Mc Graw Hill Education

The Foundations: Logic and Proofs

Chapter 1, Part I: Propositional Logic

Propositional Logic Summary

The Language of Propositions

- Connectives
- Truth Values
- Truth Tables

Applications

- Translating English Sentences
- System Specifications
- Logic Puzzles

Logical Equivalences

- Important Equivalences
- Showing Equivalence
- Satisfiability

Propositional Logic

Section 1.1

Section Summary

Propositions

Connectives

- Negation
- Conjunction
- Disjunction
- Implication; contrapositive, inverse, converse
- Biconditional

Truth Tables

Propositions

A *proposition* is a declarative sentence that is either true or false.

Examples of propositions:

a) The Moon is made of green cheese.
b) Trenton is the capital of New Jersey.
c) Toronto is the capital of Canada.
d) $1 + 0 = 1$
e) $0 + 0 = 2$

Examples that are not propositions.

a) Sit down!
b) What time is it?
c) $x + 1 = 2$
d) $x + y = z$

Practice Problem(s)

For each of the following, is this a proposition? If yes, what's its truth value?

1. Elephants are bigger than mice
2. $520 < 111$
3. $y > 5$
4. Today is September 13 and $99 < 5$
5. Today is January 1 or $99 > 5$

Practice Problem(s) (cont.)

6. Please do not fall asleep
7. If elephants were red, they could hide in cherry trees
8. $x < y$ if and only if $y > x$

Propositional Logic

Constructing Propositions

- Propositional Variables: $p, q, r, s, \ldots$
- Compound Propositions; constructed from logical connectives and other propositions
  - Negation $\neg$
  - Conjunction $\wedge$
  - Disjunction $\vee$
  - Implication $\rightarrow$
  - Biconditional $\leftrightarrow$

Compound Propositions: Negation

The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

Example: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Practice Problem(s)

What is the negation of each of these propositions?

a) Mei has an MP3 player.

b) There is no pollution in New Jersey.

c) $2 + 1 = 3$.

d) The summer in Maine is hot and sunny.

Conjunction

The *conjunction* of propositions *p* and *q* is denoted by $p \wedge q$ and has this truth table:

p	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: If *p* denotes "I am at home." and *q* denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

The *disjunction* of propositions *p* and *q* is denoted by $p \vee q$ and has this truth table:

p	q	$P \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: If *p* denotes "I am at home." and *q* denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

In English "or" has two distinct meanings.

- "Inclusive Or" - In the sentence "Students who have taken CS202 or Math120 may take this class," we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \lor q$ to be true, either one or both of p and q must be true.
- "Exclusive Or" - When reading the sentence "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."

In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication [1]

In $p \rightarrow q$ there does not need to be any connection between the hypothesis or the conclusion. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.

These implications are perfectly fine, but would not be used in ordinary English.

- "If the moon is made of green cheese, then I have more money than Bill Gates. "
- "If the moon is made of green cheese then I'm on welfare."
- "If $1 + 1 = 3$, then your grandma wears combat boots."

Understanding Implication 2

One way to view the logical conditional is to think of an obligation or contract.

- "If I am elected, then I will lower taxes."
- "If you get 100% on the final, then you will get an A."

If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Different Ways of Expressing $p \rightarrow q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

a necessary condition for p **is** q

a sufficient condition for q **is** p

Converse, Contrapositive, and Inverse

From $p \to q$ we can form new conditional statements .

- $q \to p$ is the **converse** of $p \to q$
- $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
- $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

contrapositive: If I go to town, then it is not raining.

inverse: If it is not raining, then I will go to town.

Practice Problems

Find the converse, inverse, and contrapositive of the following:

- The home team wins whenever it is raining.
- If the number is positive, then its square is positive.

Biconditional

If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q ." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

Some alternative ways "p if and only if q" is expressed in English:

- p **is necessary and sufficient for** q
- **if** p **then** q**, and conversely**
- p **iff** q

Practice Problem(s)

Let p and q be the propositions

p : I bought a lottery ticket this week.

q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$ b) $p \lor q$ c) $p \rightarrow q$

d) $p \land q$ e) $p \leftrightarrow q$ f) $\neg p \rightarrow \neg q$

g) $\neg p \land \neg q$ h) $\neg p \lor (p \land q)$

Practice Problem(s)

Let p and q be the propositions

p :You drive over 65 miles per hour.

q :You get a speeding ticket.

Write these propositions using p and q and logical connectives (including negations).

a) You do not drive over 65 miles per hour.

b) You drive over 65 miles per hour, but you do not get a speeding ticket.

Practice Problem(s) (cont.)

c) You will get a speeding ticket if you drive over 65 miles per hour.

d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.

e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.

f) You get a speeding ticket, but you do not drive over 65 miles per hour.

g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

Truth Tables For Compound Propositions

Construction of a truth table:

Rows

- Need a row for every possible combination of values for the atomic propositions.

Columns

- Need a column for the compound proposition (usually at far right)
- Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
  - This includes the atomic propositions

Example Truth Table

Construct a truth table for $p \lor q \rightarrow \neg r$

p	q	r	$\neg r$	$p \lor q$	$p \lor q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

Two propositions are *equivalent* if they always have the same truth value.

Example: Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Applications of Propositional Logic

Section 1.2

Applications of Propositional Logic: Summary

Translating English to Propositional Logic

System Specifications

Logic Puzzles

Translating English Sentences

Steps to convert an English sentence to a statement in propositional logic

- Identify atomic propositions and represent using propositional variables.
- Determine appropriate logical connectives

"If I go to Harry's or to the country, I will not go shopping."

- *p*: I go to Harry's
- *q*: I go to the country.
- *r*: I will go shopping.

If *p* or *q* then not *r*.

$$\left(p \vee q\right) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let *a*, *c*, and *f* represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$a \to (c \lor \neg f)$$

System Specifications

System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let *p* denote "The automated reply can be sent" and *q* denote "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let p denote "The diagnostic message is stored in the buffer." Let q denote "The diagnostic message is retransmitted" The specification can be written as: $p \vee q$, $\neg p$, $p \to q$. When p is false and q is true all three statements are true. So the specification is consistent.

- What if "The diagnostic message is not retransmitted" is added.

  Solution: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

Logic Puzzles

Raymond Smullyan (Born 1919)

An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.

You go to the island and meet A and B.

- A says "B is a knight."
- B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Propositional Equivalences

Section 1.3

Section Summary 2

Tautologies, Contradictions, and Contingencies.

Logical Equivalence

- Important Logical Equivalences
- Showing Logical Equivalence

Propositional Satisfiability

Tautologies, Contradictions, and Contingencies

A *tautology* is a proposition which is always true.

- Example: $p \vee \neg p$

A *contradiction* is a proposition which is always false.

- Example: $p \wedge \neg p$

A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

This truth table shows that $\neg p \lor q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences [1]

Identity Laws: $p \wedge T \equiv p, \qquad p \vee F \equiv p$

Domination Laws: $p \vee T \equiv T, \qquad p \wedge F \equiv F$

Idempotent laws: $p \vee p \equiv p, \qquad p \wedge p \equiv p$

Double Negation Law: $\neg(\neg p) \equiv p$

Negation Laws: $p \vee \neg p \equiv T, \qquad p \wedge \neg p \equiv F$

Key Logical Equivalences 2

Commutative Laws: $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$

Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$

$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption Laws: $p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Constructing New Logical Equivalences

We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.

To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs[1]

Example: Show that $\neg\left(p \vee \left(\neg p \wedge q\right)\right)$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$\begin{aligned}
\neg\left(p \vee \left(\neg p \wedge q\right)\right) &\equiv \neg p \wedge \neg\left(\neg p \wedge q\right) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge \left[\neg\left(\neg p\right) \vee \neg q\right] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge \left(p \vee \neg q\right) && \text{by the double negation law} \\
&\equiv \left(\neg p \wedge p\right) \vee \left(\neg p \wedge \neg q\right) && \text{by the second distributive law} \\
&\equiv F \vee \left(\neg p \wedge \neg q\right) && \text{because } \neg p \wedge p \equiv F \\
&\equiv \left(\neg p \wedge \neg q\right) && \text{By the identity law for } \mathbf{F}
\end{aligned}$$

Equivalence Proofs [2]

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and commutative laws laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Propositional Satisfiability

A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true. When no such assignments exist, the compound proposition is *unsatisfiable*.

A compound proposition is unsatisfiable if and only if its negation is a tautology.

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to *p* and ***F*** to *q*.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.