Exam page 413

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We obtain

$$P*(z,d) = \frac{zP(z,0) - \mu(1-z)P_0^*(s)}{sz - (1-z)(\mu - \lambda z)}.$$

Thus we have transformed the set of differential-difference equations for $p_n(t)$ both on discrete variable n and on the continuous variable t. It is now necessary to turn to Rouche's theorem to determine the unknown function $P_0^*(s)$ by appealing to the analyticity of the transform.After much work, this leads to an explicit expression for the double transform which must then be inverted on both transform variables. The final solution for the transient solution of the M/M/1 queue is obtained as

$$p_n(t) = e^{-(\lambda+\mu)t}\left[\rho^{(n-i)/2}I_{n-i}(at) + \rho^{(n-i-1)/2}In + i + 1(at) + (1-\rho)\rho^n \sum_{j=n+i+2}^{\infty} \rho^{-j/2}I_j(at)\right]$$

where *i* is the initial system size (i.e., $p_n(0) = 1$ if $n = 1$, ans $p_n(0) = 0$ if n=i, $\rho = \lambda/\mu$, $a = 2\mu\rho^{1/2}$, and

$$I_n(x) \sum_{m=0}^{\infty} \frac{(x/2)^n + 2m}{(n+m)!m!}$$

is the modified Bessel function of the first kind and of order *n*.

11.3 General Birth-Death Processes

A birth-death process may be viewed as a generalization of the M/M/1 queueing system. Perhaps more correctly, the M/M/1 queue is a special type of birth-death process. Whereas the birth and death rates in the M/M/1 queue (λ and μ, respectively) are the same irrespective of the number of customers in the system, a general birth-death process allows for different rates depending on the number of customers present. Arrivals to the system continue to be referred to as *births* and departures as *deaths* but now we introduce a birth rate λ_n, which is defined as the rate at which births occur when the population is of size *n*, and a death rate μ_n defined as the rate at which deaths occur when the population is of size *n*. Notice that, for all *n*, both λ_n and μ_n are independent of *t* which means that our concern is with continuous-time homogeneous Markov chains. On the other hand, the parameters λ_n, and μ_n, can, and frequently do, depend on the currently occupied state of the system, namely, state *n*.

11.3.1 Derivation of the State Equations

As for the M/M/1 queue, a state of the system at any time is characterized completely by specifying the size of the population at that time. Let pn(t) be the probability that the population is of size n at time t. We assume that births and deaths are independent and that

$$\begin{aligned}
\text{Prob}\{\text{One birth in } (t, t+\Delta t]|n \text{ in population}) &= \lambda_n \Delta t + o(\Delta t),\\
\text{Prob}\{\text{One death in } (t, t+\Delta t]|n \text{ in population}) &= \mu_n \Delta t + o(\Delta t),\\
\text{Prob}\{\text{Zero birth in } (t, t+\Delta t]|n \text{ in population}) &= 1 - \lambda_n \Delta t + o(\Delta t),\\
\text{Prob}\{\text{Zero deaths in } (t, t+\Delta t]|n \text{ in population}) &= 1 - \mu_n \Delta t + o(\Delta t),
\end{aligned}$$

These assumptions mean that the probability of two or more births, of two or more deaths, or of near simultaneous births and deaths in some small interval of time (Δt) is negligibly small, i.e., of order $o(\Delta t)$.

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\usepackage{geometry}
\usepackage{fancyhdr}
\usepackage{amsmath}
\usepackage{tcolorbox}
\usepackage{xcolor}
\usepackage{times}
\usepackage{amssymb}
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\usepackage{parskip}
\usepackage{fancyhdr}
\usepackage{graphicx}
\usepackage{textcomp}
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 bottom=40mm}
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\vspace{0.9mm}
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\author{Viktorija Jegorova}
\title{Exam page 413}
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\renewcommand{\baselinestretch}{0.7}
\begin{document}
\maketitle
\pagebreak
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