Assignment 2

Question 1

Give regular expressions generating the following languages over the alphabet $\Sigma = \{0,1\}$:

1. **L1 = the set of all strings that start with 1 or have odd length**
    1. $1(0|1)^*|0((0|1)(0|1))^*$
2. **L2 = the set of all strings that start with 0 and have even length**
    2. $1(0|1)^*|0((0|1)(0|1))^*$
3. **L3 = the set of all strings that end with 1 and have even length**
    1. $(0|1)((0|1)(0|1))^*1$
4. **L1∩L2**
    2. $\emptyset$
5. **L2∪L3**
    1. $0(0|1)((0|1)(0|1))^*|(0|1)((0|1)(0|1))^*1$
6. **L2∩L3**
    2. $0((0|1)(0|1))^*1$
7. **The set of all strings such that every occurrence of 1 is followed by at least two 0s, e.g., 0001000100, 100, 0, 000000000100000000100100 are in this language, but 1011, 1, 101 are not.**
    1. $(0|1000^*)^*$
8. **The set of all strings that does not contain pattern 0110.**
    2. $(1+0(0+10)^*(\epsilon+1+11))^*(\epsilon+0(0+10)^*(\epsilon+1+11))$
9. **The set of all strings except 100 and 01.**
    1. $\epsilon|0|1|00|10|11|000|001|010|011|101|110|111|(0|1)4(0|1)^*$

Question 2

Use the procedure described in Lemma 1.60 to convert the following NFAs to regular expressions:

Part A

Start state q_1, accepting state q_2

	0	1	ϵ
$\rightarrow q_1$	$\{q_2\}$	$\{q_1, q_2\}$	$\emptyset$
$*q_2$	$\{q_1\}$	$\emptyset$	$\emptyset$

Convert NFA to GNFA

States:$\{q_s, q_1, q_2, q_a\}$, where$q_s$is the start state and$q_a$is the accept state.

GNFA G Transitions:

- $q_s \to q_1$:ϵ
- $q_s \to q_2$:$\emptyset$
- $q_s \to q_a$:$\emptyset$
- $q_1 \to q_1$: 1
- $q_1 \to q_2$:$0|1$
- $q_1 \to q_a$:$\emptyset$
- $q_2 \to q_1$: 0
- $q_2 \to q_2$:$\emptyset$
- $q_2 \to q_a$:ϵ

Eliminate States

Since the GNFA has 4 states ($k = 4 > 2$), we eliminate states that are neither the start (q_s) nor accept (q_a) state, updating transitions using the formula:

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup R_4$$

where:
-$R_1 = \delta(q_i, q_{\text{rip}})$
-$R_2 = \delta(q_{\text{rip}}, q_{\text{rip}})$
-$R_3 = \delta(q_{\text{rip}}, q_j)$
-$R_4 = \delta(q_i, q_j)$
-$q_{\text{rip}}$is the state being removed.

GNFA G' Transitions:
-$q_s \to q_2$:$1^*(0|1)$
-$q_s \to q_a$:$\emptyset$
-$q_2 \to q_2$:$01^*(0|1)$
-$q_2 \to q_a$:ϵ

Remove q_2:

- New states:$\{q_s, q_a\}(k = 2)$.
- Update$q_s \to q_a$:
  -$R_1 = \delta(q_s, q_2) = 1^*(0|1)$
  -$R_2 = \delta(q_2, q_2) = 01^*(0|1)$
  -$R_3 = \delta(q_2, q_a) = \epsilon$
  -$R_4 = \delta(q_s, q_a) = \emptyset$
    - New label:$(1^*(0|1))(01^*(0|1))^*(\epsilon) \cup \emptyset = 1^*(0|1)(01^*(0|1))^*$

Final GNFA: $q_s \to q_a$ labeled $1^*(0|1)(01^*(0|1))^*$.

Since $k = 2$, the regular expression is the label:

$$1^*(0|1)(01^*(0|1))^*$$

Part B

start state q_1, accepting state q_2

	0	1	ϵ
$\rightarrow q_1$	$\emptyset$	$\emptyset$	$\{q_3\}$
$*q_2$	$\{q_2, q_3\}$	$\{q_3\}$	$\emptyset$
q_3	$\{q_3\}$	$\{q_2\}$	$\{q_3\}$

Convert NFA to GNFA

States:$\{q_s, q_1, q_2, q_3, q_a\}$,$q_s$ start, q_a accept.

GNFA G Transitions:

- $q_s \rightarrow q_1$:ϵ
- $q_1 \rightarrow q_3$:ϵ
- $q_3 \rightarrow q_3$:$0|\epsilon$
- $q_3 \rightarrow q_2$: 1
- $q_2 \rightarrow q_3$:$0|1$
- $q_2 \rightarrow q_2$: 0
- $q_2 \rightarrow q_a$:ϵ
- Others:$\emptyset$

Eliminate States

GNFA G' Transitions:
-$q_s \rightarrow q_3$:ϵ
-$q_3 \rightarrow q_3$:$0|\epsilon$
-$q_3 \rightarrow q_2$: 1
-$q_2 \rightarrow q_3$:$0|1$
-$q_2 \rightarrow q_2$: 0
-$q_2 \rightarrow q_a$:ϵ

Remove q_3:

- New states:$\{q_s, q_2, q_a\}$.
- $q_s \rightarrow q_2$:
  -$R_1 = \epsilon$,$R_2 = 0|\epsilon$,$R_3 = 1$,$R_4 = \emptyset$
  -$(\epsilon)(0|\epsilon)^*(1) \cup \emptyset = (0|\epsilon)^*1 = 0^*1$
- $q_2 \rightarrow q_2$:
  -$R_1 = 0|1$,$R_2 = 0|\epsilon$,$R_3 = 1$,$R_4 = 0$
  -$(0|1)(0|\epsilon)^*(1) \cup 0 = (0|1)0^*1 \cup 0$

GNFA G'' Transitions:
-$q_s \rightarrow q_2$:0^*1
-$q_2 \rightarrow q_2$:$(0|1)0^*1 \cup 0$
-$q_2 \rightarrow q_a$:ϵ

Remove q_2:

- New states:$\{q_s, q_a\}$.

- $q_s \to q_a$:
  -$R_1 = 0^*1$,$R_2 = (0|1)0^*1 \cup 0$,$R_3 = \epsilon$,$R_4 = \emptyset$
  -$(0^*1)[(0|1)0^*1 \cup 0]^*(\epsilon) \cup \emptyset = 0^*1[(0|1)0^*1 \cup 0]^*$

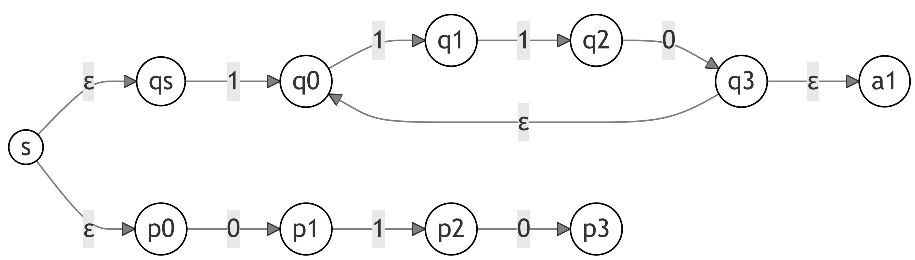
Final Regular Expression:

$$0^*1[(0|1)0^*1 \cup 0]^*$$

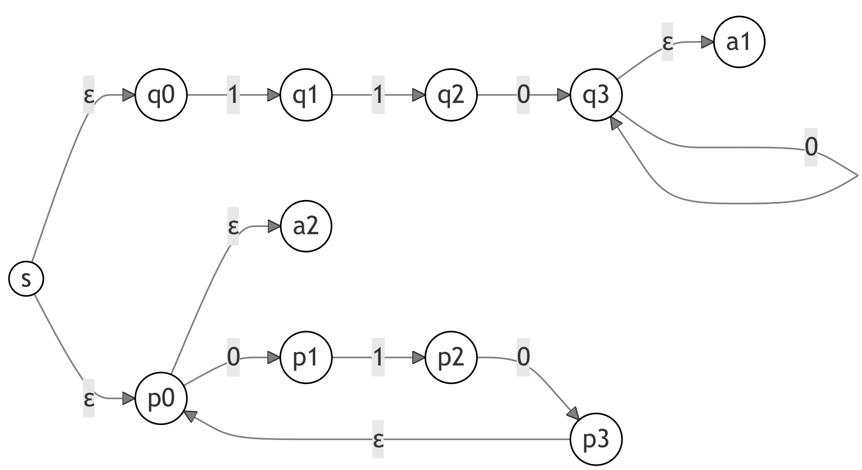
Question 3

Convert the following regular expressions to NFAs using procedure given in Theorem 1.54. In all parts $\Sigma = \{0, 1\}$.

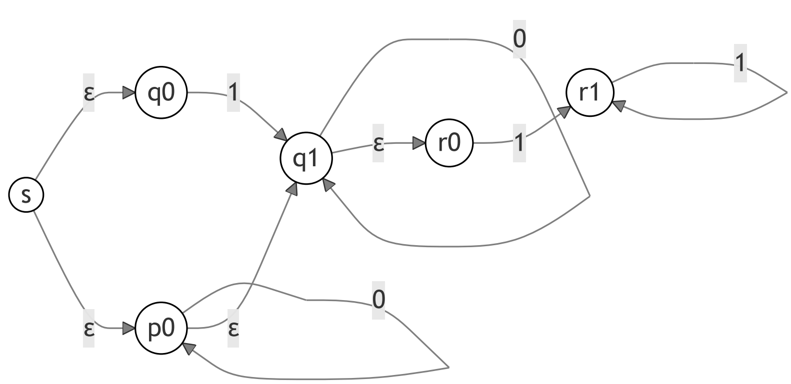
10. $1(110)^* \cup 010$



11. $110^+ \cup (010)^*$



12. $(1 \cup 0^*)0^*1^+$



Question 4

Part A

Consider language $L = \{10^n10^n|n > 0\}$ over alphabet$\Sigma = \{0, 1\}$. Using Pumping Lemma prove that this language is not regular.

To prove that the language $L = \{10^n10^n \mid n > 0\}$ over the alphabet $\Sigma = \{0, 1\}$ is not regular, we use the Pumping Lemma for regular languages. The Pumping Lemma states that if L is regular, there exists a pumping length p such that for any string $w \in L$ with $|w| \geq p$, we can divide w into three parts, $w = xyz$, satisfying:

13. $|y| > 0$,
14. $|xy| \leq p$,
15. For all $k \geq 0$, $xy^kz \in L$.

We proceed by contradiction. Assume L is regular, and let p be the pumping length. Choose the string $w = 10^p10^p$, which is in L since it is of the form 10^n10^n with $n = p > 0$. The length of w is $|w| = 2p + 2$ (since it has p 0's, a 1, then p 0's, and a 1), and since $p \geq 1$, $|w| \geq p$, satisfying the length condition.

Now, divide $w = 10^p10^p$ into xyz such that $|xy| \leq p$ and $|y| > 0$. Since $|xy| \leq p$, the substring xy must be within the first p characters of w. Writing $w = 10^p10^p$, the first p characters are 10^{p-1} (the first 1 followed by $p - 1$ 0's, since the p-th 0 is at position $p + 1$, followed by the second 1). Thus, xy is a prefix of 10^{p-1}, and y is a non-empty substring within these p characters. We consider the possible cases for y:

- **Case 1: y is some 0's from the first block of 0's.** Suppose $x = 10^a$ where $a \geq 0$, and $y = 0^b$ where $b > 0$, with $|xy| = 1 + a + b \leq p$, so $a + b \leq p - 1$. Then $z = 0^{p-a-b}10^p$, and $w = xyz = 10^a0^b0^{p-a-b}10^p = 10^p10^p$. Pumping with $k = 2$:

$$xy^2z = 10^a(0^b)^20^{p-a-b}10^p = 10^a0^{2b}0^{p-a-b}10^p = 10^{a+2b+(p-a-b)}10^p = 10^{p+b}10^p$$

This string has $p + b$ 0's between the two 1's and p 0's after the second 1. Since $b > 0$, $p + b > p$, so $10^{p+b}10^p \neq 10^m10^m$ for any m, as the counts of 0's are unequal. Thus, $xy^2z \notin L$.

- **Case 2: y includes the first 1 and possibly some 0's.** Suppose $x = \epsilon$ (empty string), $y = 10^a$ where $a \geq 0$, and $|y| = 1 + a \leq p$ (so $a \leq p - 1$), with $z = 0^{p-a}10^p$. Then $w = xyz = 10^a0^{p-a}10^p = 10^p10^p$. Pumping with $k = 2$:

$$xy^2z = y^2z = (10^a)(10^a)0^{p-a}10^p = 10^a10^a0^{p-a}10^p$$

This has three 1's, but strings in L have exactly two 1's, so $xy^2z \notin L$. Alternatively, for $k = 0$:

$$xy^0z = xz = z = 0^{p-a}10^p$$

Since $a \leq p - 1$, $p - a \geq 1$, so this starts with a 0, but all strings in L start with 1, hence $xy^0z \notin L$.

Since $|xy| \leq p$, and the second 1 is at position $p + 2 > p$, y cannot include the second 1 or any part of the second block of 0's. In all cases, pumping produces a string not in L, contradicting the Pumping Lemma. Thus, L is not regular.

Part B

Using result of (Part A) prove that language $B = \{ww | w \text{ from } \Sigma^*\}$ is not regular. DO NOT USE PUMPING LEMMA! Use closure properties of regular languages instead.

Using the result that $L = \{10^n10^n \mid n > 0\}$ is not regular, we prove that $B = \{ww \mid w \in \Sigma^*\}$ over $\Sigma = \{0, 1\}$ is not regular, relying on closure properties of regular languages instead of the Pumping Lemma.

Consider the regular language $R = 10^+10^+ = \{10^m10^n \mid m \geq 1, n \geq 1\}$, defined by the regular expression 10^+10^+, which includes strings starting with a 1, followed by one or more 0's, then a 1, and one or more 0's (e.g., 1010, 10010, but not 11). Compute the intersection $B \cap R$:

- A string in B is of the form ww, where $w \in \Sigma^*$.
- For ww to be in R, it must be of the form 10^a10^b with $a \geq 1$, $b \geq 1$, and equal to ww.

Test possible forms of w:

- If $w = 10^k$ with $k \geq 1$, then $ww = 10^k10^k = 10^k10^k$, which is in R since $k \geq 1$.
- If $w = 1$ (i.e., $k = 0$), then $ww = 11$, but 11 is not in R because 10^+ requires at least one 0 after the first 1.
- If $w = 0^a$ (no 1's), $ww = 0^{2a}$, which starts with 0, not in R.
- If $w = 0^a10^b$ with $a \geq 1$, $ww = 0^a10^b0^a10^b$, which starts with 0, not in R.
- If $w = 11$, $ww = 1111$, which has four 1's, but R has exactly two 1's.

Thus, w must have exactly one 1 and start with 1, so $w = 10^k$ with $k \geq 1$. Hence:

$$B \cap R = \{10^k10^k \mid k \geq 1\}$$

This is exactly $L = \{10^n10^n \mid n > 0\}$, since $n > 0$ corresponds to $k \geq 1$. Now, if B were regular, and R is regular, then $B \cap R$ would be regular (since regular languages are closed under intersection). But $B \cap R = L$, and from Part 1, L is not regular. This is a contradiction. Therefore, B is not regular.

Question 5

Variation of Problem 1.53: Let $\Sigma = 0, 1, 2, 3, \ldots, 9, -, =$ and $SUB = \{x - y = z |$ x,y,z are unsigned integers, and z is the difference of x and y $\}$. For example, string "99-21=78" is in SUB, while string "99-21=77" is not in SUB. Prove that SUB is not a regular language.

Assume SUB is regular. Then, for $w = 10^p - 0 = 10^p$ with $|w| \geq p$, we can write:

$$w = xyz$$

where:

- $|xy| \leq p$,
- $|y| > 0$,
- For all $k \geq 0$, $xy^kz \in SUB$.

Positions in w:

- Positions 1 to $p + 1$: "1" + "0"^p (the x part),

- Position $p+2$: "-",
- Position $p+3$: "0",
- Position $p+4$: "=",
- Positions $p+5$ to $2p+5$: "1" + "0"^p.

Since $|xy| \leq p$, the substring xy lies within the first p characters of w, which are "1" + "0"^{p-1} (e.g., if $p = 3$, $w = "1000 - 0 = 1000"$, first 3 characters are "100"). Thus, xy is entirely within the first 10^p (the x part), and y is a non-empty substring of "1" + "0"^{p-1}".

Since xy occupies the first $|x| + |y| \leq p$ characters, and the first $p+1$ characters are "1" + "0"^p, consider:

- $x = "1" + "0"^a$ (for some $a \geq 0$),
- $y = "0"^b$ (for some $b > 0$),
- $z = "0"^{p-a-b} + " - 0 = 1" + "0"^p$,
  where $|x| + |y| = (a+1) + b \leq p$, so $a + b \leq p - 1$, and $a + b + (p - a - b) = p$, matching the length of the first part.

The original x part is $"1" + "0"^a + "0"^b + "0"^{p-a-b} = "1" + "0"^p$, representing 10^p.

Pumped string:

$$w_k = xy^k z = "1" + "0"^a + ("0"^b)^k + "0"^{p-a-b} + " - 0 = 1" + "0"^p$$

- New $x' = "1" + "0"^a + "0"^{bk} + "0"^{p-a-b}$,
- $y' = "0"$,
- $z' = "1" + "0"^p$.

Simplify the new x':

- Exponents: $a + bk + (p - a - b) = p + (k-1)b$,
- So, $x' = "1" + "0"^{p+(k-1)b}$, representing the number $10^{p+(k-1)b}$,
- $y' = 0$,
- $z' = 10^p$.

Check if $w_k \in SUB$:

- $x' - y' = 10^{p+(k-1)b} - 0 = 10^{p+(k-1)b}$,
- Must equal $z' = 10^p$,
- $10^{p+(k-1)b} = 10^p$ requires $p + (k-1)b = p$, so $(k-1)b = 0$.

Since $b > 0$:

- $k - 1 = 0$,
- $k = 1$.

For $k = 1$, $w_1 = w$, which is in SUB, as expected. Test other values:

- $k = 2$: $w_2 = "1" + "0"^{p+b} + " - 0 = 1" + "0"^p$,
  - $x' = 10^{p+b}$,
  - $y' = 0$,
  - $z' = 10^p$,

    - $10^{p+b} - 0 = 10^{p+b} \neq 10^p$ (since $b > 0$, $p + b > p$),
    - Not in SUB.
- $k = 0$: $w_0 = "1" + "0"^a + "0"^{p-a-b} + " - 0 = 1" + "0"^p$,
    - $x' = "1" + "0"^{p-b}$,
    - Since $a + b \leq p - 1$, $p - a - b \geq 1$, and $b > 0$, so $p - b < p$,
    - $x' = 10^{p-b}$,
    - $10^{p-b} - 0 = 10^{p-b} \neq 10^p$,
    - Not in SUB.

Case when y includes the "1":
If $x = ""$ (empty), $y = "1"$, $z = "0"^p - 0 = 1" + "0"^p$:

- $w_k = "1"^k + "0"^p + " - 0 = 1" + "0"^p$,
- $k = 2$: "11" + "0"^p + " - 0 = 1" + "0"^p$,
    - $x' = 11 \cdot 10^p$,
    - $11 \cdot 10^p - 0 \neq 10^p$,
    - Not in SUB.
- $k = 0$: "0"^p + " - 0 = 1" + "0"^p$,
    - $x' = 0^p$ (if $p > 1$, leading zeros, still 0),
    - $0 - 0 = 0 \neq 10^p$ (since $p \geq 1$),
    - Not in SUB.

In all cases, for $k \neq 1$, $xy^kz \notin SUB$, violating the Pumping Lemma.

Since $|xy| \leq p$ restricts y to the first x part, pumping y alters x without adjusting z (or y, which remains "0"), breaking the equality $z = x - y$. This contradiction shows that SUB cannot satisfy the Pumping Lemma, so it is not regular.

Thus, SUB is not a regular language.