Problem 14 (a) in Chapter 3, on Page 144 of the following textbook: Richard O. Duda, Peter E. Hart, David G. Stork, "Pattern Classification", 2nd Edition, Wiley-Interscience, October 2000.

Problem 14 (a) is given as below:

Suppose that $p(\mathbf{x}|\mu_\mathbf{i}, \mathbf{\Sigma}, w_i) \sim N(\mu_\mathbf{i}, \mathbf{\Sigma})$, where $\mathbf{\Sigma}$ is a common covariance matrix of all c classes. Let n samples $\mathbf{x_1}, ..., \mathbf{x_n}$ be drawn as usual, and let $l_1, ..., l_n$ be their labels, so that $l_k = i$ if the state of nature of $\mathbf{x_k}$ was w_i. Show that

$$p(\mathbf{x_1}, ..., \mathbf{x_n}, \mathbf{l_1}, ..., \mathbf{l_n}|\mu_\mathbf{1}, ..., \mu_\mathbf{c}, \mathbf{\Sigma}) = \frac{\prod_{k=1}^{n} P(w_{l_k})}{(2\pi)^{nd/2}|\mathbf{\Sigma}|^{n/2}} exp\left[-\frac{1}{2}\sum_{k=1}^{n}(\mathbf{x_k} - \mu_{\mathbf{l_k}})^\mathbf{t}\mathbf{\Sigma}^{-\mathbf{1}}(\mathbf{x_k} - \mu_{\mathbf{l_k}})\right].$$

Ans. =

Using Bayes rule we can conclude that,

$$P(x_1 ---- x_n, l_1 ---- l_n | \mu_1 ----- \mu_c, \Sigma)$$
$$= P(x_1, ----- x_n | \mu_1, ---- \mu_c, l_1 --- l_n, \Sigma) P(l_1 --- l_n)$$

we know that distribution $l_1, l_2 ---- l_n$ is independent of $\mu_1, \mu_2 ---- \mu_c$ or Σ

Hence we can also write,

$$P(x_1, x_2 --- x_n | \mu_1 ---- \mu_c, l_1, l_2 --- l_n, \Sigma)$$
$$= \prod_{k=1}^{n} P(x_k | \mu_1, ---- \mu_c, \Sigma, l_k)$$

$$= \prod_{k=1}^{n} \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} exp\left[-\frac{1}{2}(x_k - \mu_{l_k})^t \Sigma^{-1} (x_k - \mu_{l_k})\right]$$

↳ equation ①

Since $(l_1, l_2 \ldots l_i)$ are independent Hence the Probability density could be defined as the product of individual Probabilities. Hence we can write,

$$P(l_1, l_2 \ldots l_n) = \prod_{k=1}^{n} P(l_k) = \prod_{k=1}^{n} P(\omega_{1k})$$

$\hookrightarrow$ equation (2)

after combining equation (1) and (2), we can write the expression as =

$$P(x_1 \ldots x_n, l_1 \ldots l_n \mid \mu_1 \ldots \mu_c, \Sigma)$$

$$= \frac{\prod_{k=1}^{n} P(\omega_{1k})}{(2\pi)^{\frac{nd}{2}} |\Sigma|^{n/2}} \exp\left[-\frac{1}{2} \sum_{k=1}^{n} (x_k - \mu_{1k})^t \Sigma^{-1} (x_k - \mu_{1k})\right]$$