Online Multivariate Optimization

Course Project - CS773A

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Introduction

Background

Performance Measures

In many cases, we would like to evaluate our algorithm performance using specific "measures"

- What do these look like?
- How do we optimize them?

Problem Setting

Binary classification problem with linear classifiers ($\mathcal{W} \subset \mathbb{R}^d$)
$\mathcal{X} \subset \mathbb{R}^d$ and $\mathcal{Y} \subset \{+1, -1\}$

Notation

- $r^+(w, x, y) = \frac{1}{p} r(y, w^T x) 1(y = 1)$
- Concave performance measure : $P_\psi(w) = \psi(P(w), N(w))$.
- Fenchel Dual : $\psi^*(\alpha, \beta) = \inf_{u,v}\{\alpha u + \beta v - \psi(u, v)\}$
- An example performance measure is : H-mean - $\frac{2PN}{P+N}$

Relevant literature

Generic performance metrics

- [4] specifies a generic method that can solve for arbitrary performance measures
- Their step per iteration might be forced to solve an intractable problem
- This gives us a worst case solution to work with anyway

Problem specific

- [2] provides a batch-SGD method that works with ranking objectives.
- It proves sublinear regrets for adversarial non-stochastic settings
- [3] provides a SGD method that works with the measures of our choice, but the regret bounds proved are for stochastic settings.

This motivates us to find a technique that can generalize to adversarial settings as well.

Our goals

Learning even with adversaries

Try and adapt the existing techniques to work with an adversarial setting. This could mean reformulating or looking at different proof techniques

Learning with delays

Analyze the performance of the current algorithms when faced with delays.

Optimizing in the face of adversaries

Mirror Descent

Theoretical basis

- General technique similar to GD, but works on a "mirror map"
- Choose iterative updates based on proximity, not by a distance, but by a divergence
- Makes no assumption about choice of adversary

Claim

For a convex, L-Lipschitz objective function f, with an α strongly convex mirror map, MD will satisfy,

$$f\left(\frac{1}{t}\sum_{i=1}^{t} x_i\right) - \min f(x) \leq O\left(\frac{1}{\sqrt{t}}\right)$$

Mirror Descent - Proof

Outline

$$
\begin{aligned}
f(x_i) - f(x) &\leq g_i^T (x_i - x) && \text{(Convexity)} \\
&\leq \frac{1}{\eta}(\nabla\Phi(x_i) - \nabla\Phi(y_{i+1}))^T (x_i - x) && \text{(Update rule)} \\
&\leq \frac{1}{\eta}(B_\Phi(x, x_i) + B_\Phi(x_i, y_{i+1}) - B_\Phi(x, y_{i+1})) && \text{(Definition } B_\Phi\text{)}
\end{aligned}
$$

Telescoping and reducing

$$
\sum_{i=1}^{t}(f(x_i) - f(x)) \leq \frac{B_\Phi(x, x_1)}{\eta} + \frac{\eta L^2 t}{2\alpha}
$$

Saddle point optimization

Setting

Consider two sets, $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$. We are given an objective function, $\phi(.,.)$ such that $\phi(., y)$ is convex, $\phi(x, .)$ is concave.

Goal

Find $z^* = (x^*, y^*)$ such that,

$$\phi(x^*, y^*) = \inf_{x \in X} \sup_{y \in Y} \phi(x, y) = \sup_{y \in Y} \inf_{x \in X} \phi(x, y)$$

Gap in solution

The quality of a solution (x', y') can be measured by,

$$\sup_{y \in Y} \phi(x', y) - \inf_{x \in X} \phi(x, y')$$

Mirror Descent in Saddle points

Let us make an observation,

$$\phi(x,y) - \phi(x',y) \leq g_x^T(x-x')$$
$$-\phi(x,y) - (-\phi(x,y')) \leq (-g_y)^T(y-y')$$
$$\text{If we take } g_z = (g_x, -g_y)$$
$$\max_{y'\in Y}\phi(x,y') - \min_{x'\in X}\phi(x',y) \leq g_z^T(z-z')$$

Converting to MD setup

Suggests usage of Mirror Descent in the combined space

Analysis: Overview

MD-SP setting

Let Φ_X, Φ_Y be mirror maps defined on both the spaces X, Y. Consider the combined map, $\Phi(x, y) = \Phi_X(x) + \Phi_Y(y)$. This works on the "combined" space z.

Claim

Mirror descent with $\eta = O(\frac{1}{\sqrt{t}})$ satisfies,

$$\max_{y \in Y} \phi(\frac{1}{t}\sum_{i=1}^{t} x_i, y) - \min_{x \in X} \phi(x, \frac{1}{t}\sum_{i=1}^{t} y_i) \leq O(\frac{1}{\sqrt{t}})$$

Mirror Descent for Saddle points

First, consider the norm on the combined field,

$$\|z\|_Z = \sqrt{\frac{\|x\|_X^2}{\kappa_X} + \frac{\|y\|_Y^2}{\kappa_Y}}$$

$$\|g_t\|_Z \leq \sqrt{\frac{L_x^2}{\kappa_X} + \frac{L_Y^2}{\kappa_Y}}$$

We can proceed in the same manner,

$$\begin{aligned}\phi(\frac{1}{t}\sum_{i=1}^{t} x_i, y) - \phi(x, \frac{1}{t}\sum_{i=1}^{t} y_i) &\leq \frac{1}{t}\sum_{i=1}^{t} \phi(x_i, y) - \phi(x, y_i) \\ &\leq \frac{1}{t}\sum_{i=1}^{t} g_i^T(z_i - z)\end{aligned}$$

Algorithm

Algorithm 1: Online Primal Dual Method

$w_0 \leftarrow 0,\ t \leftarrow 1$
while *data stream has points* **do**
  Receive data point (x_t, y_t)
  if $y_t > 0$ **then**
    $((\alpha, \beta), w_{t+1}) \leftarrow \mathrm{MD}(\psi^*, r^+)$
  end
  else
    $((\alpha, \beta), w_{t+1}) \leftarrow \mathrm{MD}(\psi^*, r^-)$
  end
  $t \leftarrow t + 1$
end
return $\bar{w} = \frac{1}{t} \sum_{\tau=1}^{t} w_\tau$

Simple mirror descent steps on the combined function.

Analysis for our setting

Similarity with MD-SP

We can see that our setting is identical to the MD-SP setting. So we can proceed with MD style updates, to obtain regret for our algorithm as,

$$\begin{aligned} \phi(x, y) &= \phi((\alpha, \beta), w) \\ &= \psi^*(\alpha, \beta) + P_\psi(w) \end{aligned}$$

$$\sum \phi((\hat{\alpha}, \hat{\beta}), w^*) - \inf \phi((\alpha, \beta), \hat{w}) \leq O(\sqrt{t})$$

We can then follow the analysis in [3] to finally obtain regret bounds.

Optimizing in the face of delays

Approach

- Analyzing the mirror descent under delayed feedback
- Recent work by P Joulani et al. “Delay-Tolerant Online Convex Optimization: United Analysis and Adaptive Gradient Algorithms”

Algorithm

Algorithm 2: Single-Instance Online Learning In Delayed environments (SOLID) [1]

```
Set x ← first prediction of BASE
for each time step t = 1, 2, ... do
    Set x_t ← x as the prediction for the current time step.
    Receive the set of feedback H_t that arrives at the end of time step t.
    for each f_s ∈ H_t do
        Update BASE with f_s.
        x ← the next prediction of BASE.
    end
end
```

Regret

From the results of [1] for Mirror Descent, the regret becomes

$$R_n \leq \frac{2R^2}{\tilde{\eta}_n} + \frac{G^2}{2} \sum_{s=1}^{n} \tilde{\eta}_s \left(1 + 2\tilde{\tau}_s\right)$$

Where R is constant such that $\tilde{\eta}_n \sum_{s=1}^{n} B_\Phi(x^*, x_s) \leq 2R^2$, G is the upper bound on the norm of the gradient, $\tilde{\tau}_s$ is difference between the time of s_{th} prediction by the BASE and the time of feedback received for that prediction.

Final Regret: $\mathcal{O}(\sqrt{T + 2\mathcal{T}})$ where $\mathcal{T} = \sum_{t=1}^{T} \tilde{\tau}_t$

Future Work

Future work

- Analysis of non-Lipschitz function where direct MD can't be applied.
- Experimental validation of regret bounds in case of delays
- Introduction of a smoothness/strongly convex surrogate for these rewards, and to see if this can lead to better rates of convergence.

Questions?

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