Problem 14 is given as below:

Let $\mathbf{x_1}, ..., \mathbf{x_n}$ be n d-dimensional samples and $\mathbf{\Sigma}$ be any nonsingular d-by-d matrix. Show that the vector $\mathbf{x}$ that minimizes

$$\sum_{k=1}^{m} (\mathbf{x_k} - \mathbf{x})^t \mathbf{\Sigma}^{-1} (\mathbf{x_k} - \mathbf{x})$$

is the sample mean, $\bar{\mathbf{x}} = 1/n \sum_{k=1}^{n} \mathbf{x_k}$.

Ans. = for n samples $x_1, x_2, x_3 --- x_n$ we can find the mean value as below:

$$\text{Mean} = \frac{\text{Sum of all samples}}{\text{Total number of samples}}$$

Hence we can write,

$$\text{Mean} = \boxed{\bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k}$$

from the given equation ⇒ ⟶ ①

$$\frac{1}{n} \sum_{k=1}^{n} (x_k - x)^t \Sigma^{-1} (x_k - x)$$

lets expand By introducing $\bar{x}$ in the bracket:

$$\frac{1}{n} \sum_{k=1}^{n} (x_k - \bar{x} + \bar{x} - x)^t \Sigma^{-1} (x_k - \bar{x} + \bar{x} - x)$$

⟶ Equation ②

This Equation could be written as =

$$\frac{1}{n}\sum_{K=1}^{n}(x_K - x)^{t}\,\Sigma^{-1}(x_K - x) =$$

$$\frac{1}{n}\sum_{K=1}^{n}(x_K - \bar{x} + \bar{x} - x)^{t}\,\Sigma^{-1}(x_K - \bar{x} + \bar{x} - x)$$

Solving Right hand side =

$$\frac{1}{n}\sum_{K=1}^{n}(x_K - \bar{x} + \bar{x} - x)^{t}\,\Sigma^{-1}(x_K - \bar{x} + \bar{x} - x)$$

$$\Rightarrow \frac{1}{n}\Big[\sum_{K=1}^{n}(x_K - \bar{x})^{t}\,\Sigma^{-1}(x_K - \bar{x})$$

$$+ 2(\bar{x} - x)^{t}\,\Sigma^{-1}\sum_{K=1}^{n}(x_K - \bar{x}) + n(\bar{x} - x)'\,\Sigma^{-1}(\bar{x} - x)\Big]$$

$$= \frac{1}{n}\sum_{K=1}^{n}(x_K \bar{x})\,\Sigma^{-1}(x_K \bar{x}) + (\bar{x} - x)^{t}\,\Sigma^{-1}(\bar{x} - x)$$

$$\geq \frac{1}{n}\sum_{K=1}^{n}(x_K - \bar{x})^{t}\,\Sigma^{-1}(x_K - \bar{x})$$

we know that =

$$\sum_{K=1}^{n}(x_K - \bar{x}) = \Big(\sum_{K=1}^{n}x_K\Big) - n\bar{x} = n\bar{x} - n\bar{x}$$

$$= 0$$

also we know that Σ is positive definite hence we can write =

$$(\bar{x}-x)^{t}\Sigma^{-1}(\bar{x}-x) \geq 0$$

with strict inequality holding if and only if $x \neq \bar{x}$.

Hence function from equation ①

$$\frac{1}{n}\sum_{k=1}^{n}(x_k-x)^{t}\Sigma^{-1}(x_k-x)$$

is minimized when $(x=\bar{x})$.

which is =

$$x=\bar{x}=\frac{1}{n}\sum_{k=1}^{n}x_k$$