hg$  and  $Hg'$  refers to the same coset.

**Claim 13.5.** Let  $H \triangleleft G$  and  $Hg_1, Hg_2$  be right cosets of  $H$  in  $G$ . Let  $g'_1 \in Hg_1, g'_2 \in Hg_2$ . Then

$$\text{If } Hg_1 = Hg'_1 \text{ and } Hg_2 = Hg'_2 \text{ then } Hg_1g_2 = Hg'_1g'_2$$

*Proof.*

$$\begin{aligned} Hg_1g_2 &= Hg'_1g_2 \quad [\text{Since } Hg'_1 = Hg_1] \\ &= g'_1Hg_2 \quad [\text{Since } H \triangleleft G] \\ &= g'_1Hg'_2 \quad [\text{Since } Hg'_2 = Hg_2] \\ &= Hg'_1g'_2 \quad [\text{Since } H \triangleleft G] \end{aligned}$$

□

We want to emphasize two things here: firstly, the above claim shows that  $*$  is well-defined whenever  $H \triangleleft G$ . Secondly, note that this claim holds only when  $H \triangleleft G$ . Otherwise, the operation  $*$  on the set of cosets is in general not well-defined.

It is natural to ask if the above claim characterizes normal subgroups. That is is the converse of the above claim true ?

**Claim 13.6.** Let  $H \leq G$ . Let  $g_1, g'_1, g_2, g'_2 \in G$  be such that if  $Hg_1 = Hg'_1$  and  $Hg_2 = Hg'_2$  implies  $Hg_1g_2 = Hg'_1g'_2$ . Then  $H \triangleleft G$ .

*Proof.* Let us first argue that  $\forall g \in G, \forall h \in H, gHg^{-1} \subseteq H$ . As  $gHg^{-1} \subseteq H$  we have  $Hg \subseteq gH$  and  $gH \subseteq Hg$ . Therefore  $H \triangleleft G$ . □

Having geared up with the basic notions of Normal Subgroups, we can now ask the following computational question.

**Problem 13.7 (Group Normality).** Given two groups  $G$  and  $H$  via their generating sets  $A$  and  $B$  ( $\langle A \rangle = G$  and  $\langle B \rangle = h$ ), check if  $H \triangleleft G$ .

### 13.2.1 Normalizer and Normal Closure of a Group

Suppose for some groups  $H$  and  $G$  where  $H \leq G$ ,  $H \not\triangleleft G$ . This happens because there is at least one  $g \in G$  such that  $Hg \neq gH$ . Let us collect all such  $g \in G$  such that  $Hg = gH$ . This leads us to the following definition

**Definition 13.8.** The set  $N_G(H) = \{g \in G \mid Hg = gH\}$  is called the **Normalizer** of  $H$  in  $G$ .

Observe that for any  $H \leq G$ ,  $N_G(H) \neq \emptyset$  as  $H \in N_G(H)$ . AS every group is a normal subgroup of itself we have  $|H| \leq |N_G(H)| \leq |G|$ .

**Observation 13.9.**  $(N_G(H), *)$  forms a group.

where  $*$  is defined in Definition 13.4.

One more computational question that arises in this context.

**Problem 13.10 (Normalizer Computation).** Given two groups  $G$  and  $H$  via their generating sets  $\langle A \rangle$  and  $\langle B \rangle$  respectively, output the generating set of  $N_G(H)$ .

Clearly, the normalizer  $N_G(H)$  is the largest subgroup of  $G$  in which  $H$  is a normal subgroup. Then a natural question that arises is the smallest normal subgroup of  $G$  that contains  $H$ .

**Definition 13.11.** The smallest normal subgroup of  $G$  that contains  $H$  is called the **Normal Closure** of  $H$  in  $G$ .

Keeping the above definition in our mind another question pops up naturally regarding the computability of normal closure of a group as defined formally below.

**Exercise 13.12.** Given  $G, H$  via their generating sets and let  $H \leq G$ , compute the generating set of the normal closure of a subgroup  $H$  of  $G$ ?

## 13.3 Group Intersection Problem in the Context of Normalizers

Let  $G, H \leq S_n$  and  $N_{S_n}(H)$  be the normalizer of  $H$  in  $S_n$ .

We say  $G$  normalizes  $H$  when  $\forall g \in G, Hg = gH$ . This means  $G \leq N_{S_n}(H)$ . We solve the Group Intersection problem for this special case.

For general groups  $G$  and  $H$ , we are not aware of an efficient algorithm for computing the generating set of  $G \cap H$ . Before getting on to the actual algorithm for computing the generating set of  $G$  and  $H$  in the special setting let us have some more observations.

Consider the set  $GH = \{gh \mid g \in G, h \in H\}$ .

**Observation 13.13.** Let  $G, H \leq S_n$ . If  $G$  normalizes  $H$ , then  $GH \leq S_n$ .

- **Closure** Let  $g_1h_1, g_2h_2 \in GH$ , then

$$\begin{aligned} g_1h_1 * g_2h_2 &= g_1g_2.h'_1h_2 \quad [\text{Since } G \text{ normalizes } H] \\ &= gh \in GH. \end{aligned}$$

- **Existence of Identity** Setting  $g = h = e$  it is trivial to observe that  $e \in GH$ .



- **Existence of Inverse**  $\forall gh \in GH, h^{-1}g^{-1} \in GH$ . Thus,

$$\begin{aligned}
gh.(h^{-1}g^{-1}) &= g(hh^{-1})g^{-1} \\
&= g(e)g^{-1} \\
&= gg^{-1} \\
&= e \in GH.
\end{aligned}$$

- **Associativity** *Follows naturally from the definition.*

We are now just a few steps away from describing the algorithm for Problem 13.1. Although the algorithm will be given explicitly in the next lecture, the first idea is to construct a tower of subgroups such that we can get the generating set of  $G \cap H$  by applying Schreier's lemma. Observe that  $G^{(i)}$  is the set of all permutations of  $S_n$  which fix the element  $i$ . As  $G^{(i)}$  normalizes  $H$ , by Observation 13.13, we infer that  $G^{(i)}H \leq S_n$ .

Now we are ready to describe the construction of the *Tower of Subgroups* as follows:

$$G \cap G^{(0)}H \geq G \cap G^{(1)}H \geq G \cap G^{(2)}H \geq \dots \geq G \cap G^{(n-1)}H$$

Two quick observations are:

**Observation 1.** On the right end of the above chain  $G^{(n-1)}$  is the subgroup which *fixes* all the elements  $1, 2, \dots, n-1$ . Hence it gives the identity permutation in  $S_n$  exactly. Thus  $G \cap G^{(n-1)}H = G \cap H$ .

**Observation 2.** On the left end of the above chain  $G \cap G^{(0)}H = G \cap GH$ . As  $G \subseteq GH$ , we have  $G \cap G^{(0)}H = G \cap GH = G$ .

Thus the above tower of subgroups really look like that as shown below.

$$G \geq G \cap G^{(1)}H \geq G \cap G^{(2)}H \geq \dots \geq G \cap H$$

## Group Intersection Problem under Special Cases-II

### 14.1 Recap

In the previous lecture, we were devising the tools that we need for solving Problem 13.1 under the special setting of a group  $G$  acting as a **normalizer** for group  $H$ . We discussed **Normal Subgroups** along with the concepts of **Normalizers** and **Normal Closure**. Then we looked at the Tower of Subgroups that we are going to use to give the algorithm for Problem 13.1. The Tower of Subgroups look like the following:

$$G \geq G \cap G^{(1)}H \geq G \cap G^{(2)}H \geq \dots \geq G \cap H$$

### 14.2 Getting to the Algorithm

In the above chain of subgroups we are already given the generating set of  $G$ . We can apply **Schreier's Lemma** to this chain of subgroups to get the generating set of  $G \cap H$ . But before doing that we must ensure of three things:

1. Computation of the coset representatives of the subgroups at each level in the chain.
2. Testing membership in the subgroups.
3. Ensuring that the number of cosets at all level is small.

These three properties must be satisfied in order to get a *strong generating set*.

- We can apply our known algorithm to find the set of coset representatives of the subgroup  $G^{(i)}H$  in  $G^{(i-1)}H$ .
- For the second property, since we already have a generating set  $\langle A \rangle$  for  $G$  as input, we have to find the generating set of  $G^{(1)}H$  such that the membership testing of elements of the subgroup can be done efficiently. Now  $G^{(1)}H$  is defined as the product of all the elements in  $G^{(1)}$  and  $H$ . Hence the generating set of  $G^{(1)}H$  is the *union* of the generating set  $S_1$  of  $G^{(1)}$  and that of  $H$  i.e.  $\langle S_1 \rangle \cup \langle B \rangle$ . Once we get these, we can easily apply our *Membership Testing algorithm* which was done in a previous lecture.

4: JS says: A detail is missing here: why is the union the generating set?

- For an assurance of the third property we need to prove the following lemma.

**Claim 14.1.** *The index of  $G \cap G^{(i)}H$  in  $G \cap G^{(i-1)}H$  is at most  $(n - i)$ .*

The crux of the proof of the above claim lies in the following arguments about how a coset of  $G^{(i)}H$  actually looks like which is inside  $G^{(i-1)}H$ . In other words,  $G^{(i)}Hg$  is a coset of  $G^{(i)}H$  inside  $G^{(i-1)}H$  if and only if  $g \in G^{(i-1)}H$ . This implies that  $g = g'h'$ , where  $g' \in G^{(i-1)}$  and  $h' \in H$ . Thus we see that

$$\begin{aligned} G^{(i)}Hg &= G^{(i)}Hg'h' \\ &= G^{(i)}g'Hh' \quad [\text{Since } G^{(i-1)} \text{ normalizes } H], Hg' = g'H \\ &= G^{(i)}g'H \quad [\text{As } h' \in H, Hh' = H] \\ &= (G^{(i)}g')H. \end{aligned}$$

The rest of the proof is left as an exercise. What remains to be argued is that after the intersection with  $G$  number of cosets still bounded above by  $(n - i)$  in the  $i$ -th level of the tower of subgroups. As an intermediate observation, we claim the following:

**Claim 14.2.**  $H \triangleleft GH$ .

*Proof.* Let  $g' \in GH$ . Then  $g' = gh$  for some  $g \in G, h \in H$ . For deducing  $H \triangleleft GH$ , we need to show that  $\forall g' \in GH, Hg' = g'H$ . So we proceed as follows:

$$\begin{aligned} Hg' &= Hgh \\ &= gHh \quad [\text{Since } G \text{ normalizes } H] \\ &= ghH \quad [\text{Since } Hh = hH] \\ &= g'H. \end{aligned}$$

□

### 14.2.1 The Algorithm

Now we are in a perfect set up to describe the algorithm for computing the generating set of  $G \cap H$ . The algorithm just applies **Schreier's Lemma** *recursively* at all the  $(n - 1)$  levels in the tower of subgroups to compute the generating set of  $G \cap G^{(i)}H$ ,  $\forall i = 1, 2, 3, \dots, (n - 1)$ .

We know that the number of cosets in the  $i$ -th level is bounded above by a small quantity  $(n - i)$  as Claim 14.1 shows. From the REDUCE algorithm that we saw in a previous lecture we know that the size of the generating set at each of the  $(n - 1)$  levels of the tower of subgroups computed by **Schreier's Lemma** is  $O(n^2)$ . Therefore the asymptotic running time of this algorithm is  $O(n^3)$ .

## 14.3 From Setwise Stabilizers to Point-wise Stabilizers : Bounded Color Class Graph Isomorphism Problem

We have already mentioned of the fact that  $GA \leq \text{SetStab} \equiv \text{Group Intersection Problem}$ . Graph Isomorphism problem is hard in general. In this context we have seen that  $\text{COLOR} - \text{GI} \leq \text{GA}$ .

When the number of colours,  $k = 1$ , colored graph isomorphism is the same problem of Graph Isomorphism. Thus instead of getting our attention to the *number of colours*, we focus on the size of each colour class, i.e.  $|\psi^{-1}(i)|, i \in \{1, 2, \dots, k\}$ . We assume that  $\forall i \in \{1, 2, \dots, k\}, |\psi^{-1}(i)| \leq b, b \in \mathbb{N}$  i.e. the size of each colour class is bounded by a **constant**  $b$ . Now the question we ask is the following:

**Problem 14.3.** *Given two graphs  $X_1(V_1, E_1)$  and  $X_2(V_2, E_2)$  with sizes of the color classes bounded by a constant, can we check if they are isomorphic?*

We reduce colored graph isomorphism problem to the colored graph automorphism problem.

Since we know that checking Isomorphism between  $X_1$  and  $X_2$  can be reduced to checking Automorphism group of  $X = X_1 \cup X_2$ . This additional constraint of colour imposes an association between an edge in the graph with another edge whose vertices are of same colour so that it becomes a colour preserving map. In other words,

$$\text{Aut}(X) \leq \text{Sym}(\Psi_1^{-1}) \times \text{Sym}(\Psi_2^{-1}) \times \text{Sym}(\Psi_3^{-1}) \times \dots \times \text{Sym}(\Psi_k^{-1})$$

where  $k$  is number of colours. We must note that all elements in  $\text{Sym}(\Psi_1^{-1}) \times \text{Sym}(\Psi_2^{-1}) \times \text{Sym}(\Psi_3^{-1}) \times \dots \times \text{Sym}(\Psi_k^{-1})$  may not be elements in  $\text{Aut}(X)$ .

Now the idea is to use the known *SetStab* reduction so that we can compute the necessary stabilizer set efficiently. From that context we denote  $E_i = E \cap \binom{C_i}{2}$  to be the set of all **intracolour edges** for each colour  $i$  and  $E_{ij} = E \cap (C_i \times C_j)$  to be the set of all **intercolour edges** between any two *distinct* colour classes  $i$  and  $j$ . Note that an edge inside  $E_i$  is mapped within  $E_i$  again and an edge inside  $E_{ij}$  is mapped within itself too, thereby obtaining a *colour preserving map*. In other words, any permutation  $g \in \text{Sym}(C_i)$  is an automorphism if and only if  $E_i^g = E_i$  and  $E_{ij}^g = E_{ij}$  i.e. we have got the desired *SetStab* form.

Let the group be  $\text{Sym}(\Psi_1^{-1}) \times \text{Sym}(\Psi_2^{-1}) \times \text{Sym}(\Psi_3^{-1}) \times \dots \times \text{Sym}(\Psi_k^{-1})$ . All we need to do is to find a subgroup of the above group that pointwise stabilizes the sets  $E_i$  and  $E_{ij}$ .

Thus we view  $S_n$  as acting on the following  $\Omega$  given by

$$\Omega = \left( \bigcup_i C_i \right) \cup \left( \bigcup_i 2^{\binom{C_i}{2}} \right) \cup \left( \bigcup_i 2^{C_i \times C_j} \right)$$

where the term  $2^{\binom{C_i}{2}}$  gives the potential subsets of all *intracolour edges* while the term  $2^{C_i \times C_j}$  gives the potential subsets of all *intercolour edges*. Now we bound the size of  $\Omega$  as follows:

$$|\Omega| \leq n + k \cdot 2^{\binom{b}{2}} + \binom{b}{2} \cdot 2^{b^2}$$

Since  $b$  is a constant,  $|\Omega|$  is polynomial in  $n$ .

**\*Title of Lecture\***

Example 1:

Consider the complete binary tree of depth  $k$ . Since any isomorphism will send leaves to leaves,  $G = \text{Aut}(X)$  acts on the leaves and we observe that given any 2 leaves we can get an isomorphism which sends one leaf to the other.

Example 2:

Consider  $X = k$  triangles.  $\text{Aut}(X)$  acts on the vertices and here too given any two vertices there is a permutation in  $\text{Aut}(X)$  that sends one to the other.

These 2 examples motivate the following definition:

**Definition 14.1.** *Transitivity:  $G$  acts on  $\Omega$  transitively if there is only one orbit, i.e.  $\forall \alpha, \beta \in \Omega$ ,  $\exists g \in G$ , such that  $\alpha^g = \beta$ .*

In example 2, any  $g \in \text{Aut}(x)$  will send a triangle to itself or another triangle. Similarly for example 1, this property holds for the leaves taken in pairs, where the pair has all ancestors common.

**Definition 14.2.** *Block:  $\Delta$  is said to be a block if  $\forall g \in G$ ,  $\Delta^g = \Delta$  or  $\Delta^g \cap \Delta = \emptyset$ .*

We observe that trivially  $\Omega$  and the singletons form blocks. We say that the group action is primitive if there are no non-trivial blocks.

Examples of block structure:

1)  $G = S_n$

Action is trivially transitive. No non-trivial blocks hence primitive.

2)  $G \leq S_n$  acts on itself (right-multiplication,  $\Omega = G$ ).

Transitive action.  $gg' = h$

Not primitive: Consider a non-trivial subgroup  $H$ .  $Hg = H$  or  $Hg$  is a coset, i.e.  $Hg \cap H = \emptyset$ .

Converse holds.

**Definition 14.3.** *Block System: A partition of  $\Omega$  into sets such that each part is a block under the action of  $G$ .*

Claim 1:  $\Delta^g$  is a block if  $\Delta$  is a block.

*Proof.* Suppose not. Implies  $\exists h \in G, \Delta^{gh} \neq \Delta^g$  and  $\Delta^{gh} \cap \Delta^g \neq \phi$ .

Consider the action of  $g' = (gh)^{-1}$  on  $\Delta^{gh}, \Delta^g$ . We have that  $(\Delta^{gh})^{g'} = \Delta$ .

Since  $\Delta^{gh} \neq \Delta^g \Rightarrow \exists \beta \in \Delta^{gh}, \beta \notin \Delta^g$ . Since any group element permutes  $\Omega$ ,  $x^g = y^g \Leftrightarrow x = y$ . So  $\beta^{g'} \in \Delta$  and  $\notin \Delta^{gg'}$ .

But  $\Delta^{gh} \cup \Delta^g \neq \phi \Rightarrow \exists \alpha \in \Delta^{gh}, \alpha \in \Delta^g$ . This means that  $\alpha^{g'} \in \Delta$  and  $\in \Delta^{gg'}$ .

Combining these two we get that  $\Delta \neq \Delta^{gg'}$  and  $\Delta \cap \Delta^{gg'} \neq \phi$ , which contradicts the fact that  $\Delta$  is a block. □

Claim 2:  $|\Delta^g| = |\Delta|$

This follows from the fact that  $g \in G$  permutes the elements of  $\Omega$ .

Claim 3:  $\bigcup_g \in G\Delta^g = \Omega$

Let  $k \in \Delta$  and  $k' \in \Omega$ . Since the action is transitive  $\exists g, k^g = k'$ . So  $k' \in \Delta^g$ . Hence  $\Omega \subseteq \bigcup_g \in G\Delta^g$ . The other containment is trivial and hence they are equal.

Claim 4:  $\Delta^{g_1} \cap \Delta^{g_2} \neq \phi \Rightarrow \Delta^{g_1} = \Delta^{g_2}$ .

This is true because  $\Delta^{g_1}$  is a block (Claim 1).

Claim 4: For any block  $\Delta$ ,  $|\Delta|$  must divide  $|\Omega|$ .

The above claims prove that  $\Omega$  is partitioned into subsets with equal cardinality, one of which is  $\Delta$ . So  $|\Delta|$  must divide  $|\Omega|$ .

Just an Observation: If  $|\Omega|$  is prime, the group action is bound to be primitive.

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**Definition 14.4.** *Maximal Subgroup:  $H \leq G$  is a maximal subgroup if  $\nexists H', H < H' < G$ .*

**Lemma 14.5.** *Let  $G$  act transitively on  $\Omega$ . The action is primitive iff  $\forall \alpha \in G, G_\alpha$  is a maximal subgroup of  $G$ .*

*Proof.*  $\Leftarrow$

We will prove the contrapositive that is: Action is not primitive  $\Leftarrow \exists \alpha G_\alpha$  is not a maximal subgroup)

Suppose  $G$ 's action is not primitive  $\Rightarrow \{\alpha\} \subsetneq \Delta \subsetneq \Omega$ .

Take  $H = \text{Setstab}(\Delta)$ .  $H$  is a subgroup of  $G$ .

Let  $g \in G_\alpha$ . Since  $\alpha^g = \alpha$ ,  $\Delta^g = \Delta$  and  $g \in H$ . Hence  $G_\alpha \leq H$ .

Let  $\beta \neq \alpha \in \Delta$  and  $g, \alpha^g = \beta$ .  $g \notin G_\alpha$  but  $g \in H$ .

Hence  $G_\alpha \subsetneq H$  and  $G_\alpha$  is not maximal.

$\Rightarrow$

$G_\alpha \subsetneq H \subsetneq G$ . To produce a  $\Delta$ ,  $\{\alpha\} \subsetneq \Delta \subsetneq \Omega$ , such that  $\Delta$  is a block.

Define  $\Delta = \alpha^H$ .

$\alpha \subsetneq \Delta$

Since  $G_\alpha \not\leq \dots$

$\Delta \not\subseteq \Omega$

$G = \bigcup_{g\beta: \alpha \rightarrow \beta} G_\alpha g\beta$ .

Obs:  $\beta \in \Delta \Leftrightarrow G_\alpha g\beta \subseteq H$ .

$\Delta$  is a block.

$\Delta^g \cap \Delta \neq \phi$

$\exists h \in H, \alpha^h \in \Delta, \alpha^h$

□