Module-I
Ordinary Differential equations-I

Differential equation:
An Equation Containing an independent Variable dependent Variable and derivatives of one or mo[re] dependent Variables w.r.t to one or more Indepen[dent] Variables is called Differential equation.

eg: ① $\frac{dy}{dx} = 2xy$ ② $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \lambda \frac{d^2y}{dx^2}$

③ $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$

Ordinary differential equation: A differential eq which contains derivatives of one dependent Variable w.r.t to only one independent Variable is called ordinary D.E.

Partial differential equation: A differential eq Containing Partial derivatives of one dependent Variable w.r.t to two or more than two independe[nt] Variables is called P.D.E.

$\frac{d^2y}{dx^2} = 4y$ - ① $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^{2x}$ - ②

$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$ - ③ $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ - ④

$x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial y^2} = 0$ - ⑤ $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 0$ - ⑥

In this D.E from ① to ③ is ordinary D.E
& From ④ to ⑥ P.D.E

Order and degree of a D.E

Order of a D.E is the order of highest derivative which occours in it

equation ①, ③ are of order two

equation ② is of order 3.

Now

Degree of a D.E: The degree of a D.E is the Power of highest order derivative which occours in it.

The D.E from ① to ⑤ each of degree 1 & eq ⑥ is of degree 2

$$y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$y - x\frac{dy}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$$

$$y^2 + x^2\frac{d^2y}{dx^2} - 2xy\frac{dy}{dx} = 1+\left(\frac{dy}{dx}\right)^2$$

$$(x^2-1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2-1) = 0$$

Order is 1 & degree is 2

LINEAR DE:- A differential equation expressed in the form of Polynomial which contains derivatives of dependent variable y this derivative occour in first degree is said to be linear in y. and also the coefficient of various terms are either constant or function of independent variable.

A D.E which is not linear is said to be NON Linear D.E

The Linear D.E of order n is

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2 \cdots P_n$ & Q are either constant or functions of independent variable x.

$\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 9y = \cos x$ is L.D.E of order 2 & degree 1

$\left(\frac{d^3 y}{dx^3}\right)^2 - 6\left(\frac{d^2 y}{dx^2}\right)^3 - 4y = 0$ is non Linear (degree is 2)

$\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + y^2 = e^x$ is non Linear (degree of dependent variable is 2)

Solution of a Differential equation: It is the relation between the variables, constants which satisfies the given D.E

General solution: The solution which contain as many arbitary constants as the order of differential eq is called the general soln of the Differential equation

eg: $y = A\cos x + B\sin x$ is general soln of $\frac{d^2 y}{dx^2} + y = 0$ (order is 2 & there are 2 arbitary constant) but $y = A\cos x$ is not the general soln (only one arbitary constant)

(2) $y = e^x$ is the soln of $\frac{dy}{dx} = y$

Particular soln: Soln obtained by giving particular value to arbitary constant in General soln

$y = 2\cos x - 2\sin x$

LINEAR Differential Equation

A D.E of the form

$$\frac{dy}{dx} + Py = Q \quad \text{is called L.D.E}$$

where P & Q are functions of x only

To solve above D.E ① we multiply it by $e^{\int P dx}$ which is called the "integrating factor" of this equation.

Then we get $e^{\int P dx} \frac{dy}{dx} + P.y.e^{\int P dx} = Q.e^{\int P dx}$

$$\frac{d}{dx}\left[y e^{\int P dx}\right] = Q.e^{\int P dx}$$

Integrating both the sides

$$y.e^{\int P dx} = \int Q.e^{\int P dx}\, dx + C$$

This is required solution.

Working Rule

1. Suppose $\frac{dy}{dx} + Py = Q$
2. $I.F = e^{\int P dx}$
3. The solution is $y(I.F) = C + \int I.F.\, Q\, dx$

Remember: 1. We should make the coefficient of $\frac{dy}{dx}$ equal to 1 by dividing with the same if any

2. $e^{\log x} = x$, $e^{-\log x} = \frac{1}{x}$

3. If the L.D.E of the type $\frac{dx}{dy} + Px = Q$

Then $I.F = e^{\int P dy}$

Then sol^n is $x(I.F) = C + \int I.F\, dy$

① Solve $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$

Sol:→ Given D.E is $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cos x}{1+x^2}$

This is L.D.E

Here $P = \frac{2x}{1+x^2}$, $Q = \frac{\cos x}{1+x^2}$

$I.F \Rightarrow e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(x^2+1)} = x^2+1$

Then the solution is

$$y \times I.F = C + \int I.F \times Q \, dx$$

$$y(x^2+1) = C + \int \cancel{(x^2+1)} \frac{\cos x}{\cancel{1+x^2}} dx$$

$$y(x^2+1) = C + \int \cos x \, dx$$

$$y(x^2+1) = C + \sin x \Rightarrow \boxed{y = \frac{C+\sin x}{x^2+1}}$$

② Solve $(1+y^2)dx = (\tan^{-1}y - x)dy$

Sol:→ Given D.E is

$$(1+y^2)\,dx = (\tan^{-1}y - x)dy$$

Given D.E is not in the form $\frac{dy}{dx} + P.y = Q$ because which contain the terms y^2 & $\tan^{-1}y$ but can written as

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

which is the form of $\frac{dx}{dy} + P.x = Q$

Here $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

The Sol^n is $x \cdot I.F = c + \int Q \cdot I.F \, dy$

$$x e^{\tan^{-1} y} = c + \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy$$

Put $\tan^{-1} y = t \Rightarrow \frac{1}{1+y^2} dy = dt$

$$\therefore x e^{\tan^{-1} y} = c + \int e^t . t \, dt$$

$$\Rightarrow x e^{\tan^{-1} y} = c + t . e^t - e^t$$

$$\Rightarrow x e^{\tan^{-1} y} = c + e^t (t - 1)$$

$$x e^{\tan^{-1} y} = c + e^{\tan^{-1} y} (\tan^{-1} y - 1)$$

$$\boxed{x = c e^{-\tan^{-1} y} + (\tan^{-1} y - 1)}$$

(E) Solve $(1+x^2)\frac{dy}{dx}+2xy=\cos x$

Sol: Given D.E is $(1+x^2)\frac{dy}{dx}+2xy=\cos x$

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{\cos x}{1+x^2}$$

This is L.D.E

$$P=\frac{2x}{1+x^2},\quad Q=\frac{\cos x}{1+x^2}$$

$$I.F=e^{\int P\,dx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log(x^2+1)}=x^2+1$$

Then soln is

$$y\times I.F=C+\int I.F.\,Q\,dx$$

$$y(1+x^2)=C+\int\cancel{(1+x^2)}\frac{\cos x}{\cancel{(1+x^2)}}dx$$

$$y(1+x^2)=C+\sin x$$

$$\boxed{y=\frac{C+\sin x}{1+x^2}}$$

(E) Solve $\frac{dy}{dx}+\frac{y}{x}=x^2$

Sol: This is L.D.E, Here $P=\frac{1}{x}$, $Q=x^2$

$$I.F=e^{\int P\,dx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

Then the soln is

$$y\times I.F=C+\int Q\times I.F\,dx$$

$$y.x=C+\int x.x^2\,dx$$

$$yx=C+\int x^3\,dx\Rightarrow\boxed{yx+C+\frac{x^4}{4}}\quad\text{or}$$

$$y=-\frac{x^3}{4}-\frac{C}{x}$$

Bernoulli's Equation or D.E Reducible To linear form

A D.E of the form

$$\frac{dy}{dx} + Py = Q.y^n$$

is said to be BERNOULLI'S EQUATION where P & Q are function of x alone (or constant) do not contain y. Given D.E is

$$\frac{dy}{dx} + P.y = Qy^n \quad - (1)$$

Dividing both sides by y^n we get

$$\frac{1}{y^n}\frac{dy}{dx} + P.\frac{1}{y^{n-1}} = Q \quad - (2)$$

Put $\frac{1}{y^{n-1}} = v \Rightarrow -(n-1)\frac{1}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$ $\quad - (3)$

From (2) & (3)

$$-\frac{1}{(n-1)}\frac{dv}{dx} + P.v = Q$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P.v = Q(1-n)$$

This is L.D.E in which v is dependent variable.

Hence it can be solved.

Remark: If $n=1$ then the variables can be separated.

1. Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Sol: The given D.E is

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

÷ Both sides by $\cos^2 y$

$$\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \quad -(1)$$

Put $\tan y = v \Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dv}{dx} \quad -(2)$

From eq (1) & (2)

$$\frac{dv}{dx} + 2x \cdot v = x^3 \quad -(3)$$

This is L.D.E in v

$$I.F = e^{\int 2x\,dx} = e^{x^2}$$

Then the sol^n is

$$v \cdot e^{x^2} = c + \int e^{x^2} \cdot x^3\,dx$$

Put $x^2 = t$ so that $x\,dx = \frac{dt}{2}$

$$\therefore v e^{x^2} = c + \int \frac{t \cdot e^t}{2}\,dt$$

$$v \cdot e^{x^2} = c + \frac{1}{2} e^t (t-1)$$

$$\Rightarrow \boxed{\tan y\, e^{x^2} = c + \frac{1}{2} e^{x^2}(x^2-1)}$$

(2) Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$.

Sol: Given D.E is $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ — (1)

$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$ — (2)

Put $\sin y = v \Rightarrow \cos y \frac{dy}{dx} = \frac{dv}{dx}$ — (3)

From (2) & (3)

$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^x$ — (4)

This is L.D.E

Here $P = \frac{-1}{1+x}$, $Q = (1+x)e^x$

$\therefore$ I.F $= e^{\int -\frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$

$\therefore$ Soln of eq (4) is

$v \cdot I.F = c + \int I.F \, Q \, dx$

$\Rightarrow \frac{v}{1+x} = c + \int \frac{1}{1+x}(1+x)e^x dx$

$\Rightarrow \frac{\sin y}{1+x} = c + e^x$

$\Rightarrow \sin y = (c+e^x)(1+x)$

Exact Differential Equations

Definition: The D.E which can be obtained directly by differentiation of their primitive (i.e solution) without eliminating or transformation are called the exact Differential equations.

For example: The equation

$x\,dx + y\,dy = 0$ is an Exact D.E, Since it can be obtained directly by differentiating its primitive

$$x^2 + y^2 = C$$

* Primitive (Antiderivative): Function $\phi(x)$ is called primitive or antiderivative of function $f(x)$ is $\phi'(x) = f(x)$

$\frac{x^4}{4}$ is primitive of x^3 $\because \frac{d}{dx}(x^4) = x^3$

Standard form of Exact Differential Equation

The given D.E $M\,dx + N\,dy = 0$ ⓵ is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ where M & N are fun^n of x & y only.

Every D.E of the form $F_1(x)\,dx + F_2(x)\,dy = 0$ is always exact too, since they satisfy condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then the sol^n of ⓵ is $\int\limits_{y=\text{cons}} M\,dx + \int (N,\text{ nt containing } x)\,dy = C$

(1) Solve $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol: $Mdx + Ndy = 0$ ∴ $M = 1+e^{x/y}$, $N = e^{x/y}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\left[x\left(\frac{-1}{y^2}\right)\right] = -\frac{x}{y^2}e^{x/y}$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\frac{1}{y}$$

$$= \frac{1}{y}e^{x/y}\left(-1+1-\frac{x}{y}\right) = \frac{1}{y}e^{x/y}\left(-\frac{x}{y}\right)$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{1}{y}e^{x/y}\left(-\frac{x}{y}\right) = \frac{-xe^{x/y}}{y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence given eq is Exact D.E

Then soln of (1) is $\int M dx + \int N dy = c$

$$\int (1+e^{x/y})dx + \int 0\,dy = c$$

$$\boxed{x + \frac{e^{x/y}}{1/y} = C}$$

(2) Solve $(1+4xy+2y^2)dx + (1+4xy+2x^2)dy = 0$

Sol: Given D.E is

$(1+4xy+2y^2)dx + (1+4xy+2x^2)dy = 0$ — (1)

Comparing (1) with $M.dx + N.dy = 0$

$M = 1+4xy+2y^2$ $\quad N = 1+4xy+2x^2$

$$\frac{\partial M}{\partial x} = 4x + 4y \qquad \frac{\partial N}{\partial x} = 4y + 4x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given D.E is exact differential equation

Hence its sol^n is

$$\int_{y=\text{con}} M \, dx + \int (N, \text{ not containing } x) \, dy = C$$

$$\int_{y=\text{const.}} (1 + 4xy + 2y^2) \, dx + \int 1 \, . dy = C$$

$$\Rightarrow x + \frac{4x^2}{2} y + 2y^2 x + y = C$$

$$x + 2x^2 y + 2xy^2 + y = C$$

$$x(1 + 2yx) + y(2xy + 1) = C$$

$$\boxed{(x + y)(1 + 2xy) = C}$$

Differential eq reducible to Exact form
(Non Exact Differential Equation)

Equation of the form $Mdx + Ndy = 0$ (which is not exact) can be made exact by multiplying the eq by suitable factor which is function of x & y (Known as Integrating factor)

$$(I.F \times \text{Non Exact D.E}) = \text{Exact D.E}$$

Method I : Integrating factor by Inspection

(1) $y\,dx + x\,dy = d(x y)$

(2) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(3) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

(4) $\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left\{\tan^{-1}\left(\dfrac{y}{x}\right)\right\}$

(5) $\dfrac{x\,dy - y\,dx}{xy} = d\log\left(\dfrac{y}{x}\right)$

(6) $\dfrac{x\,dy - y\,dx}{x^2-y^2} = d\left\{\dfrac{1}{2}\log\left(\dfrac{x+y}{x-y}\right)\right\}$

(7) $\dfrac{x\,dy + y\,dy}{x^2+y^2} = d\left\{\dfrac{1}{2}\log\left(x^2+y^2\right)\right\}$

(8) $\dfrac{y\,dx - x\,dy}{xy} = d\log\left(\dfrac{x}{y}\right)$

Method II: If eq of the form $f_1(x,y).y\,dx + f_2(x,y)x\,dy = 0$ then I.F $= \frac{1}{Mx - Ny}$ Provided $Mx - Ny \neq 0$

Method III: Given eq $M\,dx + N\,dy = 0$ is not exact but homogeneous. If $Mx + Ny \neq 0$ then I.F $= \frac{1}{Mx+Ny}$

IV: If given eq $M\,dx + N\,dy = 0$ is not exact

(i) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a funⁿ of x alone say $f(x)$

$$I.F = e^{\int f(x)\,dx}$$

(ii) when $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$ or constant then

$$I.F = e^{\int f(y)\,dy}$$

① Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$

Sol: $(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0$

$M = xy^2 + 2x^2y^3 \qquad N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2yx + 6x^2y^2 \qquad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

It is in the form of $f_1(x,y)y\,dx + f_2(x,y)x\,dy = 0$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{(xy^2+2x^2y^3)x - (x^2y - x^3y^2)y}$$

$$= \frac{1}{x^2y^2+2x^3y^3-x^2y^2+x^3y^3} = \frac{1}{3x^3y^3}$$

Now multiply eq ① by $\frac{1}{3x^3y^3}$

$$\frac{1}{3x^3y^3}[xy^2+2x^2y^3]dx + \frac{1}{3x^3y^3}[x^2y - x^3y^2]dy = 0$$

$$= \left[\frac{1}{3x^2y} + \frac{2}{3x}\right]dx + \left[\frac{1}{3x^2y} - \frac{1}{3y}\right]dy = 0 \quad -②$$

$$M = \frac{1}{3x^2y} + \frac{2}{3x} \Rightarrow \frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2}$$

$$N = \frac{1}{3xy^2} - \frac{1}{3y} \Rightarrow \frac{\partial N}{\partial x} = -\frac{1}{3x^2y^2}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [eq ② is Exact]

Hence Soln of ② is ~~Exact~~ $\int M\,dx + \int N\,dy = c$

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy$$

$$\frac{1}{3y}\int x^{-2}dx + \frac{2}{3}\int x^{-1}dx - \frac{1}{3}\int y^{-1}dy$$

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c \Rightarrow \frac{2\log x - \log y}{3} = \frac{1}{3yx} + c$$

$$\Rightarrow \boxed{2\log x - \log y = \frac{1}{xy}}$$

(2) Solve $x\,dy - y\,dx + 2x^3\,dx = 0$

Sol:

$x\,dy = y\,dx + 2x^3\,dx \Rightarrow x\,dy = dx(y + 2x^3)$

$\Rightarrow 2x^3$

(3) $x\,dy - y\,dx = (x^2+y^2)\,dx$ — (1)

Sol:

~~$x\,dy = (x^2+y^2)\,dx + y\,dx$~~

~~$x\,dy = (x^2+y^2+y)\,dx$~~

$x\,dy - y\,dx - (x^2+y^2)\,dx = 0$

$x\,dy - (x^2+y^2+y)\,dx = 0$

$M = -(x^2+y^2+y)$, $N = x$

$\frac{\partial M}{\partial y} = -1-2y$, $\frac{\partial N}{\partial x} = 1$

$\Rightarrow \left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$ eq (1) is not exact.

$\therefore$ $x\,dy - y\,dx$ given $\frac{1}{x^2+y^2}$ is an I.F

$$\frac{x\,dy - y\,dx}{x^2+y^2} = d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

Multiply ① by $\frac{1}{x^2+y^2}$

$$\frac{x\,dy - y\,dx}{x^2+y^2} = dx \Rightarrow d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = dx$$

(By Rule on Integrating)

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) = x + C}$$

Ordinary Differential Equations of First Order & Higher Degrees (Order 1 & degree greater than 1)

The standard form of O.D.E of first order & higher degree is

$$p^n + a_1(x,y)p^{n-1} + a_2(x,y)p^{n-2} + \dots + a_{n-1}(x,y)p + a_n(x,y) = 0$$

where $a_1, a_2, \dots, a_n$ are function of x & y & $p = \frac{dy}{dx}$

Such equations can be divided into following categories:

1. Equations solvable for p.
2. Equations solvable for y.
3. Equations solvable for x.

Method 1: Equation solvable for p

If D.E solvable for p, then we have Polynomial Equation in p s.t

$$p^n + a_1(x,y)p^{n-1} + a_2(x,y)p^{n-2} + \dots + a_n(x,y) = 0$$

Now we can find n factors

$$(p-f_1)(p-f_2)\dots(p-f_n) = 0$$

$\therefore$ $p - f_1 = 0$, $p - f_2 = 0$, $p - f_n = 0$

The solution of above factors are

$$F_1(x,y,c_1) = 0, \quad F_2(x,y,c_2) = 0$$

where $c_1, c_2, \dots, c_n$ are arbitrary Constant

$\because$ the order of D.E is 1 then

$$c_1 = c_2 = c_3 = \dots = c_n = c$$

The general solution is

$$\boxed{F_1(x,y,c)\cdot F_2(x,y,c)\cdots F_n(x,y,c_n) = 0}$$

(1) Solve $p^2 - 7p + 12 = 0$

Sol: $p^2 - 3p - 4p + 12 = 0$

$p(p-3) - 4(p-3) = 0$

$(p-3)(p-4) = 0$

$p = 4$ & $p = 3$

$\frac{dy}{dx} = 4 \quad \frac{dy}{dx} = 3$

Integrating both the sides $y = 4x + c_1$ & $y = 3x + c_2$

$\boxed{(y - 4x - c_1)(y - 3x - c) = 0}$

where $c_1 = c_2 = c$

(2) Solve $x^2 = p^2(a^2 - x^2)$

Sol: The given D.E is $x^2 = p^2(a^2 - x^2)$

$p^2 = \frac{x^2}{a^2 - x^2} \Rightarrow p = \pm \frac{x}{\sqrt{a^2 - x^2}}$

$\Rightarrow \frac{dy}{dx} = \pm \frac{x}{\sqrt{a^2 - x^2}}$

Taking +ve sign $dy = \frac{x}{\sqrt{a^2 - x^2}} dx$

Integrating both the sides we get

$y = -\sqrt{a^2 - x^2} + c_1$

$\Rightarrow y + \sqrt{a^2 - x^2} - c_1 = 0$

Taking -ve sign

$dy = -\frac{x}{\sqrt{a^2 - x^2}} dx$

Integrating both the sides

$y = \sqrt{a^2 - x^2} + c_2$

$y - \sqrt{a^2 - x^2} - c_2 = 0$

Required soln is $(y + \sqrt{a^2 - x^2} - c)(y - \sqrt{a^2 - x^2} - c)$

E.

(3) Solve $x^2p^3+y(1+x^2y)p^2+y^3p=0$

Sol: $P(x^2p^2+py+x^2py^2+y^3)=0$

$P[(x^2p^2+x^2y^2p)+(yp+y^3)]=0$

$P(x^2p+y)(p+y^2)=0$ $\quad p^2x^2(p+y^2)+py(p+y^2)$

$(p^2x^2+py)(p+y^2)$

$\Rightarrow P(x^2p+y)(p+y^2)=0$

$\therefore$ Its component equations are

$p=0,\ x^2p+y=0,\ p+y^2=0$

$\frac{dy}{dx}=0,\ x^2\frac{dy}{dx}+y=0,\ \frac{dy}{dx}+y^2=0$

Seprating the Variable

$\frac{dy}{dx}=0,\ \frac{dy}{y}+\frac{dx}{x^2}=0,\ \frac{dy}{y^2}+dx=0$

Integrating these equations, Corresponding Soln are

$y=c,\ \log y-\frac{1}{x}=c,\ \frac{-1}{y}+x=c$

$y-c=0,\ x\log y-1-cx=0,\ -1+xy-cy=0$

$\therefore$ Complete soln is

$(y-c)(x\log y-1-cx)(xy-1-cy)=0$

Solvable for y

Let the given D.E be solvable for y, so that it can be put into the form $y = f(x, p)$ - (1)

Differentiating (1) w.r.t x we get

$$\frac{dy}{dx} = f\left(x, p, \frac{dp}{dx}\right) \Rightarrow p = f\left(x, p, \frac{dp}{dx}\right) \quad (2)$$

This is differential eq in the variable x & p

Let its solⁿ be $\phi(x, p, c) = 0$ - (3)

where c is an arbitary constant of integration. The elimination of p between eq (1) & (3) gives us the required solⁿ.

Note: If elimination of p b/w eq (1) & (3) is not possible, then we solve the eq (1) & (3) for x & y in terms of p. Then the two parametric eq

$$x = \phi_1(p, c)$$

$$y = \phi_2(p, c)$$ where p being parameter.

Solvable for y

(1) Solve $y = -px + x^4p^2$

Sol: Given D.E is $y = -px + x^4p^2$ — (1)

Differentiating w.r.t x we get

$$\frac{dy}{dx} = -p\cdot 1 - x\cdot\frac{dp}{dx} + x^4\cdot 2p\frac{dp}{dx} + p^2\cdot 4x^3$$

$$p = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4x^3p^2$$

$$p + p - 4x^3p^2 = \frac{dp}{dx}(2px^4 - x)$$

$$2p - 4x^3p^2 = \frac{dp}{dx}(2px^4 - x)$$

$$2p(1 - 2px^3) = -x(1 - 2px^3)\frac{dp}{dx}$$

$$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Neglecting the second factor which does not contain the derivative of p, we have

$$2p + x\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{p} + 2\frac{dx}{x} = 0$$

Integrating both the sides

$$\log p + 2\log x = \log c$$

$$\log(p\cdot x^2) = \log c$$

$$p\cdot x^2 = c \Rightarrow \boxed{p = \frac{c}{x^2}}$$

Put in eq (1) we get

$$y = -\frac{c}{x^2}\cdot x + x^4\cdot\frac{c^2}{x^4} \Rightarrow \boxed{xy = -c + c^2x}$$

This is required soln.

Equations Solvable for x.

If given D.E is solvable for x, let it be put in the form

$$x = f(y, p) \quad -(1)$$

Differentiation w.r.t y gives an eq of the form

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

$$\text{i.e } \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \quad \left[\because p = \frac{dy}{dx}\right] \quad -(2)$$

which is an eq in two variables y & p. It may be possible to obtain its soln say

$$F(y, p, c) = 0 \quad -(3)$$

where c is an arbitary constant of integration

The elimination of p between (1) & (3) gives the required soln.

Solvable for x

1 Solve $y = 2px + y^2p^3$

Sol Given D.E is $y = 2px + y^2p^3$ — (1)

$$2x = \frac{y}{p} - y^2p^2 \quad \text{— (2)}$$

Differentiating w.r.t y

$$2\frac{dx}{dy} = \left[\frac{p\cdot 1 - y\frac{dp}{dy}}{p^2}\right] - \left[y^2 2p\frac{dp}{dy} + p^2\cdot 2y\right]$$

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2y^2p\frac{dp}{dy} - 2p^2y$$

$$\frac{1}{p} + 2p^2y = -\frac{dp}{dy}\left(\frac{y}{p^2} + 2py^2\right)$$

$$\left(\frac{1+2p^3y}{p}\right) = -\frac{dp}{dy}\left(\frac{y+2p^3y^2}{p^2}\right)$$

$$\frac{1+2p^3y}{p} = \frac{-y}{p^2}\frac{dp}{dy}(1+2yp^3)$$

$$1 = \frac{-y}{p}\frac{dp}{dy}$$

$$\frac{dy}{y} = -\frac{dp}{p} \Rightarrow \frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating both the sides

$$\log y + \log p = \log c$$

$$\log(p\cdot y) = \log c$$

$$p\cdot y = c$$

$$p = \frac{c}{y}$$

Put in eq (1) $y = 2\left(\frac{c}{y}\right)x + y^2\left(\frac{c}{y}\right)^3$

(2) Solve $p = \tan\left[x - \frac{p}{1+p^2}\right]$

Sol: Given D.E is $p = \tan\left[x - \frac{p}{1+p^2}\right]$

$$x = \tan^{-1}p + \frac{p}{1+p^2} \quad -(1)$$

Differentiating w.r.t y we get

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - 2p \cdot p \cdot \frac{dp}{dy}}{(1+p^2)^2}$$

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1}{1+p^2}\frac{dp}{dy} - \frac{2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\frac{1}{p} = \left[\frac{2}{1+p^2} - \frac{2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy} \Rightarrow dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating both the sides

$$y = -\frac{1}{1+p^2} + c \quad -(2)$$

$1+p^2 = t$

$2p\,dp = dt$

$\int \frac{dt}{t^2} = \int t^{-2}dt$

$\frac{t^{-2+1}}{-2+1} = \frac{t^{-1}}{-1}$

$= \frac{-1}{t}$

The relations (1) & (2) together constitute the required soln.

LINEAR D.E OF HIGHER ORDER WITH CONSTANT COEFFICIENT

The standard form of L.D.E with n^{th} order is

$$\frac{d^n y}{dx^n} + a_1\left(\frac{d^{n-1}y}{dx^{n-1}}\right) + a_2\left(\frac{d^{n-2}y}{dx^{n-2}}\right) + \dots + a_{n-1}\left(\frac{dy}{dx}\right) + a_n y = Q \quad -(1)$$

where $a_1, a_2, a_3, \dots, a_n$ are constant & Q is the funⁿ of x only is called Linear differential eq of higher order with constant coefficient.

If we denote $D \equiv \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 \equiv \frac{d^3}{dx^3}$

Then from eq (1) we get

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n]y = Q \quad -(2)$$

$$f(D)y = Q \quad -(3)$$

where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n$

The complete solⁿ of eq (2) is

$$\boxed{y = C.F + P.I}$$ where CF = Complementary function

P.I = Particular Integral

Remark: If $Q = 0$ then we do not find out the P.I that in case the complete solⁿ is

$$y = C.F$$

Auxiliary Eq or Subsidiary eq or Characteristic Equation.

Taking $D \equiv m$ in eq (3) we get $\quad [\because Q = 0]$

$$f(m) = 0$$

$$\Rightarrow m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

The eq is called A.E

WORKING RULE For Finding C.F [Q = 0]

Let the given D.E is $[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n]y = 0$
whose A.E is

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

Case I: When all the roots of A.E are real & distinct

Say $m = m_1, m_2, m_3, \dots, m_n$

$$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

Case II: When some of the roots of A.E are equal

Say $m = m_1 = m_2 = m_3$

$$C.F = (C_1 + x C_2 + x^2 C_3) e^{mx}$$

Case III: When A.E has Imaginary roots

Say $m = \alpha + i\beta$

Rules for finding P.I (particular Integral)

If $g=0$ then P.I $=0$

(1) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ when $f(a) \neq 0$

(2) $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$

(3) $\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ provided $f(a)=0$

(4) $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$ Note $-a^2 \neq (-a)^2$ when $f(-a^2) \neq 0$

(5) $\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$ when $f(-a^2)=0$

(6) $\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$ when $f(-a^2)=0$

(7) $\frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

where $[f(D)]^{-1}$ is expanded is ascending powers of D by binomial Theorem.

Some Important Expansion

1. $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$
2. $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$
3. $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$
4. $(1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$
5. $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$

(11) $\frac{1}{f(D^2)} \sin ax = x \frac{1}{f'(-a^2)} \sin ax$ when $f(-a^2)=0$

(12) $\frac{1}{f(D^2)} \cos ax = x \frac{1}{f'(-a^2)} \cos ax$ when $f(-a^2)=0$

(13) $\frac{1}{f(D)} x . V = x \frac{1}{f(D)} . V - \frac{f'(D)}{[f(D)]^2} . V$

Special Case

$\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q \, dx \quad , \quad \frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{\alpha x} . Q \, dx$

① Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

Sol:- $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$

$m^3 - 3m^2 + 4m - 2 = 0$

$(m^2 - 2m + 2)(m - 1) = 0 \Rightarrow m = 1,\ 1 \pm i$

$C.F = c_1 e^x + (c_2 \cos x + c_3 \sin x)e^x$

$P.I = \frac{1}{f(D)} \cdot g = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x) = \frac{1}{f(D)} \cdot g$

$= \frac{1}{D^3 - 3D^2 + 4D - 2} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$

I part: $1 - 3 + 4 - 2 = -2 + 2 = 0 \Rightarrow f(D) = f(a) = 0$

$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ provided $f'(a) \neq 0$

$= x \frac{1}{3D^2 - 6D + 4} e^x = x \frac{1}{3 - 6 + 4} e^x = xe^x$

II part: $\frac{1}{D^3 - 3D^2 + 4D - 2} \cos x = \frac{1}{D^2 \cdot D - 3D^2 + 4D - 2} \cos x \quad (D^2 = -1^2)$

$= \frac{1}{-1^2 \cdot D + 3 + 4D - 2} \cos x = \frac{1}{-D + 3 + 4D - 2} \cos x$

$= \frac{1}{3 + 3D - 2} \cos x = \frac{1}{3D + 1} \cos x$

$= \frac{3D - 1}{(3D + 1)(3D - 1)} \cos x = \frac{3D - 1}{9D^2 - 1} \cos x = \frac{3D - 1}{-9 - 1} \cos x = \frac{-1}{10}(3D - 1)\cos x$

$\frac{-1}{10}[3D(\cos x)-\cos x] = \frac{-1}{10}[3\sin x-\cos x]$

(2) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$

Sol:) $(D^2-2D+1)y = xe^x\sin x$

$m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

$C.F = (c_1+c_2x)e^x$

$P.I = \frac{1}{f(D)} \cdot u = \frac{1}{(D-1)^2} e^x(x\sin x)$

$\because \frac{1}{f(D)} e^{ax}.v = e^{ax}\frac{1}{f(D+a)} \cdot v \Rightarrow e^x \frac{1}{(D+1-1)^2} x\sin x$

$= e^x \frac{1}{D^2} x\sin x$

$= e^x \frac{1}{D}\int x\sin x\,dx \Rightarrow e^x\int[-x\cos x+\sin x]\,dx$

$= e^x(-x\sin x \cos x - \cos x)$

$= e^x[-x\sin x - 2\cos x]$

$y = C.F + P.I$

$y = (c_1+c_2x)e^x + e^x[-x\sin x - 2\cos x]$

(3) Solve $(D^2+1)y = \sec x$

Sol:- $m^2+1=0 \Rightarrow m=\pm i$

$C.F = c_1 \cos x + c_2 \sin x$

$$P.I = \frac{1}{f(D)} \cdot Q = \frac{1}{(D-i)(D+i)} \sec x = \frac{1}{2i}\left[\frac{1}{(D-i)}\sec x - \frac{1}{(D+i)}\sec x\right]$$

$$= \frac{1}{2i}\left[\frac{-D+i+D+i}{(D-i)(D+i)}\right]\sec x = \frac{1}{2i}\left[\frac{2i}{(D-i)(D+i)}\right]\sec x$$

$$\because \frac{1}{D-\alpha} Q = e^{\alpha x}\int e^{-\alpha x} Q\,dx \quad \& \quad \frac{1}{D+\alpha} Q = e^{-\alpha x}\int e^{\alpha x} Q\,dx$$

$$\Rightarrow \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec x = \frac{1}{2i}\left[\frac{1}{D-i}\sec x - \frac{1}{D+i}\sec x\right]$$

$$\frac{1}{2i}\left[e^{ix}\int e^{-ix}\sec x\,dx - e^{-ix}\int e^{ix}\sec x\,dx\right]$$

$$\because e^{-i\alpha x} = \cos\alpha x - i\sin\alpha x$$

$$e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$$

$$\frac{1}{2i}\left[e^{ix}\int\left(\frac{\cos x - i\sin x}{\cos x}\right)dx - e^{-ix}\int\left(\frac{\cos x + i\sin x}{\cos x}\right)dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int(dx - i\tan x\,dx) - e^{-ix}\int(dx + i\tan x\,dx)\right]$$

$$\left[\because \int \tan x\,dx = -\log\cos x\right]$$

$$\Rightarrow \frac{1}{2i}\left[e^{ix}(x + i\log\cos x) - e^{-ix}(x - i\log\cos x)\right]$$

$$= \frac{1}{2i}\left[xe^{ix} + e^{ix}i\log\cos x - xe^{-ix} + e^{-ix}i\log\cos x\right]$$

$$\frac{1}{2i}\left[x\left(e^{ix}-e^{-ix}\right)+i\left(e^{ix}+e^{-ix}\right)\log\cos x\right]$$

$\because \sin x = \frac{e^{ix}-e^{-ix}}{2i}$ & $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$$\Rightarrow x\left(\frac{e^{ix}-e^{-ix}}{2i}\right)+i\left(\frac{e^{ix}+e^{-ix}}{2i}\right)\log\cos x$$

$$\Rightarrow x\left(\frac{e^{ix}-e^{-ix}}{2i}\right)+\left(\frac{e^{ix}+e^{-ix}}{2}\right)\log\cos x$$

$$= x\sin x+\cos\log(\cos x)$$

$$y = C.F + P.I$$

$$y = c_1\cos x + c_2\sin x + x\sin x + \cos\log(\cos x)$$

(4) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\sin x$

Sol:

$(D^2 - 2D + 1)y = x\sin x$

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 \Rightarrow m = 1, 1$

$C.F = (c_1 + c_2 x)e^x$

$P.I = \frac{1}{f(D)} \cdot y = \frac{1}{(D-1)^2} x\sin x$

$\Rightarrow \frac{1}{(D-1)^2} x\sin x = x \frac{1}{(D-1)^2}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$

$\because \frac{1}{f(D)} x.V = x\frac{1}{f(D)} \cdot V - \frac{f'(D)}{[f(D)]^2} \cdot V$

$\Rightarrow x\frac{1}{(D^2+1-2D)}\sin x - \frac{(2D-2)}{(D^2+1-2D)^2}\sin x$

$\Rightarrow \frac{x}{-1^2+1-2D}\sin x - \frac{2(D-1)}{(-1^2+1-2D)^2}\sin x$ $\quad \because \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$

when $f(-a^2) \neq 0$

$\Rightarrow \frac{x}{-2D}\sin x - \frac{2(D-1)}{4D^2}\sin x$

$\Rightarrow -\frac{x}{2}\int \sin x\,dx - \frac{2D}{4D^2}\sin x + \frac{2}{4D^2}\sin x$

$\Rightarrow -\frac{x}{2}(-\cos x) - \frac{1}{2}\int \sin x\,dx + \frac{1}{2}\frac{1}{D}\int \sin x\,dx$

$\Rightarrow \frac{x}{2}\cos x + \frac{\cos x}{2} - \frac{1}{2}\int(-\cos x)dx$

$\frac{x}{2}\cos x + \frac{\cos x}{2} - \frac{\sin x}{2} \Rightarrow \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x) = P.I$

Complete solution is

$$y = C.F + P.I$$

$$y = (C_1 + C_2 x)e^{x} + \frac{x}{2}\cos x + \frac{1}{2}[\cos x - x\sin x]$$

Solve $(D^3 + 3D^2 + 2D)y = x^2$

Sol:- $m^3 + 3m^2 + 2m = 0$

$m(m^2 + 3m + 2) = 0 \Rightarrow m^2 + 3m + 2 = 0,\ m = 0$

$m(m+2) + 1(m+2) = 0 \Rightarrow (m+2)(m+1) = 0$

$\Rightarrow m = 0, -2, -1$

$$\boxed{C.F = C_1 + C_2 e^{-x} + C_3 e^{-2x}}$$

$$P.I = \frac{1}{f(D)} \cdot Q = \frac{1}{D^3 + 3D^2 + 2D} \cdot x^2 = \frac{1}{2D\left[\frac{D^2}{2} + \frac{3D}{2} + 1\right]} \cdot x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3D + D^2}{2}\right)\right]^{-1} \cdot x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3D + D^2}{2}\right) + \left(\frac{3D + D^2}{2}\right)^2 - \cdots\right]x^2$$

$\therefore (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

$$= \frac{1}{2D}\left[1 - \left(\frac{3D + D^2}{2}\right) + \left(\frac{9D^2 + D^4 + 6D^3}{4}\right)\right]x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{6D + 2D^2 + 9D^2 + D^4 + 6D^3}{4}\right)\right]x^2$$

$$= \frac{1}{2D}\left[1 - \frac{6D}{4} - \frac{2D^2}{4} + \frac{9D^2}{4} + \frac{D^4}{4} + \frac{6D^3}{4}\right]x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3D}{2} + \frac{7D^2}{4} + \frac{D^4}{4} + \frac{3D^3}{2}\right]x^2$$

$$\frac{1}{2D}\left[x^2-\frac{3}{2}D(x^2)+\frac{7}{4}D^2(x^2)\right]$$

$$=\frac{1}{2D}\left[x^2-\frac{3}{2}(2x)+\frac{7}{4}(2)\right]$$

$$=\frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]=\frac{1}{2}\int\left(x^2-3x+\frac{7}{2}\right)dx$$

$$=\frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7}{2}x\right]$$

Solve the differential eq $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$

Sol:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

$$(D^2 + 6D + 9)y = 5e^{3x}$$

Auxiliary Equation $m^2 + 6m + 9 = 0$

$$m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$(m+3)(m+3) = 0$$

$$m = -3, -3$$

$$C.F = (C_1 + C_2 x)e^{-3x}$$

Now $P.I = \frac{1}{f(D)} g = \frac{1}{D^2 + 6D + 9} 5e^{3x}$

$$= 5 \frac{1}{(3)^2 + 6(3) + 9} e^{3x} = \frac{5}{36} e^{3x}$$

$\because \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ when $f(a) \neq 0$

Complete soln: C.F + P.I

$$(C_1 + C_2 x)e^{-3x} + \frac{5}{36} e^{3x}$$

Homogeneous Linear differential equations

Or

Cauchy's Homogeneous Linear differential equations

Homogeneous linear differential equations (H.L.D.E)

A linear D.E of the type

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

where $a_1, a_2, a_3, \dots, a_n$ are constant & Q is function of x constant is called Homogeneous linear D.E or Cauchy's linear equation.

It is also called differential equation reducible to linear equation with constant coefficients

WORKING Rule

(1) Put $x = e^z$ i.e $z = \log x$

(2) Taking $x \frac{d}{dx} = D$, $x^2 \frac{d^2}{dx^2} = D(D-1)$, $x^3 \frac{d^3}{dx^3} = D(D-1)(D-2)$

(3) Then simplify it by solving method of L.D.E with constant coefficients

(4) We get solution of D.E in y & z where z is an Independent Variable

(5) At last substitute back $\log x$ for z to get general solution

① Solve $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10(1+x^{-1})$

Sol:

Put $x=e^z \Rightarrow z=\log x \Rightarrow \frac{dz}{dx}=\frac{1}{x}$

$\therefore$ $x\frac{d}{dx}=D$, $x^2\frac{d^2}{dx^2}=D(D-1)$, $x^3\frac{d^3}{dx^3}=D(D-1)(D-2)$

Then eq ① become

$[D(D-1)(D-2)+2D(D-1)+2]y=10(1+e^{-z})$

$[D(D^2-2D-D+2)+2D^2-2D+2]y=10(1+e^{-z})$

$[D(D^2-3D+2)+2D^2-2D+2]y=10(1+e^{-z})$

$\Rightarrow [D^3-3D^2+2D+2D^2-2D+2]y=10(1+e^{-z})$

$\Rightarrow [D^3-D^2+2]y=10(1+e^{-z})$

$\Rightarrow m^3-m^2+2=0$

$m^3-2m^2+m^2-2m+2m+2=0$

$m^2(m+1)-2m(m+1)+2(m+1)=0$

$(m+1)(m^2-2m+2)=0$

$m=-1$, $m^2-2m+2=0 \Rightarrow m=\frac{2\pm 2i}{2}=1\pm i$

$m=-1,\ 1\pm i$

$C.F=c_1e^{-z}+e^z(c_2\cos z+c_3\sin z)$

$=c_1x^{-1}+x[c_2\cos(\log x)+c_3\sin(\log x)]$

$P.I=\frac{1}{f(D)}\cdot g=\frac{1}{D^3-D^2+2}\cdot 10(1+e^{-z})$

$$\Rightarrow 10\frac{1}{D^3-D^2+2} + 10\frac{1}{D^3-D^2+2}e^{-z}$$

Using $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ when $f(a) \neq 0$

$$= 10\left[\frac{1}{D^3-D^2+2}e^{0z}\right] + 10\left[\frac{1}{D^3-D^2+2}e^{-z}\right]$$

$$= 10\left[\frac{1}{0-0+2}\right] + 10\left[\frac{1}{(D+1)(D^2-2D+2)}e^{-z}\right]$$

$$= 10\left[\frac{1}{2}\right] + 10\frac{1}{D+1}\left[\frac{1}{D^2-2D+2}e^{-z}\right]$$

$$= 5 + \frac{10}{D+1}\left[\frac{1}{1+2+2}e^{-z}\right] = 5 + \frac{10}{D+1}\left[\frac{1}{5}e^{-z}\right]$$

$$= 5 + \frac{10}{5}\left[\frac{e^{-z}}{D+1}\right] \Rightarrow 5 + \frac{2(e^{-z})}{D+1-1} \Rightarrow 5 + \frac{2}{D}e^{-z}$$

$$\Rightarrow 5 + 2\int e^{-z}dz \Rightarrow 5 + 2e^{-z}.z$$

$$P.I = 5 + 2x^{-1}\log x$$

$$y = C.F + P.I = c_1x^{-1} + x\left[c_2\cos(\log x) + c_3\sin(\log x)\right]$$

$$= 5 + 2x^{-1}\log x$$

(Q) Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^2+2\log x$

Sol:-

$$D(D-1)-2D-4)y=x^2+2\log x$$

$$(D^2-D-2D-4)y=x^2+2\log x$$

$$(D^2-3D-4)y=x^2+2\log x$$

$$\Rightarrow (D^2-3D-4)y=e^{2z}+2z$$ (L.D.E with constant coefficient)

$$m^2-3m-4=0 \Rightarrow m=-1,4$$

$$C.F=C_1e^{-z}+C_2e^{4z}=C_1x^{-1}+C_2x^4$$

$$P.I=\frac{1}{D^2-3D-4}(e^{2z}+2z)=\frac{1}{D^2-3D-4}e^{2z}+2\frac{1}{D^2-3D-4}\cdot z$$

$$=\frac{1}{4-6-4}e^{2z}+2\left(\frac{-1}{4}\right)\left[\frac{1}{-\frac{D^2}{4}+\frac{3D}{4}+1}\right]\cdot z$$

$$=-\frac{e^{2z}}{6}-\frac{2}{4}\left[1-\left(\frac{D^2-3D}{4}\right)\right]^{-1}z$$

$$\because (1-D)^{-1}=1+D+D^2+D^3+\dots$$

$$\Rightarrow -2\left[1+\frac{D^2-3D}{4}+\dots\right]z=\frac{-2}{4}\left[z+\frac{1}{4}D^2(z)-\frac{3}{4}D(z)\right]$$

$$=\frac{-e^{2z}}{6}-\frac{2}{4}\left[z-\frac{3}{4}\right]=-\frac{e^{2z}}{6}-\frac{z}{2}+\frac{3}{8}$$

$$y=C_1x^{-1}+C_2x^{-4}-\frac{e^{2z}}{6}-\frac{z}{2}+\frac{3}{8}$$

Simultaneous Linear Differential equation

In this, we discuss differential equation in which there is one Independent Variable & two or more than two dependent Variables.

To solve such equations completely, we require as many simultaneous equations as the number of dependent Variable.

$\therefore$ D.E of the form

$$F_1 D(x) + F_2(D)y = T_1 \quad -(1)$$

$$\& \quad \phi_1(D)x + \phi_2(D)y = T_2 \quad -(2)$$

are called Ordinary differential (Simultaneous) equation with constant coefficient, where x & y are two dependent Variables & t is an Independent Variable & T_1 & T_2 are functions of t.

$$D = \frac{d}{dt}$$

There are two methods for the solution of simultaneous linear D.E

1. D-Operator method (Symbolic method)
2. Method of differentiation.

Method 1: D-operator

Suppose the simultaneous D.E is

$$F_1 D(x) + F_2(D)y = T_1$$

$$\phi_1(D)x + \phi_2(D)y = T_2 \quad \text{where } D = \frac{d}{dt}$$

Now we eliminate y from (1) & (2) we get D.E in term of x & we have to solve the D.E for x & putting the value of x & further operation in eq (1) or eq (2) we find value of y

$\therefore$ x & y is required general solution.

Method 2: Sometimes given simultaneous D.E are differentiated w.r.t to independent variable t & two more simultaneous equations are obtained. Then solve them by eliminating one of the dependent variable x & y & then the other dependent variable is find out.

(1) Solve $\frac{dx}{dt} + y = \sin t$ where $D = \frac{d}{dt}$

$\frac{dy}{dt} + x = \cos t$ given that $x = 2$ & $y = 0$

when $t = 0$

Sol:- $Dx + y = \sin t$ — (1)

$Dy + x = \cos t$ — (2)

Now multiply (1) by D & (2) by 1, eliminating y

$$D^2x + Dy = D\sin t$$

$$\underline{-x \ \underset{-}{+} Dy = \cos t}$$

$$D^2x - x = D\sin t - \cos t$$

$\Rightarrow x(D^2-1) = \cos t - \cos t \Rightarrow x(D^2-1) = 0$

A.E is $m^2 - 1 = 0 \Rightarrow m = 1, -1$

Now $x = C.F + P.I = c_1e^t + c_2e^{-t} + 0$ — (1)

Now $\frac{dx}{dt} = c_1e^t - c_2e^{-t}$

from $\frac{dx}{dt} + y = \sin t$

$c_1e^t - c_2e^{-t} + y = \sin t$

$$\boxed{y = \sin t - c_1e^t + c_2e^{-t}}$$

Using given condition

$t=0$ when $x=2$

$t=0$ when $y=0$

Then $0=-c_1+c_2 \Rightarrow$

from ① $2=c_1+c_2$

Then $0=-\cancel{c_1}+c_2$

$2=\cancel{c_1}+c_2$

$2=2c_2$ $\Rightarrow$ $\boxed{c_2=1,\ c_1=1}$

Hence complete soln is

$x=e^t+e^{-t}$

$y=\sin t-e^t+e^{-t}$

(2) Solve $\frac{dx}{dt} + 2y = e^t$

$\frac{dy}{dt} - 2x = e^{-t}$

Soln: $Dx + 2y = e^t$ — (1) (Multiply by D)

$Dy - 2x = e^{-t}$ — (2) (Multiply by 2)

$D^2x + 2Dy = De^t$

$2Dy - 4x = 2e^{-t}$

$\Rightarrow D^2x + 2Dy = De^t$

$\underset{+}{-}4x + 2Dy = \underset{-}{2}e^{-t}$

$D^2x + 4x = De^t - 2e^{-t}$

$x(D^2+4) = e^t - 2e^{-t}$ — (3)

$m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$C.F = C_1 \cos 2t + C_2 \sin 2t$

$P.I = \frac{1}{D^2+4}(e^t - 2e^{-t}) = \frac{1}{D^2+4} e^t - 2\frac{1}{D^2+4} e^{-t}$

Using $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ when $f(a) \neq 0$

$= \frac{1}{1^2+4} e^t - 2\frac{1}{(-1)^2+4} e^{-t} \Rightarrow \frac{1}{5}e^t - \frac{2}{5}e^{-t} = \frac{1}{5}(e^t - 2e^{-t})$

$\boxed{x = C.F + P.I}$

$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{5}(e^t - 2e^{-t})$

$\therefore \frac{dx}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{5}(e^t + 2e^{-t})$

Put in eq (1)

$$\frac{dx}{dt} + 2y = e^t$$

$$-2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{5}(e^t + 2e^{-t}) + 2y = e^t$$

$$2y = 2c_1 \sin 2t - 2c_2 \cos 2t - \frac{e^t}{5} - \frac{2e^{-t}}{5} + e^t$$

$$2y = 2c_1 \sin 2t - 2c_2 \cos 2t - \frac{e^t}{5} - \frac{2e^{-t}}{5} + e^t$$

$$2y = 2c_1 \sin 2t - 2c_2 \cos 2t - \frac{2e^{-t}}{5} + \frac{4e^t}{5}$$

$$\boxed{y = c_1 \sin 2t - c_2 \cos 2t - \frac{e^{-t}}{5} + \frac{2e^t}{5}}$$