Assignment 3

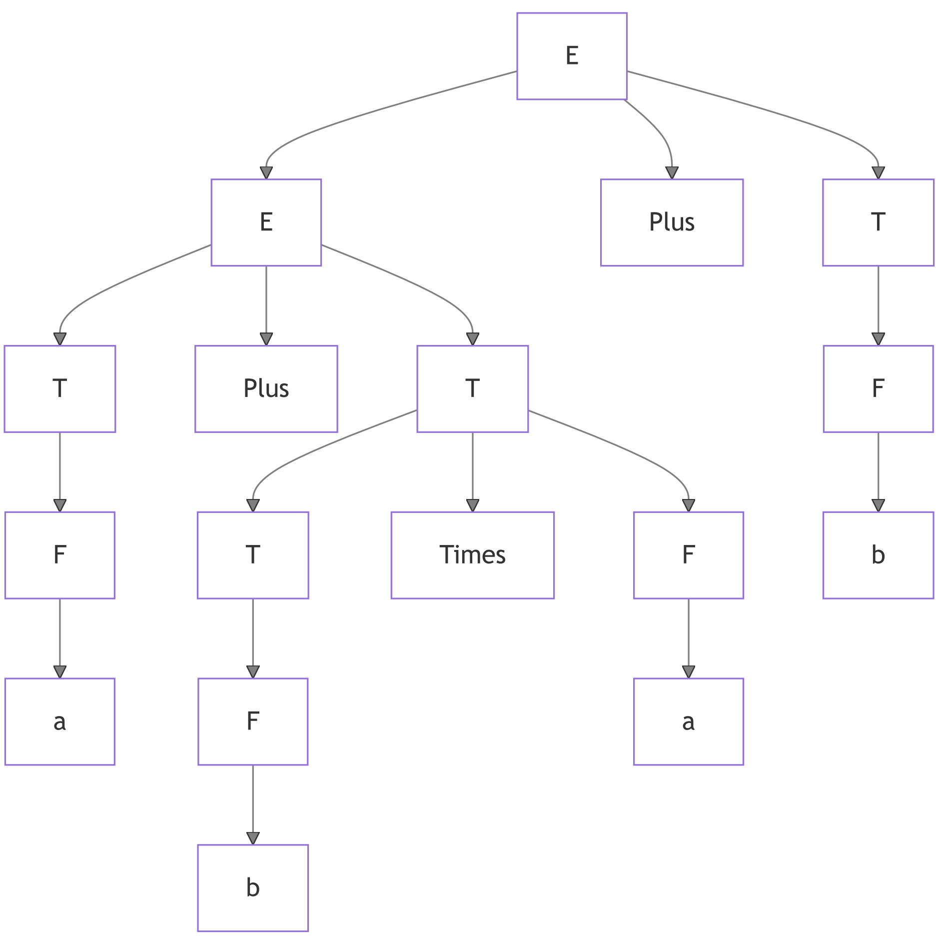
Question 1

1.A: $a + b \times a + b$

Leftmost Derivation

1. $E \to E + T$
2. $E + T \to (E + T) + T$
3. $(E + T) + T \to (T + T) + T$
4. $(T + T) + T \to (F + T) + T$
5. $(F + T) + T \to (a + T) + T$ ($F \to a$)
6. $(a + T) + T \to (a + (T \times F)) + T$
7. $(a + (T \times F)) + T \to (a + (F \times F)) + T$
8. $(a + (F \times F)) + T \to (a + (b \times F)) + T$
9. $(a + (b \times F)) + T \to (a + (b \times a)) + T$
10. $(a + (b \times a)) + T \to (a + (b \times a)) + F$
11. $(a + (b \times a)) + F \to (a + (b \times a)) + b$ ($F \to b$)

Parse Tree

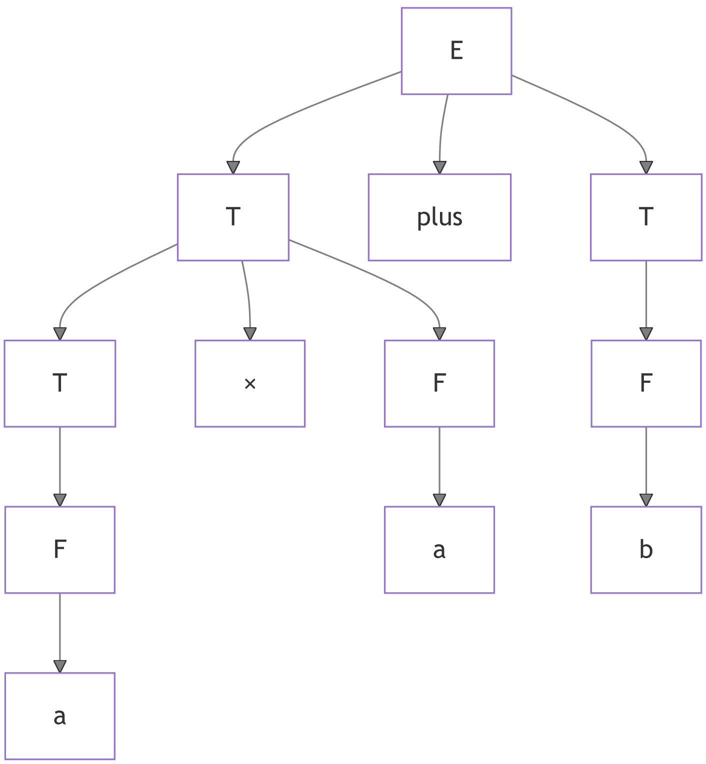


1.B: $a \times a + b$

Leftmost Derivation

1. $E \to E + T$
2. $E + T \to T + T$
3. $T + T \to (T \times F) + T$
4. $(T \times F) + T \to (F \times F) + T$
5. $(F \times F) + T \to (a \times F) + T$
6. $(a \times F) + T \to (a \times a) + T$
7. $(a \times a) + T \to (a \times a) + F$
8. $(a \times a) + F \to (a \times a) + b$ ($F \to b$)

Parse Tree



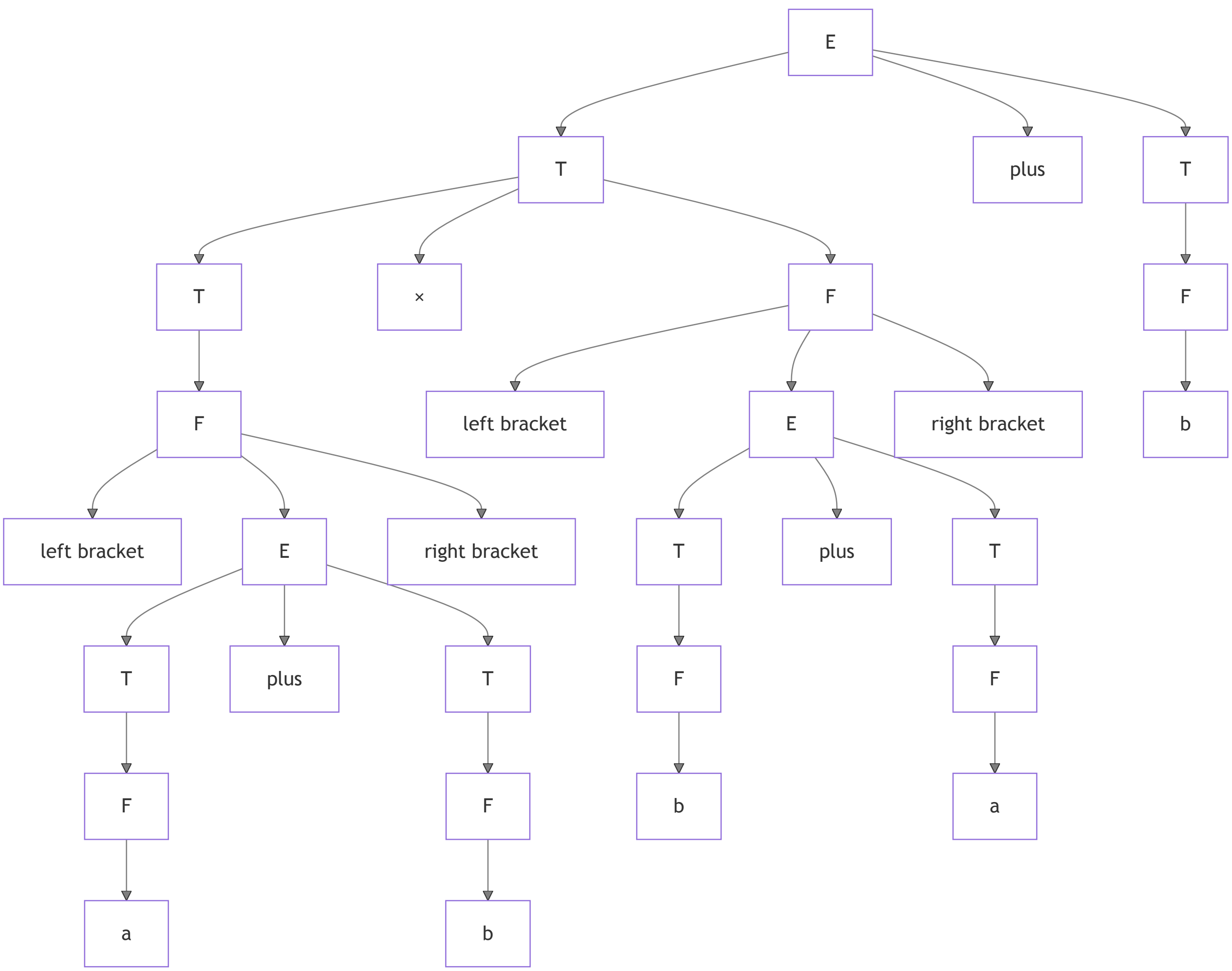
1.C: $(a + b) \times (b + a) + b$

Leftmost Derivation

1. $E \to E + T$ (for $+b$)
2. $E + T \to T + T$
3. $T + T \to (T \times F) + T$
4. $(T \times F) + T \to (F \times F) + T$
5. $(F \times F) + T \to ((E) \times F) + T$ ($F \to (E)$)
6. $((E) \times F) + T \to ((E + T) \times F) + T$ ($E \to E + T$ for $a + b$)
7. $((E + T) \times F) + T \to ((T + T) \times F) + T$ ($E \to T$)

8. $((T+T)\times F)+T \to ((F+T)\times F)+T$ ($T \to F$ for a)
9. $((F+T)\times F)+T \to ((a+T)\times F)+T$ ($F \to a$)
10. $((a+T)\times F)+T \to ((a+F)\times F)+T$ ($T \to F$ for b)
11. $((a+F)\times F)+T \to ((a+b)\times F)+T$ ($F \to b$)
12. $((a+b)\times F)+T \to ((a+b)\times (E))+T$ ($F \to (E)$ for $(b+a)$)
13. $((a+b)\times (E))+T \to ((a+b)\times (E+T))+T$ ($E \to E+T$)
14. $((a+b)\times (E+T))+T \to ((a+b)\times (T+T))+T$ ($E \to T$)
15. $((a+b)\times (T+T))+T \to ((a+b)\times (F+T))+T$ ($T \to F$ for b)
16. $((a+b)\times (F+T))+T \to ((a+b)\times (b+T))+T$ ($F \to b$)
17. $((a+b)\times (b+T))+T \to ((a+b)\times (b+F))+T$ ($T \to F$ for a)
18. $((a+b)\times (b+F))+T \to ((a+b)\times (b+a))+T$ ($F \to a$)
19. $((a+b)\times (b+a))+T \to ((a+b)\times (b+a))+F$ ($T \to F$ for b)
20. $((a+b)\times (b+a))+F \to ((a+b)\times (b+a))+b$ ($F \to b$)

Parse Tree

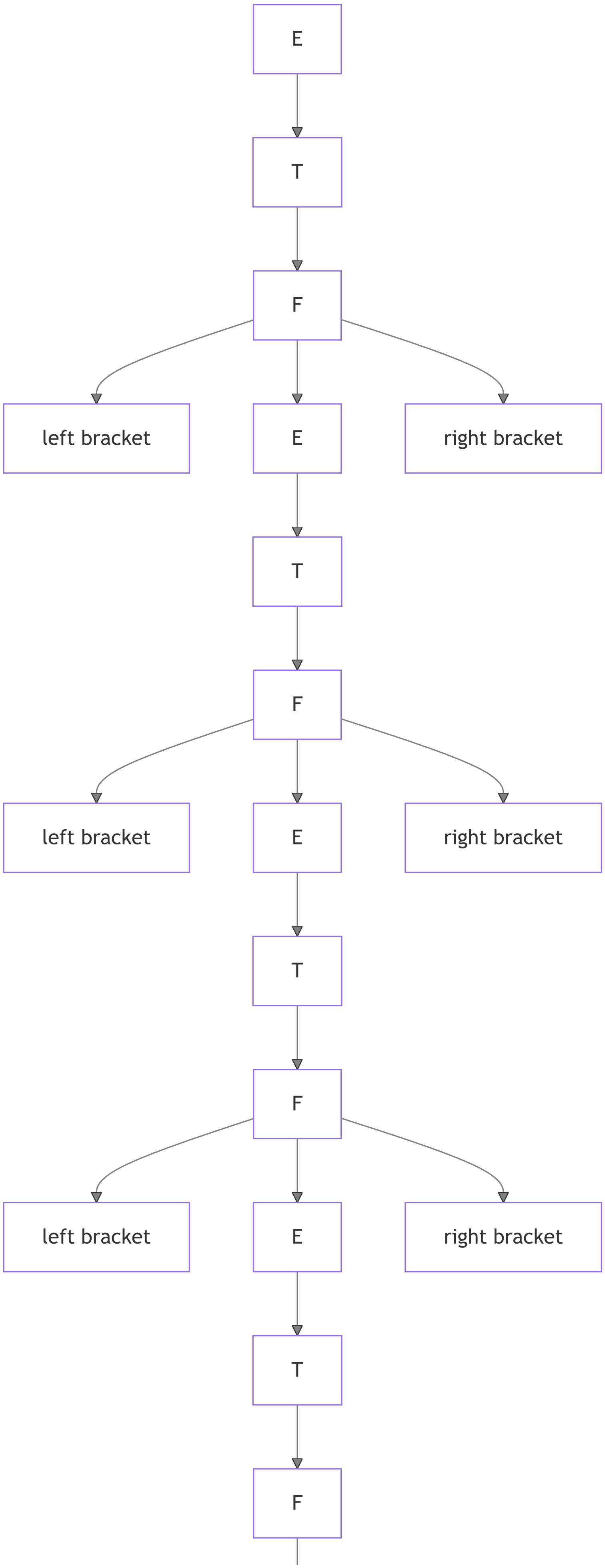


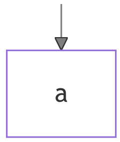
1.D: $(((a)))$

Leftmost Derivation

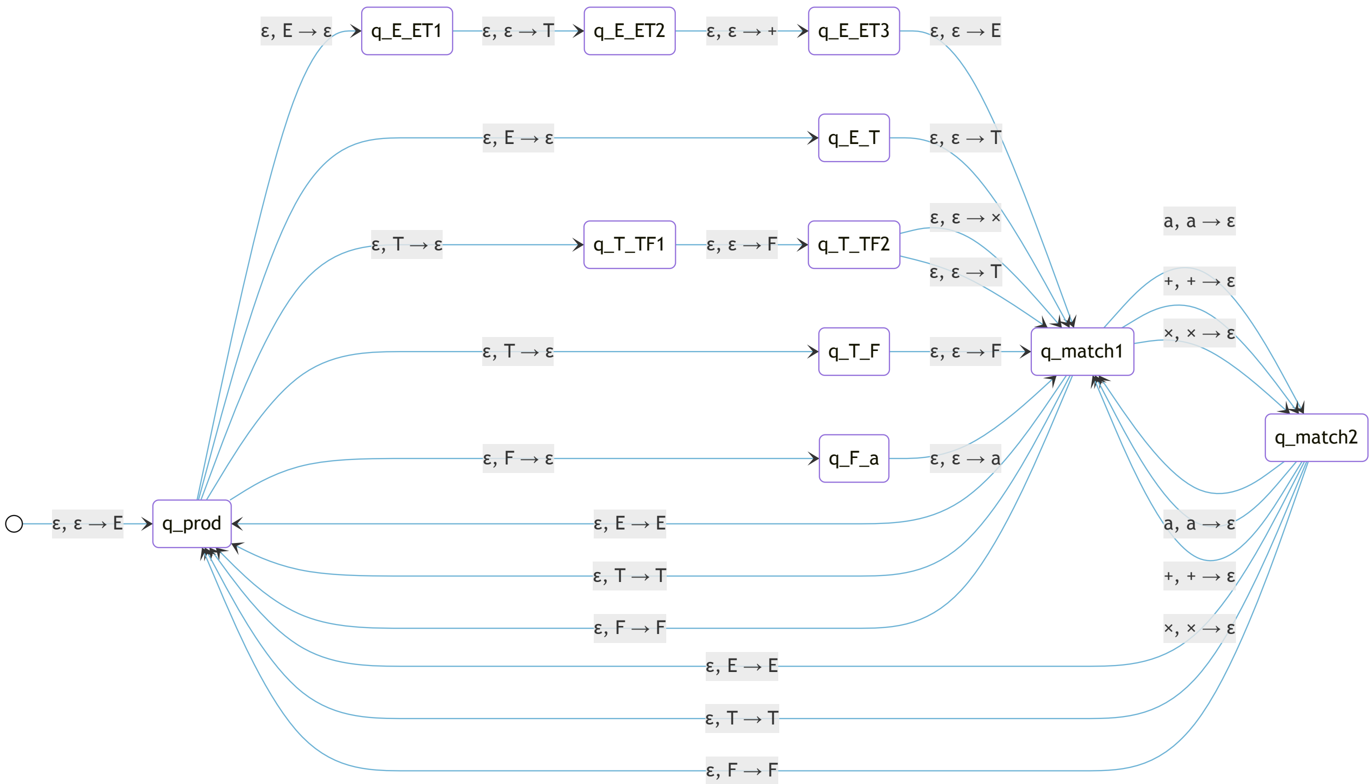
1. $E \to T$
2. $T \to F$
3. $F \to (E)$
4. $(E) \to (T)$ ($E \to T$)
5. $(T) \to (F)$ ($T \to F$)
6. $(F) \to ((E))$ ($F \to (E)$)
7. $((E)) \to ((T))$ ($E \to T$)
8. $((T)) \to ((F))$ ($T \to F$)
9. $((F)) \to (((E)))$ ($F \to (E)$)
10. $(((E))) \to (((T)))$ ($E \to T$)
11. $(((T))) \to (((F)))$ ($T \to F$)
12. $(((F))) \to (((a)))$ ($F \to a$)

Parse Tree


E
T
F
left bracket
E
right bracket
T
F
left bracket
E
right bracket
T
F
left bracket
E
right bracket
T
F



Question 2



Question 3

The grammar is ambiguous. The string 00111 has two different leftmost derivations:

First Leftmost Derivation:

$$S \to A \to 0S1S \to 0B1S \to 00S1S \to 0011S \to 00111$$

Second Leftmost Derivation:

$$S \to B \to 0S \to 0A \to 00S1S \to 0011S \to 00111$$

These derivations correspond to different parse trees, proving the grammar's ambiguity.

Question 4

New Start Variable

New start variable S'

- $S' \to A$

Grammar becomes:

- $S' \to A$

- $A \to ABC \mid AC \mid B$
- $C \to BC1 \mid B \mid \epsilon$
- $B \to 00 \mid \epsilon$

Remove Nullable Variables

- $B \to \epsilon$, so B is nullable.
- $C \to \epsilon$, so C is nullable.
- $S' \to A$:
  - A is not directly nullable yet, so this remains $S' \to A$.
  - However, since A can derive ϵ (e.g., $A \to B \to \epsilon$), and ϵ is in the language, add $S' \to \epsilon$ after adjusting A's productions.
- $A \to ABC$:
  - B is nullable, C is nullable.
  - All combinations:
    - ABC (both present)
    - AC (omit B)
    - AB (omit C)
    - A (omit both B and C)
  - So, $A \to ABC \mid AB \mid AC \mid A$.
- $A \to AC$:
  - C is nullable.
  - Combinations:
    - AC (present)
    - A (omit C)
  - So, $A \to AC \mid A$
- $A \to B$:
  - B is nullable.
  - Combinations:
    - B (present)
    - ϵ (omit B)
- $C \to BC1$:
  - B is nullable.
  - Combinations:
    - $BC1$ (present)
    - $C1$ (omit B)
  - So, $C \to BC1 \mid C1$.
- $C \to B$:
  - B is nullable.
  - Combinations:
    - B (present)
    - ϵ (omit B)

- $C \to \epsilon$:
  - Remove this production.
- $B \to 00$:
  - No nullable variables, so $B \to 00$.
- $B \to \epsilon$:
  - Remove this production.

Grammar becomes:

- $S' \to A \mid \epsilon$
- $A \to ABC \mid AB \mid AC \mid A \mid B$
- $C \to BC1 \mid C1 \mid B$
- $B \to 00$

Remove Unit Productions

Eliminate $S' \to A$:

- $A \to ABC \mid AB \mid AC \mid A \mid B$
- $S' \to ABC \mid AB \mid AC \mid A \mid B \mid \epsilon$.

Eliminate $A \to A$:

- $A \to ABC \mid AB \mid AC \mid B$.

Eliminate $A \to B$:

- $B \to 00$
- Replace $A \to B$ with $A \to 00$.
- $A \to ABC \mid AB \mid AC \mid 00$.

Eliminate $C \to B$:

- $B \to 00$
- Replace $C \to B$ with $C \to 00$.
- $C \to BC1 \mid C1 \mid 00$.

Grammar after eliminating unit productions:

- $S' \to ABC \mid AB \mid AC \mid 00 \mid \epsilon$
- $A \to ABC \mid AB \mid AC \mid 00$
- $C \to BC1 \mid C1 \mid 00$
- $B \to 00$

Eliminate Useless Symbols

Derivation to Terminals:

- $B \to 00$ (terminals).
- $C \to 00$ (terminals), $C \to C1$ (since $C \to 00$, it can reach terminals), $C \to BC1$ (since $B \to 00$, reachable).
- $A \to 00$ (terminals), $A \to AB, AC, ABC$ (all use B, C, which reach terminals).
- $S' \to 00$ (terminals), $S' \to ABC, AB, AC$ (reachable).

No useless symbols exist.

Grammar Remains unchanged:

- $S' \to ABC \mid AB \mid AC \mid 00 \mid \epsilon$
- $A \to ABC \mid AB \mid AC \mid 00$
- $C \to BC1 \mid C1 \mid 00$
- $B \to 00$

Convert to Chomsky Normal Form

Handle Terminals:
Replace terminals in productions with more than one symbol by introducing new variables:

- Define $E \to 0$ (for terminal 0).
- Define $D \to 1$ (for terminal 1).
- $A \to 00$:
  - Replace 00 with EE (since $E \to 0$).
  - $A \to EE$.
- $C \to 00$:
  - $C \to EE$.
- $B \to 00$:
  - $B \to EE$.
- $S' \to 00$:
  - $S' \to EE$.
- $C \to BC1$:
  - Replace 1 with D: $C \to BCD$.
- $C \to C1$:
  - Replace 1 with D: $C \to CD$.

Productions with More Than Two Symbols:

- $S' \to ABC$:
  - Three variables: introduce $P \to BC$, then $S' \to AP$.
- $A \to ABC$:
  - Three variables: use $P \to BC$, then $A \to AP$.
- $C \to BCD$:

- Three variables: introduce $Q \to CD$, then $C \to BQ$.

Grammar in CNF

- $S' \to AP \mid AB \mid AC \mid EE \mid \epsilon$
- $A \to AP \mid AB \mid AC \mid EE$
- $C \to BQ \mid CD \mid EE$
- $B \to EE$
- $P \to BC$
- $Q \to CD$
- $E \to 0$
- $D \to 1$

Question 5

Let L be a context-free grammar (CFG) that generates the language of all strings over the alphabet $\{0, 1\}$ where the number of 0s is equal to the number of 1s plus 2:

- **Start variable**: S
- **Productions**:
  - $S \to 0A0 \mid 0B \mid C0$
  - $A \to 0A1 \mid 1A0 \mid \epsilon$
  - $B \to 0A \mid 1A0$
  - $C \to 0A1 \mid 1A$

Justification

The language we need to generate is $L = \{w \in \{0, 1\}^* \mid n_0(w) = n_1(w) + 2\}$, where $n_0(w)$ is the number of 0s and $n_1(w)$ is the number of 1s in the string w. This means every string in the language must have exactly two more 0s than 1s. The grammar above achieves this by ensuring that each generated string consists of a balanced part (with an equal number of 0s and 1s) plus exactly two additional 0s.

- **Base Case**: $S \to 0B \to 00$ generates 00 (2 0s, 0 1s), which satisfies the condition.
- **Inductive Step**: Each production either:
  - Adds a 0 without a 1 ($S \to 0S \mid S0$, $B \to 0$, $C \to 0$),
  - Balances 0s and 1s (A, $B \to 0A1$, etc.),
  - Ensures the net excess of 0s over 1s remains exactly 2 by combining one extra 0 from B or C with another from S.

Thus, every derivation from S produces a string where $n_0 = n_1 + 2$, and all such strings can be generated by placing two extra 0s around or within balanced substrings, which this grammar achieves efficiently.

Question 6

Language $L_1 = \{x^n y^m x^m z^n \mid n > 0, m > 0\}$ is context-free.
Language $L_2 = \{x^n y^n x^n z^m \mid n > 0, m \geq 0\}$ is not context-free.

Proof that L_1 is Context-Free

Context-Free Grammar for L_1:

- **Start symbol**: S
- **Productions**:
  - $S \to xAz$
  - $A \to xAz \mid B$
  - $B \to yBy \mid yy$

This CFG generates all strings in L_1, so L_1 is context-free.

Proof that L_2 is Not Context-Free

To show that L_2 is not context-free, we use the **Pumping Lemma for context-free languages**.

Pumping Lemma Statement:

If L_2 is context-free, there exists a constant p such that for any string $w \in L_2$ with $|w| \geq p$, we can write $w = uvxyz$ where:

1. $|vxy| \leq p$
2. $|vy| \geq 1$
3. For all $k \geq 0$, $uv^k xy^k z \in L_2$.

Proof by Contradiction:

Assume L_2 is context-free. Let p be the pumping constant.

Choose $w = x^p y^p x^p$ (where $n = p > 0, m = 0 \geq 0$):

- $w \in L_2$
- $|w| = p + p + p = 3p \geq p$

Now, divide $w = uvxyz$ with $|vxy| \leq p$ and $|vy| \geq 1$. Since $|vxy| \leq p$, vxy cannot span all three parts (x^p, y^p, x^p). We consider possible cases:

- **Case 1:** vxy **is within the first** x^p
  - v and y are x's, $u = x^a, v = x^b, x = x^c, y = x^d, z = x^{p-a-b-c-d} y^p x^p$ (where $b + c + d \leq p, b + d \geq 1$).
  - Pump with $k = 2$: $uv^2 xy^2 z = x^{p+b+d} y^p x^p$.
  - This has more x's in the first part than y's or the second x^p, so it's not in L_2.
- **Case 2:** vxy **is within** y^p
  - v and y are y's, $u = x^p y^a, v = y^b, x = y^c, y = y^d, z = y^{p-a-b-c-d} x^p$.
  - Pump with $k = 2$: $uv^2 xy^2 z = x^p y^{p+b+d} x^p$.
  - The number of y's exceeds n, so it's not in L_2.
- **Case 3:** vxy **is within the second** x^p
  - v and y are x's, $u = x^p y^p x^a, v = x^b, x = x^c, y = x^d, z = x^{p-a-b-c-d}$.
  - Pump with $k = 2$: $uv^2 xy^2 z = x^p y^p x^{p+b+d}$.

- The third part has more x's than the first, so it's not in L_2.

- **Case 4:** vxy **straddles** $x^p y^p$ **or** $y^p x^p$
  - Similar analysis shows pumping disrupts the $x^n y^n x^n$ equality (e.g., adding x's and y's unevenly).

In all cases, pumping produces a string not in L_2, contradicting the assumption. Thus, L_2 is not context-free.