Problem - 5.2 = In the given Question, The Intern is right, Supervi-nor is wrong, Intern did not make any mistake, It is possible that the returns of google stock is linear combination of other returns

But that is only possible when the return vectors are linearly Dependent.
400 Stocks Observed for 250 Trading days.

It forms 400 different 250-vectors. for each stock.

vectors = $a_1, a_2, a_3 ---- a_k$
where $k = 1, 2, 3 ---- 400$

also $n = 250$

we know that any collection of $(n+1)$ or more n-vectors is linearly Dependent.

Here $n < K$

$(250 < 400)$

Hence these vectors are linearly Dependent.

When a collection of vectors in linearly dependent then atleast one of the vectors could be expressed as a linear combination of the other vectors.

Hence Intern is right the Return of google stock could be expressed as the linear combination of the other return vectors.

<u>Problem 5.B</u> = Options (A), (C), (D) are correct.

Early termination of gram-Schmidt algorithm at 5th iteration.

That means that vector a_5 could be represented as the linear combination of vectors a_1, a_2, a_3, a_4.
Hence vectors a_1, a_2, a_3, a_4, a_5 are linearly Dependent.

a_1, a_2, a_3, a_4 are linearly independent.
Hence a_2, a_3, a_4 are also linearly independent b/c its a subset.

a_4 cant be zero b/c its one of the vectors in linear independent group.

Problem 5.9 =

Total flop count for k iterations is given by formula —

Total no. of flops $= (4n-1)\dfrac{k(k-1)}{2} + 3nk$

$\approx 2nk^2$

when $k = 1000$, $n = 10{,}000$

Then Total no. of flops =

$2nk^2 = 2 \cdot (10{,}000)(1000)^2$
$= 2 \times 10^{10}$ flops

These flops took 2 seconds to complete. In other words 2×10^3 milliseconds to complete.

Time taken by 1 flop $= \dfrac{2 \times 10^3}{2 \times 10^{10}}$

$= 10^{-7}$ milliseconds

Total number of flops when $n = 1000$, $k = 500$

Then $2nk^2 = 2 \cdot (1000)(500)^2$
$= 2 \cdot 1000 \cdot 500 \cdot 500$
$= 1000 \cdot 1000 \cdot 500$
$= 5 \times 10^8$ flops

$$\text{Time Taken} = \begin{pmatrix}\text{no. of} \\ \text{flops}\end{pmatrix} \times \begin{pmatrix}\text{Time Taken} \\ \text{By 1 flop}\end{pmatrix}$$

$$= 5 \times 10^{8} \times 10^{-7}$$

$$= 5 \times 10$$

$$= 50 \text{ milliseconds.}$$

Hence Gram-Schmidt algorithm will take 50 milliseconds for $\tilde{k} = 500$, $\hat{n} = 1000$.

Time Taken = (50 milliseconds)

or

(0.05 seconds.)

PROBLEM 6.8 = Cash flow to Bank Accoun Balance.

t

T-vector $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_T \end{bmatrix}$

T-vector b denotes the bank account balance in the T-periods.

given that $\boxed{b_1 = c_1}$

also $b_t = (1+r) b_{t-1} + c_t$
where $t = 2, 3, \ldots T$

$$b_1 = c_1$$
$$b_2 = (1+r) b_{2-1} + c_2$$
$$= (1+r) b_1 + c_2$$
$$= (1+r) c_1 + c_2$$

$$b_3 = (1+r)\, b_{3-1} + c_3$$
$$= (1+r)\, b_2 + c_3$$
$$= (1+r)\left[(1+r)c_1 + c_2\right] + c_3$$
$$= (1+r)^2 c_1 + (1+r) c_2 + c_3$$

$$b_4 = (1+r)\, b_{4-1} + c_4$$
$$= (1+r)\, b_3 + c_4$$
$$= (1+r)\left[(1+r)^2 c_1 + (1+r) c_2 + c_3\right] + c_4$$
$$= (1+r)^3 c_1 + (1+r)^2 c_2 + (1+r) c_3 + c_4$$

Then we can figure out b_T as well.

$$b_T = (1+r)^{T-1} c_1 + (1+r)^{T-2} c_2 + (1+r)^{T-3} c_3$$
$$- - - - - - - (1+r)\, c_{T-1} + c_T$$

now T- vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_T \end{bmatrix}$

b could also be represented as below =

$b = Ac$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_T \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ (1+r) & 1 & 0 & 0 & \cdots & 0 \\ (1+r)^2 & (1+r) & 1 & 0 & \cdots & 0 \\ \vdots \\ (1+r)^{T-1} & (1+r)^{T-2} & \cdots & & (1+r) & 1 \end{bmatrix}}_{(T \times T) \text{ Matrix } A.} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_T \end{bmatrix}$$

we could represent Matrix A as below =

$$\boxed{a_{ij} = \begin{cases} 0, & \text{if } i < j \\ 1, & \text{if } i = j \\ (1+r)^{i-j}, & \text{if } i > j \end{cases}}$$

PROBLEM 6.13 ⇒

Given that →

$p(t) = c_1 + c_2 t + c_3 t^2 + \dots c_n t^{n-1}$

Then $p'(t) = \frac{d(p(t))}{dt}$

$p'(t) = \frac{d}{dt}\left(c_1 + c_2 t + c_3 t^2 \dots c_n t^{n-1}\right)$

$p'(t) = \frac{d}{dt} c_1 + \frac{d}{dt} c_2 t + \frac{d}{dt} c_3 t^2 + \dots$
$+ \frac{d}{dt} c_n t^{n-1}$

$p'(t) = 0 + c_2 \frac{d}{dt} t + c_3 \frac{d}{dt} t^2 + \dots$
$+ c_n \frac{d}{dt} t^{n-1}$

$p'(t) = 0 + c_2 \cdot 1 + 2 c_3 t + \dots$
$+ (n-1) c_n t^{n-2}$

$$\boxed{p'(t) = c_2 + 2c_3 t + \cdots + (n-1)c_n t^{n-2}}$$

↳ Equation ①.

also we are given that →

$$p'(t) = d_1 + d_2 t + \cdots + d_{n-1} t^{n-2}$$

Comparing the given equation with equation ① we can say that =

$$d_1 = c_2$$

$$d_2 = 2c_3$$

$$\vdots$$

$$d_{n-1} = (n-1)c_n$$

we have to find a matrix D for which

$$d = Dc$$

$$\begin{bmatrix} c_2 \\ 2c_3 \\ \vdots \\ (n-1)c_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & (n-1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Matrix d with Dimension $(n-1) \times 1$
rows column

Matrix D with Dimensions $(n-1) \times n$
rows columns

Matrix C with dimension $n \times 1$
rows column

Hence we can write matrix D in more generic form as $\rightarrow$

$$D \Rightarrow \quad d_{ij} = \begin{cases} i, & \text{if } (i+1) = j \\ 0, & \text{otherwise.} \end{cases}$$

Dimensions of D are $\rightarrow$ $(n-1) \times (n)$
rows columns

PROBLEM 6.18 ⟹

To show that the columns of a Vandermonde matrix are linearly independent if the numbers $t_1, t_2, \ldots t_m$ are distinct.

Vandermonde matrix (V) $=$

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & & \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{bmatrix}$$

we are also given that $(m \geq n)$.

multiplying the n-vector c by the Vandermonde matrix V is the same as evaluating the polynomial of degree less than n with coefficients $c_1, c_2 \ldots c_n$ at points $t_1, t_2 \ldots t_m$.

$$V \cdot C = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

if the columns of matrix V are independent then the linear combination of column values should be equal to 0.

Hence,

$$c_1 + c_2 t_1 + c_3 t_1^2 - - - - - + c_n t_1^{n-1} = 0$$

$$c_1 + c_2 t_2 + c_3 t_2^2 - - - - - + c_n t_2^{n-1} = 0$$

$$\vdots$$

$$c_1 + c_2 t_m + c_3 t_m^2 - - - - - + c_n t_m^{n-1} = 0$$

Hence we can write $\Rightarrow$

$$P(t_i) = c_1 + c_2 t_i + c_3 t_i^2 - - - + c_n t_i^{n-1}$$

where $t_i = t_1, t_2 ---- t_m$

for linear independence of all columns.

$$P(t_i) = 0$$

for all $t_i = t_1, t_2, t_3 ---- t_m$.

That means $t_1, t_2, t_3 ---- t_m$ are distinct roots of $P(t_i)$.

but we are given that $(m \geq n)$

That means polynomial $P(t_i)$ which has degree less than n, has n or more roots.

Hence in order to make,

$$P(t_i) = 0$$

$$c_1 + c_2 t_i + c_3 t_i^2 + ---- c_n t_i^{n-1} = 0$$

all of its coefficients should be zero.

when all coefficients are zero. Then the linear combination of different column values is 0.

Which proves that columns of a Vandermonde matrix are linearly independent when $t_1, t_2 --- t_m$ are distinct.

Problem 6.22 $\Rightarrow$

Compute $z = (A+B)(x+y)$

where A, B are $m \times n$ matrices.
x, y are n-vectors.

(a) = Compute flop count when we evaluate z as it is expressed.

$(A+B) \Rightarrow$ will require mn additions. so mn flops.

$(x+y) \Rightarrow$ will require n additions, so n flops.

Multiplication of $(A+B)(x+y) \Rightarrow$

$$P = \begin{bmatrix} C \end{bmatrix}_{m \times n} \begin{bmatrix} D \end{bmatrix}_{n} \qquad \begin{bmatrix} A+B = C \\ x+y = D \end{bmatrix}$$

$$P = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

To calculate one element entry of P. we have to do.

n Products + $(n-1)$ additions.
$= 2n-1$ flops.

Hence to calculate m entries.

$= m(2n-1)$ flops.

Hence Total Number of flops =

$$mn + n + m(2n-1)$$

Total flops = $mn + n + 2mn - m$

Total flops = $3mn + n - m$ flops.

(b) = flop count when,

$$Z = Ax + Ay + Bx + By$$

4 vector multiplies $[m,n] \times [n]$

3 vector additions $[m] + [m]$

1 vector multiplication will take

$= m(2n-1)$ flops.

Hence 4 vector multiplications will take = $4m(2n-1)$ flops

1 vector addition will take = m flops.

3 vector addition will take = $3m$ flops.

Total Number of flops =

$$4m(2n-1) + 3m$$

$$= 8mn - 4m + 3m$$

$$= 8mn - m$$

$$\boxed{\text{Total number of flops} = 8mn - m}$$

(c) = if m and n are non zero positive integers then →

$$n \leq mn$$

adding 7mn (+ve value) both sides.

$$7mn + n \leq 8mn$$

Subtracting m (+ve value) both sides

$$\boxed{7mn + n - m \leq 8mn - m} \to \text{Equation (1)}.$$

We can say that =

$$\boxed{3mn + n - m < 7mn + n - m}$$

Hence

$$\boxed{3mn + n - m < 8mn - m}$$

Hence we can say method (a) requires fewer flops than method (b).

$$\boxed{\text{No. of flops method (a)} < \text{no. of flops required in method (b)}}$$

PROBLEM 7.15 =

It is given that →

$$y = c * u \longrightarrow \text{equation (1)}$$

and $z = h * y \longrightarrow$ equation (2)

putting value of y in equation (1) from equation (2) we get ⇒

$$z = h * (c * u)$$

Since convolution is associative we can write ⇒

$$z = (h * c) * u$$

But it in given that

$$\boxed{(h * c) \approx e_1}$$

Hence we can write =

$$\boxed{z = e_1 * u}$$

Hence we can nay that the received equalized signal (z)

can be obtained by convolving unit vector of length $(n+k-1)$

with the transmitted signal u.