Problem 10.15 =

A is an $m \times n$ matrix. where $a_1, a_2 \cdots a_n$ are its columns.

$$[a]_{m\times n} = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ \text{column } a_1 & \text{column } a_2 & \cdots & \text{colum } a_n \end{array} \end{array}$$

Then we can write =

$$a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \\ \vdots \\ a_{mi} \end{bmatrix}, \quad a_J = \begin{bmatrix} a_{1J} \\ a_{2J} \\ a_{3J} \\ \vdots \\ a_{mJ} \end{bmatrix}$$

Then Gram matrix $G = A^T A$

$$G = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & ---- & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & ---- & a_{mn} \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & ---- & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & ---- & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{21}^2 + --a_{m1}^2 & a_{11}a_{12} + a_{21}a_{22} + --- a_{m1}a_{m2} & ---- \\ a_{12}a_{11} + a_{22}a_{21} + --- a_{m2}a_{m1} & a_{12}^2 + a_{22}^2 + --- a_{m2}^2 & --- \\ \vdots & \vdots & \end{bmatrix}$$

By seeing this matrix we can conclude =

$$G_{ii} = (a_{1i}^2 + a_{2i}^2 + a_{3i}^2 ----- + a_{mi}^2)$$

$$G_{jj} = (a_{1j}^2 + a_{2j}^2 + a_{3j}^2 ---- + a_{mj}^2)$$

$$G_{ij} = (a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} --- a_{mi}a_{mj})$$

for matrix $[A]_{m \times n} \ni$

$$a_i - a_j = \begin{bmatrix} a_{1i} - a_{1j} \\ a_{2i} - a_{2j} \\ a_{3i} - a_{3j} \\ \vdots \\ a_{mi} - a_{mj} \end{bmatrix}$$

Then $\| a_i - a_j \| =$

$$= \sqrt{(a_{1i}-a_{1j})^2 + (a_{2i}-a_{2j})^2 + \text{----} (a_{mi}-a_{mj})^2}$$

$$= \sqrt{a_{1i}^2 + a_{1j}^2 - 2a_{1i}a_{1j} + a_{2i}^2 + a_{2j}^2 - 2a_{2i}a_{2j} \text{ ------ } a_{mi}^2 + a_{mj}^2 - 2a_{mi}a_{mj}}$$

$$= \sqrt{(a_{1i}^2 + a_{2i}^2 + a_{3i}^2 \text{ ---- } a_{mi})^2 + (a_{1j}^2 + a_{2j}^2 + a_{3j}^2 \text{ --- } a_{mj})^2 - 2(a_{1i}a_{1j} + a_{2i}a_{2j} \text{ ----- } a_{mi}a_{mj})}$$

$$= \sqrt{(G_{ii})^2 + (G_{jj})^2 - 2(G_{ij})}$$

Hence =

$$\|a_i - a_j\| = \sqrt{(G_{ii})^2 + (G_{jj})^2 - 2\,(G_{ij})}$$

PROBLEM 10.16 =

lets consider $[A]_{n \times k}$ matrix =

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nk} \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ \text{column } a_1 & \text{column } a_2 & \text{column } a_3 & & \text{column } a_k \end{array} \end{array}$$

So $[A]_{n \times k} = \left[a_1 \;\; a_2 \;\; \cdots \;\; a_k \right]$

(a) = k vector μ gives the means of the columns.

$\mu_i = \text{avg}(a_i)$ and $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_k \end{bmatrix}$

Then $\mu_1 = \text{avg}(a_1)$

$$= \frac{a_{11} + a_{21} + a_{31} + \cdots + a_{n1}}{n}$$

$$\mu_2 = \text{avg}(a_2) = \frac{a_{12} + a_{22} + a_{32} + \cdots a_{n2}}{n}$$

$$\mu_3 = \text{avg}(a_3) = \frac{a_{13} + a_{23} + a_{33} + \cdots a_{n3}}{n}$$

$$\vdots$$

$$\mu_k = \text{avg}(a_k) = \frac{a_{1k} + a_{2k} + a_{3k} + \cdots + a_{nk}}{n}$$

Then we can express μ as $\Rightarrow$

$$\mu = \frac{1}{n} \begin{bmatrix} a_{11} + a_{21} + a_{31} + \cdots + a_{n1} \\ a_{12} + a_{22} + a_{32} + \cdots a_{n2} \\ a_{13} + a_{23} + a_{33} + \cdots a_{n3} \\ \vdots \\ a_{1k} + a_{2k} + a_{3k} + \cdots a_{nk} \end{bmatrix}$$

we can express μ in terms of A as =

$$\mu = \frac{1}{n}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nk} \end{bmatrix}^T_{n \times k} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_n \right)$$

$$\mu = \frac{1}{n}\left[A^T \mathbb{1} \right]$$

$$\boxed{\mu = \frac{(A^T \mathbb{1})}{n}}$$

(b) $\Rightarrow$ $\tilde{a}_1, \tilde{a}_2 ---- \tilde{a}_K$ are demeaned versi-ons of $a_1, a_2 --- a_K$.

Then Matrix $\tilde{A} = [\tilde{a}_1, \tilde{a}_2, a_3 --- \tilde{a}_K]$

$$\tilde{a}_1 = a_1 - \begin{bmatrix} \mu_1 \\ \mu_1 \\ \vdots \\ \mu_1 \end{bmatrix}_n$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_1 \\ \mu_1 \\ \vdots \\ \mu_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \mu_1 \\ a_{21} - \mu_1 \\ a_{31} - \mu_1 \\ \vdots \\ a_{n1} - \mu_1 \end{bmatrix}$$

Then similarly $\Rightarrow$

$$\tilde{a}_2 = \begin{bmatrix} a_{12} - u_2 \\ a_{22} - u_2 \\ a_{32} - u_2 \\ \vdots \\ a_{n2} - u_2 \end{bmatrix}$$

$\vdots$

$$\tilde{a}_k = \begin{bmatrix} a_{1k} - u_k \\ a_{2k} - u_k \\ a_{3k} - u_k \\ \vdots \\ a_{nk} - u_k \end{bmatrix}$$

Then following =

$$\tilde{A} = [\tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \dots \tilde{a}_k]$$

we can represent $\tilde{A}$ as =

$$\begin{bmatrix} a_{11}-\mu_1 & a_{12}-\mu_2 & \cdots & a_{1k}-\mu_k \\ a_{21}-\mu_1 & a_{22}-\mu_2 & \cdots & a_{2k}-\mu_k \\ a_{31}-\mu_1 & a_{32}-\mu_2 & \cdots & a_{3k}-\mu_k \\ \vdots & \vdots & & \vdots \\ a_{n1}-\mu_1 & a_{n2}-\mu_2 & \cdots & a_{nk}-\mu_k \end{bmatrix}_{n \times k}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nk} \end{bmatrix} - \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_k \\ \mu_1 & \mu_2 & & \vdots \\ \vdots & \vdots & & \\ \mu_1 & \mu_2 & \cdots & \mu_k \end{bmatrix}_{n \times k}$$

$$= A - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1} \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_k \end{bmatrix}_{1 \times k}$$

$$= A - 1 \cdot \mu^T$$

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$$\boxed{\hat{A} = A - 1 . \mu^T}$$

where 1 is a n-vector with all values as 1.

(c) = In the previous part we calculated $\tilde{A}$ ⇒

$$\tilde{A} = \begin{bmatrix} a_{11}-\mu_1 & a_{12}-\mu_2 & \cdots & a_{1k}-\mu_k \\ a_{21}-\mu_1 & a_{22}-\mu_2 & \cdots & a_{2k}-\mu_k \\ a_{31}-\mu_1 & a_{32}-\mu_2 & \cdots & a_{3k}-\mu_k \\ \vdots & \vdots & & \vdots \\ a_{n1}-\mu_1 & a_{n2}-\mu_2 & \cdots & a_{nk}-\mu_k \end{bmatrix}_{n \times k}$$

Then covariance matrix ⇒

$$\Sigma = \frac{1}{N} \tilde{A}^T \tilde{A}$$

$$= \frac{1}{N} \begin{bmatrix} a_{11}-\mu_1 & a_{12}-\mu_2 & \cdots & a_{1k}-\mu_k \\ a_{21}-\mu_1 & a_{22}-\mu_2 & \cdots & a_{2k}-\mu_k \\ a_{31}-\mu_1 & a_{32}-\mu_2 & \cdots & a_{3k}-\mu_k \\ \vdots & \vdots & & \vdots \\ a_{n1}-\mu_1 & a_{n2}-\mu_2 & \cdots & a_{nk}-\mu_k \end{bmatrix}^T \begin{bmatrix} a_{11}-\mu_1 & a_{12}-\mu_2 & \cdots & a_{1k}-\mu_k \\ a_{21}-\mu_1 & a_{22}-\mu_2 & \cdots & a_{2k}-\mu_k \\ a_{31}-\mu_1 & a_{32}-\mu_2 & \cdots & a_{3k}-\mu_k \\ \vdots & \vdots & & \vdots \\ a_{n1}-\mu_1 & a_{n2}-\mu_2 & \cdots & a_{nk}-\mu_k \end{bmatrix}$$

$$
= \begin{bmatrix}
\dfrac{(a_{11}-\mu_1)^2+(a_{21}-\mu_1)^2 - - - + (a_{n1}-\mu_1)^2}{N} & - - - - - - & \dfrac{(a_{11}-\mu_1)(a_{1k}-\mu_k)+(a_{21}-\mu_1)(a_{2k}-\mu_k) + - - - (a_{n1}-\mu_1)(a_{nk}-\mu_k)}{N} \\
\dfrac{(a_{12}-\mu_2)(a_{11}-\mu_1)+(a_{22}-\mu_2)(a_{21}-\mu_1) + - - - - + (a_{n2}-\mu_2)(a_{n1}-\mu_1)}{N} & \dfrac{(a_{12}-\mu_2)^2+(a_{22}-\mu_2)^2 + - - - (a_{n2}-\mu_2)^2}{N} & - - - - - \\
\vdots & & \\
\vdots & &
\end{bmatrix}
$$

In the above matrix $\Rightarrow$

$$\Sigma_{11} = \frac{(a_{11}-\mu_1)^2+(a_{21}-\mu_1)^2 - - - + (a_{n1}-\mu_1)^2}{N}$$

$$= \text{std}(a_1)^2$$

Similarly =

$$\Sigma_{22} = \frac{(a_{12}-\mu_2)^2+(a_{22}-\mu_2)^2 + - - - (a_{n2}-\mu_2)^2}{N}$$

$$= \text{std}(a_2)^2$$

Hence we can say that =

$$\Sigma_{ij} = std(a_i)^2, \quad \text{when } i = j$$

when $i \neq j$ for Example in above matrix =

$$\Sigma_{21} = \frac{(a_{12}-\mu_2)(a_{11}-\mu_1) + (a_{22}-\mu_2)(a_{21}-\mu_1) + \dots + (a_{n2}-\mu_2)(a_{n1}-\mu_1)}{N}$$

also $std(a_1)\, std(a_2)\, \rho_{12} =$

$$std(a_1) = \sqrt{\frac{(a_{11}-\mu_1)^2 + (a_{21}-\mu_1)^2 \dots + (a_{n1}-\mu_1)^2}{N}}$$

$$std(a_2) = \sqrt{\frac{(a_{12}-\mu_2)^2 + (a_{22}-\mu_2)^2 + \dots (a_{n2}-\mu_2)^2}{N}}$$

$$\rho_{12} = \frac{\tilde{a}_1^T \tilde{a}_2}{\|\tilde{a}_1\| \, \|\tilde{a}_2\|}$$

$$= \frac{(a_{12}-\mu_2)(a_{11}-\mu_1) + (a_{22}-\mu_2)(a_{21}-\mu_1) + \dots + (a_{n2}-\mu_2)(a_{n1}-\mu_1)}{\sqrt{(a_{11}-\mu_1)^2 + (a_{21}-\mu_1)^2 \dots + (a_{n1}-\mu_1)^2}\ \sqrt{(a_{12}-\mu_2)^2 + (a_{22}-\mu_2)^2 + \dots (a_{n2}-\mu_2)^2}}$$

Then $\text{std}(a_1)\,\text{std}(a_2)\,\rho_{12} =$

$$\Rightarrow \frac{(a_{12}-\mu_2)(a_{11}-\mu_1)+(a_{22}-\mu_2)(a_{21}-\mu_1)+\cdots+(a_{n2}-\mu_2)(a_{n1}-\mu_1)}{N}$$

which is equal to Σ_{12}.

Hence we can say that

$$\Sigma_{12} = \text{std}(a_1)\,\text{std}(a_2)\,\rho_{12}$$

or

$$\Sigma_{ij} = \text{std}(a_i)\,\text{std}(a_j)\,\rho_{ij}$$

when $i \neq j$

Hence

$$\Sigma_{ij} = \begin{cases} \text{std}(a_i)^2 & \text{when } i = j \\ \text{std}(a_i)\,\text{std}(a_j)\,\rho_{ij} & \text{when } i \neq j \end{cases}$$

(d)= standardized vector of a_1 =

$$z_1 = \frac{\tilde{a}_1}{\operatorname{std}(a_1)}$$

$$= \frac{1}{\operatorname{std}(a_1)} \begin{bmatrix} a_{11} - \mu_1 \\ a_{21} - \mu_1 \\ \vdots \\ a_{n1} - \mu_1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{a_{11} - \mu_1}{\operatorname{std}(a_1)} \\ \dfrac{a_{21} - \mu_1}{\operatorname{std}(a_1)} \\ \vdots \\ \dfrac{a_{n1} - \mu_1}{\operatorname{std}(a_1)} \end{bmatrix}$$

$$z_2 = \frac{\tilde{a}_2}{std(a_2)}$$

$$= \begin{bmatrix} \dfrac{a_{12} - \mu_2}{std(a_2)} \\ \dfrac{a_{22} - \mu_2}{std(a_2)} \\ \vdots \\ \dfrac{a_{n2} - \mu_2}{std(a_2)} \end{bmatrix}$$

Similarly

$$z_K = \frac{\tilde{a}_K}{std(a_K)} = \begin{bmatrix} \dfrac{a_{1K} - \mu_K}{std(a_K)} \\ \dfrac{a_{2K} - \mu_K}{std(a_K)} \\ \vdots \\ \dfrac{a_{nK} - \mu_K}{std(a_K)} \end{bmatrix}$$

Then we can write matrix Z as =

$$Z = \begin{bmatrix} \frac{a_{11}-\mu_1}{std(a_1)} & \frac{a_{12}-\mu_2}{std(a_2)} & \cdots & \frac{a_{1K}-\mu_K}{std(a_K)} \\ \frac{a_{21}-\mu_1}{std(a_1)} & \frac{a_{22}-\mu_2}{std(a_2)} & \cdots & \frac{a_{2K}-\mu_K}{std(a_K)} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}-\mu_1}{std(a_1)} & \frac{a_{n2}-\mu_2}{std(a_2)} & & \frac{a_{nK}-\mu_K}{std(a_K)} \end{bmatrix}$$

$$= \left[\begin{bmatrix} \frac{1}{std(a_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{std(a_2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{std(a_3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{1}{std(a_K)} \end{bmatrix} \begin{bmatrix} a_{11}-\mu_1 & a_{12}-\mu_2 & \cdots & a_{1K}-\mu_K \\ a_{21}-\mu_1 & a_{22}-\mu_2 & \cdots & a_{2K}-\mu_K \\ \vdots & \vdots & & \vdots \\ a_{n1}-\mu_1 & a_{n2}-\mu_2 & \cdots & a_{nK}-\mu_K \end{bmatrix}^T \right]^T$$

$$= \left[\begin{bmatrix} \frac{1}{std(a_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{std(a_2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{std(a_3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{1}{std(a_K)} \end{bmatrix} \left(A - 1 \cdot \mu^T\right)^T \right]^T$$

$$= \left(\left(A - \mathbb{1} \cdot \mu^T \right)^T \right)^T \begin{bmatrix} \frac{1}{std(a_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{std(a_2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{std(a_3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{1}{std(a_k)} \end{bmatrix}^T$$

$$= \left(A - \mathbb{1} \cdot \mu^T \right) \begin{bmatrix} \frac{1}{std(a_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{std(a_2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{std(a_3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{1}{std(a_k)} \end{bmatrix}$$

PROBLEM 10.36 =

A is $n \times n$ matrix, lets consider A $\Rightarrow$

$$[A]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

x is n-vector $\Rightarrow$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Calculating $x^T A x$ $\Rightarrow$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \cdots a_{n1}x_n) & (a_{12}x_1 + a_{22}x_2 + \cdots a_{n2}x_n) & \cdots & (a_{1n}x_1 + a_{2n}x_2 + \cdots a_{nn}x_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

now lets calculate =

$$(x^T A x) = \left(a_{11}x_1^2 + a_{21}x_1x_2 + a_{31}x_1x_3 + \cdots + a_{n1}x_1x_n \right)$$

$$+ \left(a_{12}x_1x_2 + a_{22}x_2^2 + a_{32}x_2x_3 + \cdots + a_{n2}x_2x_n \right)$$

$$\vdots$$

$$+ \left(a_{1n}x_1x_n + a_{2n}x_2x_n + \cdots + a_{nn}x_n^2 \right)$$

Hence we can write that:

$$x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

(b) Show that $x^T(A^T)x = x^T A x$

we have $x^T (A^T) x$

lets take transpose =

$$\left(x^T (A^T) x\right)^T$$

$$= \left[(x^T) \left((A^T) x\right)\right]^T$$

$$= \left[\left((A^T) x\right)^T (x^T)^T\right]$$

$$\left\{ \begin{array}{l} \text{we know} \\ (AB)^T = B^T A^T \end{array} \right\}$$

$$= \left[\left(x^T (A^T)^T \right) (x) \right]$$

$$= \left[x^T A x \right]$$

Hence,

$$\boxed{x^T (A^T) x = x^T A x}$$

(C) = Show that =

$$x^T\left((A+A^T)/2\right)x = x^TAx$$

we have $x^T\left[\frac{(A+A^T)}{2}\right]x$

$$= \frac{1}{2}x^T(A+A^T)x$$

$$= \frac{1}{2}(x^TA + x^TA^T)x$$

$$= \frac{1}{2}(x^TAx + x^TA^Tx)$$

we proved that $(x^TAx = x^TA^Tx)$

$$= \frac{1}{2}(2x^TAx)$$

$$= x^TAx$$

Hence $x^T\left[\frac{(A+A^T)}{2}\right]x = x^TAx$

(d) $= 2x_1^2 - 3x_1x_2 - x_2^2$

lets consider A which is a symmetric matrix.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & bx_1 + cx_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= ax_1^2 + bx_1x_2 + bx_1x_2 + cx_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

comparing $ax_1^2 + 2b x_1 x_2 + c x_2^2$

with $2x_1^2 - 3x_1 x_2 - x_2^2$

Then we know =)

1) = $a = 2$

2) = $2b = -3$

$b = -3/2$

3) = $c = -1$

Hence

$$A = \begin{bmatrix} 2 & -3/2 \\ -3/2 & -1 \end{bmatrix}$$

Then Quadratic form =

$$x^T \begin{bmatrix} 2 & -3/2 \\ -3/2 & -1 \end{bmatrix} x$$

PROBLEM 10.42 =

Ax of an $n \times n$ matrix A and n-vector x takes $2n^2$ flops.

formulate faster method in linear time (n) for matrix vector multiplication with matrix $A = I + ab^T$

$$Ax = (I + ab^T)x$$

$a, b, x =$ n-vectors.

$$Ax = (I + ab^T)x$$

$$Ax = Ix + ab^Tx \qquad [Ix = x]$$

$$Ax = x + a(b^Tx) \qquad [\text{assume } b^Tx = G]$$

$$Ax = x + (a(G))$$

$$Ax = (x + (aG)) \qquad [\text{assume } aG = T]$$

$$Ax = (x + T)$$

complexity =

(1) $= Ix = x$, $[I]_{n\times n}[x]_{n\times 1} = [x]_{n\times 1}$

complexity $= n$

(2) $= b^T x = G$, $[b^T]_{1\times n}[x]_{n\times 1} = [G]_{1\times 1}$

complexity = (n multiplications) + (n-1) sum

$= n + n - 1$

$= 2n - 1$

(3) $= aG = T$, $[a]_{n\times 1}[G]_{1\times 1} = [T]_{n\times 1}$

complexity = n multiplications = n

(4) $= x + T = [x]_{n\times 1} + [T]_{n\times 1}$

complexity = n additions $= n$

Total complexity $= n + (2n-1) + n + n$

$= 5n - 1$

Hence Ax was calculated in linear time.

PROBLEM 17.3 =

(a) = Give an example showing that $A \neq 0$ is not enough to conclude that $x = y$,

when $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

and $x = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

and $y = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$

In that case $Ax = Ay$

$$\begin{bmatrix} 1+0 & 1+0 \\ 1+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1+0 & 1+0 \\ 1+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

But $\boxed{x \neq y}$

Hence $A \neq 0$ is not sufficient to conclude that $\boxed{x = y}$ when $Ax = Ay$.

(b) = if A is left invertible then,

There would exist a Matrix B.

so that $BA = I$

now we have $Ax = Ay$

multiplying both sides by matrix B.

$$BAx = BAy$$

$$= Ix = Iy$$

$$\left[\begin{array}{l} Ix = x \\ Iy = y \end{array} \right]$$

$$\Rightarrow \boxed{x = y}$$ proved

(c) = lets assume, $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

we can see,

$|A| = 6 \times 2 - 4 \times 3 = 12 - 12 = 0$

Determinant of $A = 0$ Hence A is not invertible. Hence A is also not left invertible.

lets consider $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$

Then $Ax = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Then $Ay = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

we can see $Ax = Ay$

But $X \neq Y$.

PROBLEM 11.9 =

(a) = given that $y = Ax$

Hence $x = A^{-1} y$

$(I + BA) x = 0$

$\Rightarrow (I + BA)(A^{-1} y) = 0$

$= I A^{-1} y + B A A^{-1} y = 0$

$= I A^{-1} y + B I y = 0$

$= A^{-1} y + B y = 0$

multiply both side with A =

$A A^{-1} y + A B y = 0$

$= y + A B y = 0$

$= (I + AB) y = 0$ Proved

Hence $(I + BA)$ is invertible.

(D) = $B(I+AB) = (I+BA)B$

multiplying left with $(I+BA)^{-1}$

multiplying Right with $(I+AB)^{-1}$

$$(I+BA)^{-1} B(I+AB)(I+AB)^{-1}$$

$$= (I+BA)^{-1}(I+BA)B(I+AB)^{-1}$$

$$= (I+BA)^{-1} B I = I B (I+AB)^{-1}$$

$$= (I+BA)^{-1} B = B \; I+AB)^{-1}$$

Hence proved

PROBLEM 11.27 =

(1) = for $n = 500$.

Time taken to calculate $x = A^{-1}B$.

Time taken = 0.010175 seconds.

$$\text{Processor speed} = \frac{\text{no. of flops}}{\text{Time taken}}$$

$$= \frac{2n^3}{0.010175}$$

$$= \frac{2 \times (500)^3}{0.01}$$

$$= \frac{2 \times 125 \times 10^6}{0.01}$$

$$= \frac{250 \times 10^6}{0.01}$$

$$= 25 \times 10^9 \text{ flops/sec}$$

Processor speed = 25 Gflops/sec.

[1 Gflop = 10^9 flops]

(2) = for n = 1000

Time taken to calculate $x = A^{-1}B$.

Time taken = 0.02055 seconds.

$$\text{Processor speed} = \frac{\text{no. of flops}}{\text{Time taken}}$$

$$= \frac{2n^3}{0.02055} = \frac{2(1000)^3}{0.02}$$

$$= \frac{2 \times 10^9}{0.02}$$

$$= 100 \times 10^9 \text{ flops/sec}$$

Processor speed = 100 Gflops/sec.

(3) = for $n = 2000$

Time taken to calculate $x = A^{-1}B$

Time taken = 0.103796 seconds.

$$\text{Processor speed} = \frac{\text{no. of flops} \times 10^{-9}}{\text{Time taken}}$$

$$= \frac{2n^3 \times 10^{-9}}{0.10}$$

$$= \frac{2 \times (2000)^3 \times 10^{-9}}{0.10}$$

$$= \frac{2 \times 8}{0.1}$$

$$= \frac{16}{0.1}$$

$$= 160 \text{ Gflops/sec.}$$

Value of n	Processor Speed (Gflops/sec)
500	25
1000	100
2000	160

NOTE: Those values should be ideally close but on my laptop I tried running the script various times and thats the result I obtained.

Please check the attached screen shots for Time taken by program for different values of n.

Please see attached PDF file named as $\boxed{\text{11.27 Timing test}}$

PROBLEM 12.13 =

(a) = Given that,

$$x^{k+1} = x^k$$

$$\therefore\ x^{(k+1)} = x^k - \mu A^T (Ax^k - b)$$

$$\Rightarrow\ 0 = \mu A^T (Ax^k - b)$$

$$\Rightarrow\ 0 = \mu A^T A x^k - \mu A^T b$$

as given $\mu = \frac{1}{\|A\|^2}$

$$\Rightarrow\ 0 = \frac{A^T A x^k}{\|A\|} - \frac{A^T b}{\|A\|^2}$$

$$\Rightarrow\ 0 = \frac{A^T A x^k}{A^T A} - \frac{A^T b}{A^T A}$$

$$\Rightarrow\ 0 = x^k - (A^T A)^{-1} A^T b$$

$$\Rightarrow \quad x^k = (A^T A)^{-1} A^T b$$

given that,

$$x^k = A^+ b$$

$$A^+ = (A^T A)^{-1} A^T$$

(b) = $\quad x^{k+1} = x^k - \mu A^T (Ax^k - b)$

$$x^{k+1} = x^k - \mu A^T A x^k + \mu A^T b$$

$$x^{k+1} = x^k (I - \mu A^T A) + \mu A^T b$$

comparing this equation with

$$x^{k+1} = x^k u + v$$

where $u = (I - \mu A^T A)$

$v = \mu A^T b$

(C) = Please see the code, graph in the attached PDF file named as $\boxed{12.37 \ (c)}$.