Quaternionic Remeshing During Surface Evolution

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Source

Source material

- A. Gruber, E. Aulisa, "Computational p-Willmore Flow with Conformal Penalty." (To appear in *ACM Transactions on Graphics*).
- A. Gruber, E. Aulisa, "Quaternionic Remeshing During Surface Evolution". *ICNAAM 2020 Proceedings.*

  Preprints available at http://myweb.ttu.edu/agruber

Outline

What are geometric flows?

A geometric flow is a procedure which changes the metric and curvature data of a surface (M, g) in a prescribed way.

Flows are usually formulated as systems of nonlinear PDE, since the relevant differential operators depend on the evolving surface data.

Examples: Let $\mathbf{r} : M \to \mathbb{R}^3$ be an immersion with mean curvature H and Gauss map $N : M \to S^2$.

- The mean curvature flow
$$\frac{d}{dt}\mathbf{r} = -2HN.$$
- The Ricci flow
$$\frac{d}{dt}g = -2\mathrm{Ric}(g).$$
- The Willmore flow
$$\frac{d}{dt}\mathbf{r} = -\left(\Delta_g H - 2H(H^2 - K)\right) N.$$

Why do we like flows?

Highly useful for many diverse mathematical and scientific applications.

- The Ricci flow is used extensively in the proofs of the Poincaré and Thurston geometrization conjectures.
- Mean curvature flow is a reasonable approximation to diffusion through porous membranes.
- The Willmore flow is used in computer graphics to smooth rough surface data.

Last (but not least), they lead to beautiful and compelling illustrations!

How are flows simulated?

Running computational flows relies on formulating a reasonable discretization of the continuous problem.

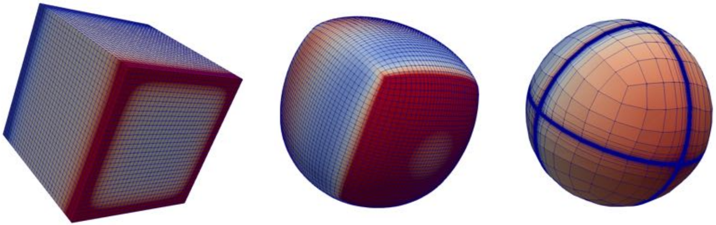
One popular technique for this purpose is finite-element approximation. Here, a discrete surface is represented as a collection of elements (usually simplices) with a piecewise-linear nodal basis.

This brings new issues which have to be dealt with, as things which are harmless in the smooth setting become problematic in the discrete setting...

- Cannot choose a preferential frame in which to calculate derivatives; no natural adaptation (e.g. moving frame) is possible.
- Cannot "mindlessly" apply geometric identities—smooth relationships are satisfied only in the limit of refinement!
- Cannot directly implement geometric quantities such as K and $\nabla_g N$ which rely on explicit second derivatives of position.

The issue of mesh quality

Even more troublesome, natural tangential motion during the flow can ruin a perfectly good mesh!



This is a problem common to all geometric flow algorithms, and different techniques have been developed to “solve” it.

This talk will explore a method for preventing such degradation based on quaternionic surface theory.

How has this problem been addressed?

Some documented approaches to the problem of preserving mesh quality include the following.

- Make flow algorithms fast enough that using a very fine mesh is feasible (brute force approach).
- Build conformality directly into the flow equations as an integrability condition (very interested, but difficult and limited by initial mesh quality).
- Remesh the evolving surface along the flow according to some minimization procedure (this is our approach).

By carefully remeshing at each step of the flow, it is possible to control mesh quality along the evolution.

Outline

Quaternionic remeshing procedure

To correct mesh sliding at each time step, our goal is to enforce Cauchy-Riemann equations on the tangent bundle TM.

- $u : M \to \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$, oriented immersion of M.
- J, complex structure (rotation operator $J^2 = -\mathrm{Id}_{TM}$) on TM.
- $*\alpha = \alpha \circ J$, minus the usual Hodge star on differential forms.

Thm: Kamberov, Pedit, Pinkall [1]

The immersion u is conformal if and only if there is a Gauss map $N : M \to \mathrm{Im}\,\mathbb{H}$ such that $*du = N\,du$.

This says that du takes normalized orthogonal bases on TM to normalized orthogonal bases on the image.

It can be shown that this is equivalent to the classical Cauchy-Riemann equations when $u : \mathbb{C} \to \mathbb{C}$.

Quaternionic remeshing procedure (2)

How do we enforce this condition locally?

Let $X : V \subset \mathbb{R}^2 \to M$ be a parametrization, and $u = \mathrm{Id}_M$. Choose x^1, x^2 as coordinates on V, then:

- $\boldsymbol{\partial}_1 := \boldsymbol{\partial}_{x^1}$ and $\boldsymbol{\partial}_2 := \boldsymbol{\partial}_{x^2}$ are an ON basis for TV.
- $dX(\boldsymbol{\partial}_1) := \mathbf{X}_1$ and $dX(\boldsymbol{\partial}_2) := \mathbf{X}_2$ are a (not usually ON) basis for TM.
- $J\,\boldsymbol{\partial}_1 = \boldsymbol{\partial}_2$, $J\,\boldsymbol{\partial}_2 = -\boldsymbol{\partial}_1$ on TV.
- $du(\mathbf{X}_i) = dX(\boldsymbol{\partial}_i)$ since $u \circ X = X$.

Remeshing is then accomplished through minimization of the *conformal distortion* functional,

$$\mathcal{CD}(u) = \frac{1}{2} \int_M \left| {*du} - N\, du \right|^2 d\mu_g .$$

Quaternionic remeshing procedure (3)

If $\{\mathbf{e}_K\}$ is a basis for $\mathrm{Im}\,\mathbb{H}$ and $u_i^K = \langle du(\boldsymbol{\partial}_i), \mathbf{e}_K\rangle_{\mathbb{R}}$, we can define the operators Q_1, Q_2 through (Einstein summation assumed)

$$\left(Q_1 u^K\right)\mathbf{e}_K = *du(\boldsymbol{\partial}_1) - N\,du(\boldsymbol{\partial}_1) = \left(u_2^K - N\,u_1^K\right)\mathbf{e}_K,$$
$$\left(Q_2 u^K\right)\mathbf{e}_K = *du(\boldsymbol{\partial}_2) - N\,du(\boldsymbol{\partial}_2) = -\left(u_1^K + N\,u_2^K\right)\mathbf{e}_K.$$

Note that all products are assumed to operate over the field of quaternions $\mathbb{H}$. (No assumption that N is normal to the image of du!)

The variation of the conformal distortion (under fixed metric g) is then

$$\delta\mathcal{CD}(u)\varphi = \int_M \langle *du - N\,du, *d\varphi - N\,d\varphi\rangle_g\,d\mu_g := \int_M \langle Q(u), d\varphi\rangle_g\,d\mu_g.$$

Quaternionic remeshing procedure (4)

With the notions,

- $\langle a, b\rangle_{\mathbb{R}} = \mathrm{Re}(a\bar{b})$.
- $\bar{\mathbf{v}} = -\mathbf{v}$ for $\mathbf{v} \in \mathrm{Im}\,\mathbb{H}$.
- $\mathbf{e}_0 = 1$ and $\mathbf{e}_{-K} = -\mathbf{e}_K$,

the inner product $\langle Q(f), d\varphi\rangle_g$ can be organized according to

$$\langle Q(f), d\varphi\rangle_g = \left\langle \left(Q_i f^K\right)\mathbf{e}_K, \left(Q_j\varphi^L\right)\mathbf{e}_L\right\rangle_g = -g^{ij}\,\mathrm{Re}\left(Q_i f^K\,\mathbf{e}_{K\times L}\,\overline{Q_j\varphi^L}\right).$$

Consideration of $\mathcal{CD}$ alone is sufficient for maintaining mesh quality, but not for preserving the surface geometry.

We need to appropriately constrain this minimization in order to preserve the shape of the surface.

Remeshing constraint

Consider $x \in M$ and a curve $u(\mathbf{x} + t\mathbf{v})$ for some $\mathbf{v} \in T\Sigma$.

Taylor expansion then shows

$$u(\mathbf{x} + t\mathbf{v}) = u(\mathbf{x}) + t\, du_{\mathbf{x}}(\mathbf{v}) + \frac{t^2}{2}(\nabla du)_{\mathbf{x}}(\mathbf{v}, \mathbf{v}) + O(t^3),$$
$$N\left(\mathbf{x} + \frac{t}{2}\mathbf{v}\right) = N(\mathbf{x}) + \frac{t}{2} dN_{\mathbf{x}}(\mathbf{v}) + O(t^2).$$

Moreover, since $\langle N, du(\mathbf{v})\rangle = 0$ for all $\mathbf{v} \in T\Sigma$, differentiation shows that

$$\langle dN(\mathbf{v}), du(\mathbf{v})\rangle + \langle N, (\nabla du)(\mathbf{v}, \mathbf{v})\rangle = 0.$$

Therefore,

$$\left\langle u(\mathbf{x} + t\mathbf{v}) - u(\mathbf{x}),\ N\left(\mathbf{x} + \frac{t}{2}\mathbf{v}\right)\right\rangle = 0 + O(t^3).$$

Remeshing constraint (2)

This calculation motivates the following constraint.

<u>Idea</u>: Let $u(\mathbf{x})$ be the old solution and $\hat{u}(\mathbf{x})$ be the new solution. If $\hat{u}(\mathbf{x})$ is close to $u(\mathbf{x})$, then $(u - \hat{u})(\mathbf{x})$ should be approximately orthogonal to $\tilde{N}(\mathbf{x}) = \frac{1}{2}\left(N(\mathbf{x}) + \hat{N}(\mathbf{x})\right)$.

Given $\varepsilon > 0$, the quaternionic remeshing problem is to compute:

- a function $v : M \to \mathbb{R}^3$,
- a Lagrange multiplier $\rho : M \to \mathbb{R}$,

so that the new immersion $\hat{u} = u + v$ solves

$$\min_v \left(\mathcal{CD}(u+v) + \frac{\varepsilon}{2} \int_M |\rho|^2 \, d\mu_g + \int_M \rho \left\langle v, N \right\rangle \, d\mu_g \right).$$

This provides something close to a tangential reparametrization of M.

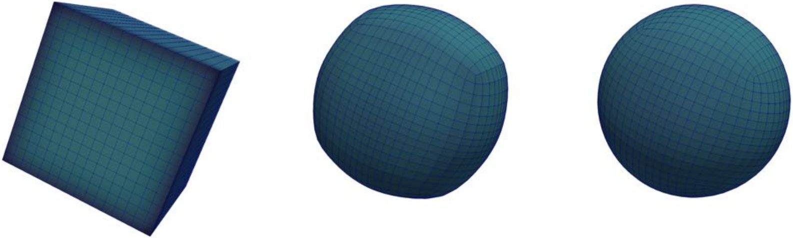
(The penalty term involving ε is beneficial for mixed FEM saddle point problems.)

Remeshing constraint (3)

Formulated weakly, we need a new immersion $\hat{u} = u + v$ and a multiplier ρ satisfying

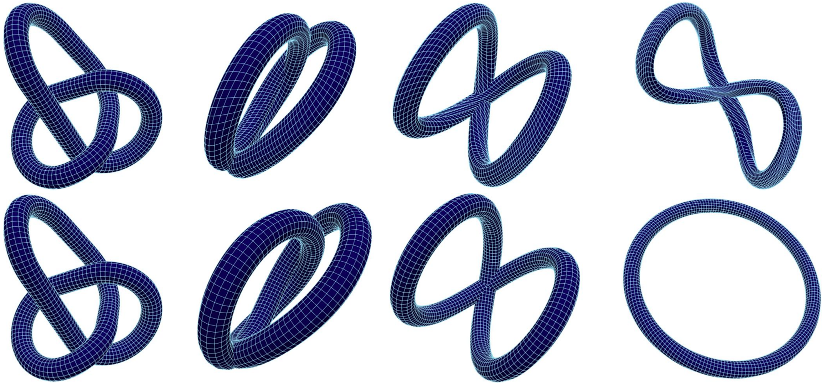
$$0 = \int_M \langle Q(u+v), d\varphi \rangle_g \, d\mu_g + \int_M \rho \langle \varphi, N \rangle \, d\mu_g,$$
$$0 = \int_M \psi \langle v, N \rangle \, d\mu_g + \varepsilon \int_\Sigma \psi \rho \, d\mu_g,$$

for all $\varphi \in H^1(M; \mathbb{R}^3)$ and for all $\psi \in H^1(M; \mathbb{R})$.



Example

This procedure stops an evolving surface from failing to reach a minimizing configuration due to artificial degeneration.



The finite-element discretization

We assume the smooth surface M is polygonally approximated by nondegenerate simplices T_h:

$$M_h = \bigcup_{T_h \in \mathcal{T}_h} T_h.$$

- $\{a_j\}_{j=1}^N$, nodes of the triangulation.
- $\{\phi_i\}$, standard nodal basis on $M_h(t)$, so $\phi_i(a_j, t) = \delta_{ij}$.
- $\mathbb{P}_1(T_h)$, the space of linear polynomials on T_h.

The space of piecewise-linear finite elements on $M_h(t)$ is then

$$S_h(t) = \mathrm{Span}\{\phi_i\} = \{\phi \in C^0(M_h(t)) \,:\, \phi|_{T_h} \in \mathbb{P}_1(T_h), T_h \in \mathcal{T}_h\},$$

In practice, we allow both triangulations and quadrangulations of M.

Discrete conformal penalty regularization

Problem (Discrete conformal penalty regularization)

Let $\varepsilon > 0$ be fixed, let $\hat{u}, u, N, \rho$ be as before, and let $\tilde{N} = (1/2)\left(N + \hat{N}\right)$. Given u_h^{k+1}, N_h^{k+1}, solving the discrete conformal penalty regularization problem means finding functions $\hat{u}_h^{k+1}, \rho_h$ which satisfy the system

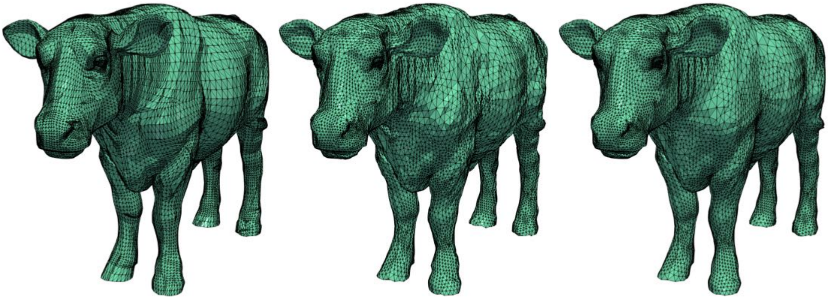
$$
\begin{aligned}
0 &= \int_{M_h^{k+1}} \left\langle \hat{Q}_h^{k+1}, d\varphi_h \right\rangle_{g_h} d\mu_{g_h} + \int_{M_h^{k+1}} \rho_h \left\langle \varphi_h, \tilde{N}_h^{k+1} \right\rangle d\mu_{g_h}, \\
0 &= \int_{M_h^{k+1}} \psi_h \left\langle \left(\hat{u}_h^{k+1} - u_h^{k+1}\right), \tilde{N}_h^{k+1} \right\rangle d\mu_{g_h} + \varepsilon \int_{M_h^{k+1}} \psi_h \, \rho_h \, d\mu_{g_h},
\end{aligned}
$$

for all $\varphi_h, \psi_h \in S_h$ and where $\left\langle \hat{Q}_h^{k+1}, d\varphi_h \right\rangle_{g_h}$ refers to the discretization on the known surface M_h^{k+1} of the analogous continuous quantity, which involves components of the known normal N_h^{k+1} and derivatives of the unknown immersion $\hat{u}_h^{k+1}$, computed with respect to M_h^{k+1}.

Linear vs. Nonlinear regularization

- Only nonlinearity in the conformal regularization comes from $\tilde{N}$.
- Method can be easily modified to require only a linear solve (use known N).

Performance comparison on a cow with 34.5k triangles. Original mesh (left), linear algorithm (middle), nonlinear algorithm (right).



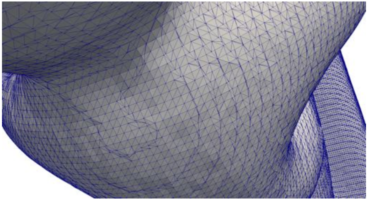
Conformal to what?

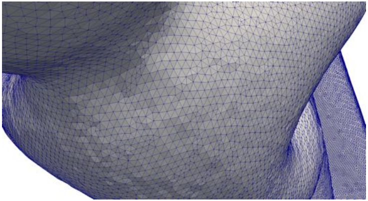
In practice, it is important to specify a reference triangulation to represent the desired "conformal class".

This is done by letting the largest angle of each element adjust the others.

Algorithm 1 Generation of target angles

Require: Reference triangulation $\mathcal{T}$ of the closed surface M.
  for $T \in \mathcal{T}$ **do**
    for vertex $1 \leq i \leq 3$ **do**
      Compute $m_i = \#$ of adjacent elements
      $\alpha_i \leftarrow \alpha_i / m_i$
    end for
    Determine maximum vertex angle α_i.
    if $\alpha_i > \alpha_j$ for all $j \neq i$ **then**
      for vertices $j \neq i$ **do**
        $\alpha_j \leftarrow \alpha_j \, (\pi - \alpha_i) \, / \left(\sum_{k \neq i} \alpha_k \right)$
      end for
    else
      for vertices $1 \leq j \leq 3$ **do**
        $\alpha_j \leftarrow \alpha_j \, \pi \, / \left(\sum_{k=1}^{3} \alpha_k \right)$.
      end for
    end if
  end for





Outline

The p-Willmore energy

Let $u : M \to \mathbb{R}^3$ be a smooth immersion of the closed surface M. Then, the p-Willmore energy is defined as

$$\mathcal{W}^p(u) = \int_M |H|^p \, d\mu_g, \qquad p \geq 1$$

where $H = (1/2)(\kappa_1 + \kappa_2)$ is the mean curvature of the immersed surface $u(M)$, and $d\mu_g$ is the induced area element.

We will say that the immersion $u : M \to \mathbb{R}^3$ is p-Willmore provided it is stationary (to first order) with respect to $\mathcal{W}^p$. Alternatively, M is a **p-Willmore surface**.

Question: How does the p-Willmore functional relate to its namesake? Moreover, how do we visualize p-Willmore surfaces for theoretical and computational study?

Model problem: p-Willmore flow

Consider the problem of minimizing $\mathcal{W}^p$ over **closed** surfaces subject to physical area/volume constraints

Precisely, this means computing an immersion $u : M \to \mathbb{R}^3$ with Gauss map $N : M \to \mathbb{S}^2$ satisfying

$$\begin{aligned}
&\min_u \left(\mathcal{W}^p(u) + \lambda\mathcal{V}(u) + \nu\mathcal{A}(u)\right), \\
&\quad s.t. \\
&\quad 3V_0 = \int_M \langle u, N\rangle \; d\mu_g := \mathcal{V}(u), \\
&\quad A_0 = \int_M d\mu_g := \mathcal{A}(u),
\end{aligned}$$

for Lagrange multipliers $\lambda, \nu : M \to \mathbb{R}$.

A suitable weak formulation of this will allow for implementation using piecewise-linear finite elements.

The p-Willmore flow system

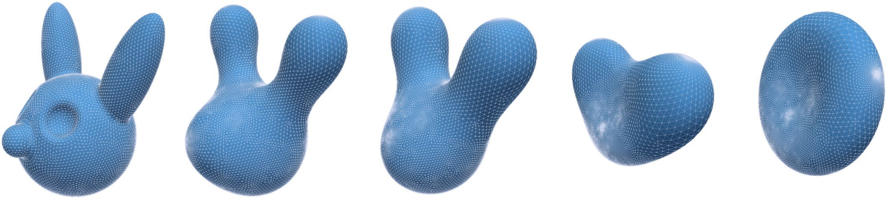
Problem (Closed surface p-Willmore flow with constraint)

Let $p \geq 1$, $D(\varphi) = \nabla_g \varphi + (\nabla_g \varphi)^T$, *and* $W := |Y|^{p-2} Y$. *Determine a family* $u : M \times (0, T] \to \mathbb{R}^3$ *of surface immersions with* $M(t) = u(M, t)$ *such that* $M(0)$ *has initial volume* V_0, *initial surface area* A_0, *and for all* $t \in (0, T]$ *the equations*

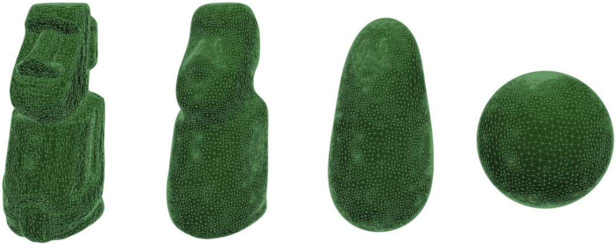
$$\begin{aligned}
0 &= \int_M \langle \dot{u}, \varphi \rangle \, d\mu_g + \int_M \nu \, \langle du, d\varphi \rangle_g \, d\mu_g + \int_M \lambda \, \langle \varphi, N \rangle \, d\mu_g \\
&\quad + \int_M \left((1-p)|Y|^p - p \, \mathrm{div}_g W \right) \mathrm{div}_g \varphi \, d\mu_g \\
&\quad + \int_M p \left(\langle D(\varphi) du, dW \rangle_g - \langle d\varphi, dW \rangle_g \right) d\mu_g, \qquad (1) \\
0 &= \int_M \langle Y, \psi \rangle \, d\mu_g + \int_M \langle du, d\psi \rangle_g \, d\mu_g, \\
0 &= \int_M \left\langle W - |Y|^{p-2} Y, \xi \right\rangle d\mu_g, \\
A_0 &= \int_M 1 \, d\mu_g, \qquad (2) \\
3V_0 &= \int_M \langle u, N \rangle \, d\mu_g, \qquad (3)
\end{aligned}$$

are satisfied for some piecewise-constant λ, ν *and all* $\varphi, \psi, \xi \in H^1\left(M(t); \mathbb{R}^3\right)$.

Examples of p-Willmore flow



Area and volume constrained 2-Willmore flow of a rabbit-dog.



Volume-constrained MCF on an Easter Island head.

Some implementation details

Nonlinear systems are solved using two-step Newton iteration:

- $\mathcal{R}(\mathbf{v}_h) = 0$ formal solution to system.
- $\mathbf{v}_h^0$ an initial guess.

$$\mathbf{v}_h^i = \mathbf{v}_h^{i-1} - \mathcal{J}^{-1}\left(\mathbf{v}_h^{i-1}\right)\mathcal{R}\left(\mathbf{v}_h^{i-1}\right) \quad \text{for} \quad i \geq 1,$$

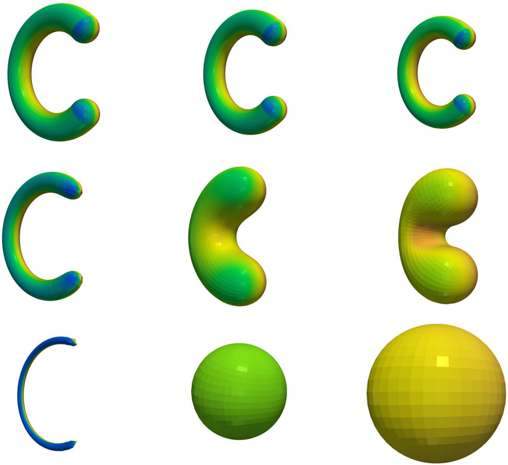
where $\mathcal{J}(\mathbf{v}_h)$ is the Jacobian of $\mathcal{R}$ in $\mathbf{v}_h$:

$$\mathcal{J}\left(\mathbf{v}_h^i\right) = \frac{\partial \mathcal{R}}{\partial \mathbf{v}_h}\left(\mathbf{v}_h^i\right).$$

- Integrals are computed using a 7^{th}-order tensor product quadrature rule.
- Jacobian evaluation is done using ADEPT automatic differentiation.

Unconstrained flow comparison on a letter C

MCF (0-Willmore)	Willmore flow (2-Willmore)	4-Willmore flow

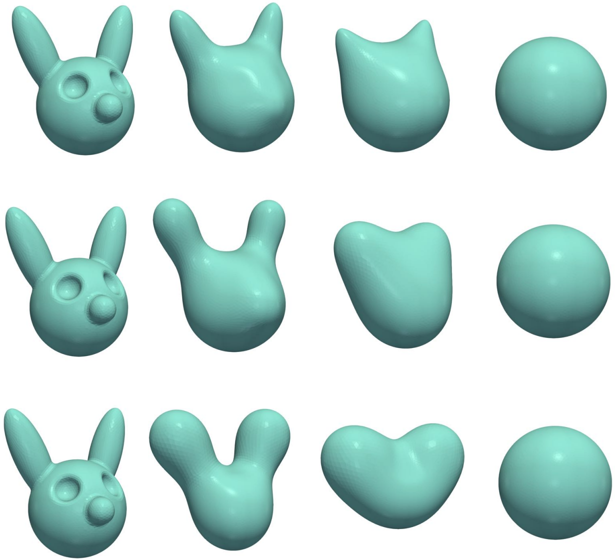


Volume-constrained flow comparison on a rabbit-dog

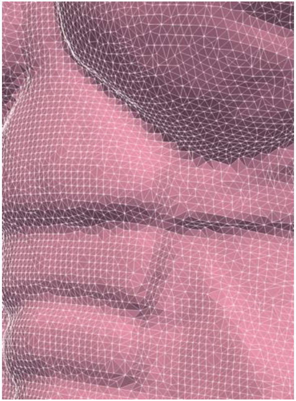
MCF
(0-
Willmore)

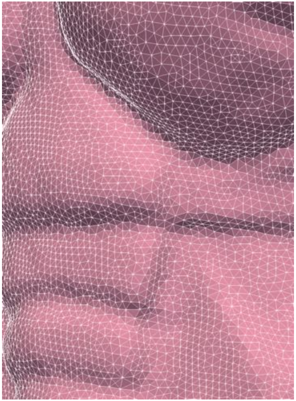
Willmore
flow
(2-
Willmore)

4-
Willmore
flow



Mesh edit of a cartoon armadillo





Area preserving 2-Willmore flow of a cow

Area and volume preserving 2-Willmore flow of a cow

Almost isometric 2-Willmore flow of a torus knot

Almost-isometric 2-Willmore flow of a torus knot (again)

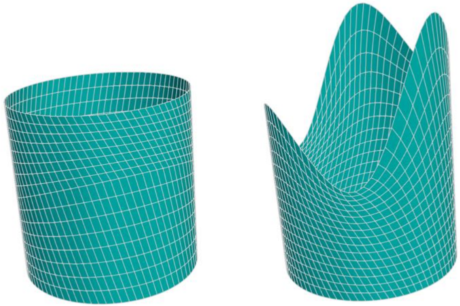
Coming soon!

Computing quasiconformal immersions $u : M \to \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$. Solutions to

$$du^- = \mu\, du^+.$$

- $du^\pm$ holomorphic / antiholomorphic parts of du w.r.t a predefined conformal immersion.
- $\mu : TM \to TM$ complex-valued tensor $\mu\, d\bar{z} \otimes \frac{\partial}{\partial z}$ (Beltrami coefficient).

Allows us to consider boundary conditions!



Thank you!

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