CP214: Discrete Structures
Sets, Functions, and Relations

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Sets
Section 2.1

Set

- A set is an unordered collection of objects.
- $a \in A$ "a is an element of A"
  "a is a member of A"
- $a \notin A$ "a is not an element of A"
- $A = \{a_1, a_2, \ldots, a_n\}$ "A contains…"
- Order of elements is meaningless
- It does not matter how often the same element is listed.

Set Equality

- Sets A and B are equal if and only if they contain exactly the same elements.
- Examples:

$A = \{9, 2, 7, -3\}, B = \{7, 9, -3, 2\}$: $A = B$

$A = \{dog, cat, horse\}$,
$B = \{cat, horse, squirrel, dog\}$: $A \neq B$

$A = \{dog, cat, horse\}$,
$B = \{cat, horse, dog, dog\}$: $A = B$

Some Important Sets

- **N** = *natural numbers* = {0,1,2,3….}
- **Z** = *integers* = {…,-3,-2,-1,0,1,2,3,…}
- **Z**$^+$ = *positive integers* = {1,2,3,…..}
- **R** = set of *real numbers*
- **R**$^+$ = set of *positive real numbers*
- **C** = set of *complex numbers*
- **Q** = set of rational numbers

Examples of Sets

- $A = \emptyset = \{\}$ "empty set/null set"
- $A = \{\{x, y\}\}$
  Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid x \in \mathbf{N} \wedge x > 7\} = \{8, 9, 10, \ldots\}$
  "set builder notation"

Subsets

- $A \subseteq B$ "A is a subset of B"
- $A \subseteq B$ if and only if every element of A is also an element of B.
- $A \subseteq B$ holds if and only if $\forall x\ (x \in A \rightarrow x \in B)$
- Examples:

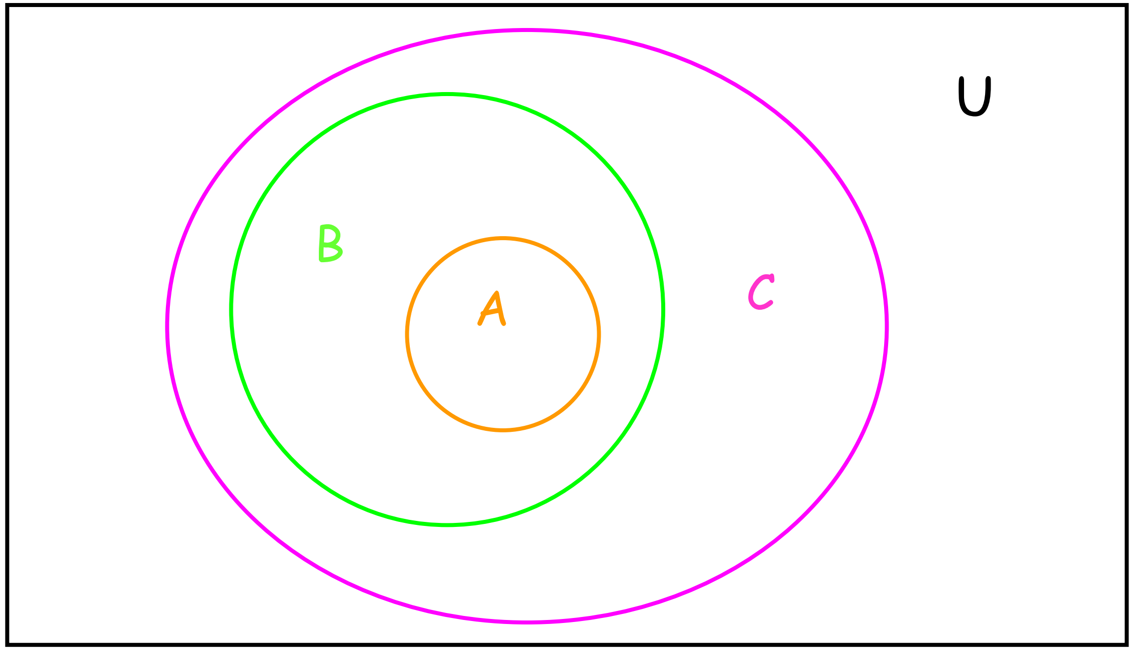
$A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? true

$A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$? true

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$? false

Subsets

- Useful rules:
  - $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
  - $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (see Venn Diagram)



Subsets

- Useful rules:
  - $\varnothing \subseteq A$ for any set A
  - $A \subseteq A$ for any set A

- Proper subsets:
- $A \subset B$ "A is a proper subset of B"
- $A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$

Cardinality of Sets

- If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.

- Examples:
- A = {Mercedes, BMW, Porsche}, $|A| = 3$

$B = \{1, \{2, 3\}, \{4, 5\}, 6\}$ $|B| = 4$

$C = \emptyset$ $|C| = 0$

$D = \{ x \in \mathbf{N} \mid x \leq 7000 \}$ $|D| = 7001$

$E = \{ x \in \mathbf{N} \mid x \geq 7000 \}$ E is infinite!

The Power Set

- 2^A or $P(A)$ "power set of A"
- $2^A = \{B \mid B \subseteq A\}$ (contains all subsets of A)
- Examples:
- $A = \{x, y, z\}$
- $2^A = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$
- $A = \varnothing$
- $2^A = \{\varnothing\}$

Cardinality of power sets:
$| 2^A | = 2^{|A|}$

Cartesian Product

- The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of objects.
- Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e. $a_i = b_i$ for $1 \le i \le n$.
- The Cartesian product of two sets is defined as:

  $$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

- Example: $A = \{x, y\}$, $B = \{a, b, c\}$
  $A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

Cartesian Product

- Note that:
  - $A \times \varnothing = \varnothing$
  - $\varnothing \times A = \varnothing$
  - For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
  - $|A \times B| = |A| \cdot |B|$

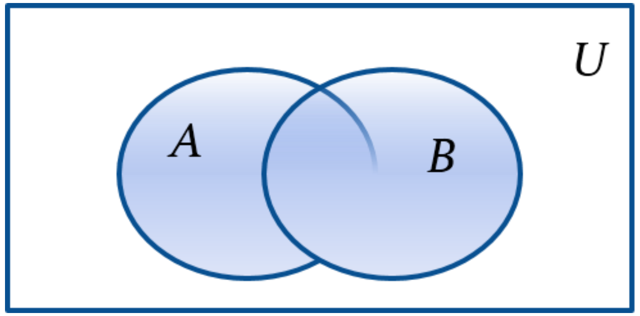
- The Cartesian product of two or more sets is defined as:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$$

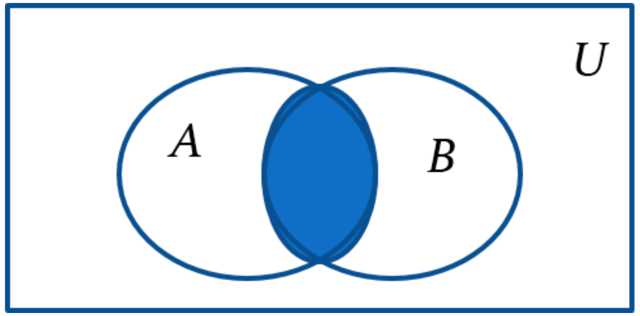
Set Operations
Section 2.2

Set Operations

- **Union:** $A \cup B = \{x \mid x \in A \vee x \in B\}$
- **Example:** $A = \{a, b\}$, $B = \{b, c, d\}$
  $A \cup B = \{a, b, c, d\}$



- **Intersection:** $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- **Example:** $A = \{a, b\}$, $B = \{b, c, d\}$
  $A \cap B = \{b\}$

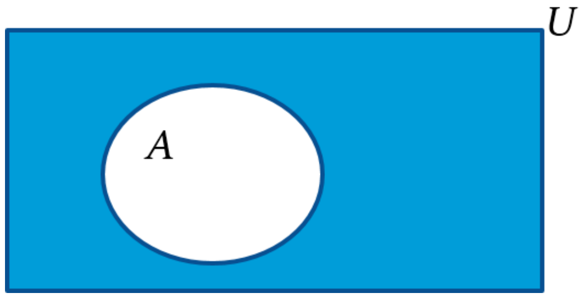


Set Operations

- The complement of a set *A* contains exactly those elements under consideration that are not in *A*:

  -*A* = U-*A*

  $$\bar{A} = \{x | x \in U \wedge x \notin A\}$$



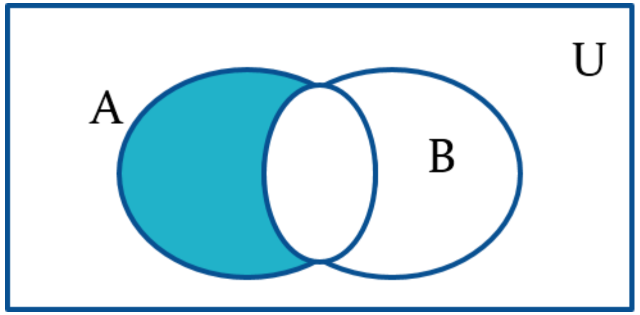
- Example: U = **N**, *A* = {250, 251, 252, …}

  -*A* = {0, 1, 2, …, 248, 249}

Set Operations

- Two sets are called disjoint if their intersection is empty, that is, they share no elements:
- $A \cap B = \emptyset$

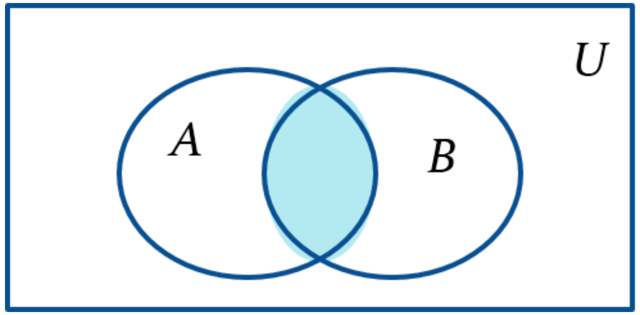
- The difference between two sets A and B contains exactly those elements of A that are not in B:
- $A - B = \{x \mid x \in A \land x \notin B\}$
  Example: $A = \{a, b\}$, $B = \{b, c, d\}$, $A - B = \{a\}$



The Cardinality of the Union of Two Sets

Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Review Questions

- **Example**: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$
  $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$
   Solution: $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$
   Solution: $\{4,5\}$
3. $\bar{A}$
   Solution: $\{0,6,7,8,9,10\}$
4. $\bar{B}$
   Solution: $\{0,1,2,3,9,10\}$
5. $A - B$
   Solution: $\{1,2,3\}$
6. $B - A$
   Solution: $\{6,7,8\}$

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proving Set Identities

- There are different ways to prove set identities.
- We will discuss the following two.
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

Proof of Second De Morgan Law[1]

- **Example**: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- **Solution**: We prove this identity by showing that:

$$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \qquad \text{and}$$

$$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

Proof of Second De Morgan Law 2

- These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg\left(\left(x \in A\right) \wedge \left(x \in B\right)\right)$	by defn. of intersection
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Proof of Second De Morgan Law 3

- These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \neg\left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Membership Table

- **Example**: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- **Solution**:

A	B	C	B∩C	A∪(B∩C)	A∪B	A∪C	(A∪B)∩ (A∪C)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Exercises

- **Question 1:**
- Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $|\, 2^A \times 2^B \,|$?

- **Question 2:**
- Is it true for all sets A and B that $(A \times B) \cap (B \times A) = \emptyset$? Or do A and B have to meet certain conditions?

- **Question 3:**
- For any two sets A and B, if $A - B = \emptyset$ and $B - A = \emptyset$, can we conclude that $A = B$?

Exercises

- **Question 1:**
- Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $|\ 2^A \times 2^B\ |$?

- **Answer:**
- $|\ 2^A \times 2^B\ | = |\ 2^A\ | \cdot |\ 2^B\ | = 2^{|A|} \cdot 2^{|B|} = 8 \cdot 16 = 128$

Exercises

- **Question 2:**
- Is it true for all sets A and B that $(A \times B) \cap (B \times A) = \emptyset$? Or do A and B have to meet certain conditions?

- **Answer:**
- If A and B share at least one element x, then both $(A \times B)$ and $(B \times A)$ contain the pair (x, x) and thus are not disjoint.
- Therefore, for the above equation to be true, it is necessary that $A \cap B = \emptyset$.

Exercises

- **Question 3:**
- For any two sets A and B, if $A - B = \emptyset$ and $B - A = \emptyset$, can we conclude that $A = B$?

- **Answer:**
- Yes, we can conclude $A = B$.

Functions
Section 2.3

Functions

- A function f from a set A to a set B, denoted $f: A \to B$ is an assignment of exactly one element of B to each element of A.
- We write
- $f(a) = b$
- if b is the unique element of B assigned by the function f to the element a of A.

Functions

- If $f:A \rightarrow B$, we say that A is the domain of f and B is the codomain of f.
- If $f(a) = b$, we say that b is the image of a and a is the pre-image of b.
- The range of $f:A \rightarrow B$ is the set of all images of elements of A.
- We say that $f:A \rightarrow B$ maps A to B.

Functions

- Let us take a look at the function $f{:}P \to C$ with
- P = {Linda, Max, Kathy, Peter}
- C = {Boston, New York, Hong Kong, Moscow}

- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = New York

- Here, the range of f is C.

Functions

- Let us re-specify f as follows:

- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = Boston

P = {Linda, Max, Kathy, Peter}
C = {Boston, New York, Hong Kong, Moscow}

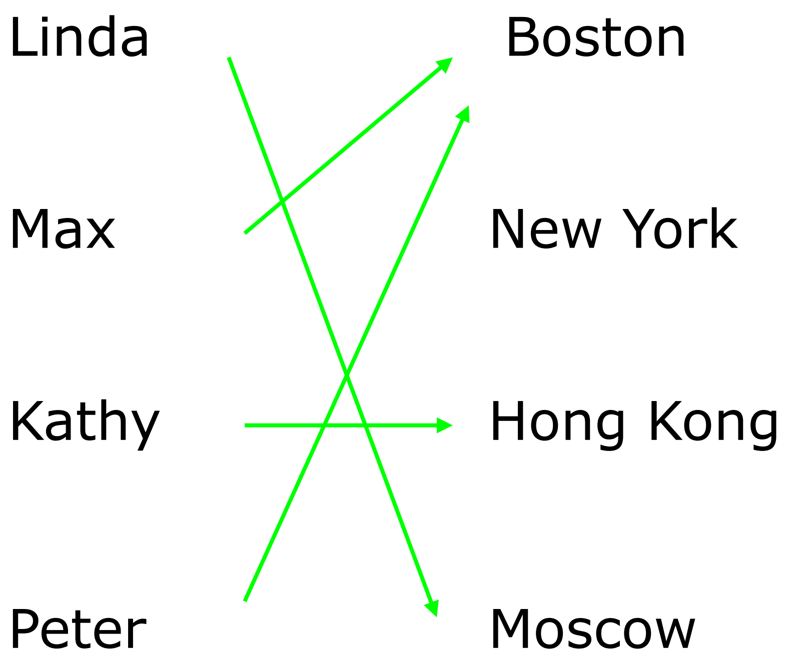
- Is f still a function? yes

  What is its range? {Moscow, Boston, Hong Kong}

Functions

- Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Functions

- If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:
- $f:\mathbf{R} \rightarrow \mathbf{R}$
- $f(x) = 2x$
- This leads to:
- $f(1) = 2$
- $f(3) = 6$
- $f(-3) = -6$
- …

Functions

- Let f_1 and f_2 be functions from A to **R**.
- Then the sum and the product of f_1 and f_2 are also functions from A to **R** defined by:
- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1f_2)(x) = f_1(x)\, f_2(x)$

- Example:
- $f_1(x) = 3x, \quad f_2(x) = x + 5$
- $(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$
- $(f_1f_2)(x) = f_1(x)\, f_2(x) = 3x\,(x + 5) = 3x^2 + 15x$

Functions

- We already know that the range of a function $f:A \to B$ is the set of all images of elements $a \in A$.
- If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.
- We denote the image of S by f(S):
- $f(S) = \{f(s) \mid s \in S\}$

Functions

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

- What is the image of S = {Linda, Max} ?
- f(S) = {Moscow, Boston}

- What is the image of S = {Max, Peter} ?
- f(S) = {Boston}

Properties of Functions

- A function $f: A \to B$ is said to be one-to-one (or injective), if and only if
- $\forall x, y \in A\ (f(x) = f(y) \to x = y)$
- In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

Properties of Functions

f(Linda) = Moscow
f(Max) = Boston
f(Kathy) = Hong Kong
f(Peter) = Boston

Is f one-to-one?

No, Max and Peter are mapped onto the same element of the image.

g(Linda) = Moscow
g(Max) = Boston
g(Kathy) = Hong Kong
g(Peter) = New York

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

- A function $f:A \rightarrow B$ with $A,B \subseteq \mathbf{R}$ is called strictly increasing, if
- $\forall x,y \in A\ (x < y \rightarrow f(x) < f(y))$,
- and strictly decreasing, if
- $\forall x,y \in A\ (x < y \rightarrow f(x) > f(y))$.

- Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties of Functions

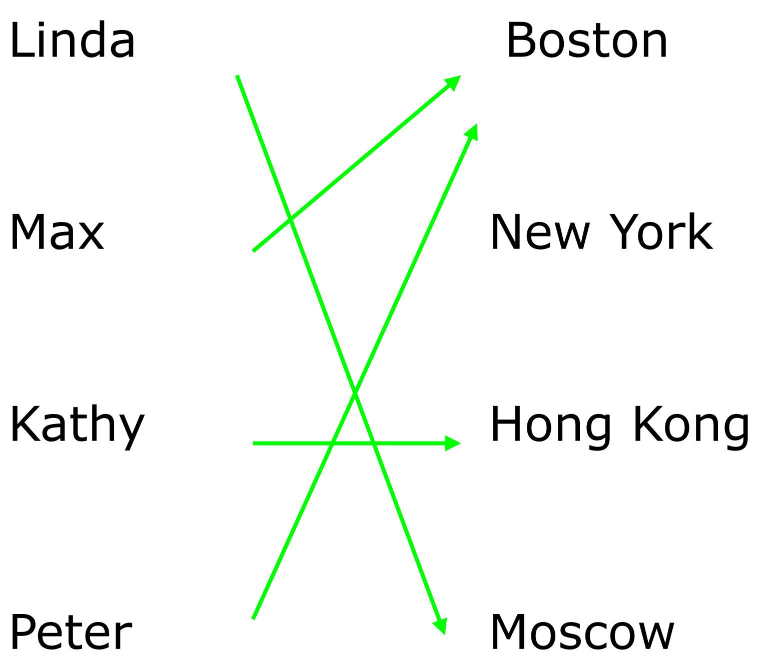
- A function $f:A\rightarrow B$ is called onto, or surjective, if and only if for every element $b\in B$ there is an element $a\in A$ with $f(a) = b$.
- In other words, f is onto if and only if its range is its entire codomain.
- A function $f: A\rightarrow B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.
- Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

Properties of Functions

- Examples:
- In the following examples, we use the arrow representation to illustrate functions $f{:}A \to B$.
- In each example, the complete sets A and B are shown.

Properties of Functions

- Is f injective?
- No.
- Is f surjective?
- No.
- Is f bijective?
- No.



Properties of Functions

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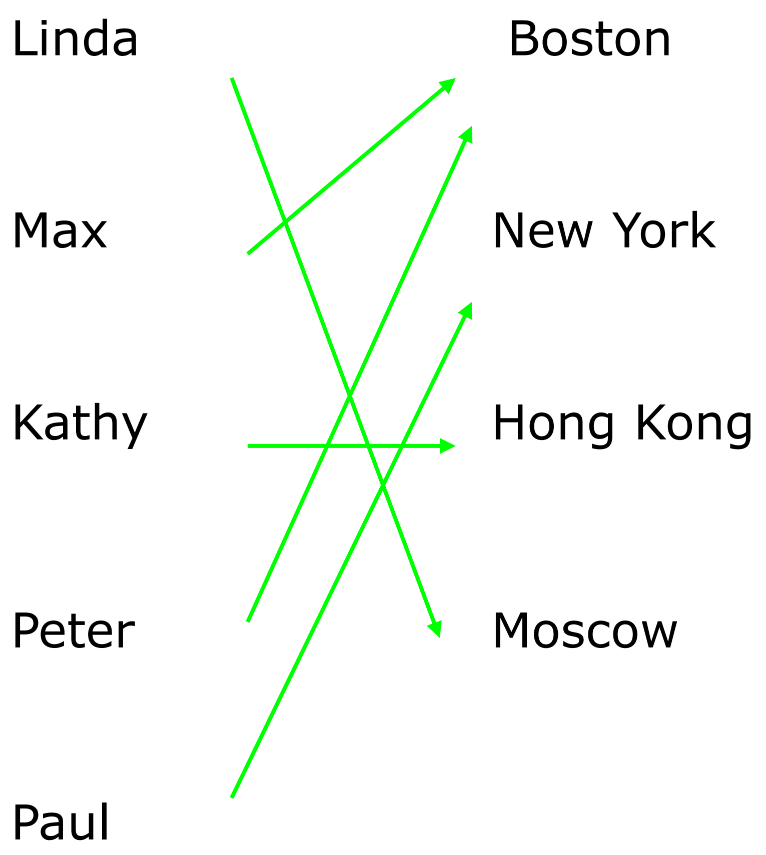

Properties of Functions

- Is f injective?
- Yes.
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- No.
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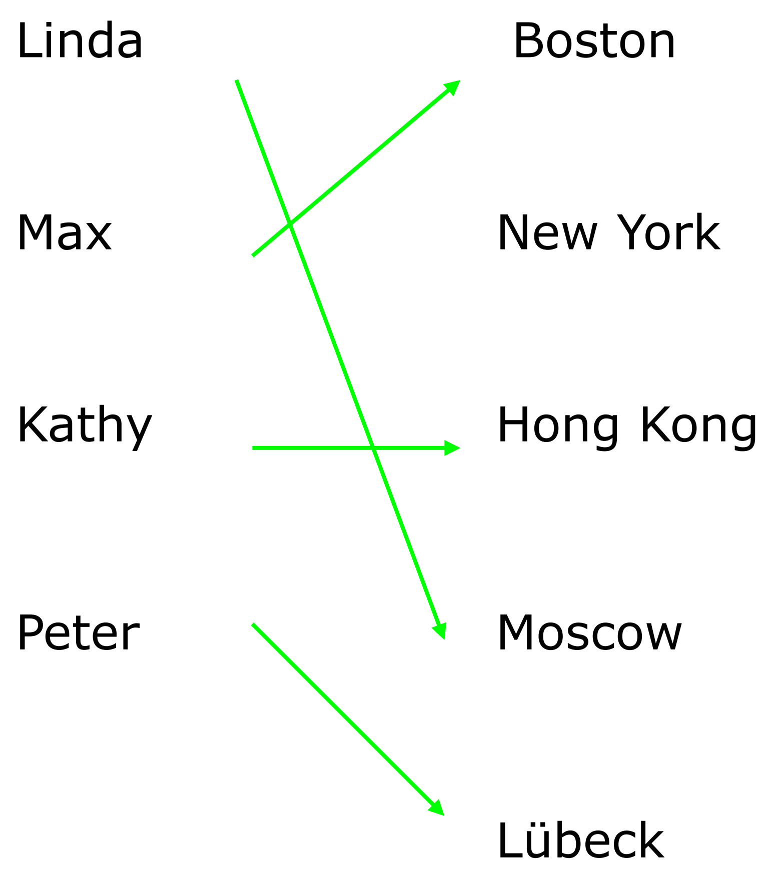
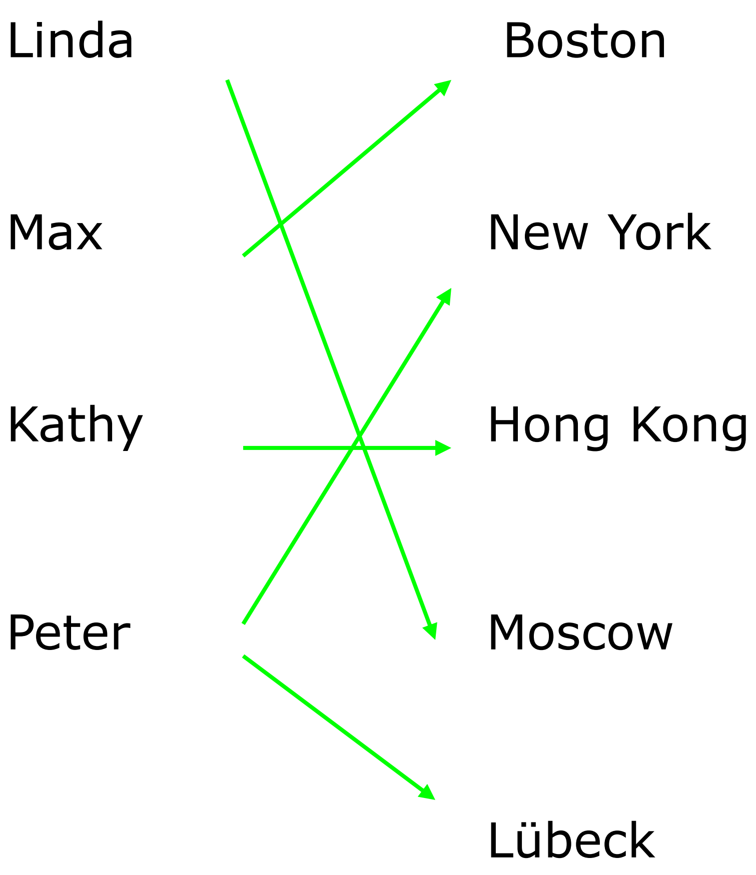


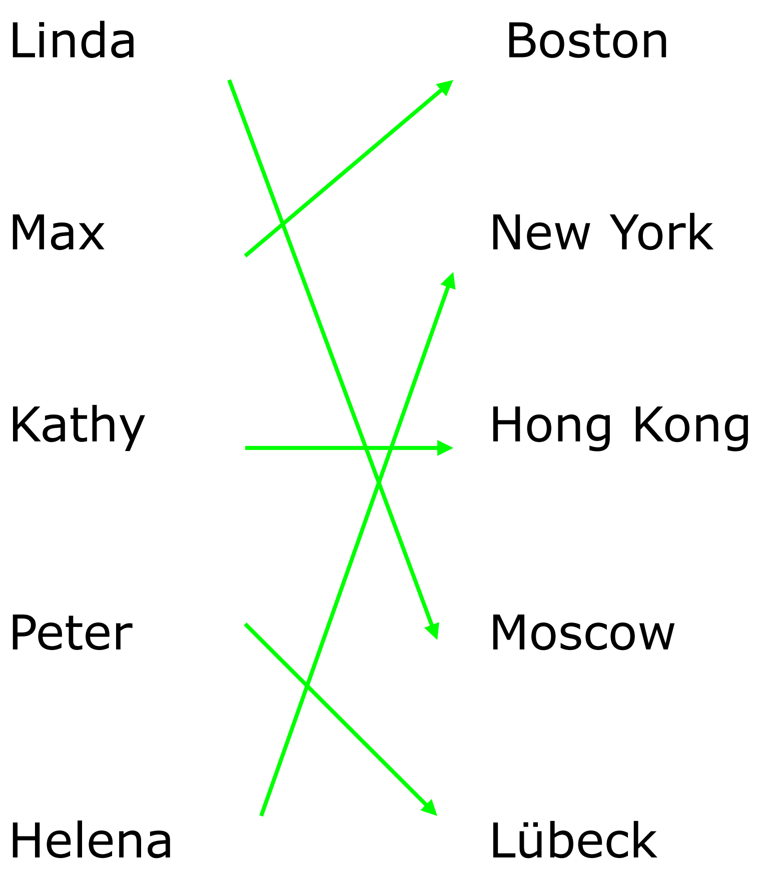
Properties of Functions

- Is f injective?
- No! f is not even a function!



Properties of Functions

- Is f injective?
- Yes.
- Is f surjective?
- Yes.
- Is f bijective?
- Yes.

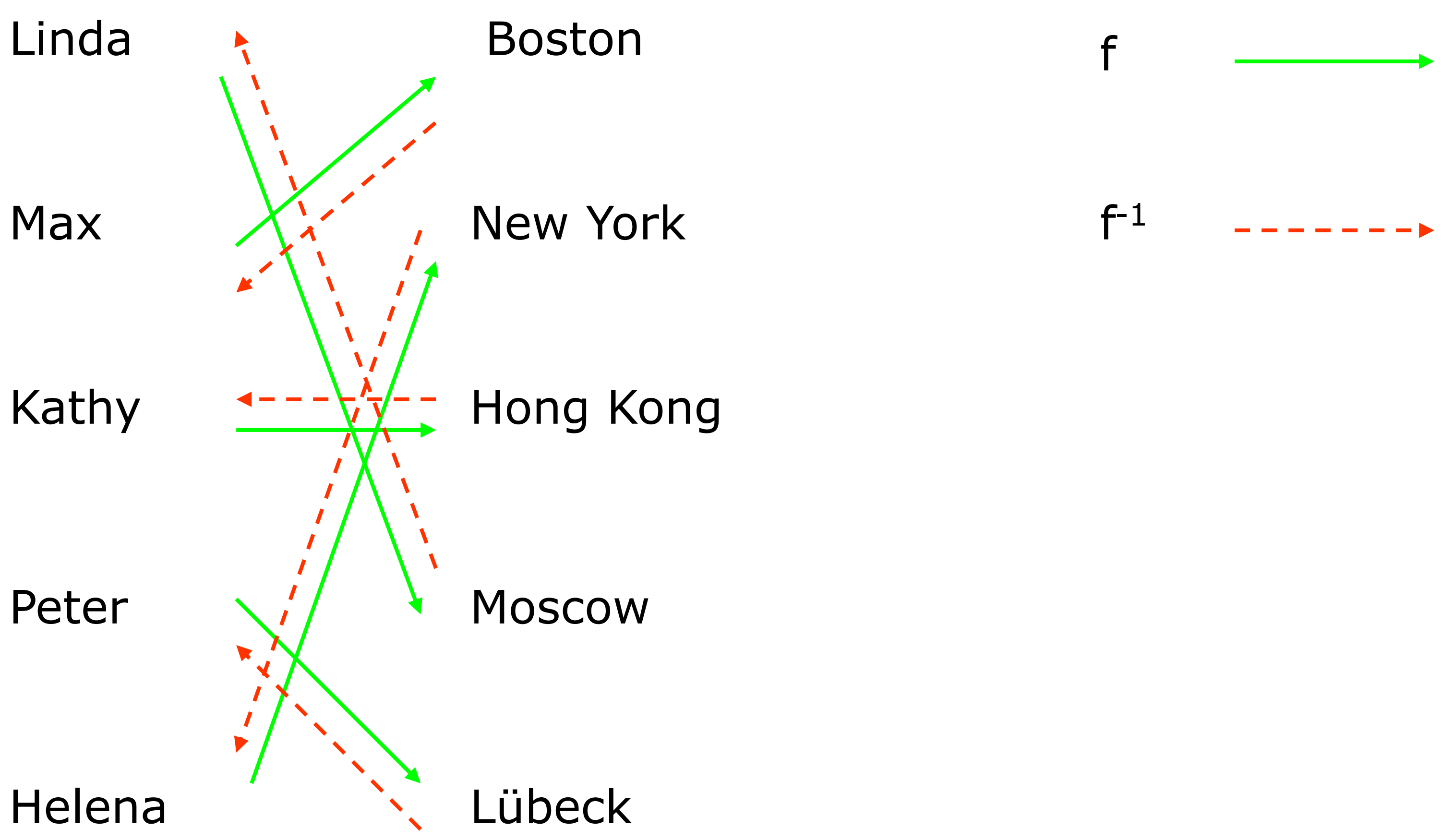


Inversion

- An interesting property of bijections is that they have an inverse function.

- The inverse function of the bijection $f{:}A \rightarrow B$ is the function $f^{-1}{:}B \rightarrow A$ with
- $f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion


Linda
Max
Kathy
Peter
Helena
Boston
New York
Hong Kong
Moscow
Lübeck
f
f^{-1}

Inversion

Example:

f(Linda) = Moscow
f(Max) = Boston
f(Kathy) = Hong Kong
f(Peter) = Lübeck
f(Helena) = New York

Clearly, f is bijective.

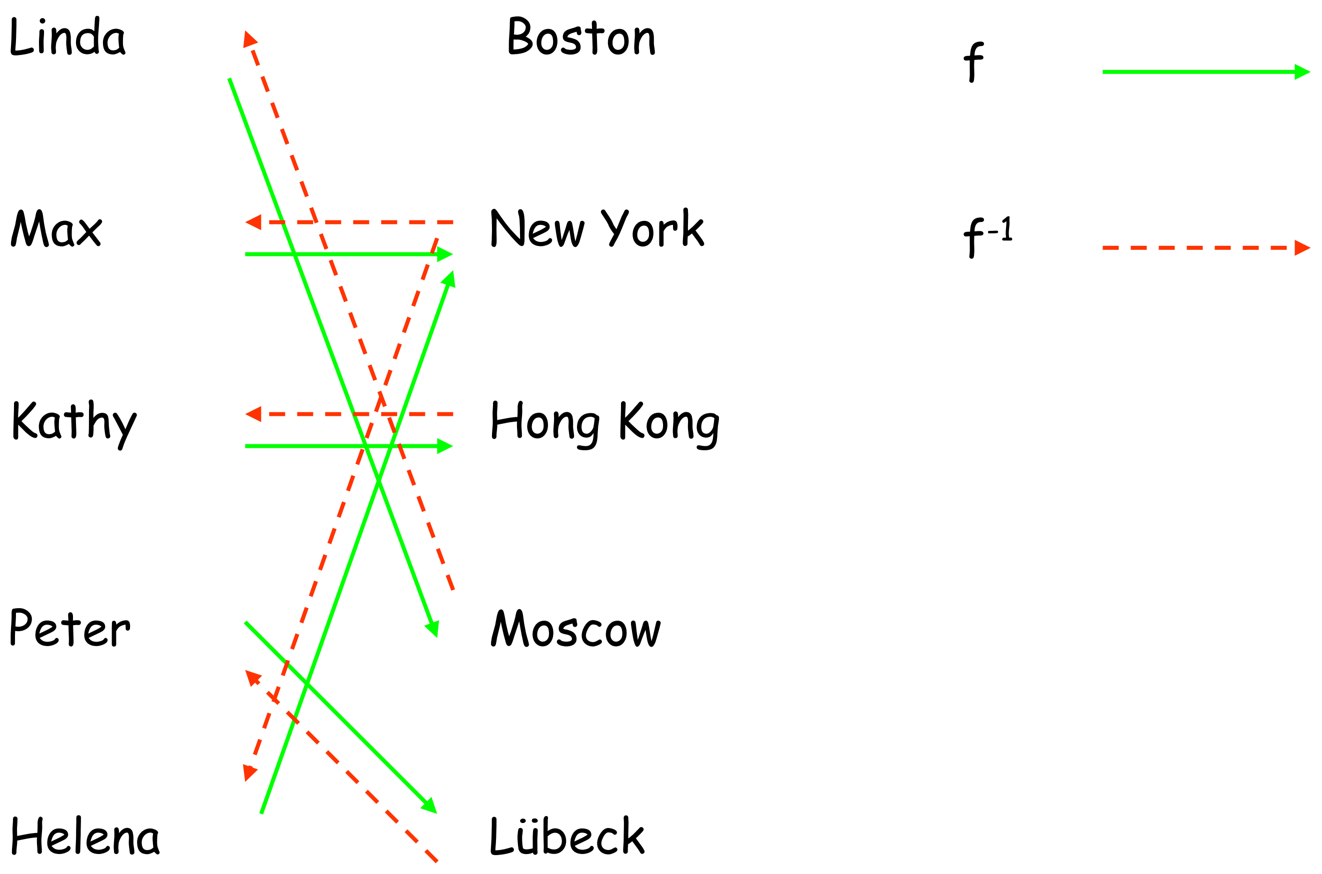
The inverse function f^{-1} is given by:

f^{-1}(Moscow) = Linda
f^{-1}(Boston) = Max
f^{-1}(Hong Kong) = Kathy
f^{-1}(Lübeck) = Peter
f^{-1}(New York) = Helena

Inversion is only possible for bijections
(= invertible functions)

Inversion

- $f^{-1}: C \rightarrow P$ is no function, because it is not defined for all elements of C and assigns two images to the pre-image New York.



Composition

- The composition of two functions $g:A \rightarrow B$ and $f:B \rightarrow C$, denoted by $f \circ g$, is defined by
- $(f \circ g)(a) = f(g(a))$
- This means that
- first, function g is applied to element $a \in A$, mapping it onto an element of B,
- then, function f is applied to this element of B, mapping it onto an element of C.
- Therefore, the composite function maps from A to C.

Composition

- Example:
- $f(x) = 7x - 4$, $g(x) = 3x$,
- $f:\mathbf{R}\to\mathbf{R}$, $g:\mathbf{R}\to\mathbf{R}$
- $(f \circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$
- $(f \circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

- Composition of a function and its inverse:
- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$
- The composition of a function and its inverse is the **identity function** $i(x) = x$.

Floor and Ceiling Functions

- The floor and ceiling functions map the real numbers onto the integers ($R \to Z$).
- The floor function assigns to $r \in R$ the largest $z \in Z$ with $z \leq r$, denoted by $\lfloor r \rfloor$.
- Examples: $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$
- The ceiling function assigns to $r \in R$ the smallest $z \in Z$ with $z \geq r$, denoted by $\lceil r \rceil$.
- Examples: $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Relations

Sections 9.1, 9.5, 9.6

Relations

- If we want to describe a relationship between elements of two sets A and B, we can use ordered pairs with their first element taken from A and their second element taken from B.
- Since this is a relation between two sets, it is called a binary relation.
- Definition: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.
- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$.

Relations

- When (a, b) belongs to R, a is said to be related to b by R.
- Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).
- P = {Carl, Suzanne, Peter, Carla},
- C = {Mercedes, BMW, tricycle}
- D = {(Carl, Mercedes), (Suzanne, Mercedes), (Suzanne, BMW), (Peter, tricycle)}
- This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

Functions as Relations

- You might remember that a function f from a set A to a set B assigns a unique element of B to each element of A.
- The graph of f is the set of ordered pairs (a, b) such that b = f(a).
- Since the graph of f is a subset of $A \times B$, it is a relation from A to B.
- Moreover, for each element a of A, there is exactly one ordered pair in the graph that has a as its first element.

Functions as Relations

- A function is a relation which has an additional property:
  - if both (x,y) and (x,z) are in the relation then $y=z$.
- Every function is a relation.
- Every relation is not a function.

Relations on a Set

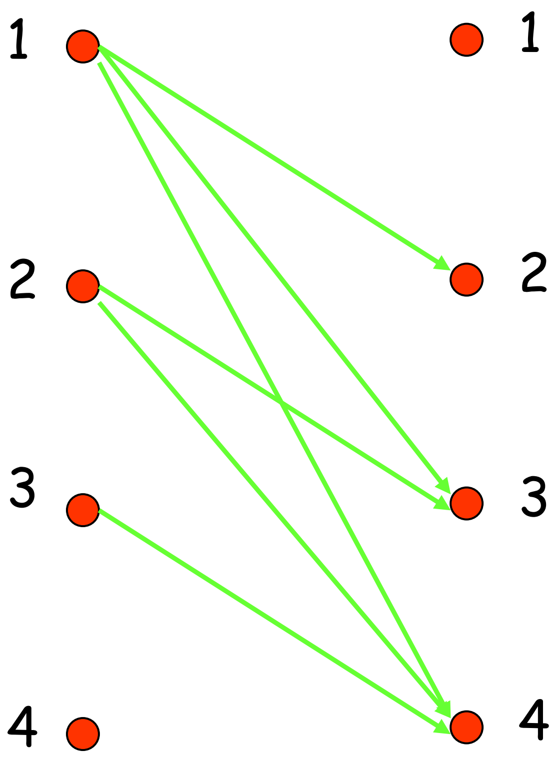
- **Definition:** A relation on the set A is a relation from A to A.

- In other words, a relation on the set A is a subset of $A \times A$.

- **Example:** Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Relations on a Set

- **Solution:** R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}



R	1	2	3	4
1		x	x	x
2			x	x
3				x
4				

Relations on a Set

- How many different relations can we define on a set A with n elements?
- A relation on a set A is a subset of $A \times A$.
- How many elements are in $A \times A$?
- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?
- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.
- Answer: We can define 2^{n^2} different relations on A.

Properties of Relations

- We will now look at some useful ways to classify relations.
- **Definition:** A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.
- Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

$R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$ No.

$R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$ Yes.

$R = \{(1, 1), (2, 2), (3, 3)\}$ No.

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

- **Definitions:**
- A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.

Properties of Relations

- Are the following relations on {1, 2, 3, 4} **symmetric**?
- R1 ={(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}, No.
- R2 ={(1, 1), (1, 2), (2, 1)}, Yes
- R3 ={(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}, Yes
- R4 ={(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}, No.
- R5 ={(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}, No.
- R6 ={(3, 4)}. No.

Properties of Relations

- Are the following relations on {1, 2, 3, 4} **antisymmetric**?
- R1 ={(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}, No.
- R2 ={(1, 1), (1, 2), (2, 1)}, No
- R3 ={(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}, No
- R4 ={(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}, Yes
- R5 ={(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}, Yes
- R6 ={(3, 4)}. Yes

Properties of Relations

- **Definition:** A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.
- Are the following relations on {1, 2, 3, 4} transitive?

R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)}	Yes.
R = {(1, 3), (3, 2), (2, 1)}	No.
R = {(2, 4), (4, 3), (2, 3), (4, 1)}	No.

Equivalence Relations

- **Equivalence relations** are used to relate objects that are similar in some way.
- **Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

- Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.
- Since R is **reflexive**, every element is equivalent to itself.
- Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

- Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

- **Example:** Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?
- **Solution:**
- R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
- R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
- R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.
- R is an equivalence relation.

Partial Orderings

- Sometimes, relations do not specify the equality of elements in a set, but define an **order** on them.
- **Definition:** A relation R on a set S is called a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive.
- A set S together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R).

Partial Orderings

- **Example:** Consider the "greater than or equal" relation $\geq$ (defined by $\{(a, b) \mid a \geq b\}$).
- Is $\geq$ a **partial ordering** on the set of integers?
- $\geq$ is **reflexive**, because $a \geq a$ for every integer a.
- $\geq$ is **antisymmetric**, because if $a \neq b$, then $a \geq b \wedge b \geq a$ is false.
- $\geq$ is **transitive**, because if $a \geq b$ and $b \geq c$, then $a \geq c$.
- Consequently, $(Z, \geq)$ is a partially ordered set.

Partial Orderings

- **Another example:** Is the "inclusion relation" $\subseteq$ a **partial ordering** on the power set of a set S?
- $\subseteq$ is **reflexive**, because $A \subseteq A$ for every set A.
- $\subseteq$ is **antisymmetric**, because if $A \neq B$, then $A \subseteq B \wedge B \subseteq A$ is false.
- $\subseteq$ is **transitive**, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Consequently, $(P(S), \subseteq)$ is a partially ordered set.

Summary

- In this part, we discussed:
  - Set Theory
  - Functions
  - Relations

References

- These slides are largely based on the following two sources:
  - Slides by Marc Pomplun, Umass Boston
  - Official McGraw Hill's Slides for our textbook