# Computing stationary solutions to p-Willmore flow

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**Source material**

- A. Gruber, E. Aulisa, "Computational p-Willmore Flow with Conformal Penalty" (to appear in ACM Transactions on Graphics).

# Outline

1. Introduction
2. The continuous setting
3. The discrete setting
4. Implementation & results

# The p-Willmore energy

Let $u : M \to \mathbb{R}^3$ be a smooth immersion of the closed surface $M$. Then, the p-Willmore energy is defined as

$$\mathcal{W}^p(u) = \int_M |H|^p \, d\mu_g, \qquad p \geq 1$$

where $H = (1/2)(\kappa_1 + \kappa_2)$ is the mean curvature of the immersed surface $u(M)$.

We will say that the immersion $u : M \to \mathbb{R}^3$ is p-Willmore provided it is stationary (to first order) with respect to $\mathcal{W}^p$. Alternatively, $M$ is a **p-Willmore surface**.

Question: How does the p-Willmore functional relate to its namesake? Moreover, how do we visualize p-Willmore surfaces for theoretical and computational study?

# Review: the Willmore energy

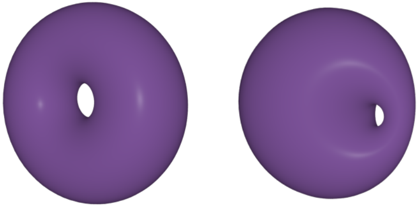
The Willmore energy $\mathcal{W}^2$ is the simplest nontrivial model for bending energy proposed by Sophie Germain in 1821, involving a symmetric and quadratic combination of $\kappa_1$ and $\kappa_2$.

Moreover, the Willmore energy is invariant under conformal transformations of $\mathbb{R}^3$ (Mobius transformations) and punishes deviation from roundness. When $M$ is closed,

$$\frac{1}{4}\int_M (\kappa_1 - \kappa_2)^2 \, d\mu_g = \int_M (H^2 - K) \, d\mu_g = \mathcal{W}(u) - 2\pi\chi(M),$$

by the Gauss-Bonnet theorem.

These tori have identical Willmore energy!



# Invariances of p-Willmore energy?

Note that the conformal invariance of the 2-Willmore energy does **not** extend to other values of $p$.

In fact, the p-Willmore energy is generally not even scale-invariant, since a dilation $u \mapsto (1/t)u$ induces the change $(t > 0)$

$$\int_M H^p \, d\mu_g \mapsto \int_M (tH)^p \, \frac{1}{t^2} d\mu_g = \int_M t^{p-2} H^p \, d\mu_g$$

and so the p-Willmore energy depends on the scale factor $t$ when $p \neq 2$.

This has significant consequences on immersed p-Willmore surfaces for $p \neq 2$, and is also seen in the gradient flow.

Particularly, an immersed surface can decrease its p-Willmore energy by **shrinking** uniformly when $p = 0$ and **growing** otherwise.

# Model problem

We consider the problem of minimizing the p-Willmore energy subject to physical constraints on surface area and enclosed volume.

More precisely, this means computing an immersion $u : M \to \mathbb{R}^3$ satisfying

$$\min_u \left(\mathcal{W}^p(u) + \lambda\mathcal{V}(u) + \nu\mathcal{A}(u)\right),$$

$$s.t.$$

$$V_0 = \int_M \langle u, N\rangle \, d\mu_g := \mathcal{V}(u),$$

$$A_0 = \int_M d\mu_g := \mathcal{A}(u),$$

where $N : M \to \mathbb{S}^2$ is the outward-directed unit normal field.

A suitable weak formulation of this will allow for implementation using piecewise-linear finite elements.

# Why constrain p-Willmore?

Aside from the fact that many useful physical models have minimizers which are naturally constrained in these ways, there is the following result.

**Theorem: G., Toda, Tran [1]**

When $p > 2$, any $p$-Willmore surface $M \subset \mathbb{R}^3$ satisfying $H = 0$ on $\partial M$ is minimal.

More precisely, let $p > 2$ and $u : M \to \mathbb{R}^3$ be a p-Willmore immersion of the surface $M$ with boundary $\partial M$. If $H = 0$ on $\partial M$, then $H \equiv 0$ everywhere on $M$.

Since there are no closed minimal surfaces in $\mathbb{R}^3$, this means there are no closed unconstrained p-Willmore minima when $p > 2$.

We need additional constraints in order to have (the possibility of) stationary solutions.

# Outline

# The p-Willmore equation

It can be shown that any p-Willmore surface satisfies the equation

$$0 = -\frac{p}{2}\Delta_g\left(H|H|^{p-2}\right) - pH|H|^{p-2}\left(2H^2 - K\right) + 2H|H|^p.$$

However, this equation is $4^{th}$-order in the immersion $u$, and so not suitable for finite-element modeling.

Instead, we must compute a weak expression which:

- Is at most first-order in the immersion $u$.
- Does not require a preferential frame in which to calculate derivatives (no moving frame).
- Considers general variations $\varphi : M \to \mathbb{R}^3$, which may have tangential as well as normal components.
- Avoids geometric terms that are not easily discretized, such as $K$ and $\nabla_g N$.

# The variational framework

To compute a usable expression for $\delta\mathcal{W}^p$, we adopt the framework of Dziuk and Elliott [2].

Consider a parametrization $X_0 : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ of (a portion of) the surface $M$, and let $u_0 : M \to \mathbb{R}^3$ be identity on $M$, so $u \circ X = X$.

A variation of the immersion $u$ is a smooth function $\varphi : M \to \mathbb{R}^3$ and a 1-parameter family $u(x,t) : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that $u(x,0) = u_0$ and

$$u(x,t) = u_0(x) + t\varphi(x).$$

Note that this pulls back to a variation $X : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$,

$$X(v,t) = X_0(v) + t\Phi(v),$$

where $\Phi = \varphi \circ X$. Note further that (since $u$ is identity on $X(t)$) the time derivatives are related by

$$\dot{u} = \frac{d}{dt}u(X,t) = \nabla_{\mathbb{R}^3} u \cdot \dot{X} + u_t = \dot{X}.$$

# Calculating the first variation

To study p-Willmore flow with area and volume constraints, our goal is to develop a suitable weak-form expression of the equation

$$\dot{u} = -\delta\mathcal{W}^p(u) - \lambda\delta\mathcal{V}(u) - \nu\delta\mathcal{A}(u).$$

First, note that the components of the induced metric on $M$ are

$$g_{ij} = \left\langle \partial_{x_i} X, \partial_{x_j} X \right\rangle = \langle X_i, X_j \rangle$$

so that the surface gradient of a function $f$ defined on $M$ can be expressed on $V$ as (Einstein summation assumed)

$$(\nabla_g f) \circ X = g^{ij} F_i X_j,$$

where $F = f \circ X$ and $g^{ik} g_{kj} = \delta^i_j$. It follows that for $f, \varphi : M \to \mathbb{R}^3$,

$$\langle df, d\varphi \rangle_g \circ X = g^{ij} F_i \Phi_j,$$

and the Laplace-Beltrami operator on $M$ is

$$(\Delta_g f) \circ X = (\mathrm{div}_g \nabla_g f) \circ X = \frac{1}{\sqrt{\det g}} \partial_j \left( \sqrt{\det g}\, g^{ij} F_i \right).$$

# Calculating the first variation (2)

The trick to finding a good expression for the variation of $\mathcal{W}^p$ is to exploit the geometry of the problem.

In particular, there is the identity $\Delta_g u = 2HN$ for (twice) the mean curvature vector, so Dziuk noticed in [3] that the $4^{th}$-order Willmore equation can be split by defining $Y = \Delta_g u$.

Weakly, we have

$$0 = \int_M \langle Y, \psi \rangle \, d\mu_g + \int_M \langle du, d\psi \rangle_g \, d\mu_g,$$

for all $\psi \in H^1(M; \mathbb{R}^3)$.

With this idea, the ($2^p$-scaled) p-Willmore functional becomes

$$\mathcal{W}^p(M) = \int_M |Y|^p \, d\mu_g,$$

which involves no explicit derivatives of $u$.

# Calculating the first variation (3)

## Problem (Closed surface p-Willmore flow with constraint)

*Let $p \geq 1$, $D(\varphi) = \nabla_g\varphi + (\nabla_g\varphi)^T$, and $W := |Y|^{p-2}Y$. Determine a family $u : M \times (0, T] \to \mathbb{R}^3$ of surface immersions with $M(t) = u(M, t)$ such that $M(0)$ has initial volume $V_0$, initial surface area $A_0$, and for all $t \in (0, T]$ the equations*

$$\begin{aligned}
0 &= \int_M \langle \dot{u}, \varphi\rangle \, d\mu_g + \int_M \nu \langle du, d\varphi\rangle_g \, d\mu_g + \int_M \lambda \langle \varphi, N\rangle \, d\mu_g \\
&\quad + \int_M \left((1-p)|Y|^p - p \operatorname{div}_g W\right) \operatorname{div}_g \varphi \, d\mu_g \\
&\quad + \int_M p \left(\langle D(\varphi) du, dW\rangle_g - \langle d\varphi, dW\rangle_g\right) d\mu_g, \qquad (1) \\
0 &= \int_M \langle Y, \psi\rangle \, d\mu_g + \int_M \langle du, d\psi\rangle_g \, d\mu_g, \\
0 &= \int_M \left\langle W - |Y|^{p-2}Y, \xi\right\rangle d\mu_g, \\
A_0 &= \int_M 1 \, d\mu_g, \qquad (2) \\
3V_0 &= \int_M \langle u, N\rangle \, d\mu_g, \qquad (3)
\end{aligned}$$

*are satisfied for some piecewise-constant $\lambda, \nu$ and all $\varphi, \psi, \xi \in H^1\left(M(t); \mathbb{R}^3\right)$.*

# Continuous stability

This weak formulation can be shown to monotonically decrease the p-Willmore energy.

**Theorem: Aulisa, G.**

The closed surface p-Willmore flow is energy decreasing for $p \geq 1$, i.e.

$$0 = \int_{M(t)} |\dot{u}|^2 \, d\mu_g + \frac{d}{dt} \int_{M(t)} \left( |Y|^p + \lambda \langle u, N \rangle + \nu \right) d\mu_g$$

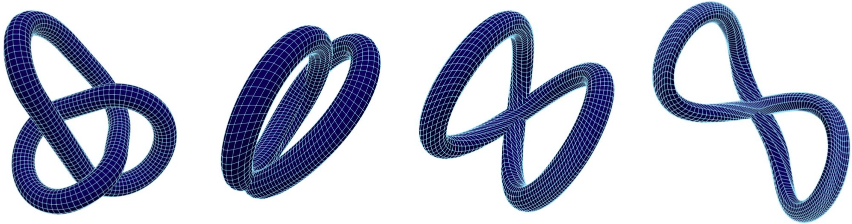
for all $t \in (0, T]$ and any piecewise-constant $\lambda, \nu$.

- This very nice property is what allows us to compute p-Willmore surfaces using the gradient flow.
- It is important to make sure that this stability property is preserved by our chosen discretization.

We now have a beautiful way to compute any p-Willmore surface that is stationary under the constrained p-Willmore flow, right? .....

# Mesh degeneration

Unfortunately, discrete computational flows typically involve some degree of *mesh sliding*, which can artificially break the simulation.



In order to prevent this, we now formulate a secondary system which will regularize the evolving surface at each time step.

This will help prevent failure which is not reflected by the continuous flow.

# Conformal correction (1)

To correct mesh sliding at each time step, the goal is to enforce "Cauchy-Riemann equations" on the tangent bundle $TM$.

Let $u : M \to \operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$ be an oriented immersion of $M$, and $J$ be a complex structure (rotation operator $J^2 = -\mathrm{Id}_{TM}$) on $TM$. Then, if $*\alpha = \alpha \circ J$ is the usual Hodge star on forms,

**Thm: Kamberov, Pedit, Pinkall [4]**

The immersion $u$ is conformal if and only if there is a Gauss map $N : M \to \operatorname{Im} \mathbb{H}$ such that $*du = N\, du$.

Note that,

- $N \perp du(v)$ for all tangent vectors $v \in TM$.
- $v, w \in \operatorname{Im} \mathbb{H} \longrightarrow vw = -v \cdot w + v \times w$.

**Conclusion:** conformality may be enforced by requiring $*du(v) = N \times du(v)$ on a basis for $TM$!

# Conformal correction (2)

How is this accomplished on the parametrization domain $V$? Choose $x^1, x^2$ as coordinates on $V$, then:

- $\partial_1 := \partial_{x^1}$ and $\partial_2 := \partial_{x^2}$ are an ON basis for $TV$.
- $dX(\partial_1) := X_1$ and $dX(\partial_2) := X_2$ are a (not usually ON) basis for $TM$.
- $J\,\partial_1 = \partial_2$, $J\,\partial_2 = -\partial_1$ on $TV$.
- $du(X_i) = dX(\partial_i)$ since $u$ is identity on $M$.

We proceed through minimization. Define the **conformal distortion** functional,

$$\mathcal{CD}(u) = \frac{1}{2}\int_M |du\,J - N \times du|^2\, d\mu_g,$$

and notice that

$$\begin{aligned}&|du\,J - N \times du|^2 \circ X \\ &\qquad = g^{ij}\left\langle (dX\,J(\partial_i) - N \times dX(\partial_i)), (dX\,J(\partial_j) - N \times dX(\partial_j))\right\rangle .\end{aligned}$$

# Conformal correction (3)

Careful rearrangement of the variation of this expression (assuming no change in the metric!!) yields the dyadic product $\left\langle \hat{Q}, \mathrm{Jac}\,\varphi \right\rangle$, where (indices $1 \leq i \leq 3$, mod 3),

$$\begin{aligned}\hat{Q}^i_1 &= g^{22}\,W_i + g^{11}\,(n_{i+1}\,V_{i+2} - n_{i+2}\,V_{i+1}) \\ &\quad + g^{12}\,(n_{i+2}\,W_{i+1} - n_{i+1}\,W_{i+2} - V_i)\,, \\ \hat{Q}^i_2 &= g^{11}\,V_i + g^{22}\,(n_{i+2}\,W_{i+1} - n_{i+1}\,W_{i+2}) \\ &\quad + g^{12}\,(n_{i+1}\,V_{i+2} - n_{i+2}\,V_{i+1} - W_i)\,.\end{aligned}$$

Here, the $n_i$ represent the components of $N$ on $(TV)^{\perp}$ and $V, W$ are vectors involving quadratic combinations of the $n_i$ and $X^i_j = \langle dX(\partial_j), e_i \rangle$.

Since (at least formally) there is a tensor $Q \in T^*V \otimes TM$ such that $\hat{Q}^{lj} = g^{kj} Q^l_k$, we conclude

$$\delta\mathcal{CD}(u)\varphi = \int_M \langle Q, d\varphi \rangle_g \, d\mu_g\,.$$

# Conformal correction (4)

Minimizing $\mathcal{CD}(u)$ directly will not respect the current surface geometry; we also need a constraint.

Idea: If $u(x)$ (old) and $\hat{u}(x)$ (new) are close, then $(u - \hat{u})(x)$ should be orthogonal to $N(x)$ (to first order).

More formally, given $\varepsilon > 0$ and an immersion $u$ with outer normal field $N$, the mesh regularization problem is to find a function $v : M \to \mathbb{R}^3$ and a Lagrange multiplier $\rho : M \to \mathbb{R}$, so that the new immersion $\hat{u} = u + v$ is the solution to

$$\min_v \left( \mathcal{CD}(u + v) + \varepsilon |\rho|^2 \mathcal{A}(u) + \rho \int_M \langle v, N \rangle \, d\mu_g \right).$$

This provides something close to a tangential reparametrization of $M$.

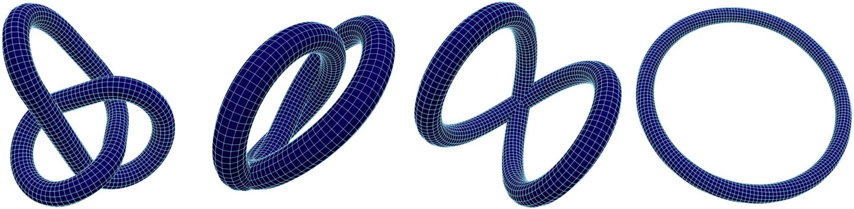
Note the penalty term in the minimization, which is beneficial for saddle point problems involving a mixture of linear and constant finite-elements.

# Conformal correction (5)

Formulated weakly, the goal of this procedure is find a new immersion $\hat{u} = u + v$ and a multiplier $\rho$ which satisfy the system

$$\begin{aligned} 0 &= \int_M \rho \langle \varphi, N \rangle \, d\mu_g + \int_M \langle Q, d\varphi \rangle_g \, d\mu_g, \\ 0 &= \int_M \psi \langle v, N \rangle \, d\mu_g + \varepsilon \int_M \psi \rho \, d\mu_g, \end{aligned}$$

for all $\varphi, \psi \in H^1(M; \mathbb{R}^3)$.



As can be seen, this works quite well in keeping meshes well-behaved as they evolve.

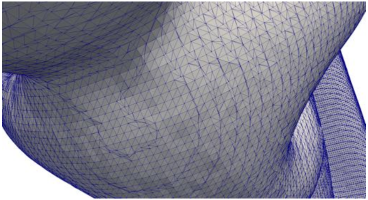
# Conformal to what?

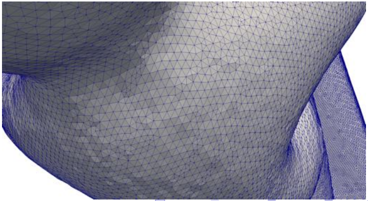
To make full use of the conformal penalty regularization procedure, it is important to specify a reference triangulation.

In practice this is done by letting the largest angle of each element adjust the others.

**Algorithm 1** Generation of target angles

**Require:** Reference triangulation $\mathcal{T}$ of the closed surface $M$.
  **for** $T \in \mathcal{T}$ **do**
    **for** vertex $1 \leq i \leq 3$ **do**
      Compute $m_i = \#$ of adjacent elements
      $\alpha_i \leftarrow \alpha_i / m_i$
    **end for**
    Determine maximum vertex angle $\alpha_i$.
    **if** $\alpha_i > \alpha_j$ for all $j \neq i$ **then**
      **for** vertices $j \neq i$ **do**
        $\alpha_j \leftarrow \alpha_j \left(\pi - \alpha_i\right) / \left(\sum_{k \neq i} \alpha_k\right)$
      **end for**
    **else**
      **for** vertices $1 \leq j \leq 3$ **do**
        $\alpha_j \leftarrow \alpha_j \, \pi / \left(\sum_{k=1}^{3} \alpha_k\right)$.
      **end for**
    **end if**
  **end for**





# Outline

# The spatial discretization

We assume the smooth surface $M$ is polygonally approximated by nondegenerate simplices $T_h$, so that

$$M_h = \bigcup_{T_h \in \mathcal{T}_h} T_h.$$

Denoting the nodes of this triangulation by $\{a_j\}_{j=1}^N$, the standard nodal basis $\{\phi_i\}$ on $M_h(t)$ satisfies $\phi_i(a_j, t) = \delta_{ij}$.

The space of piecewise-linear finite elements on $M_h(t)$ is then

$$S_h(t) = \mathrm{Span}\{\phi_i\} = \{\phi \in C^0(M_h(t)) \,:\, \phi|_{T_h} \in \mathbb{P}_1(T_h), T_h \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(T_h)$ denotes the space of linear polynomials on $T_h$.

Note that in practice we allow not only triangulations, but also quadrangulations of the continuous surface $M$.

# What about the temporal discretization?

A good discretization of the continuous systems discussed should:

- Preserve (at least empirically) the energy-decrease of the p-Willmore flow.
- Be robust to noise and other numerical artifacts.
- Be relatively fast to implement.

The simplest thing to do is to linearize the problem at each time step, effectively pushing the nonlinearities into the time domain. This is the standard strategy of Dziuk and Elliott in [2].

Instead, we choose our discretization "as centrally as possible". More precisely, let $\tau > 0$ be a fixed temporal stepsize, and denote $u_h^k = u_h(\cdot, k\tau)$.

Then, $M_h^{k+\frac{1}{2}}$ is the image of the immersion $u_h^{k+\frac{1}{2}} = (1/2)\left(u_h^k + u_h^{k+1}\right)$, and for any field quantity $F$,

$$F_h^{k+\frac{1}{2}} = \frac{1}{2}\left(F_h^k + F_h^{k+1}\right).$$

# Discrete p-Willmore flow

**Problem**

*Let $u, Y, W, \lambda, \nu$ be as in Problem 1. Given the discrete data $u_h^k, Y_h^k, W_h^k$ at time $t = k\tau$, the p-Willmore flow problem is to find functions $u_h^{k+1}, Y_h^{k+1}, W_h^{k+1}, \lambda_h, \nu_h$ which satisfy the system of equations*

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle Y_h^{k+\frac{1}{2}}, \psi_h \right\rangle d\mu_{g_h} + \int_{M_h^{k+\frac{1}{2}}} \left\langle du_h^{k+1}, d\psi_h \right\rangle_{g_h} d\mu_{g_h},$$

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle \left( W_h^{k+\frac{1}{2}} - \left| Y_h^{k+\frac{1}{2}} \right|^{p-2} Y_h^{k+\frac{1}{2}} \right), \xi_h \right\rangle d\mu_{g_h}, \tag{4}$$

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle du_h^{k+\frac{1}{2}}, \left( du_h^{k+1} - du_h^k \right) \right\rangle_{g_h} d\mu_{g_h}, \tag{5}$$

$$0 = \int_{M_h^{k+\frac{1}{2}}} \left\langle \left( u_h^{k+1} - u_h^k \right), N_h^{k+\frac{1}{2}} \right\rangle d\mu_{g_h}, \tag{6}$$

$$\begin{aligned} 0 = &\int_{M_h^{k+\frac{1}{2}}} \frac{\left\langle \left( u_h^{k+1} - u_h^k \right), \varphi_h \right\rangle}{\tau} d\mu_{g_h} + \int_{M_h^{k+\frac{1}{2}}} \lambda_h \left\langle \varphi_h, N_h^{k+\frac{1}{2}} \right\rangle d\mu_{g_h} \\ &+ \int_{M_h^{k+\frac{1}{2}}} \nu_h \left\langle du_h^{k+\frac{1}{2}}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} + (1-p) \int_{M_h^{k+\frac{1}{2}}} \left| Y_h^{k+\frac{1}{2}} \right|^p \left\langle du_h^{k+\frac{1}{2}}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} \\ &- p \int_{M_h^{k+\frac{1}{2}}} \left( \mathrm{div}_{g_h} W_h^{k+\frac{1}{2}} \right) \left\langle du_h^{k+\frac{1}{2}}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} - p \int_{M_h^{k+\frac{1}{2}}} \left\langle dW_h^{k+1}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} \\ &+ p \int_{M_h^{k+\frac{1}{2}}} \left\langle D(\varphi_h) du_h^k, dW_h^k \right\rangle_{g_h} d\mu_{g_h}, \end{aligned} \tag{7}$$

*for all $\varphi_h, \psi_h, \xi_h \in S_h$.*

# Discrete conformal penalty regularization

## Problem (Discrete conformal penalty regularization)

*Let $\varepsilon > 0$ be fixed, let $\hat{u}, u, N, \rho$ be as before, and let $\tilde{N} = (1/2)\left(N + \hat{N}\right)$. Given $u_h^{k+1}, N_h^{k+1}$, solving the discrete conformal penalty regularization problem means finding functions $\hat{u}_h^{k+1}, \rho_h$ which satisfy the system*

$$0 = \int_{M_h^{k+1}} \rho_h \left\langle \varphi_h, \tilde{N}_h^{k+1} \right\rangle d\mu_{g_h} + \int_{M_h^{k+1}} \left\langle \hat{Q}_h^{k+1}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h},$$

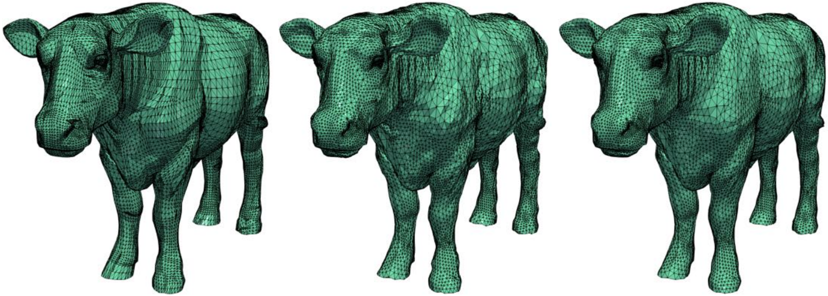
$$0 = \int_{M_h^{k+1}} \psi_h \left\langle \left( \hat{u}_h^{k+1} - u_h^{k+1} \right), \tilde{N}_h^{k+1} \right\rangle d\mu_{g_h} + \varepsilon \int_{M_h^{k+1}} \psi_h \, \rho_h \, d\mu_{g_h},$$

*for all $\varphi_h, \psi_h \in S_h$ and where $\left\langle \hat{Q}_h^{k+1}, d\varphi_h \right\rangle_{g_h}$ refers to the discretization on the known surface $M_h^{k+1}$ of the analogous continuous quantity, which involves components of the known normal $N_h^{k+1}$ and derivatives of the unknown immersion $\hat{u}_h^{k+1}$, computed with respect to $M_h^{k+1}$.*

# Linear vs. Nonlinear regularization

Since the only nonlinearity in the discrete conformal penalty regularization comes from $\tilde{N}$, it is easy to adapt our method to require only a linear solve by instead using the known $N$.

Below is a performance comparison on a cow with 34.5k triangles. Original mesh (left), linear algorithm (middle), nonlinear algorithm (right).



# Outline

# Main algorithm

Fist, note the following (linear) systems used to generate the initial curvature data.

$$0 = \int_{M_h^k} \left\langle Y_h^k, \psi_h \right\rangle d\mu_{g_h} + \int_{M_h^k} \left\langle du_h^k, d\psi_h \right\rangle_{g_h} d\mu_{g_h}, \tag{8}$$

$$0 = \int_{M_h^k} \left\langle \left( W_h^k - \left| Y_h^k \right|^{p-2} Y_h^k \right), \xi_h \right\rangle d\mu_{g_h}. \tag{9}$$

**Algorithm 2** p-Willmore flow with conformal penalty

**Require:** Closed, oriented surface immersion $u_h^0 : M_h^0 \to \mathbb{R}^3$; real numbers $\varepsilon, \tau > 0$, integer $k_{\max} \geq 1$.

**while** $0 \leq k \leq k_{\max}$ **do**

Solve (8) for $Y_h^k$

Solve (9) for $W_h^k$

Solve the p-Willmore flow problem for $u_h^{k+1}, Y_h^{k+1}, W_h^{k+1}, \lambda_h, \mu_h$

Solve the mesh regularization problem for $\hat{u}_h^{k+1}, \rho_h$

$u_h^{k+1} = \hat{u}_h^{k+1}$

$k = k + 1$

**end while**

# Some implementation details

The nonlinear systems in our algorithm are solved using two-step Newton iteration.

If $\mathcal{R}(\mathbf{v}_h) = 0$ represents (formally) one such system and $\mathbf{v}_h^0$ is an initial guess, this involves computing

$$\mathbf{v}_h^i = \mathbf{v}_h^{i-1} - \mathcal{J}^{-1}\left(\mathbf{v}_h^{i-1}\right)\mathcal{R}\left(\mathbf{v}_h^{i-1}\right) \qquad \text{for} \qquad i \geq 1,$$

where $\mathcal{J}\left(\mathbf{v}_h\right)$ is the Jacobian of the residual $\mathcal{R}$ in $\mathbf{v}_h$, evaluated through

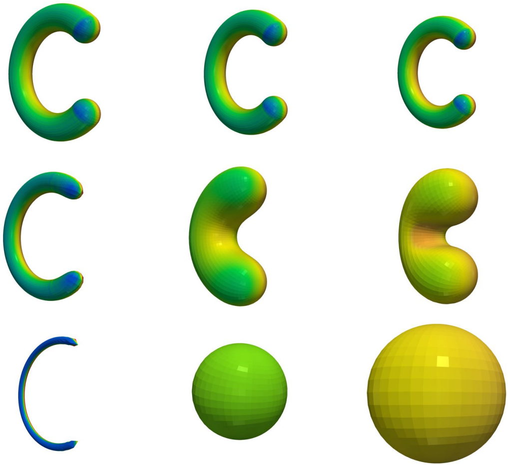
$$\mathcal{J}\left(\mathbf{v}_h^i\right) = \frac{\partial \mathcal{R}}{\partial \mathbf{v}_h}\left(\mathbf{v}_h^i\right).$$

The necessary integrals are computed using a $7^{th}$-order tensor product quadrature rule, and the Jacobian evaluation is done using the automatic differentiation library ADEPT.

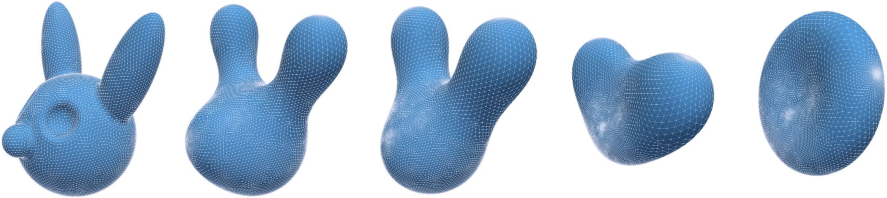
This allows for arbitrarily accurate derivative calculations, which improves stability of the simulations.

# Unconstrained flow comparison on a letter C

| MCF (0-Willmore) | Willmore flow (2-Willmore) | 4-Willmore flow |
|---|---|---|



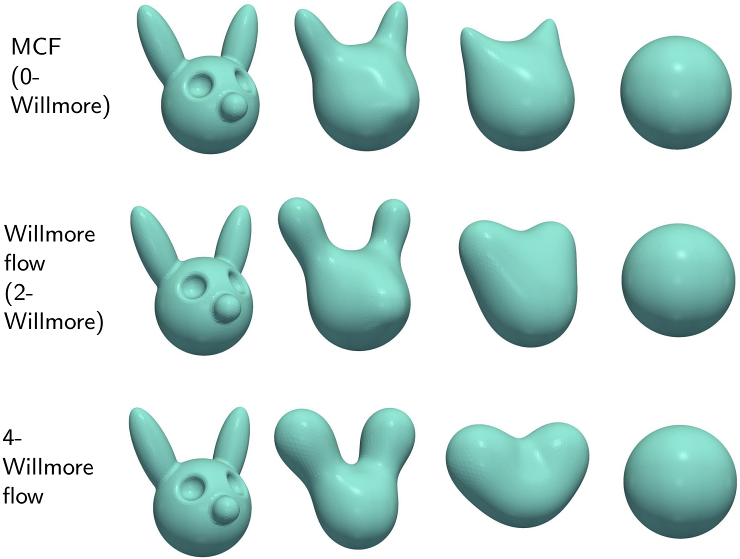
# Constrained 2-Willmore flow



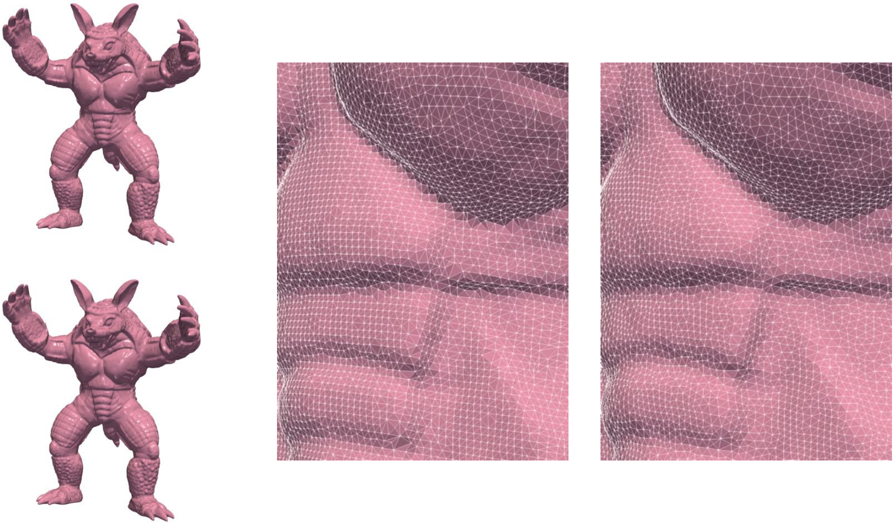
Area and volume constrained 2-Willmore flow of a rabbit-dog.

Volume constrained 2-Willmore flow of a genus 4 statue mesh.

# Volume-constrained flow comparison on a rabbit-dog



# Mesh edit of a cartoon armadillo



# Area preserving 2-Willmore flow of a cow

# Area and volume preserving 2-Willmore flow of a cow

# Almost isometric 2-Willmore flow of a torus knot

# Almost-isometric 2-Willmore flow of a torus knot (again)

# Future work

Challenges:

- Find a reasonable way to conformally-correct on the surface itself, not just its tangent space (higher-order approximation).
- Stabilize the p-Willmore flow for higher values of $p$.

Ideas to investigate:

- Build conformality directly into the flow equations.
- Compute a time-dependent holomorphic 1-form basis for $T^*M$ and use it to conformally parametrize at each step.
- Extend these ideas to other curvature flows of interest (Ricci flow, Yamabe flow, etc.)

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