PROBLEM 8.3 ⇒ a and x are 3-vectors.

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

cross Product $f(x) = a \times x$ is given as =

$$a \times x = \begin{bmatrix} a_2 x_3 - a_3 x_2 \\ a_3 x_1 - a_1 x_3 \\ a_1 x_2 - a_2 x_1 \end{bmatrix}$$

we can represent this cross Product in the form of following Matrix Multiplication ⇒

$$a \times x = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Hence $a \times x = Ax$

where $A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Proof that $f(x) = Ax = a \times x$ is a linear function for all x.

for any function to be linear it has to follow the Principle of Superposition.

for any linear function $\Rightarrow$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Should be true.

lets assume $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

are 3-vectors. and α and β are two scalars.

Then $\alpha x + \beta y = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix} + \begin{bmatrix} \beta y_1 \\ \beta y_2 \\ \beta y_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}$$

Then cross product $\Rightarrow$

$$f(\alpha x + \beta y) = A(\alpha x + \beta y)$$

$$f(\alpha x + \beta y) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}$$

also $\Rightarrow$

$$\alpha f(x) + \beta f(y) = \alpha [Ax] + \beta (A \cdot y)$$

$$= A\alpha x + A\beta y \quad \text{(commutative property)}$$

$$= A(\alpha x + \beta y) \quad \text{(Distributive Property)}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}$$

we can see that =

$$f(\alpha x+\beta y) = \alpha f(x) + \beta f(y)$$

Hence function

$$f(x) = a \times x = Ax$$

is a linear function for fixed a.

and $A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Problem 8.4 =

given 3X3 image is = lets say its I.

$$I = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 & 9 \\ \hline \end{array}$$

Then the 9-vector x will be =

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

(column major)

This vector x represents Image I in vector form.

given that $y = f(x)$

where y represents the transformed image. y is also a 9-vector.

for each operation we have to calculate the 9×9 matrix A for which $y = Ax$.

(a)= Turn the original image x upside down.

given Image I =

1	4	7
2	5	8
3	6	9

when its turned upside down then resultant Image =

$I_{upside/down}$ =

3	6	9
2	5	8
1	4	7

Then resultant 9-vector y after transformation could be represented as =

$$y = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 6 \\ 5 \\ 4 \\ 9 \\ 8 \\ 7 \end{bmatrix}$$

(formed using the transformed Image matrix)

In order to get =

$$y = Ax$$

which is =

$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 6 \\ 5 \\ 4 \\ 9 \\ 8 \\ 7 \end{bmatrix} = [A]_{9\times 9} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

$\Rightarrow [A]_{9\times 9}$ matrix has to be $\Rightarrow$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(B) = Rotate the original image x clockwise 90°.

Original Image I =

1	4	7
2	5	8
3	6	9

rotate image I By 90° clockwise ⇒

I rotated By 90° =

3	2	1
6	5	4
9	8	7

Then resultant 9-vector y =

$$y = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 2 \\ 5 \\ 8 \\ 1 \\ 4 \\ 7 \end{bmatrix}$$

In this case =

$$y = A x$$

$$\begin{bmatrix} 3 \\ 6 \\ 9 \\ 2 \\ 5 \\ 8 \\ 1 \\ 4 \\ 7 \end{bmatrix} = [A]_{9\times 9} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

Then $[A]_{9\times 9} =$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(C) ⇒ Translate the image up by 1 pixel and to the right by 1 pixel. In the translated image assign value $y_i = 0$ to the pixels in the first column and the last row.

Original image I =

1	4	7
2	5	8
3	6	9

Then transformed image =

$I_{\text{Translated}}$ =

0	2	5
0	3	6
0	0	0

In that case 9-vector $y, x \Rightarrow$

$$y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 3 \\ 0 \\ 5 \\ 6 \\ 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

also $y = Ax$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 3 \\ 0 \\ 5 \\ 6 \\ 0 \end{bmatrix} = [A]_{9 \times 9} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$$

Then matrix A should be :

$$[A]_{9\times 9} \Rightarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(D) = Set each pixel value y_i to be the average of the neighbors of pixel i in the original image.

original image I =

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{bmatrix}$$

where $P_1 = 1$, $P_2 = 4$, $P_3 = 7$

$P_4 = 2$, $P_5 = 5$, $P_6 = 8$

$P_7 = 3$, $P_8 = 6$, $P_9 = 9$

9-vector $x = \begin{bmatrix} P_1 \\ P_4 \\ P_7 \\ P_2 \\ P_5 \\ P_8 \\ P_3 \\ P_6 \\ P_9 \end{bmatrix}$

The translated image I' could be formed by taking average of neighbor pixels in original image.

lets represent the translated image by I' and the new pixels By p'.

$$I' = \begin{bmatrix} p_1' & p_2' & p_3' \\ p_4' & p_5' & p_6' \\ p_7' & p_8' & p_9' \end{bmatrix}$$

$$p_1' = \frac{p_2 + p_4}{2}$$

$$p_2' = \frac{p_1 + p_3 + p_5}{3}$$

$$p_3' = \frac{p_2 + p_6}{2}$$

$$p_4' = \frac{p_1 + p_5 + p_7}{3}$$

$$p_5' = \frac{p_2 + p_4 + p_6 + p_8}{4}$$

$$p_6' = \frac{p_3 + p_5 + p_9}{3}$$

$$P_7' = \frac{P_4 + P_8}{2}, \quad P_8' = \frac{P_5 + P_7 + P_9}{3}, \quad P_9' = \frac{P_6 + P_8}{2}$$

Then translated image 9-vector $Y \Rightarrow$

$$Y = \begin{bmatrix} P_1' \\ P_4' \\ P_7' \\ P_2' \\ P_5' \\ P_8' \\ P_3' \\ P_6' \\ P_9' \end{bmatrix} = \begin{bmatrix} \frac{P_2 + P_4}{2} \\ \frac{P_1 + P_5 + P_7}{3} \\ \frac{P_4 + P_8}{2} \\ \frac{P_1 + P_3 + P_5}{3} \\ \frac{P_2 + P_4 + P_6 + P_8}{4} \\ \frac{P_5 + P_7 + P_9}{3} \\ \frac{P_2 + P_6}{2} \\ \frac{P_3 + P_5 + P_9}{3} \\ \frac{P_6 + P_8}{2} \end{bmatrix}$$

In that case

$$y = Ax$$

$$\begin{bmatrix} \frac{p_2+p_4}{2} \\ \frac{p_1+p_5+p_7}{3} \\ \frac{p_4+p_8}{2} \\ \frac{p_1+p_3+p_5}{3} \\ \frac{p_2+p_4+p_6+p_8}{4} \\ \frac{p_5+p_7+p_9}{3} \\ \frac{p_2+p_6}{2} \\ \frac{p_3+p_5+p_9}{3} \\ \frac{p_6+p_8}{2} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}_{9\times 9} \begin{bmatrix} p_1 \\ p_4 \\ p_7 \\ p_2 \\ p_5 \\ p_8 \\ p_3 \\ p_6 \\ p_9 \end{bmatrix}$$

Then Matrix $[A]_{9 \times 9} \Rightarrow$

$$
\begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{bmatrix}
$$

PROBLEM 8.12 ⇒

(a) = given that ⇒

$$\alpha = \int_{-1}^{1} f(x)\,dx$$

also standard method to calculate α ⇒

$$\hat{\alpha} = \omega_1 f(t_1) + \omega_2 f(t_2) \ldots \omega_n f(t_n)$$

when $\boxed{f(x) = 1}$

$$\alpha = \int_{-1}^{1} 1 \cdot dx = [x]_{-1}^{1} = 1 - (-1) = 2$$

$$\hat{\alpha} = \omega_1 \cdot 1 + \omega_2 \cdot 1 \;\; ---- \;\; \omega_n \cdot 1$$

$$\hat{\alpha} = 1 \cdot \omega_1 + 1 \cdot \omega_2 + \cdots\cdots 1 \cdot \omega_n$$

we have, $\hat{\alpha} = \alpha$, hence we get

$$\boxed{1 \cdot \omega_1 + 1 \cdot \omega_2 + \cdots\cdots 1 \cdot \omega_n = 2} \longrightarrow (1)$$

when $\boxed{f(x) = x}$ then,

$$\alpha = \int_{-1}^{1} x \cdot dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = \left[\frac{1}{2} - \left(\frac{1}{2}\right)\right]$$
$$= 0$$

$$\hat{\alpha} = w_1 t_1 + w_2 t_2 + \text{----} \; w_n t_n$$

Since $\hat{\alpha} = \alpha$ we can write.

$$\boxed{t_1 w_1 + t_2 w_2 + \text{----} \; t_n w_n = 0} \rightarrow (2)$$

Similarly when $\boxed{f(x) = x^2}$ then,

$$\alpha = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right]$$
$$= \frac{2}{3}$$

$$\hat{\alpha} = w_1 t_1^2 + w_2 t_2^2 + \text{-----} \; w_n t_n^2$$

Since $\hat{\alpha} = \alpha$ we can write $\Rightarrow$ $\rightarrow (3)$

$$\boxed{t_1^2 w_1 + t_2^2 w_2 + \text{----} \; t_n^2 w_n = \frac{2}{3}}$$

when $\boxed{f(x) = x^3}$ then,

$$\alpha = \int_{-1}^{1} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \left[1/4 - (1/4)\right]$$

$$= 0$$

$$\hat{\alpha} = w_1 t_1^3 + w_2 t_2^3 + \cdots w_n t_n^3$$

$\boxed{\hat{\alpha} = \alpha}$ Hence we can write =

$$\boxed{t_1^3 w_1 + t_2^3 w_2 + \cdots w_n t_n^3 = 0}$$

eqn (4)

when $\boxed{f(x) = x^d}$ then

$$\alpha = \int_{-1}^{1} x^d \, dx = \left[\frac{x^{d+1}}{d+1}\right]_{-1}^{1} = \left[\frac{1}{d+1} - \frac{(-1)^{d+1}}{d+1}\right]$$

$$= \left[\frac{1 - (-1)^{d+1}}{d+1}\right]$$

$$\hat{\alpha} = w_1 t_1^d + w_2 t_2^d + \cdots w_n t_n^d$$

Since $\boxed{\hat{\alpha} = \alpha}$ we can write that =

$$\boxed{t_1^d w_1 + t_2^d w_2 + \cdots t_n^d w_n = \left[\frac{1-(-1)^d}{d+1}\right]}$$

↳ eqn. (5)

writing all equations together =

$$
\begin{aligned}
1 \cdot w_1 + 1 \cdot w_2 + \cdots\cdots 1 \cdot w_n &= 2 \\
t_1 w_1 + t_2 w_2 + \cdots\cdots t_n w_n &= 0 \\
t_1^2 w_1 + t_2^2 w_2 + \cdots\cdots t_n^2 w_n &= 2/3 \\
t_1^3 w_1 + t_2^3 w_2 + \cdots\cdots w_n t_n^3 &= 0 \\
\vdots \qquad\quad \vdots \qquad\quad & \vdots \qquad \vdots \\
t_1^d w_1 + t_2^d w_2 + \cdots t_n^d w_n &= \left[\frac{1-(-1)^{d+1}}{d+1}\right]
\end{aligned}
$$

Then we can represent those equations in following matrix form =

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ t_1^3 & t_2^3 & \cdots & t_n^3 \\ \vdots & \vdots & & \vdots \\ t_1^d & t_2^d & \cdots & t_n^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ \vdots \\ \frac{1-(-1)^{d+1}}{d+1} \end{bmatrix}$$

$\boxed{Aw = b}$ form.

(b) = Show that following Quadrature Methods have order 1, 2 and 3 respectively.

(i) = Trapezoid rule ⇒

$n=2$, $t_1=-1$, $t_2=1$ and $w_1=1$, $w_2=1$

lets calculate $\hat{\alpha}$ and compare with α values.

when $n=2$

$$\underset{\text{values vector}}{\hat{\alpha}} = \begin{bmatrix} 1 & 1 \\ t_1 & t_2 \\ t_1^2 & t_2^2 \\ \vdots & \vdots \\ t_1^d & t_2^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ \vdots & \\ (-1)^d & (1)^d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underset{\text{values vector}}{\hat{\alpha}} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ \vdots \\ (-1)^d + 1 \end{bmatrix}$$

compare $\hat{\alpha}$ and α vectors.

$$\underset{\text{vetor}}{\hat{\alpha}} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ \vdots \\ (-1)^d + 1 \end{bmatrix} \qquad \underset{\text{vector}}{\alpha} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ \vdots \end{bmatrix}$$

for trapazoid rule.

$\hat{\alpha} = \alpha$ for $f(x) = 1$

$f(x) = x$

but for $f(x) = x^2$, $\hat{\alpha} \neq \alpha$

Hence trapazoid rule is of order 1.

(2) = Simpsons rule ⇒

$n = 3$, $t_1 = -1$, $t_2 = 0$ and $t_3 = 1$

$w_1 = 1/3$, $w_2 = 4/3$, $w_3 = 1/3$

$$\underset{\text{value vector}}{\hat{\alpha}} = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^3 & t_2^3 & t_3^3 \\ t_1^4 & t_2^4 & t_3^4 \\ \vdots & \vdots & \vdots \\ t_1^d & t_2^d & t_3^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\hat{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ \vdots & & \\ (-1)^d & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

$$\hat{\alpha} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 2/3 \\ \vdots \\ \frac{(-1)^d + 1}{3} \end{bmatrix}$$

when $d = 4$, $\alpha = \dfrac{1 - (-1)^{d+1}}{d+1}$

$$= \frac{1 - (-1)^{4+1}}{4+1} = \frac{2}{5}$$

Comparing $\hat{\alpha}$ and α vectors.

$$\hat{\alpha} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 2/3 \\ \vdots \\ \frac{(-1)^d + 1}{3} \end{bmatrix} \qquad \alpha = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 2/5 \\ \vdots \\ \frac{1 + (-1)^{d+1}}{d+1} \end{bmatrix}$$

$\hat{\alpha} = \alpha$ for $f(x) = 1$

$f(x) = x$

$f(x) = x^2$

$f(x) = x^3$

but for $f(x) = x^4$, $\hat{\alpha} \neq \alpha$

Hence Simpsons rule holds for order 3 hence it also holds for order 2.

(3) = Simpsons $3/8$ rule =

$n=4, \; t_1=-1, \; t_2=-1/3, \; t_3=1/3, \; t_4=1$

$\omega_1 = 1/4, \; \omega_2 = 3/4, \; \omega_3 = 3/4, \; \omega_4 = 1/4$

$$\hat{\alpha} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \\ t_1^4 & t_2^4 & t_3^4 & t_4^4 \\ \vdots & \vdots & \vdots & \vdots \\ t_1^d & t_2^d & t_3^d & t_4^d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

$$\hat{\alpha} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1/3 & 1/3 & 1 \\ 1 & 1/9 & 1/9 & 1 \\ -1 & \frac{-1}{27} & \frac{1}{27} & 1 \\ 1 & \frac{1}{81} & \frac{1}{81} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^d & (-1/3)^d & (1/3)^d & (1)^d \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \\ 3/4 \\ 1/4 \end{bmatrix}$$

$$\hat{\alpha} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 28/54 \\ \vdots \\ (-1)^d \, 1/4 + (-1/3)^d \, 3/4 + (1/3)^d \, 3/4 + 1/4 \end{bmatrix}$$

Comparing $\hat{\alpha}$ and α vectors -

$$\hat{\alpha} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 14/27 \\ \vdots \\ \vdots \end{bmatrix} \qquad \alpha = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 2/5 \\ \vdots \\ \vdots \\ \frac{1+(-1)^{d+1}}{d+1} \end{bmatrix}$$

$\hat{\alpha} = \alpha$ for $f(x) = 1$
$f(x) = x$
$f(x) = x^2$
$f(x) = x^3$

But $\hat{\alpha} \neq \alpha$ for $f(x) = x^4$

Hence Simpsons $3/8$ rule order $= 3$.

Problem 6.16 ⇒

$$f(u,v) = \theta_1 + \theta_2 u + \theta_3 v + \theta_4 uv$$

① $= f(x_1, y_1) = F_{11}$

$$\theta_1 + \theta_2 x_1 + \theta_3 y_1 + \theta_4 x_1 y_1 = f_{11}$$

② $= f(x_1, y_2) = F_{12}$

$$\theta_1 + \theta_2 x_1 + \theta_3 y_2 + \theta_4 x_1 y_2 = F_{12}$$

③ $= f(x_2, y_1) = f_{21}$

$$\theta_1 + \theta_2 x_2 + \theta_3 y_1 + \theta_4 x_2 y_1 = F_{21}$$

④ $= f(x_2, y_2) = f_{22}$

$$\theta_1 + \theta_2 x_2 + \theta_3 y_2 + \theta_4 x_2 y_2 = f_{22}$$

writing all four equations together

$$\theta_1 + \theta_2 x_1 + \theta_3 y_1 + \theta_4 x_1 y_1 = F_{11}$$

$$\theta_1 + \theta_2 x_1 + \theta_3 y_2 + \theta_4 x_1 y_2 = F_{12}$$

$$\theta_1 + \theta_2 x_2 + \theta_3 y_1 + \theta_4 x_2 y_1 = F_{21}$$

$$\theta_1 + \theta_2 x_2 + \theta_3 y_2 + \theta_4 x_2 y_2 = f_{22}$$

Then we can represent those equations in matrix form as =

$$A\theta = b$$

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \end{bmatrix}$$

$[A]_{4\times4}$ 4-vector b

PROBLEM 9.) ⇒

lets consider the initial states of three compartments at time t are =

compartment 1 = $(x_t)_1$

compartment 2 = $(x_t)_2$

compartment 3 = $(x_t)_3$

1 = 10% of material in comp. 1 moves to comp. 2.

comp. 1 state = $(x_t)_1 - 0.10\,(x_t)_1$

2 = 5% of the material in comp. 2 moves to comp. 3.

comp 2 state = $(x_t)_2 - 0.05\,(x_t)_2 + 0.10\,(x_t)_1$

(3) = 5% of the material in compartment 3 moves to compartment 2.

$$\text{comp (3) state} = (x_t)_3 - 0.05(x_t)_3 + 0.05(x_t)_2$$

(4) = 5% of the material in compartment 3 is eliminated.

$$\text{comp(3) state} = (x_t)_3 - 0.05(x_t)_3 - 0.05(x_t)_3 + 0.05(x_t)_2$$

final states of all compartments at time (t+1) =

$$(x_{t+1})_1 = 0.90(x_t)_1 + 0.05(x_t)_3$$

$$(x_{t+1})_2 = 0.10(x_t)_1 + 0.95(x_t)_2$$

$$(x_{t+1})_3 = 0.05(x_t)_2 + 0.90(x_t)_3$$

Matrix form =

$$\begin{bmatrix} (x_{t+1})_1 \\ (x_{t+1})_2 \\ (x_{t+1})_3 \end{bmatrix} = \begin{bmatrix} 0.90 & 0 & 0.05 \\ 0.10 & 0.95 & 0 \\ 0 & 0.05 & 0.90 \end{bmatrix} \begin{bmatrix} (x_t)_1 \\ (x_t)_2 \\ (x_t)_3 \end{bmatrix}$$

$$x_{t+1} = A x_t$$

where $A = \begin{bmatrix} 0.90 & 0 & 0.05 \\ 0.10 & 0.95 & 0 \\ 0 & 0.05 & 0.90 \end{bmatrix}$

PROBLEM 9.3 =

we are given that

$$x_{t+1} = Ax_t + c \longrightarrow \text{eqtn (1)}$$

when $x_1 = z$ then $x_2 = z$, $x_3 = z$ ----

i.e. if system starts in state z then it stays in state z.

then we can say that =

that if $\boxed{x_t = z}$

then $\boxed{x_{t+1} = z}$

Putting this value in equation (1) ⇒

$$x_{t+1} = Ax_t + c$$

$$z = Az + c$$

we can write ⇒

$$I.z = A.z + c$$

[I = identity matrix
$I.z = z$]

$$\Rightarrow I.z - A.z = c$$

$$= (I - A)z = c$$

Comparing Equations :-

$(I-A)z = c$ and $Fz = g$

we can say that

$$\boxed{F = I - A}$$

(I = identity matrix)

and $\boxed{g = c}$

PROBLEM 10.11 ⇒

Trace of matrix = sum of the diagonal entries of a square matrix is called the trace of the matrix.

(a) = A and B are $m \times n$ matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

lets calculate matrix product $A^T B$

$$A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n \times m}$$

Then Product $P = A^T B$

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n \times m} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

$$= \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

To find the trace of matrix Product $(A^T B) = P$, we have to Sum all the diagonal elements of Product matrix P. that means we only need to calculate diagonal elements of matrix P.

P_{11} = multiply (1st row of matrix A^T with 1st column of matrix B)

$= a_{11} b_{11} + a_{21} b_{21} + a_{31} b_{31} \cdots + a_{m1} b_{m1}$

P_{22} = multiply (2nd row of matrix AT with 2nd column of matrix B)

$= a_{12} b_{12} + a_{22} b_{22} + \cdots a_{m2} b_{m2}$

$\vdots$

P_{nn} = multiply (nth row of A^T matrix with nth column of matrix B)

$= a_{1n} b_{1n} + a_{2n} b_{2n} + \cdots a_{mn} b_{mn}$

Then trace $(A^T B) \Rightarrow P_{11} + P_{22} + P_{33} --- P_{nn}$

$$tr(A^T B) = a_{11} b_{11} + a_{21} b_{21} + a_{31} b_{31} ---+ a_{m1} b_{m1}$$
$$+ a_{12} b_{12} + a_{22} b_{22} + ----- a_{m2} b_{m2}$$
$$\vdots$$
$$+ a_{1n} b_{1n} + a_{2n} b_{2n} + ----- a_{mn} b_{mn}$$

This could be written as =

$$tr(A^T B) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$$

* = Time complexity to calculate $tr(A^T B) \Rightarrow$

complexity to evaluate 1 diagonal element =
(m products) + (m-1) additions.
$= m + m - 1 = 2m - 1$

complexity to evaluate n diagonal elements =
$= n(2m-1)$

Total complexity = $n(2m-1)$ + $(n-1)$ additions
$= 2mn - n + n - 1$
$= 2mn - 1$

(b) = Show that $tr(A^TB) = tr(B^TA)$

$$A^TB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T_{m\times n} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n\times m} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots \\ P_{n1} & \hat{P}_{n2} & \cdots & \hat{P}_{nn} \end{bmatrix}_{n\times n}$$

(final product representath A^TB)

Similarly ⇒

$$B^TA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}^T_{m\times n} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{m1} \\ b_{12} & b_{22} & \cdots & b_{m2} \\ \vdots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{mn} \end{bmatrix}_{n\times m} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$B^T A = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & & & \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}_{n \times n}$$

In the Product, $P = A^T B$ and $q = B^T A$ we can see that diagonal elements are same =

$$P_{11} = q_{11} = a_{11} b_{11} + a_{21} b_{21} + a_{31} b_{31} \cdots + a_{m1} b_{m1}$$

$$P_{22} = q_{22} = a_{12} b_{12} + a_{22} b_{22} + \cdots\cdots a_{m2} b_{m2}$$

$$\vdots$$

$$P_{nn} = q_{nn} = a_{1n} b_{1n} + a_{2n} b_{2n} + \cdots\cdots\cdots a_{mn} b_{mn}$$

we can write =

$$P_{11} + P_{22} + \cdots + P_{nn} = q_{11} + q_{12} + \cdots q_{nn}$$

Sum of diagonal of $(A^T B)$ = Sum of diagonal $(B^T A)$

$$\boxed{tr(A^T B) = tr(B^T A)}$$

Proved

Since the diagonals of Matrices =

$A^T B$ and $B^T A$ are identical.

The Sum of diagonals of $A^T B$ and $B^T A$ will also be identical.

Hence $\boxed{tr(A^T B) = tr(B^T A)}$

(c)= Show that $tr(A^T A) = ||A||^2$

$$\text{Matrix } A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}_{m \times n}$$

$$||A||^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2$$

$$\left.\begin{aligned} ||A||^2 = & \ a_{11}^2 + a_{12}^2 + ---- a_{1n}^2 \\ & + a_{21}^2 + a_{22}^2 + ---- a_{2n}^2 \\ & \vdots \\ & + a_{m1}^2 + a_{m2}^2 + ---- a_{mn}^2 \end{aligned}\right\} \to \text{eqtion } (1)$$

$$A^T A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}^T_{m\times n} \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}_{m\times n}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{21} & --- & a_{m1} \\ a_{12} & a_{22} & --- & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n\times m} \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} P_{11} & P_{12} & --- & P_{1n} \\ P_{21} & P_{22} & --- & P_{2n} \\ \vdots & & & \\ P_{n1} & \hat{P}_{n2} & -- & \hat{P}_{nn} \end{bmatrix}_{n\times n}$$

(let's say P represent the final product matrix)

To find trace we have to sum only diagonal elements $(P_{11}, P_{22} --- P_{nn})$.

P_{11} = multiply 1^{st} row of A^T with 1^{st} column of A.

$$= a_{11}^2 + a_{21}^2 + a_{31}^2 \; --- + a_{m1}^2$$

P_{22} = multiply 2nd row of A^T with 2nd column of A

$$= a_{12}^2 + a_{22}^2 + a_{32}^2 + \cdots + a_{m2}^2$$

$\vdots$

P_{nn} = multiply nth row of A^T with nth column of A.

$$= a_{1n}^2 + a_{2n}^2 + a_{3n}^2 + \cdots + a_{mn}^2$$

$$tr(A^TA) = P_{11} + P_{22} + P_{33} + \cdots P_{nn}$$

$$tr(A^TA) = a_{11}^2 + a_{21}^2 + a_{31}^2 \cdots + a_{m1}^2$$
$$+ a_{12}^2 + a_{22}^2 + a_{32}^2 + \cdots + a_{m2}^2$$
$$\vdots$$
$$+ a_{1n}^2 + a_{2n}^2 + a_{3n}^2 + \cdots + a_{mn}^2$$

$tr(A^TA) = \|A\|^2$ (from equation (1) comparision)

(d) = Show that $tr(A^T B) = tr(BA^T)$

$$A^T B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T_{m \times n} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

$$A^T B = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n \times m} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

$$A^T B = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots \\ \hat{P}_{n1} & \hat{P}_{n2} & \cdots & \hat{P}_{nn} \end{bmatrix}_{n \times n}$$

(final product representath $A^T B$)

$$tr(A^T B) = P_{11} + P_{22} + P_{33} + \cdots P_{nn}$$

$$\begin{aligned} tr(A^T B) = {} & a_{11} b_{11} + a_{21} b_{21} + a_{31} b_{31} \cdots + a_{m1} b_{m1} \\ & + a_{12} b_{12} + a_{22} b_{22} + \cdots a_{m2} b_{m2} \\ & \vdots \\ & + a_{1n} b_{1n} + a_{2n} b_{2n} + \cdots a_{mn} b_{mn} \end{aligned}$$

$\hookrightarrow$ Equation (1)

$$BA^T = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T_{m \times n}$$

$$BA^T = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}_{n \times m}$$

$$BA^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}_{m \times m}$$

↳ Product BA^T representation.

lets evaluate BA^T matrix Diagonal Elements.

a_{11} = (multiply 1st row of B with 1st column of A^T)

$= a_{11} b_{11} + a_{12} b_{12} \cdots\cdots a_{1n} b_{1n}$

a_{22} = (multiply 2nd row of B with 2nd column of A^T)

$= a_{21} b_{21} + a_{22} b_{22} + \cdots\cdots a_{2n} b_{2n}$

$\vdots$

$q_{mm} =$ (multiply mth row of B with mth column of A^T)

$= a_{m1} b_{m1} + a_{m2} b_{m2} + \cdots a_{mn} b_{mn}$

$$tr(BA^T) = q_{11} + q_{22} + \cdots q_{mm}$$

$$= a_{11} b_{11} + a_{12} b_{12} \cdots a_{1n} b_{1n}$$
$$+ a_{21} b_{21} + a_{22} b_{22} + \cdots a_{2n} b_{2n}$$
$$\vdots$$
$$+ a_{m1} b_{m1} + a_{m2} b_{m2} + \cdots a_{mn} b_{mn}$$

$\hookrightarrow$ equation 2

On comparing equation 1 and equation 2

we find that $tr(A^T B) = tr(BA^T)$

PROBLEM 10.13 $\Rightarrow$

we know that Dirichlet energy is given by =

$$D(v) = \|A^T v\|^2$$

we also know that for any matrix =

$$B^T B = \|B\|^2$$

Hence we can write =

$$D(v) = \|A^T v\|^2$$

$$D(v) = (A^T v)^T (A^T v)$$

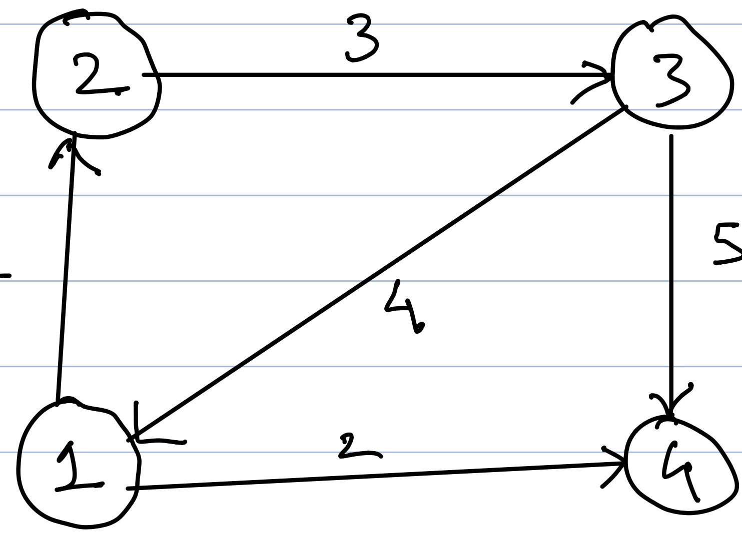
$$D(v) = [v^T (A^T)^T] (A^T v)$$

$$D(v) = (v^T A)(A^T v)$$

$$D(v) = v^T (A A^T) v$$

$$D(v) = v^T L v \quad (\text{given that } AA^T = L)$$

(b) = lets consider a graph to describe entries of L.



In that case we can say =

degree of node 1 = 3

degree of node 2 = 2

degree of node 3 = 3

degree of node 4 = 2

for this graph $\Rightarrow$

$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 5}$$

$$A^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}_{5\times 4}$$

Laplacian matrix $L = AA^T$

$AA^T =$

$$\begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 5} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}_{5\times 4}$$

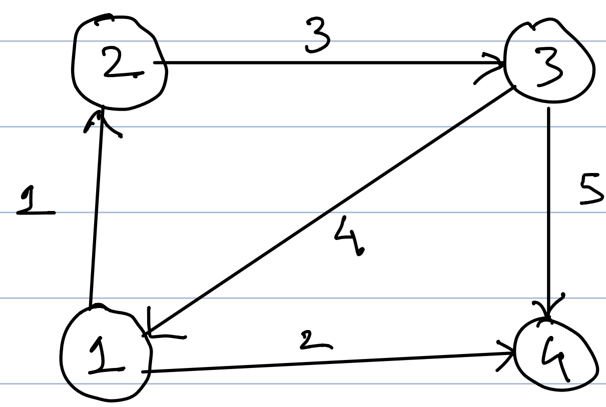
$$\Rightarrow \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}_{4\times 4}$$

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}_{4\times 4}$$

(1) = we can see that diagonal of Laplacian matrix represents the degrees of the nodes.

(2) = other entries tell whether there is an edge between two vertices or not. -1 explains that there

is an edge present between two vertices (nodes). 0 explains that there is no edge present between the nodes.



In this graph ⇒

$L_{12} = -1$ (edge present)
$L_{21} = -1$ (edge present)

$L_{23} = -1$ (edge present)
$L_{32} = -1$ (edge present)

$L_{13} = -1$ (edge present)
$L_{31} = -1$ (edge present)

$L_{14} = -1$ (edge present)
$L_{41} = -1$ (edge present)

$L_{34} = -1$ (edge present)

$L_{43} = -1$ (edge present)

But $L_{24} = 0$ (no edge present)

$L_{42} = 0$ (no edge present