A STUDY ON FUNDAMENTAL GROUP

A Project Report Submitted to the Central University of Punjab

By

Vikas

(Reg. no.: 18msmath15)

in partial fulfillment for the Degree of

Master of Science

In

Mathematics

Under the Supervision of

Dr. Sandeep Kaur

(Assistant professor)

DEPARTMENT OF MATHEMATICS AND STATISTICS

SCHOOL OF BASIC AND APPLIED SCIENCES

CENTRAL UNIVERSITY OF PUNJAB, BATHINDA

DECLARATION

I, Vikas hereby declare that the Project work entitled "**A STUDY ON FUNDAMENTAL GROUP**" submitted to Central University of Punjab, Bathinda is a record of review work done by me in partial fulfillment of requirements for the award of degree of Master of Science in Mathematics under the supervision of **Dr. Sandeep Kaur**, Assistent professor, Department of Mathematics and Statistics, School of Basic and Applied Sciences, Central University of Punjab, Bathinda. All help received by me from various sources have been duly acknowledged.
The result of this report have not been submitted to any other University or Institute for the award of any degree or diploma in the present form.

Vikas
M. Sc. Mathematics
IV Semester
Reg. No.: 18msmath15
Department of Mathematics and Statistics
Central University of Punjab, Bathinda

CERTIFICATE

This is to certify that the Project report entitled **"A STUDY ON FUNDAMENTAL GROUP"**, submitted to the Department of Mathematics and Statistics, Central University of Punjab, Bathinda, in partial fulfillment for the course Project work (MAT.599) in the 4^{th} Semester of M.Sc Mathematics, is a record of review work carried out by Vikas (Reg. no.: 18msmath15) under my supervision.

Dr. Sandeep Kaur
(Assistent professor)
Department of Mathematics and Statistics,
Central University of Punjab, Bathinda.

ACKNOWLEDGEMENT

I would like to thank the people who have been enormously helpful in the preparation of this report. Firstly, I express my sincere gratitude to my supervisor.
I would like to devote this work to MAA SARASWATI, the goddess of music and wisdom.
I take this opportunity to thank my supervisor Dr.Sandeep Kaur , for both suggesting this topic and assisting me throughout. I am very much thankful for her expert guidance and uplifting inspiration, continuous support and cooperation throughout my report without which the present work would not have been possible. In spite of her busy schedule and so many responsibilities, she always extended full attention, patience hearing and complete support to me. It was a great chance for learning under her guidance.
I would like to thank my family and friends for all of their support throughout my time in this university. I dedicate this work to my parents.

Vikas
M. Sc. Mathematics
Central University of Punjab, Bathinda

ABSTRACT

Chapter 1 contains a brief introduction and the origanation of the fundamental group. How fundamenatal group can play a vital role to distinction of the topological spaces. It also intoroduce the idea how to find a the fundamental group of topological spaces

Chapter 2 contains the properties of various topologically spaces and find the fundamental groups of some famous known topologically space. It also give the information whether there exist any homeomorphism between the given spaces or not.

Finally, in chapter 3, we study the theorem named Seifert-Van Kampen theorem which make it easier to find the fundamental group of some complex topological spaces.

Contents

Chapter 1

THE FUNDAMENTAL GROUP

1.1 Introduction

In the mathematical field of algebraic topology, the fundamental group is a mathematical group associated to any given pointed topological space that provides a way to determine when two paths, starting and ending at a fixed base point, can be continuously deformed into each other. It records information about the basic shape, or holes, of the topological space. The fundamental group is the first and simplest homotopy group. The fundamental group is a topological invariant: homeomorphic topological spaces have the same fundamental group. Henri Poincare defined the fundamental group in 1895 in his paper "Analysis Situs". The concept emerged in the theory of Riemann Surfaces, in the work of Bernhard Riemann, Poincare, and Felix Klein.

1.2 Origin of Fundamental Group

One of the basic problem of topology is to determine whether two given topological spaces are homeomorphic or not. There is no method for solving this problem in general, but techniques do exist that apply in particular cases.

Showing that two spaces are homeomorphic is a matter of constructing a continuous mapping from one to other having a continuous inverse, and constructing continuous function is a problem that we have developed techniques to handle.

Notice that two spaces are not homeomorphic is a different matter. For that, one must show that a continuous map with continous inverse does not exist. If one can find some topological property that holds for one space but not for other, than the problem is solved-that spaces cannot be homeomorphic.

- $[0, 1]$ cannot be homeomorphic to to the open interval $(0, 1)$ due to compactness.
- R is not homeomorphic to the R^2 due to connectedness.

But the topological property we have studied up to now do nor carry us very far more in solving the problem. For instance

- The plane R^2 is not homeomorphic to R^3? (by which property)
- S^2, the torus T, and the double torus $T\#T$. None of the topological properties we have studied up to now will distinguish between them.

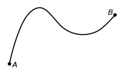
1.3 Concept of Fundamental Group

There is more general idea than of simple connectedness in which simple connectedness is included as a special case. It involves a certain group that is called **fundamental group** of the space. Two spaces that are homeomorphic have fundmental groups that are isomorphic and the condition of simple connectedness is just the condition that the fundamental group of X is the trivial(one element) group. Then S^2 and T are not homeomorphic can be rephrased by saying that the fundamental group of S^2 is trivial and the fundamental group of T is not. The fundamental group will distinguish between more spaces than the condition of simple connectedness will.

1.3.1 Path and Homotopy

A path in a topological space X is a continuous function f from the unit interval I = [0,1] to X

$$f : I \to X$$



The initial point of the path is $f(0)$ and the terminal point is $f(1)$. One often speaks of a "path from x → y " where x and y are the initial and terminal points of the path.

1.3.2 Product or Composition of Paths

One can compose paths in a topological space in an obvious manner. Suppose f is a path from x to y and g is a path from y to z. The path fg is defined as the path obtained by first traversing f and then traversing g:

$$fg(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ g(2s-1) & 1/2 \leq s \leq 1 \end{cases}$$

Clearly path composition is only defined when the terminal point of f coincides with the initial point of g. If one considers all loops based at a point x_0, then path composition is a binary operation. loops are the connected curves having starting and end point same. i.e. here I mean $f(0)$=$f(1)$

1.4 Definition

Definition: For a path f : I to X, f is a loop if $f(0) = f(1)$. We denote by $\pi(X; x_0)$ the set of all homotopy classes [f] of loops f with basepoint $x_0 = f(0)$.

Now, we are ready to construct the fundamental group.

Theorem: $\pi(X, x_0)$ is a group under the operation [f]×[g] = $[f \times g]$ for loops f, g $\epsilon(X; x_0)$.

1.5 Axioms of Group

1.5.1 Closure

As discussed already that the product of two paths is again a path, so the result is obvious. i.e. if α and β are two loops at the base point x_0 then

$$[\alpha \times \beta] = [\alpha] \times [\beta]$$

1.5.2 Identity

The constant loops that goes nowhere is identity element of the fundametal group. i.e.

$$[\alpha_0] \times [\beta] = [\beta] \times [\alpha_0] = [\beta]$$

1.5.3 Associativity

Associativity can be easily varified. i.e. if we have α, β and γ three loops in the set of equivalence class of homotopy then

$$\{\alpha \times \beta\} \times \gamma = \alpha \times \{\beta \times \gamma\}$$

1.5.4 Inverse

for a given loop $\alpha(t)$ we can a find a another loop $\beta(t)$ with $\beta(t)$=$\alpha(1-t)$ which will satisfy

$$\alpha \times \beta = e$$

where 'e' is identity or say constant loop of homotopy class group.

Thus $\pi(X, x_0)$ is group which is known as the fundamental group

Chapter 2

FUNDAMENTAL GROUP OF VARIOUS SPACES

2.1 Fundamental Group of Circle

If our topological space is the cirle S^1 then we can define a loop on circle at base point $(1,0) = x_0$ (say) then we have the following loops: $\alpha_1(t) = t$, $\alpha_2(t) = 2t(mod 1)$, $\alpha_3(t) = 3t(mod 1)$, $\alpha_{-1}(t) = -t(mod 1)$, $\alpha_{-2}(t) = -2t$ the operation can be defined as $[\alpha_1] \times [\alpha_{-2}] = [\alpha_{-1}]$ it is easy to see that

$$[\alpha_n] \times [\alpha_m] = [\alpha_{n+m}]$$

thus the fundamental group of S^1=$\pi(S^1, x_0)$$\cong$ additive group of $\mathbb{Z}$

2.2 Fundamental Group of Torus

Consider the well known structure of topology called Torus. Then we can choose α and β are two loops at the base point x_0 such that α and β can serve as the generator of the fundamental group of the Torus. If the torus cut on the side along the paths of α and β then if the formed rectangle has sides a and b the there can be a loop such that

$$\alpha \times \beta \times \alpha^{-1} \times \beta^{-1} = e$$

where 'e' is constant loop here. also it is easy to see that

$$\alpha \times \beta = \beta \times \alpha$$

and every element can be written as the form as $\alpha^n \beta^m$ where m,n$\in \mathbb{Z}$ so we duduce that fundamental group of torus T= $\pi(T, x_0) \cong$ additive group of $\mathbb{Z} \times \mathbb{Z}$

2.3 Fundamental Group of Spaces by Covering Maps

We have studied the basic need and structure of the fundamental group on some well known topological spaces. Now we are going to relate the concept of fundamental group with the spaces to study them more precisely. There are also some of topological results which is useful in the study of the fundamental group which we will see further.

Covering Spaces

We have shown that any convex subspace of $\mathbb{R}$ has a trivial fundamental group, we turn now to the task of computing some fundamental group that are not trivial. One of the most useful tools for this purpose is the notion of covering spaces.

Definition: Let p: E$\rightarrow$B be a contiuous surjective map. The open set U of B is said to be evenly covered by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphic to U.

Definition

Let p:E→B be a contiuous and surjective. If every point b of B has a neighbourhood U that is evenly covered by p, then p is called a covering map, and E is said to be a **covering spaces** of B.

Example Let X be any space. let $I : X \to X$ be the identity map. Then I is a covering map. More gnerally, let E be the space $X \times \{1, ..., n\}$ consisting of n disjoint copies of X. The map $p : E \to X$ given by $p(x, i) = x$ for i= 1,2...,n is again a covering map. In this case, we can picture the entire space E as a stack of pancakes over X.

Example The map $p : \mathbb{R} \to S^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x) \tag{2.3.1}$$

is a covering map.

Note: One can picture p as function that wraps the real line $\mathbb{R}$ around the circle S^1 and in the process maps each interval [n,n+1] onto S^1.

Proof:Consider the subset U of S^1 consisting of those points having positive first coordinate. The set $p^{-1}(U)$ consist of those points x for which $\cos 2\pi x$ is positive, that is it is the union of the intervals

$$V_n = (n - \frac{1}{4}, n + \frac{1}{4})$$

for all $n \in \mathbb{Z}$. Now, restricted to any close interval $\bar{V_n}$, the map p is injective because $\sin 2\pi x$ is strictly monotonic on such an interval. Furthermore, p carries $\bar{V_n}$ surjectively onto $\bar{U_n}$, by the intermediate value theorem. Since $\bar{V_n}$ is compact, $p|\bar{V_n}$ is a homeomorphism of $\bar{V_n}$ with $\bar{U_n}$. In particular, $p|V_n$ is a homeomorphism of V_n with U.

2.3.1 The Fundamental Group of Circle by Covering Map

Definition Let p:E→B be a map. If f is continuous mapping of some space X into B, a **lifting** of is a map $\tilde{f}$:X→E such that po$\tilde{f}$=f.

Definition Let $p : E \to B$ be a covering map, let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element [f] of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0.

Theorem 2.3.1. *Let $p : E \to B$ be a covering map, let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof: If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then f=po$\tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$ by definition. Suppose E is simply connected. Let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be lifting of f and g, respectively, to paths in E that begin at e_0, then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is path homotopy $\tilde{F}$ in E between$\tilde{f}$ and $\tilde{g}$. Then $po\tilde{F}$ is a path homotopy in B between f and g.

Theorem 2.3.2. *The fundamental group of S^1 is isomorphic to the additive group of integer.*

Proof: Let $p : E \to B$ be a covering map of (2.3.1), let $e_0 = 0$, and let $b_0 = p(e_0)$. Then $p^{-1}(b_0)$ is the set $\mathbb{Z}$ of integer. Since $\mathbb{R}$ is simply connected, the lifting correspondence

$$\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

is bijective. We show that ϕ is homomorphism, and the theorem is proved.Given $[f]$ and $[g]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective lifting to paths on $\mathbb{R}$ beginning at 0. Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$, then $\phi([f]) = n$ and $\phi([g]) = m$, by definition. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$$

on $\mathbb{R}$. Because $p(n + x) = p(x)$ for all $\mathbb{R}$, the path $\tilde{\tilde{g}}$ is a lifting of g, it begins at n. Then the product $\tilde{f} * \tilde{g}$ is defined, and it is the lifting of f*g that begins at 0. The end point of this path is $\tilde{\tilde{g}}(1) = n + m$. Then by definition,

$$\phi([f] * [g]) = n + m = \phi([f]) + \phi([g])$$

Chapter 3

The Seifert-Van Kampen Theorem

The Seifert-Van Kampen theorem of algebraic topology (named after Herbert Seifert and Egbert Van Kampen), sometimes just called Van Kampen's theorem gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known. By systematic use of this theorem one can compute the fundamental groups of a very large number of spaces.

3.1 Preliminary Part

In this part, we study a concept that plays a role for arbitrary groups similar to that played by the direct sum for abelian groups.It is called free product of groups.

Direct sum of abelian group

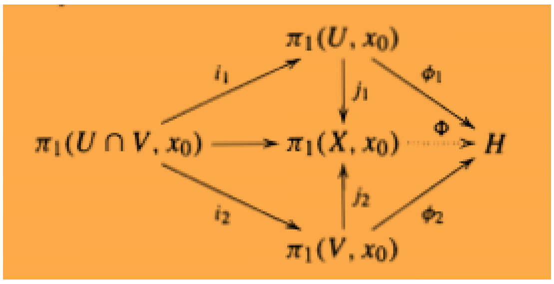
Suppose G is an abelian group, and $\{G_\alpha\}_{\alpha\in J}$ is an indexed family of subgroups of G. We say that the groups G_α generate G if every element x of G can be written as a finite sum of elements of the groups G_α.

Definition Let G be an abelian group and let $\{a_\alpha\}$ be an indexed family of elements of G; let G_α be the subgroup of G generated by a_α. If the groups G_α generate G, we also say that the elements a_α generate G. If each group G_α is infinite cyclic, and if G is the direct sum of the groups G_α, then G is said to be a free abelian group having the elements $\{a_\alpha\}$. as a basis.

Theorem 3.1.1. Seifert-Van Kampen Theorem *Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let*

$$\phi_1 : \pi_1(U, x_0) \longrightarrow H \quad \textit{and} \quad \phi_2 : \pi_1(V, x_0) \longrightarrow H$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated in the following diagram, each induced by inclusion.



If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$ This theorem says that if ϕ_1 and ϕ_2 are arbitrary homomorphisms that are "compatible on $U \cap V$ " then they induce a homomorphism of $\pi_1(X, x_0)$ into H.

Theorem 3.1.2. Seifert-Van Kampen Theorem *(classical version). Assume the hypotheses of the preceding theorem. Let*

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

be the homomorphism of the free product that extends the homomorphisms j_1 and j_2 induced by inclusion. Then j is surjective, and its kemel is the least normal *subgroup N of the free product that contains all elements represented by words of the form*

$$\left(i_1(g)^{-1}, i_2(g)\right)$$

for $g \in \pi_1(U \cap V, x_0)$ Said differently, the kemel of j is generated by all elements of the free product of the form $i_1(g)^{-1}i_2(g)$, and their conjugates.

Bibliography

[1] Barrett O'Neill, *Semi-Riemannian Geometry With Applications To Relativity, Academic Press* (1983).

[2] Duggal K. L. and Sahin B., *Differential Geometry of Lightlike Submanifolds, Birkhauser verlag AG, Berlin* (2010).

[3] B. Sahin, *Nonexistence of Warped Product Semi- Slant Submanifold of Kaehler Manifolds, Geom. Dedicata 117*(2006), 195-202.