CS771A - Assignment 0

Vikas Jain, 13788

Q1 (a) $\sum_{i=1}^{n} P(C_i) - \sum_{1\le i<j\le n} P(C_i \cap C_j) \le P\left(\bigcup_{i=1}^{n} C_i\right) \le \sum_{i=1}^{n} P(C_i)$

(i) $P\left(\bigcup_{i=1}^{n} C_i\right) \le \sum_{i=1}^{n} P(C_i)$

By induction

base case $n=1$

$P(C_1) = P(C_1)$

hypothesis: $P\left(\bigcup_{i=1}^{n} C_i\right) \le \sum_{i=1}^{n} P(C_i)$

Inductive step :- for $n+1$

$$P\left(\bigcup_{i=1}^{n+1} C_i\right) = P\left(\bigcup_{i=1}^{n} C_i \cup C_{n+1}\right)$$

$$= P\left(\bigcup_{i=1}^{n} C_i\right) + P(C_{n+1}) - P\left(\bigcup_{i=1}^{n} C_i \cap C_{n+1}\right)$$

$$\le \sum_{i=1}^{n+1} P(C_i) - P\left(\bigcup_{i=1}^{n} C_i \cap C_{n+1}\right)$$

$$\le \sum_{i=1}^{n+1} P(C_i)$$

hence $\boxed{P\left(\bigcup_{i=1}^{n+1} C_i\right) \le \sum_{i=1}^{n+1} P(C_i)}$

(ii) $\sum_{i=1}^{n} P(C_i) - \sum_{1\le i<j\le n} P(C_i \cap C_j) \le P\left(\bigcup_{i=1}^{n} C_i\right)$

By induction

base case $n=1$

$P(C_1) = P(C_1)$

hypothesis: $\sum_{i=1}^{n} P(C_i) - \sum_{1\le i<j\le n} P(C_i \cap C_j) \le P\left(\bigcup_{i=1}^{n} C_i\right)$

Inductive step :- for $n+1$

$$P\left(\bigcup_{i=1}^{n+1} C_i\right) = P\left(\bigcup_{i=1}^{n} C_i \cup C_{n+1}\right)$$

$$= P\left(\bigcup_{i=1}^{n} C_i\right) + P(C_{n+1}) - P\left(\bigcup_{i=1}^{n} C_i \cap C_{n+1}\right)$$

$$= P\left(\bigcup_{i=1}^{n} C_i\right) + P(C_{n+1}) - P\left((C_1 \cap C_{n+1}) \cup (C_2 \cap C_{n+1}) \cup \cdots \cup (C_n \cap C_{n+1})\right)$$

$$\geq \sum_{i=1}^{n} P(C_i) - \sum_{1 \leq i < j \leq n} P(C_i \cap C_j) + P(C_{n+1}) - \sum_{i=1}^{n} P(C_i \cap C_{n+1})$$

$$= \sum_{i=1}^{n+1} P(C_i) - \sum_{1 \leq i < j \leq n+1} P(C_i \cap C_j)$$

Hence $P\left(\bigcup_{i=1}^{n+1} C_i\right) \geq \sum_{i=1}^{n+1} P(C_i) - \sum_{1 \leq i < j \leq n} P(C_i \cap C_j)$

(b) $$P\left(\bigcap_{i=1}^{n} C_i\right) \geq \sum_{i=1}^{n} P(C_i) - n + 1$$

By induction

base case $n=1$

$$P(C_1) = P(C_1) - 1 + 1$$

<u>hypothesis</u> for n

$$P\left(\bigcap_{i=1}^{n} C_i\right) \geq \sum_{i=1}^{n} P(C_i) - n + 1$$

<u>Inductive step</u> for $n+1$

$$P\left(\bigcap_{i=1}^{n+1} C_i\right) = P\left(\bigcap_{i=1}^{n} C_i \cap C_{n+1}\right)$$

$$= P\left(\bigcap_{i=1}^{n} C_i\right) + P(C_{n+1}) - P\left(\bigcap_{i=1}^{n} C_i \cup C_{n+1}\right)$$

$$\geq \sum_{i=1}^{n} P(C_i) - n + 1 + P(C_{n+1}) - P\left(\bigcap_{i=1}^{n} C_i \cup C_{n+1}\right)$$

since $P\left(\bigcap_{i=1}^{n} C_i \cup C_{n+1}\right) \leq 1$

$$P\left(\bigcap_{i=1}^{n+1} C_i\right) \geq \sum_{i=1}^{n+1} P(C_i) - n + 1 - 1$$

$$= \sum_{i=1}^{n+1} P(C_i) - (n+1) + 1$$

Hence, $$\boxed{P\left(\bigcap_{i=1}^{n} C_i\right) \geq \sum_{i=1}^{n} P(C_i) - n + 1}$$

2(a)(a) $f(x,y) = \begin{cases} cx^2y, & 0<x<y<1 \\ 0, & \text{o.w} \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^1 \left(\int_0^y cx^2y\,dx\right) dy = 1$$

$$\int_0^1 \frac{cy^4}{3}\,dy = 1$$

$$\frac{c}{15} = 1 \Rightarrow \boxed{c = 15}$$

(b) Marginal pdfs.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 cx^2y\,dy = \frac{cx^2}{2}$$

$$f_X(x) = \begin{cases} cx^2/2, & 0<x<1 \\ 0, & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 cx^2y\,dx = cy/3$$

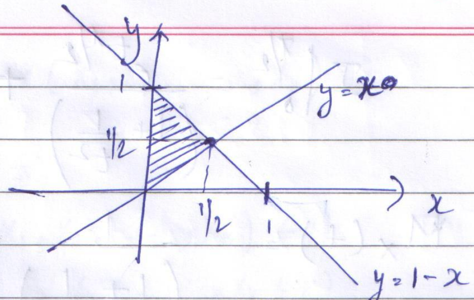
$$f_Y(y) = \begin{cases} cy/3, & 0<y<1 \\ 0, & \text{o.w.} \end{cases}$$

(c) Since $f_{X,Y}(x,y) \neq f_X(x)\,f_Y(y)$

$\Rightarrow$ X and Y are not independent

(d) $P((X+Y)<1)$

$P(Y<1-X)$



$$= \int_0^{1/2}\left(\int_0^y cx^2y\,dx\right)dy + \int_{1/2}^{1}\left(\int_0^{1-y} cx^2y\,dx\right)dy$$

$$= \int_0^{1/2} \frac{cy^4}{4}\,dy + \int_{1/2}^{1} \frac{c(1-y)^3}{3}y\,dy$$

$$= \frac{c}{20}\left|y^5\right|_0^{1/2} + \frac{c}{3}\left|\frac{t^4}{4} - \frac{t^5}{5}\right|_0^{1/2}$$

$$= \frac{c}{20\times 2^5} + \frac{c}{3}\cdot\frac{1}{2^4}\left(\frac{1}{4}-\frac{1}{10}\right) = \frac{c}{20.2^5} + \frac{c}{20.2^4}$$

$$= \frac{3c}{20.2^5} = \frac{3\times 15}{20.2^5} = \frac{9}{27}$$

2(b) $f(x) = \begin{cases} \sin x, & 0<x<\pi/2 \\ 0, & \text{o.w.} \end{cases}$

$$M_X(t) = E(e^{tX})$$

$$= \int_0^{\pi/2} e^{tx}\sin x\,dx$$

Let $I = \int e^{tx}\sin x\,dx$

$$I = \sin x\,\frac{e^{tx}}{t} - \int \cos x\,\frac{e^{tx}}{t}\,dx$$

$$I = \frac{\sin x\,e^{tx}}{t} - \frac{1}{t}\left[\cos x\,\frac{e^{tx}}{t} + \int \sin x\,\frac{e^{tx}}{t}\,dx\right]$$

$$I = \frac{\sin x\,e^{tx}}{t} - \frac{1}{t}\left[\frac{\cos x\,e^{tx}}{t} + \frac{1}{t}I\right]$$

$$I = \frac{\frac{e^{tx}}{t}\left(\sin x - \frac{\cos x}{t}\right)}{\left(1+\frac{1}{t^2}\right)} + c$$

$$I\Big|_0^{\pi/2} = \frac{e^{t\pi/2}}{\left(1+\frac{1}{t^2}\right)} + \frac{1/t}{\left(1+\frac{1}{t^2}\right)}$$

$$M_x(t) = \frac{1}{\left(1+\frac{1}{t^2}\right)}\left(e^{t\pi/2} + \frac{1}{t}\right)$$

$$E[X^r] = M_x^{(r)}(0)$$

$$E[X^3] = M_x^{(3)}(0)$$

$$M_x^{(1)}(t) = \frac{1}{\left(1+\frac{1}{t^2}\right)}\left(\frac{e^{t\pi/2}}{\pi/2} - \frac{1}{t^2}\right) + \left(e^{t\pi/2} + \frac{1}{t}\right)\frac{-1}{\left(1+\frac{1}{t^2}\right)^2}\left(\frac{-2}{t^3}\right)$$

$$M_x^{(2)}(t) = \frac{1}{\left(1+\frac{1}{t^2}\right)}\left(\frac{e^{t\pi/2}}{(\pi/2)^2} + \frac{2}{t^3}\right) + \left(\frac{e^{t\pi/2}}{\pi/2} - \frac{1}{t^2}\right)\frac{-1}{\left(1+\frac{1}{t^2}\right)^2}\left(\frac{-2}{t^3}\right)$$

$$+ 2\left[\frac{1}{(1+t^2)^2}\left(e^{t\pi/2} + \frac{t e^{t\pi/2}}{\pi/2}\right) + \left(1+te^{t\pi/2}\right)\frac{-2}{(t^2+1)^3}2t\right]$$

$$M_x^{(3)}(0) = 3\left(\frac{\pi}{2}\right)^2 - 6$$

2(C) $E[X] = 4$, $E[Y] = -4$, $E[X^2] = E[Y^2] = 20$

$\rho(X,Y) = -0.5$, $var(X) = 20 - 4^2 = 4 = var(Y)$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$$

$$cov(X,Y) = -0.5 \times \sqrt{4\times4} = -2$$

$$E[XY] - E[X]E[Y] = -2$$

$$E[XY] = -2 + (4)(-4) = -18$$

(a) $E[2X-Y] = 2E[X] - E[Y] = 2(4) - (-4) = 12$

(b) $var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$

$= 4 + 4 + 2(-2)$

$= \underline{4}$

(c) $E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[XY] = 40 - 2(18)$

$= \underline{4}$

classmate
Date
Page

3. (a), (c) let $Z = \max(X_1, X_2, \ldots, X_n)$

$$F_Z(z) = P(Z \le z)$$
$$= P(\max(X_1, X_2, \ldots, X_n) \le z)$$
$$= P(X_1 \le z, X_2 \le z, \ldots, X_n \le z)$$
$$= P(X_1 \le z)\, P(X_2 \le z) \ldots P(X_n \le z) \quad \{\text{Independent events}\}$$
$$= z \cdot z \ldots z \quad \{\text{uniform distribution over } (0,1)\}$$
$$= z^n$$

$$f_Z(z) = n z^{n-1}$$

$\frac{2}{3}$

$$\boxed{E[Z] = \int_{-\infty}^{\infty} z\, f_Z(z)\, dz = \frac{n}{n+1}}$$

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z)\, dz = \frac{n}{n+2}$$

$$\boxed{\mathrm{var}(Z) = E[Z^2] - (E[Z])^2 = \frac{n}{n+2} - \frac{n}{(n+1)^2}}$$

3. (b, d) let $Z = \min(X_1, \ldots X_n)$

$$F_Z(z) = P(Z \le z) = P(\min(X_1, \ldots, X_n) \le z)$$
$$= 1 - P(\min(X_1, \ldots, X_n) > z)$$
$$= 1 - P(X_1 > z, X_2 > z, \ldots, X_n > z) = 1 - P(X_1 > z)\, P(X_2 > z) \ldots P(X_n >$$
$$= 1 - (1 - P(X_1 \le z))(1 - P(X_2 \le z)) \ldots (1 - P(X_n \le z))$$
$$= 1 - (1-z)(1-z) \ldots (1-z)$$
$$= 1 - (1-z)^n$$

$$f_Z(z) = n(1-z)^{n-1}$$

$$\boxed{E[Z] = \int_{-\infty}^{\infty} z\, f_Z(z)\, dz = \frac{1}{n+1}}$$

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 f_Z(z)\, dz = \frac{2n}{n(n+1)(n+2)}$$

$$\boxed{\mathrm{var}(Z) = E[Z^2] - (E[Z])^2 = \frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)^2} = \frac{\cancel{n} \; -1}{(n+1)^2(n+2)}}$$

Q. 4. (a) $X \geq 0$

$$P[X \geq c] \leq \frac{E[X]}{c}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{c} x f_X(x)\, dx + \int_{c}^{\infty} x f_X(x)\, dx$$

$$= \int_{0}^{c} x f_X(x)\, dx + \int_{c}^{\infty} x f_X(x)\, dx \quad (\text{since } X \geq 0)$$

$$\geq \int_{c}^{\infty} x f_X(x)\, dx \quad \left[\text{since } \int_0^c x f_X(x)\, dx \geq 0\right]$$

$$\geq c \int_{c}^{\infty} f_X(x)\, dx$$

$$= c\, P[X \geq c]$$

$$\Rightarrow \boxed{P[X \geq c] \leq \frac{E[X]}{c}}$$

Eequality holds if $\int_0^c x f_X(x)\, dx = 0$

and $\int_c^{\infty} x f_X(x)\, dx = c \int_c^{\infty} f_X(x)\, dx$

$\Rightarrow$ $f_X(x)$ is a point function at c.

(b) from (a) $P(|X-\mu| \geq k\sigma) \leq \dfrac{E[|X-\mu|^2]}{k^2\sigma^2}$

$$\Rightarrow P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \quad \{E[|X-\mu|^2] = \sigma^2$$

(c) $P(-2 < X < 8)$, $\sigma^2 = 13 - 9 = 4$.

$= P(-5 < X-3 < 5) = P(|X-3| \leq 5)$

$= 1 - P(|X-3| \geq 5)$

$K\sigma = 5 \Rightarrow K = 5/2$

$P(|X-3| \geq 5) \leq \frac{4}{25}$

$1 - P(|X-3| \geq 5) \geq 1 - \frac{4}{25}$

$$\boxed{P(|X-3| < 5) \geq \frac{21}{25}}$$

Q5 (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A^2 = \begin{bmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \\ 102 & 126 & 150 \end{bmatrix}$

$A^3 = \begin{bmatrix} 468 & 576 & 684 \\ 1062 & 1305 & 1458 \\ 1656 & 2034 & 2412 \end{bmatrix}$

$A^3 - 15A^2 - 17A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

classmate

Date ____

Page ____

(b) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

eigen values of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 8, -1, -1$$

eigen vectors of A

$$V_1 \Rightarrow \begin{bmatrix} 3-8 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & 3-8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$V_1 = a\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad a \in \mathbb{R}$$

$$V_2 \Rightarrow \begin{bmatrix} 3+1 & 2 & 4 \\ 2 & +1 & 2 \\ 4 & 2 & 3+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$V_2 = a\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad a, b \in \mathbb{R}$$

eigen vector matrix $S = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

$$A = S \Lambda S^{-1}$$

where $\Lambda = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $S^{-1} = \frac{1}{9}\begin{bmatrix} 2 & 1 & 2 \\ 5 & -2 & -4 \\ -4 & -2 & +5 \end{bmatrix}$

Now, $A^k = S \Lambda^k S^{-1}$

where $\Lambda^k = \begin{bmatrix} 8^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix}$, $A^5 = \begin{bmatrix} 14563 & 7282 \\ 7282 & 3640 \\ 14564 & 7282 \end{bmatrix}$

classmate
Date
Page

6. $P[A_1] = 1/2, \quad P[A_2] = 1/3, \quad P[A_3] = 1/4$

$P[(A_1 \cap A_2) \cup A_3^c]$

$= P(A_1 \cap A_2) + P(A_3^c) - P(A_1 \cap A_2 \cap A_3^c)$

$= P(A_1)P(A_2) + 1 - P(A_3) - [P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)]$

$= \frac{1}{2} \times \frac{1}{3} + \left(1 - \frac{1}{4}\right) - \left[\frac{1}{2} \times \frac{1}{3} - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}\right]$

$= \underline{\underline{\frac{19}{24}}}$

Q7. $p(spam) = 0.11$

$p(non\ spam) = 0.89$

$p(s \mid spam) = 0.9$

$p(n \mid spam) = 0.1$

$p(s \mid non\ spam) = 0.15$

$p(n \mid non\ spam) = 0.85$

(a) $p(s) = p(s \mid spam) \times p(spam) + p(s \mid non\ spam) \times p(non\ spam)$

$= 0.9 \times 0.11 + 0.15 \times 0.89$

$= 0.2325$

(b) $p(spam \mid s) = \frac{p(spam \cap s)}{p(s)} = \frac{p(s \mid spam) \times p(spam)}{p(s)}$

$= \frac{0.9 \times 0.11}{0.2325}$

(c) $p(nonspam \mid n) = \frac{p(n \mid nonspam) \times p(nonspam)}{p(n)}$

$= \frac{0.85 \times 0.89}{0.7675}$

(d) $p(s \mid non\ spam) + p(n \mid spam)$

$= 0.15 + 0.1 = \underline{0.25}$

8. Using Jacobi's equation

$$\frac{\partial}{\partial t} \det(A(t)) = \det(A) \, Tr\left[A^{-1} \frac{\partial A(t)}{\partial t} \right]$$

$$\frac{\partial}{\partial X} \log|X| = \left[\frac{\partial}{\partial x_{ij}} \log|X| \right]$$

$$\frac{\partial}{\partial x_{ij}} \log|X| = \frac{1}{|X|} (-1)^{i+j} |X_{\setminus i, \setminus j}|$$

$$\frac{\partial}{\partial X} \log|X| = \frac{1}{|X|} \left[(-1)^{i+j} |X_{\setminus i, \setminus j}| \right]$$

let $A = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n$

$$f(\bar{x}) = \log|A|$$

$$\frac{\partial f(\bar{x})}{\partial x_i} = \frac{\partial}{\partial x_i} \log|A|$$

$$= \frac{1}{|A|} \frac{\partial}{\partial x_i} |A|$$

$$= \frac{1}{|A|} |A| \, Tr\left[A^{-1} \frac{\partial A}{\partial x_i} \right]$$

$$= Tr\left[A^{-1} F_i \right]$$

$$\nabla_{\bar{x}} f(\bar{x}) = \begin{bmatrix} Tr[A^{-1} F_1] \\ \vdots \\ Tr[A^{-1} F_n] \end{bmatrix}$$

9. (a) $\bar{X}_{3\times1} \sim N_3(\bar{\mu}, \Sigma)$

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Sigma = [\,cov(X_i, X_j)\,]$ $i = 1..3,\ j = 1..3$

since $cov(X_1, X_2) = 1 \neq 0$

$\Rightarrow X_1$ & X_2 are not independent.

$$N_3(\bar{\mu}, \Sigma) = \frac{1}{(2\pi)^{3/2}|\Sigma|} \exp\left\{-\frac{1}{2}(\bar{x}_3 - \bar{\mu})^T \Sigma^{-1} (\bar{x}_3 - \bar{\mu})\right\}$$

Now break $\Sigma = \begin{bmatrix} \Sigma_{12} & 0 \\ 0 & \Sigma_{33} \end{bmatrix}$ where $\Sigma_{12} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$, $\Sigma_{33} = [2]$

and $\bar{\mu} = \begin{bmatrix} \bar{\mu}_{12} \\ \mu_3 \end{bmatrix}$ where $\bar{\mu}_{12} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

Rewrite $N_3(\bar{\mu}, \Sigma)$

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{12}^{-1} & 0 \\ 0 & \Sigma_{33}^{-1} \end{bmatrix}$$

$$N_3(\bar{\mu}, \Sigma) = \frac{1}{(2\pi)^{3/2}|\Sigma_{12}||\Sigma_{33}|} \exp\left\{-\frac{1}{2}(\bar{x}_2 - \bar{\mu}_{12})^T \Sigma_{12}^{-1} (\bar{x}_2 - \bar{\mu}_{12}) - \frac{1}{2}(x_3 - \mu_3)^T \Sigma_{33}^{-1} (x_3 - \mu_3)\right\}$$

$$= \frac{1}{2\pi|\Sigma_{12}|} e^{\left\{-\frac{1}{2}(\bar{x}_2 - \bar{\mu}_{12})^T \Sigma_{12}^{-1} (\bar{x}_2 - \bar{\mu}_{12})\right\}} \cdot \frac{1}{\sqrt{2\pi}|\Sigma_{33}|} e^{\left\{-\frac{1}{2}(x_3 - \mu_3)^T \Sigma_{33}^{-1} (x_3 - \mu_3)\right\}}$$

$$= N_2(\bar{\mu}_{12}, \Sigma_{12}) \cdot N_1(\mu_3, |\Sigma_{33}|)$$

$\Rightarrow (X_1 X_2)$ & X_3 are independent.

Conditional probability $f(\bar{X}_2 | \bar{X}_1 = \bar{x}_1)$

(b) $\bar{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} \sim N_p(\bar{\mu}, \Sigma)$, $\bar{\mu} = \begin{bmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \end{bmatrix}$

$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

define new random variable $X_2' = X_2 - \Sigma_{21}\Sigma_{11}^{-1} X_1$

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2' \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}$$
$$= A\bar{X}$$

To show: X_1, X_2' are independent

$$\begin{aligned} cov(\bar{X}_1, \bar{X}_2') &= cov(X_1, X_2 - \Sigma_{21}\Sigma_{11}^{-1} X_1) \\ &= cov(X_1, X_2) - cov(X_1, X_1)\Sigma_{11}^{-1}\Sigma_{12} \\ &= \Sigma_{12} - \Sigma_{11}\Sigma_{11}^{-1}\Sigma_{12} \\ &= 0 \end{aligned}$$

Since X_1, X_2' are Normal distributions and their covariance is zero $\Rightarrow$ $\bar{X}_1, \bar{X}_2'$ are independent

$$\cancel{E[X_2 | X_1 = \bar{x}_1]} = \cancel{E[X_2' | X_1 = x_1] + \Sigma_{21}\Sigma_{11}^{-1} x_1}$$
$$= \cancel{\bar{\mu}_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \bar{\mu}_1)}$$

$$\begin{aligned} E[\bar{X}_2' | \bar{X}_1 = \bar{x}_1] &= E[\bar{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\bar{X}_1] \\ &= E[\bar{X}_2] - E[\Sigma_{21}\Sigma_{11}^{-1}\bar{X}_1] \\ &= \bar{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1}\bar{\mu}_1 \end{aligned}$$

$$\begin{aligned} E[\bar{X}_2 | \bar{X}_1 = \bar{x}_1] &= E[\bar{X}_2' | \bar{X}_1 = \bar{x}_1] + \Sigma_{21}\Sigma_{11}^{-1} x_1 \\ &= \bar{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1}\bar{\mu}_1 + \Sigma_{21}\Sigma_{11}^{-1}\bar{x}_1 \end{aligned}$$

$$\boxed{E[\bar{X}_2 | \bar{X}_1 = \bar{x}_1] = \bar{\mu}_2 + \Sigma_{22}\Sigma_{11}^{-1}(x_1 - \mu_1)}$$

$Cov(\bar{X}_2' | \bar{X}_1 = \bar{x}_1) = cov(\bar{X}_2')$

$$\begin{aligned} LHS = cov(\bar{X}_2' | \bar{X}_1 = \bar{x}_1) &= cov(\bar{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\bar{x}_1 | X_1 = \bar{x}_1) \\ &= cov(\bar{X}_2 | \bar{X}_1 = \bar{x}_1) \end{aligned}$$

$$RHS = cov(\bar{X}_2')$$

$$= cov(\bar{X}_2 - \Sigma_{21}\Sigma_{11}^{-1}\bar{X}_1)$$

$$\cancel{= cov(\bar{X}_2 \mid X_1 = \bar{x}_1)}$$

$$= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$$

$$\Rightarrow \boxed{cov(\bar{X}_2 \mid X_1 = \bar{x}_1) = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}}$$

(C) $$f_Y(\bar{y}) = \frac{1}{(2\pi)^{n/2}} |\Sigma|^{-1/2} \exp\left\{-\frac{(\bar{y}-\bar{\mu})'\Sigma^{-1}(\bar{y}-\bar{\mu})}{2}\right\}$$

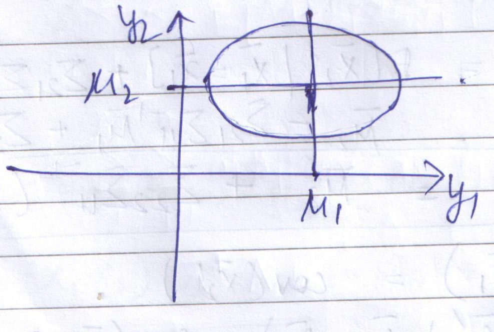
for $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\bar{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$

$$f_Y(\bar{y}) = \frac{1}{2\pi} |\Sigma|^{-1/2} \exp\left\{-\begin{bmatrix} y_1-\mu_1 \\ y_2-\mu_2 \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} y_1-\mu_1 \\ y_2-\mu_2 \end{bmatrix}\right\}$$

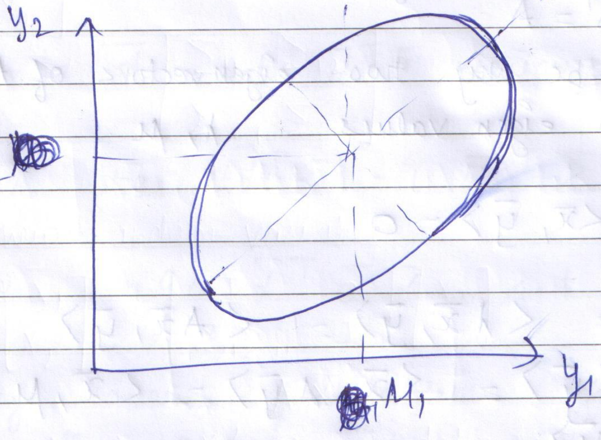
where $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_{11}\sigma_{12}}$$

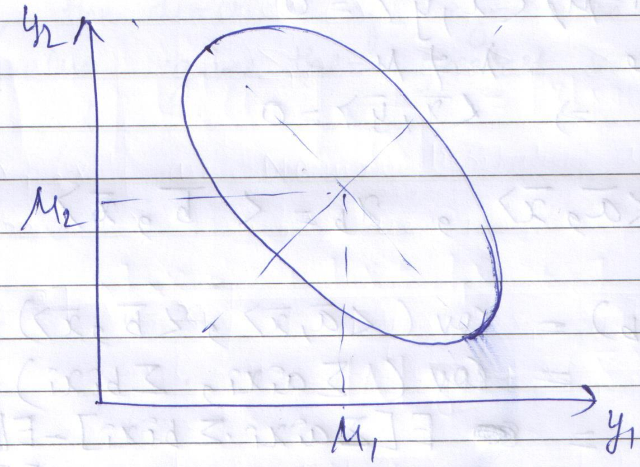
zero correlation, $\sigma_{12} = 0$



positive correlation $\sigma_{12} > 0$



negative correlation , $\sigma_{12} < 0$



classmate

Date

Page

Q10 (a) let A be real symmetric matrix.

$$A^T = A$$

let $\bar{x}, \bar{y}$ be any two eigen vectors of A.

with eigen values λ, μ.

To show $\langle \bar{x}, \bar{y} \rangle = 0$

$$\lambda \langle \bar{x}, \bar{y} \rangle = \langle \lambda \bar{x}, \bar{y} \rangle = \langle A\bar{x}, \bar{y} \rangle$$

$$= \langle \bar{x}, A^T \bar{y} \rangle = \langle \bar{x}, A\bar{y} \rangle = \langle \bar{x}, \mu \bar{y} \rangle$$

$$= \mu \langle \bar{x}, \bar{y} \rangle$$

$$\Rightarrow (\lambda - \mu) \langle \bar{x}, \bar{y} \rangle = 0$$

since $\lambda \neq \mu$

$$\Rightarrow \langle \bar{x}, \bar{y} \rangle = 0$$

(b) $x_a = \langle \bar{a}, \bar{x} \rangle$, $x_b = \langle \bar{b}, \bar{x} \rangle$

$$\operatorname{cov}(x_a, x_b) = \operatorname{cov}(\langle \bar{a}, \bar{x} \rangle, \langle \bar{b}, \bar{x} \rangle)$$

$$= \operatorname{cov}(\sum a_i x_i, \sum b_i x_i)$$

$$= E[\sum a_i x_i \cdot \sum b_i x_i] - E[\sum a_i x_i] E[\sum b_i x_i]$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i x_i b_j x_j\right] - \sum a_i E[x_i] \sum b_i E[x_i]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} E[a_i b_j x_i x_j] - \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j E[x_i] E[x_j]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \operatorname{cov}(x_i, x_j)$$

$$= \bar{a}^T A \bar{b}$$

(c) for a covariance matrix, Σ

To find $\bar{\alpha}$ s.t. $\bar{\alpha}^T \bar{x}$ maximizes $Var(\bar{\alpha}^T \bar{x})$

Now, $Var(\bar{\alpha}^T \bar{x}) = \bar{\alpha}^T \Sigma \bar{\alpha}$

Now $\max_{\bar{\alpha}} \bar{\alpha}^T \Sigma \bar{\alpha}$ s.t. $\bar{\alpha}^T \bar{\alpha} = 1$

Solving above problem gives $\bar{\alpha}$ to be the eigen vector with maximum eigen value.

Hence, the direction of eigenvector with maximum eigen value gives the greatest variance.