CP 414 Winter 2025, Assignment 4
Due: Friday, April 4

1. Recall, that for a given undirected graph **G = (V, E)**, a subset ***S*** of the set of vertices ***V*** is called an ***independent set*** if $(u,v) \notin E$ for every pair of vertices $u,v \in S$. Consider the following variations of the independent set problem for an undirected graphs:

ISD Input: a graph **G = (V, E)**, and an integer **k**.
Output: TRUE, if there is a subset **V'** of **V** with at least **k** elements such that no edge in **E** has endpoints in **V'.**

ISO Input: a graph **G = (V, E)**.
Output: integer **n** – size of the largest independent set of the graph **G**.

ISF Input: a graph **G = (V, E)**.
Output: subset **V'** – largest independent set of the graph **G**.

Show that **ISO** and **ISF** are polynomial-time Turing reducible to **ISD**, i.e., show the **pseudocode** solving **ISO** and **ISF** that uses **ISD** as a subroutine, and makes "polynomially many" calls to this subroutine. Provide detailed pseudocode solutions with justification of correctness and running time analysis.

2. The decision problem **HamPathD** is defined as follows:

- Problem instance: an undirected graph **G = (V,E)**.
- Question: Is there a **simple** path **P** in **G** having exactly **|V|-1** edges?

The finding problem **HamPathF** is defined as follows:

- Problem Instance: an undirected graph **G = (V,E)**.
- Output: a simple path **P** in **G** having exactly **|V|-1** edges (sequence of **vertices** forming this path).

a) Show that **HamPathF** is polynomial time Turing reducible to **HamPathD,** i.e., show the solution of **HamPathF** that uses **HamPathD** as a subroutine. Justify the running time of the reduction.

b) The decision problem **HamCycleD** is defined as follows:

- Problem Instance: an undirected graph **G = (V,E)**.
- Question: Is there a simple cycle **P** in **G** having exactly **|V|** edges?

Give a complete proof that **HamCycleD** $\leq_P$ **HamPathD**. Assuming that **HamCycle** is NP-complete, prove that **HamPath** is NP-complete.

3. Selecting representatives. Consider the following problem:

Reps-D: Given

- a set S of n students that we associate with the numbers $\{1, 2, \ldots, n\}$,
- a list of courses ***L***, where each course is a subset $S \subseteq \{1, 2, \ldots, n\}$ that represents the students registered in this course, and
- a positive integer k,

determine if there is a set $\mathbf{R} \subseteq \{1,2,\ldots,n\}$ of size $|\mathbf{R}| \leq k$ such that every course contains at least one of the students in **R**. This set (if exists) is a set of representatives such that each course is represented by at least one student.

Provide complete proof that **Reps-D** is in complexity class **NP**, by defining a suitable certificate and presenting and analyzing running time of verification algorithm for the certificate you defined. Then prove that it is **NP**-complete by reducing from known **NP**-complete problem considered in lectures.

4. Consider the following instance of 3-SAT problem:

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

and reduction from 3-SAT to SUBSET-SUM. What is the instance of SUBSET-SUM problem that corresponds to the above instance of 3-SAT? If the above instance of 3-SAT is YES instance, show a certificate, verifying this, and also corresponding certificate of the SUBSET-SUM problem.

5. Consider the same instance of 3-SAT problem:

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

and reduction from 3-SAT to ISD (as defined in question 2). What is the instance of ISD problem that corresponds to the above instance of 3-SAT? If the above instance of 3-SAT is YES instance, show a certificate, verifying this, and also corresponding certificate of the ISD problem.

(bonus) 6. Use pumping lemma to show that the following language is not context-free: set of palindromes over alphabet **{0,1}** with equal number of **0**s and **1**s. **Provide detailed proof.** In particular make sure that you give a counterexample string that belongs to the language and is not fixed length (as pumping length is unknown). Also, do not forget to consider all possible splits of this string under conditions 2) and 3) of the pumping lemma.