

C3-W8-FirstExam-Final

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To determine performance measures, we first compute

$$\begin{aligned} L_q &= \sum_{n=c}^K (n-c)p_n = p_0 \frac{(\lambda/\mu)^c \lambda/c\mu}{c!(1-\lambda/c\mu)^2} \left[1 - \left(\frac{\lambda}{c\mu} \right)^{K-c+1} - \left(1 - \frac{\lambda}{c\mu} \right) (K-c+1) \left(\frac{\lambda}{c\mu} \right)^{K-c} \right] \\ &= p_0 \frac{(c\rho)^c \rho}{c!(1-\rho)^2} [1 - \rho^{K-c+1} - (1-\rho)(K-c+1)\rho^{K-c}], \end{aligned}$$

where we have defined $\rho = \lambda/(c\mu)$. This allows us to obtain the average number of customers in the system (see also Exercise 11.6.1):

$$L = L_q + \left(c - \sum_{n=0}^{c-1} (c-n)p_n \right) = L_q + c - \sum_{n=0}^{c-1} (c-n) \frac{(\lambda/\mu)^n}{n!} p_0, \quad (11.12)$$

which is the number waiting in the queue plus the average number of busy servers. In the special case of $c = 1$, this reduces to

$$L = L_q + (1 - p_0) = L_q + \frac{\lambda'}{\mu},$$

where $\lambda' = \lambda(1 - p_K)$ is the effective arrival rate. The time spent in the system can now be obtained from $W = L/\lambda'$. Finally, the mean time spent in the queue waiting for service to begin is found from $W_q = W - 1/\mu$ or $W_q = L_q/\lambda'$.

Erlang's Loss Formula

For the special case when $K = c$, i.e., the $M/M/c/c$ queue, the so-called "blocked calls cleared with c servers" system, these results lead to another well-known formula associated with the name Erlang. The probability that there are n customers in the $M/M/c/c$ queue is given as

$$p_n = \frac{(\lambda/\mu)^n / n!}{\sum_{i=0}^c (\lambda/\mu)^i / i!} = p_0 \frac{(\lambda/\mu)^n}{n!}.$$

The formula for p_c is called "Erlang's loss formula" and is the fraction of time that all c servers are busy. It is written as $B(c, \lambda/\mu)$ and called "Erlang's B formula":

$$B(c, \lambda/\mu) = \frac{(\lambda/\mu)^c / c!}{\sum_{i=0}^c (\lambda/\mu)^i / i!} = p_0 \frac{(\lambda/\mu)^c}{c!}.$$

Notice that the probability that an arrival is lost is equal to the probability that all channels are busy. Erlang's loss formula is also valid for the $M/G/c/c$ queue. In other words, the steady-state probabilities are a function only of the mean service time, and not of the complete underlying cumulative distribution function. An efficient recursive algorithm for computing $B(c, \lambda/\mu)$ is given by

$$B(0, \lambda/\mu) = 1, \quad B(c, \lambda/\mu) = \frac{(\lambda/\mu)B(c-1, \lambda/\mu)}{c + (\lambda/\mu)B(c-1, \lambda/\mu)}.$$

We leave its proof as an exercise. It is also possible to express Erlang's C formula in terms of $B(c, \lambda/\mu)$ and again, we leave this as an exercise.

11.7 Finite-Source Systems—The $M/M/c/M$ Queue

The $M/M/c/M$ queue is a c -server system with a finite customer population. No longer do we have a Poisson input process with an infinite user population. There is a total of M customers and a customer is either in the system, or else is outside the system and is in some sense "arriving." When