The Foundations: Logic and Proofs

Predicate Logic and Proofs

Summary

- Predicates and Quantifiers
- Nested Quantifiers
- Rules of Inference
- Introduction to Proofs

Predicates and Quantifiers

Section 1.4

Section Summary 1

Predicates

Variables

Quantifiers

- Universal Quantifier
- Existential Quantifier

Negating Quantifiers

- De Morgan’s Laws for Quantifiers

Translating English to Logic

Propositional Logic Not Enough

If we have:

"All men are mortal."

"Socrates is a man."

- Does it follow that "Socrates is mortal?" Yes
- Can it be represented in propositional logic? No
- We will use Predicate Logic for such cases.

Introducing Predicate Logic

Predicate logic is based *Propositional functions*.

- A propositional function is a statement involving one or more variables,
  - e.g.: x-3 > 5.
- Let us call this propositional function $P(x)$, where P is the predicate and x is the variable.

Propositional Functions

- Propositional functions become propositions when their variables are each replaced by a value from the *domain*
- Domain is also called universe of discourse (*U*)

- *Let P(x) be "x-3 > 5" and U be all integers,*

What is the truth value of P(2) ? false

What is the truth value of P(8) ? false

What is the truth value of P(9) ? true

Examples of Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5)

Solution: F

R(3,4,7)

Solution: T

R(x, 3, z)

Solution: Not a Proposition

Examples of Propositional Functions

Let "x - y = z" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

Q(2,-1,3)

Solution: T

Q(3,4,7)

Solution: F

Q(x, 3, z)

Solution: Not a Proposition

Compound Expressions

Connectives from propositional logic carry over to predicate logic.

If $P(x)$ denotes "$x > 0$," find these truth values:

$P(3) \vee P(-1)$ **Solution**: T

$P(3) \wedge P(-1)$ **Solution**: F

$P(3) \rightarrow P(-1)$ **Solution**: F

$P(3) \rightarrow \neg P(-1)$ **Solution**: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \wedge P(y)$

$P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles Peirce (1839-1914)

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All men are Mortal."
- "Some cats do not have fur."

The two most important quantifiers are:

- *Universal Quantifier,* **"**For all," symbol: ∀
- *Existential Quantifier*, "There exists," symbol: ∃

We write as in $\forall x\ P(x)$ and $\exists x\ P(x)$.

$\forall x\ P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

$\exists x\ P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is:

   false

2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is:

   true

3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is:

   false

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that $P(x)$," or "For at least one x, $P(x)$."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is:

   true

2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is:

   false

3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is:

   true

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$

$\forall x\, (P(x) \vee Q(x))$ means something different.

Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic[1]

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(*x*) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \land J(x))$ is not correct. What does it mean?

Translating from English to Logic 2

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\, (S(x) \wedge J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Equivalences in Predicate Logic

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

The notation $S \equiv T$ indicates that S and T are logically equivalent.

Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Negating Quantified Expressions 1

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here $J(x)$ is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg \forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent

Negating Quantified Expressions 2

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.			
Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every *x*, *P*(*x*) is false.	There is *x* for which *P*(*x*) is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an *x* for which *P* (*x*) is false.	*P*(*x*) is true for every *x*.

The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x(S(x) \land M(x))$$

2. "Every student in this class has visited America or Mexico."

   Solution: Add $A(x)$ denoting "x has visited America."

   $$\forall x\left(S(x) \rightarrow (M(x) \lor A(x))\right)$$

Nested Quantifiers

Section 1.5

Section Summary

Nested Quantifiers

Order of Quantifiers

Translating from Nested Quantifiers into English

Translated English Sentences into Logical Expressions

Negating Nested Quantifiers

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: “Every real number has an inverse” is

$$\forall x \, \exists y (x + y = 0)$$

where the domains of x and y are the real numbers.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \ \forall y P(x,y)$ and $\forall y \ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \ \exists y Q(x,y)$ is true, but $\exists y \ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers [1]

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

$\forall x \forall y P(x,y)$ Answer:False

$\forall x \exists y P(x,y)$ Answer:True

$\exists x \forall y P(x,y)$ Answer:True

$\exists x \exists y P(x,y)$ Answer:True

Questions on Order of Quantifiers 2

Example 2: Let *U* be the real numbers,

Define *P(x,y)* : *x* / *y* = 1

What is the truth value of the following:

$\forall x \forall y P(x, y)$ Answer:False

$\forall x \exists y P(x, y)$ Answer:False

$\exists x \forall y P(x, y)$ Answer:False

$\exists x \exists y P(x, y)$ Answer:True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\left(C(x) \vee \exists y(C(y) \wedge F(x,y))\right)$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x\forall y\forall z\left(\left(F(x,y) \wedge F(x,z) \wedge (y \neq z)\right) \rightarrow \neg F(y,z)\right)$$

Solution: There is a student none of whose friends are also friends with each other.

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f " and $Q(f,a)$ be "f is a flight on a ."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

- **Example 1**: “Brothers are siblings.”
- **Solution**: $\forall x \, \forall y \, (B(\mathrm{x,y}) \rightarrow S(\mathrm{x,y}))$
- **Example 2**: “Siblinghood is symmetric.”
- **Solution**: $\forall x \, \forall y \, (S(x,y) \rightarrow S(y,x))$
- **Example 3**: “Everybody loves somebody.”
- **Solution**: $\forall x \, \exists y \, L(x,y)$
- **Example 4**: “There is someone who is loved by everyone.”
- **Solution**: $\exists y \, \forall x \, L(x,y)$
- **Example 5**: “There is someone who loves someone.”
- **Solution**: $\exists x \, \exists y \, L(x,y)$
- **Example 6**: “Everyone loves himself”
- **Solution**: $\forall x \, L(x,x)$

Negating Nested Quantifiers

Example: Recall the logical expression developed a few slides back:

$$\exists w \forall a \exists f \big(P(w,f) \land Q(f,a)\big)$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \forall a \exists f \big(P(w,f) \land Q(f,a)\big)$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg \exists w \forall a \exists f \big(P(w,f) \land Q(f,a)\big)$
2. $\forall w \neg \forall a \exists f \big(P(w,f) \land Q(f,a)\big)$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f \big(P(w,f) \land Q(f,a)\big)$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg \big(P(w,f) \land Q(f,a)\big)$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f \big(\neg P(w,f) \lor \neg Q(f,a)\big)$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Rules of Inference

Section 1.6

Section Summary [1]

Valid Arguments

Inference Rules for Propositional Logic

Using Rules of Inference to Build Arguments

Rules of Inference for Quantified Statements

Building Arguments for Quantified Statements

Revisiting the Socrates Example

We have the two premises:

- "All men are mortal."
- "Socrates is a man."

And the conclusion:

- "Socrates is mortal."

How do we get the conclusion from the premises?

The Argument

We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\begin{array}{l} \forall x\left(Man(x) \rightarrow Mortal(x)\right) \\ Man(Socrates) \\ \hline \therefore \quad Mortal(Socrates) \end{array}$$

We will see shortly that this is a valid argument.

Valid Arguments[1]

- We will show how to construct valid arguments in two stages:
  1. For propositional logic
  2. For predicate logic
- The rules of inference are the essential building block in the construction of valid arguments.

Arguments in Propositional Logic

- An *argument* in propositional logic is a sequence of propositions.
- The argument is valid if the premises imply the conclusion.
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is a tautology.

Rules of Inference for Propositional Logic: Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ \underline{p \quad\quad} \\ \therefore q \end{array}$$

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Example:

Let *p* be "It is snowing."

Let *q* be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$$\begin{array}{r} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg q$$

Example:

Let *p* be “it is snowing.”

Let *q* be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”

“I will not study discrete math.”

“Therefore , it is not snowing.”

Hypothetical Syllogism

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let *p* be "it snows."

Let *q* be "I will study discrete math."

Let *r* be "I will get an A."

"If it snows, then I will study discrete math."

"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

Corresponding Tautology:

$$(\neg p \wedge (p \vee q)) \rightarrow q$$

Example:

Let *p* be “I will study discrete math.”

Let *q* be “I will study English literature.”

“I will study discrete math or I will study English literature.”

“I will not study discrete math.”

“Therefore , I will study English literature.”

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Corresponding Tautology:

$$(p \wedge q) \rightarrow p$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$$((p) \wedge (q)) \to (p \wedge q)$$

Example:

Let *p* be "I will study discrete math."

Let *q* be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Using the Rules of Inference to Build Valid Arguments

A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

A valid argument takes the following form:

$$
\begin{array}{c}
S_1 \\
S_2 \\
. \\
. \\
. \\
S_n \\
\therefore C
\end{array}
$$

Valid Arguments 2

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that *q* is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Simplification using (1)
3. $p \rightarrow q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments 3

Example 2:

With these hypotheses:

"It is not sunny this afternoon and it is colder than yesterday."

"We will go swimming only if it is sunny."

"If we do not go swimming, then we will take a canoe trip."

"If we take a canoe trip, then we will be home by sunset."

Using the inference rules, construct a valid argument for the conclusion:

"We will be home by sunset."

Solution:

1. Choose propositional variables:

p : "It is sunny this afternoon." r : "We will go swimming." t : "We will be home by sunset."

q : "It is colder than yesterday." s : "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q, r \rightarrow p, \ \neg r \rightarrow s, \ s \rightarrow t$$

$$\text{Conclusion: } t$$

Continued on next slide →

Valid Arguments 4

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:

- Rules of Inference for Propositional Logic
- Rules of Inference for Quantified Statements

The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

"All dogs are cuddly."

"Therefore, Fido is cuddly."

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."

"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."

"Therefore, someone got an A in the class."

Using Rules of Inference [1]

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ " x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	**Reason**
1. $\forall x\left(M(x) \rightarrow L(x)\right)$	Premise
2. $M(J) \rightarrow \mathrm{L}(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference 2

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x\left(C(x) \land \neg B(x)\right) \\ \forall x\left(C(x) \to P(x)\right) \\ \hline \therefore \exists x\left(P(x) \land \neg B(x)\right) \end{array}$$

Continued on next slide →

Using Rules of Inference 3

Valid Argument:

Step	**Reason**
1. $\exists x\left(C(x) \wedge \neg B(x)\right)$	Premise
2. $C(a) \wedge \neg \mathrm{B}(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x\left(C(x) \rightarrow P(x)\right)$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x\left(P(x) \wedge \neg B(x)\right)$	EG from (8)

Returning to the Socrates Example

$$\forall x\left(Man\left(x\right)\rightarrow Mortal\left(x\right)\right)$$

$$\frac{Man\left(Socrates\right)}{\therefore \quad Mortal\left(Socrates\right)}$$

Solution for Socrates Example

Valid Argument

Step	***Reason***
1. $\forall x\left(Man(x) \rightarrow Mortal(x)\right)$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (1)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Introduction to Proofs

Section 1.7

Section Summary

Mathematical Proofs

Direct Proofs

Indirect Proofs

Terminology

- An axiom is a basic assumption about mathematical structures that needs no proof.
- We can use a proof to demonstrate that a particular statement is true. A proof consists of a sequence of statements that form an argument.
- The steps that connect the statements in such a sequence are the rules of inference.
- Cases of incorrect reasoning are called fallacies.

Terminology (cont.)

- A theorem is a statement that can be shown to be true.
- A lemma is a simple theorem used as an intermediate result in the proof of another theorem.
- A corollary is a proposition that follows directly from a theorem that has been proved.
- A conjecture is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

Arguments

- Just like a rule of inference, an argument consists of one or more hypotheses and a conclusion.
- We say that an argument is valid, if whenever all its hypotheses are true, its conclusion is also true.
- However, if any hypothesis is false, even a valid argument can lead to an incorrect conclusion.

Arguments (Cont.)

Example:

"If 101 is divisible by 3, then 101^2 is divisible by 9. 101 is divisible by 3. Consequently, 101^2 is divisible by 9."

Although the argument is valid, its conclusion is incorrect, because one of the hypotheses is false ("101 is divisible by 3.").

If in the above argument we replace 101 with 102, we could correctly conclude that 102^2 is divisible by 9.

Even and Odd Integers

Definition: The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k, such that $n = 2k + 1$.

Note that every integer is either even or odd and no integer is both even and odd.

Proving Theorems

Direct proof:

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

Example: Give a direct proof of the theorem "If n is odd, then n^2 is odd."

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems to show that q must also be true (n^2 is odd).

Direct Proof

n is odd.

Then $n = 2k + 1$, where k is an integer.

Consequently, $n^2 = (2k + 1)^2$.

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Since n^2 can be written in this form, it is odd.

Indirect Proof

Indirect proof:

An implication $p \rightarrow q$ is equivalent to its contra-positive $\neg q \rightarrow \neg p$. Therefore, we can prove $p \rightarrow q$ by showing that whenever q is false, then p is also false.

Example: Give an indirect proof of the theorem "If 3n + 2 is odd, then n is odd."

Idea: Assume that the conclusion of this implication is false (n is even). Then use rules of inference and known theorems to show that p must also be false (3n + 2 is even).

Indirect Proof (Cont.)

n is even.

Then $n = 2k$, where k is an integer.

It follows that $3n + 2 = 3(2k) + 2$

$$= 6k + 2$$

$$= 2(3k + 1)$$

Therefore, $3n + 2$ is even.

We have shown that the contrapositive of the implication is true, so the implication itself is also true (If $3n + 2$ is odd, then n is odd).