Bayesian Inference for Binomial Proportions

You flip a coin three times:

Do you think the coin is fair?

Your belief depends on your prior knowledge.

- Newborn: Heads it is!
- You: Most coins are fair.

P-value:

$$P(D_{(\text{or} > D)} | H0)$$

Posterior Probability

$P(H0|D)$

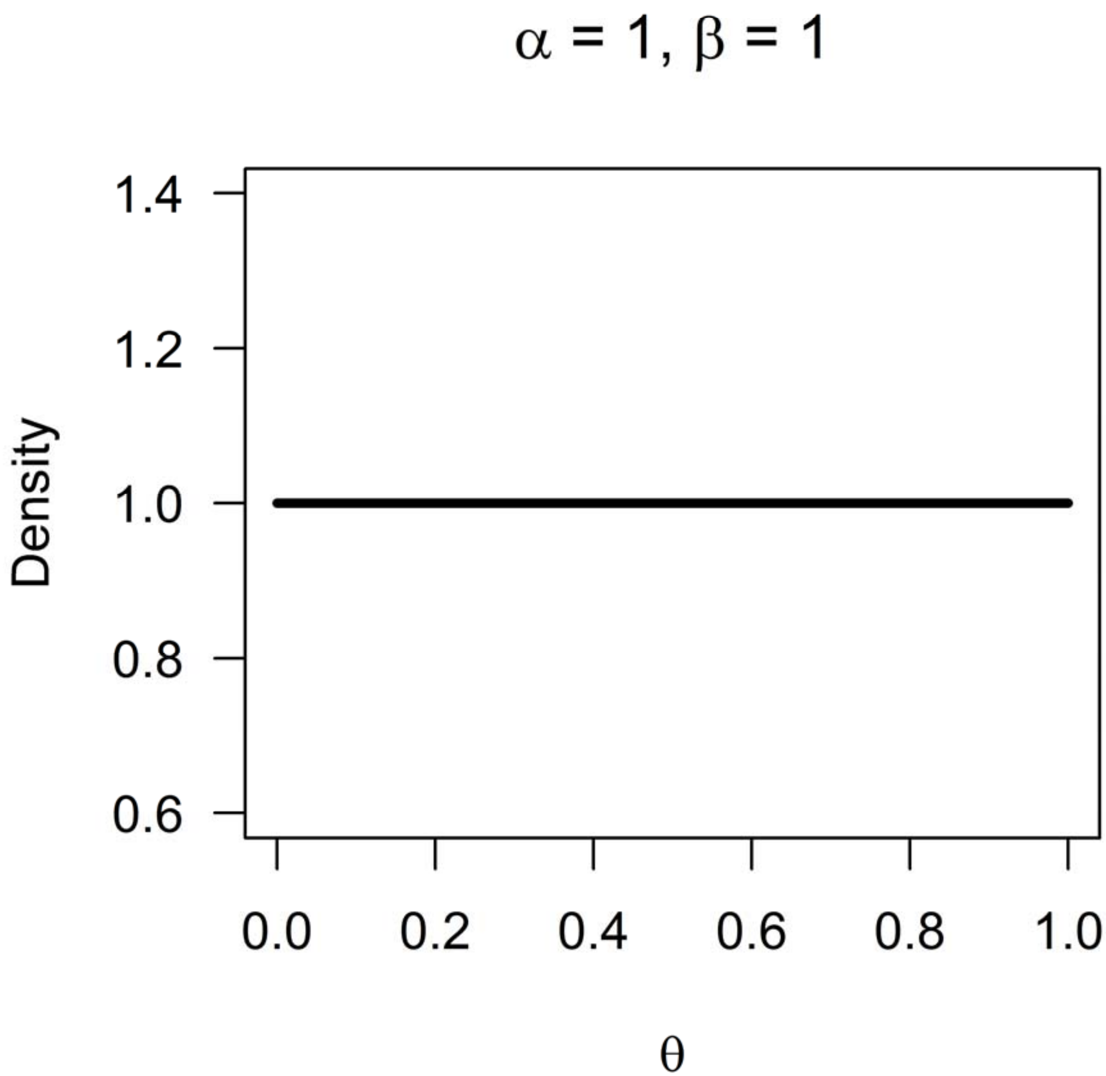
Prior Belief +
Data =
Posterior Belief

Posterior odds:

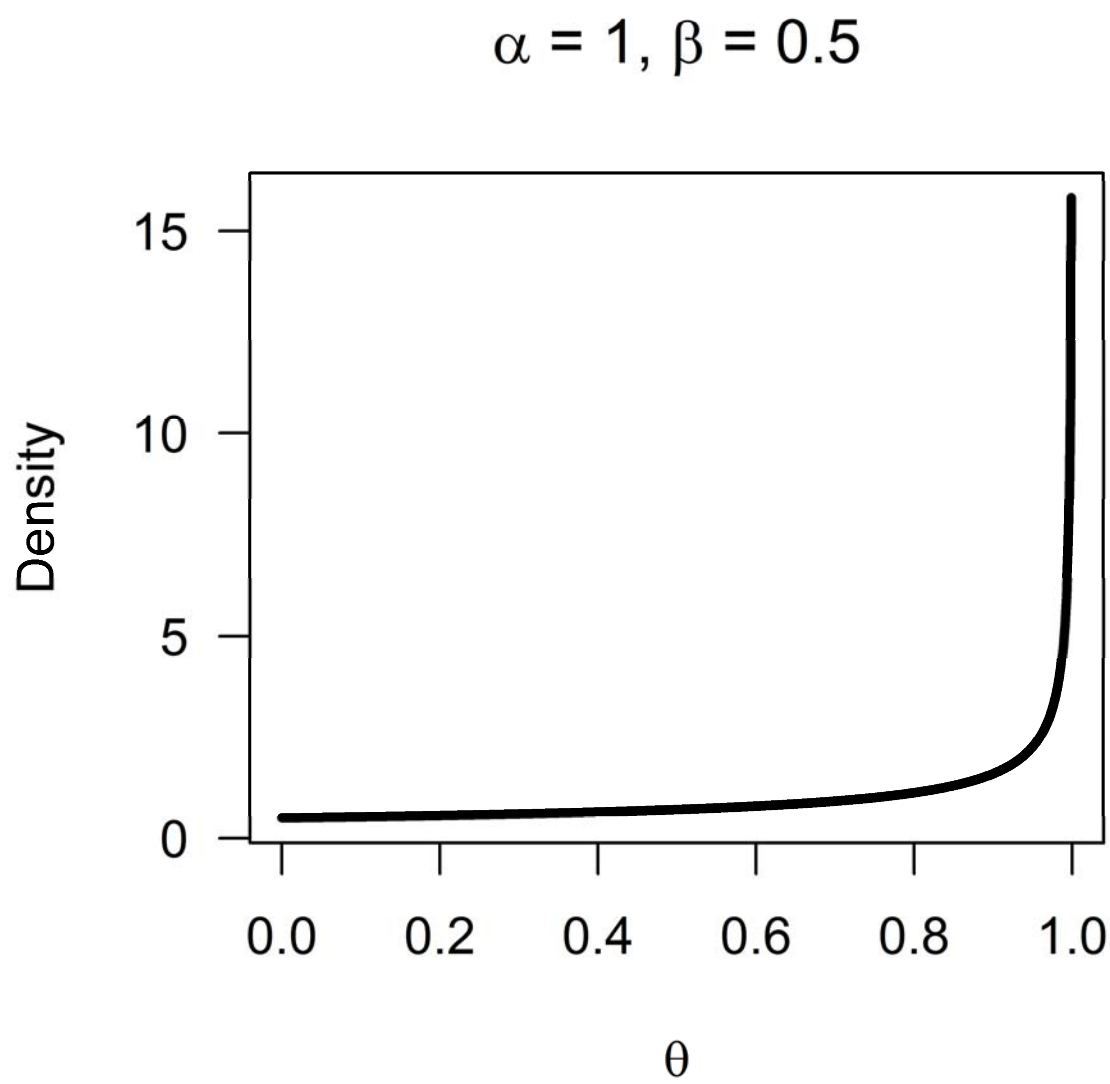
$$\frac{P(\mathrm{H1}|\mathrm{D})}{P(\mathrm{H0}|\mathrm{D})} = \frac{P(\mathrm{D}|\mathrm{H1})}{P(\mathrm{D}|\mathrm{H0})} \times \frac{P(\mathrm{H1})}{P(\mathrm{H0})}$$

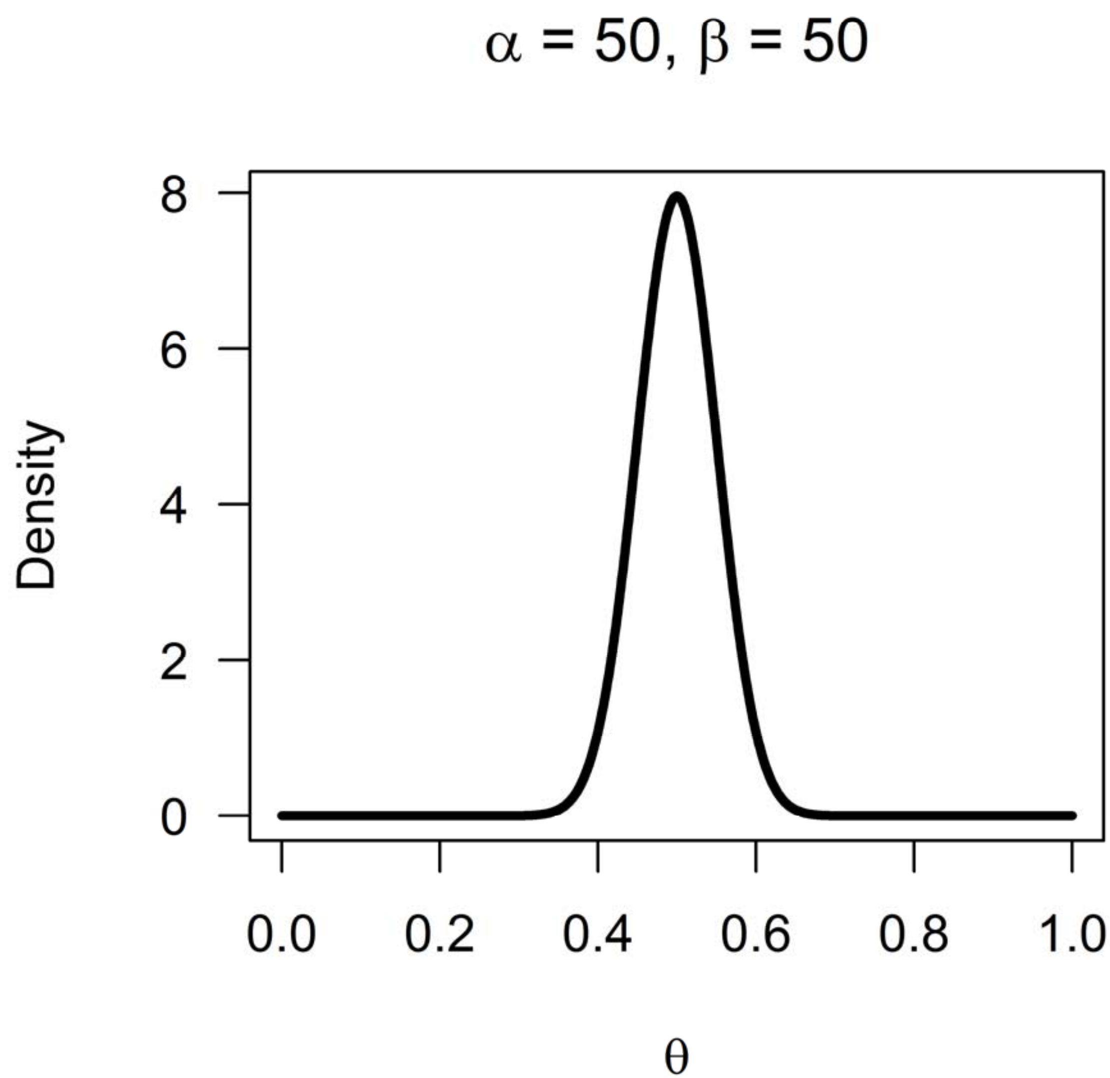
$$Posterior \ = \ Likelihood\ Ratio \ \times \ Prior$$

For the prior, a **beta distribution** is used. The beta prior is determined by α and β.



α = 1, β = 1
1.4
1.2
1.0
0.8
0.6
Density
0.0
0.2
0.4
0.6
0.8
1.0
θ


α = 1, β = 0.5
Density
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0
θ

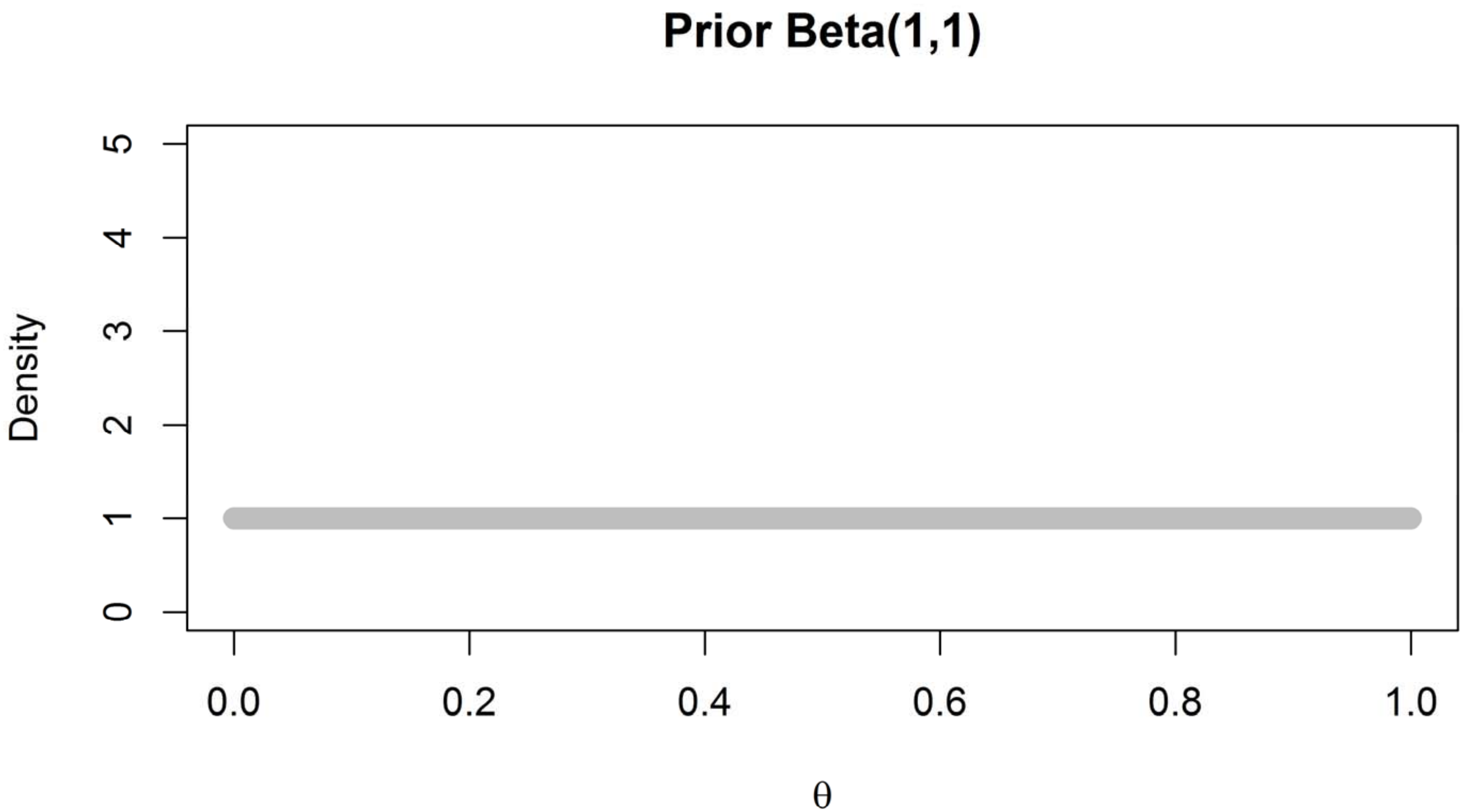

α = 50, β = 50
Density
0
2
4
6
8
0.0
0.2
0.4
0.6
0.8
1.0
θ

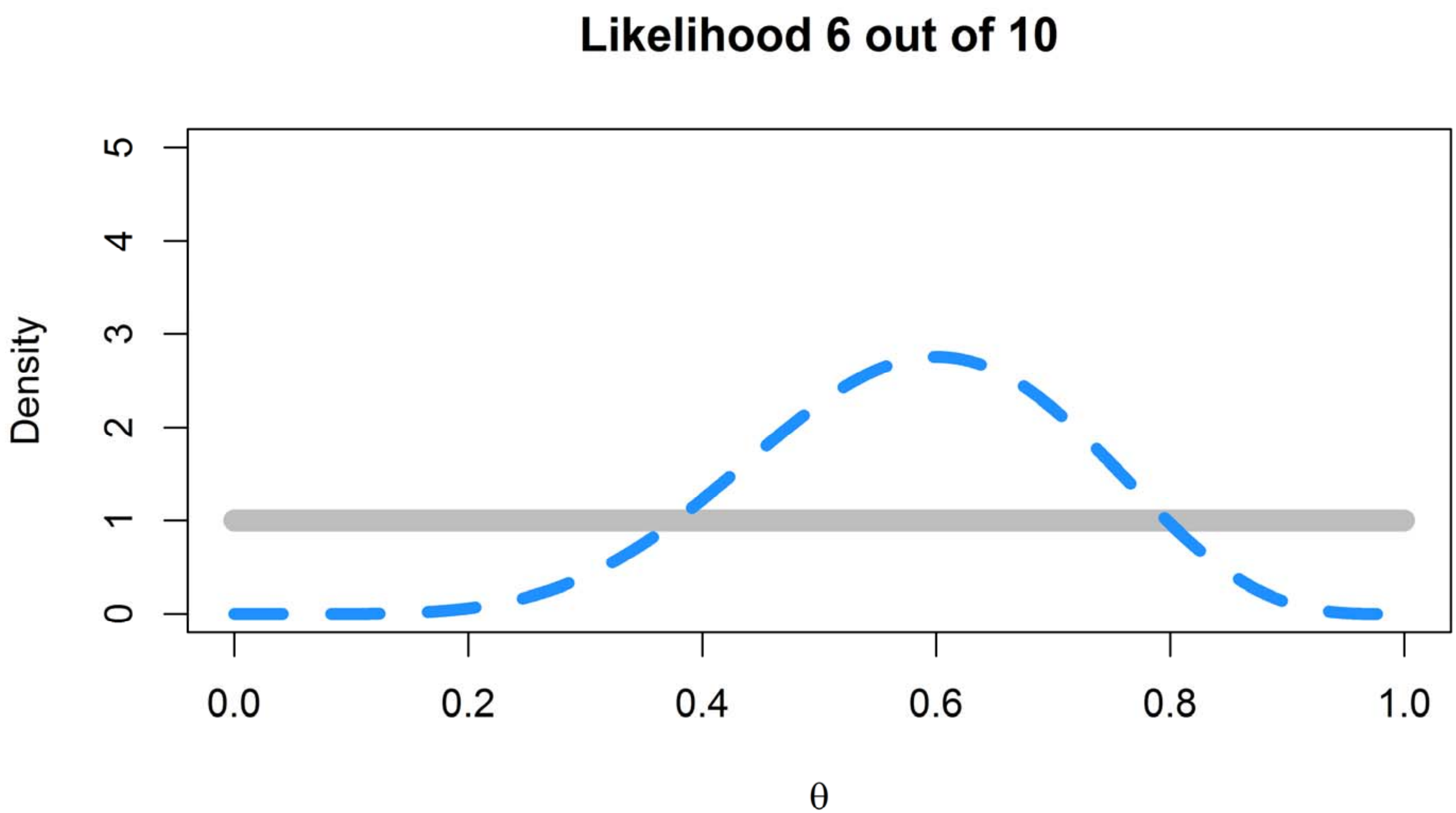
Now let's combine our prior belief with the data!

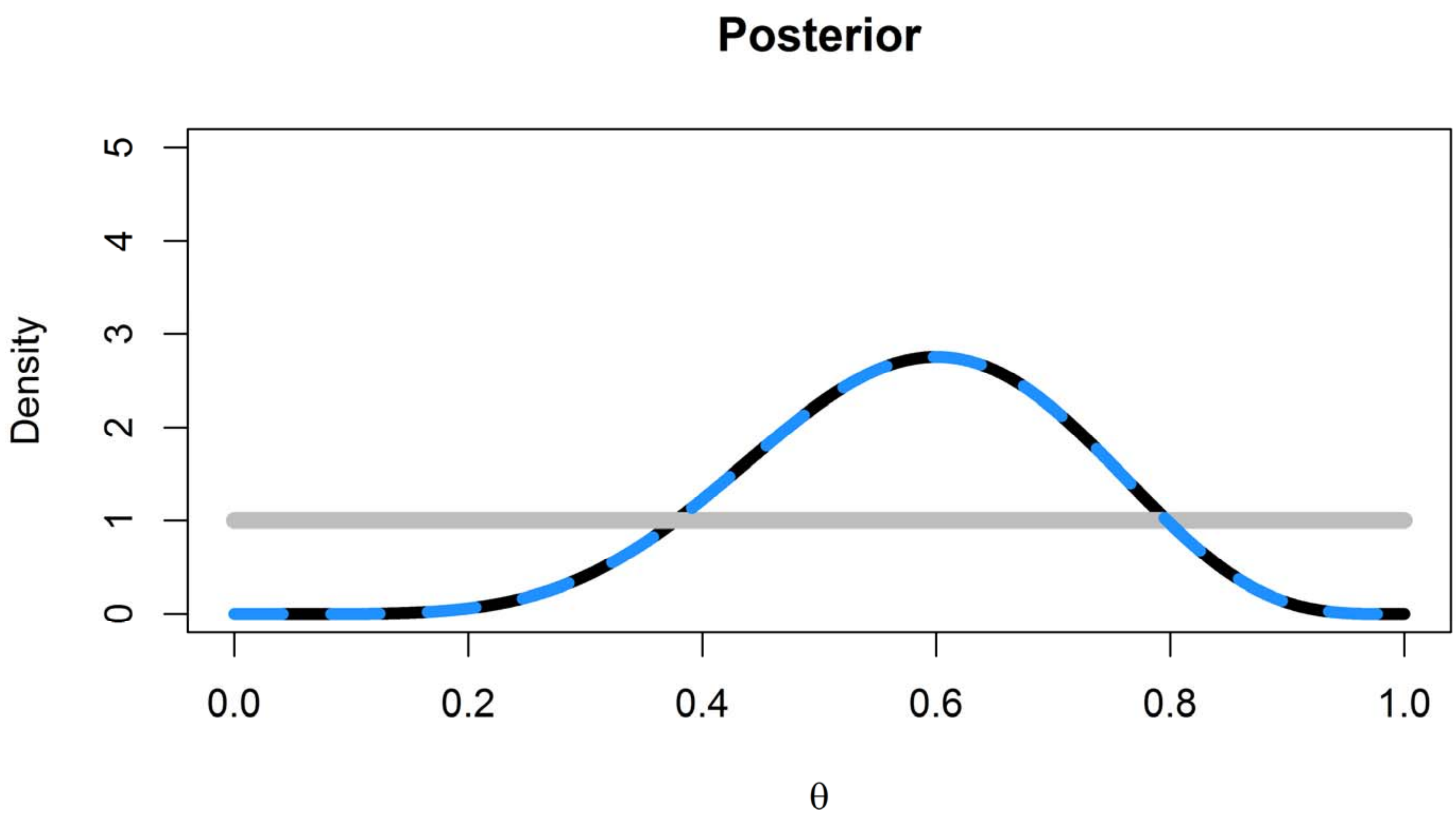
The posterior is a Beta(α^*, β^*) distribution:

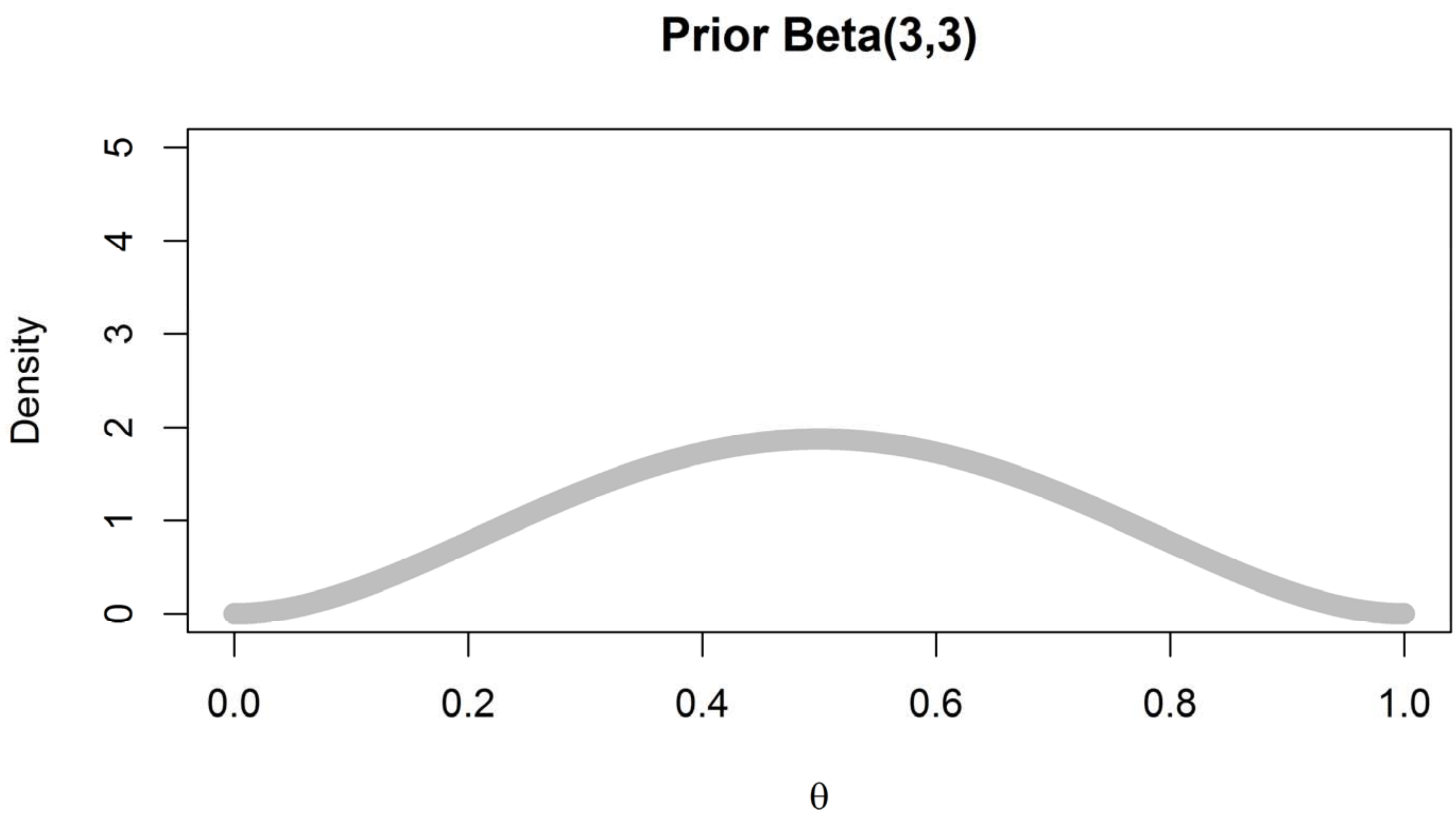
$$\alpha^* = \alpha_{prior} + \alpha_{likelihood} - 1$$

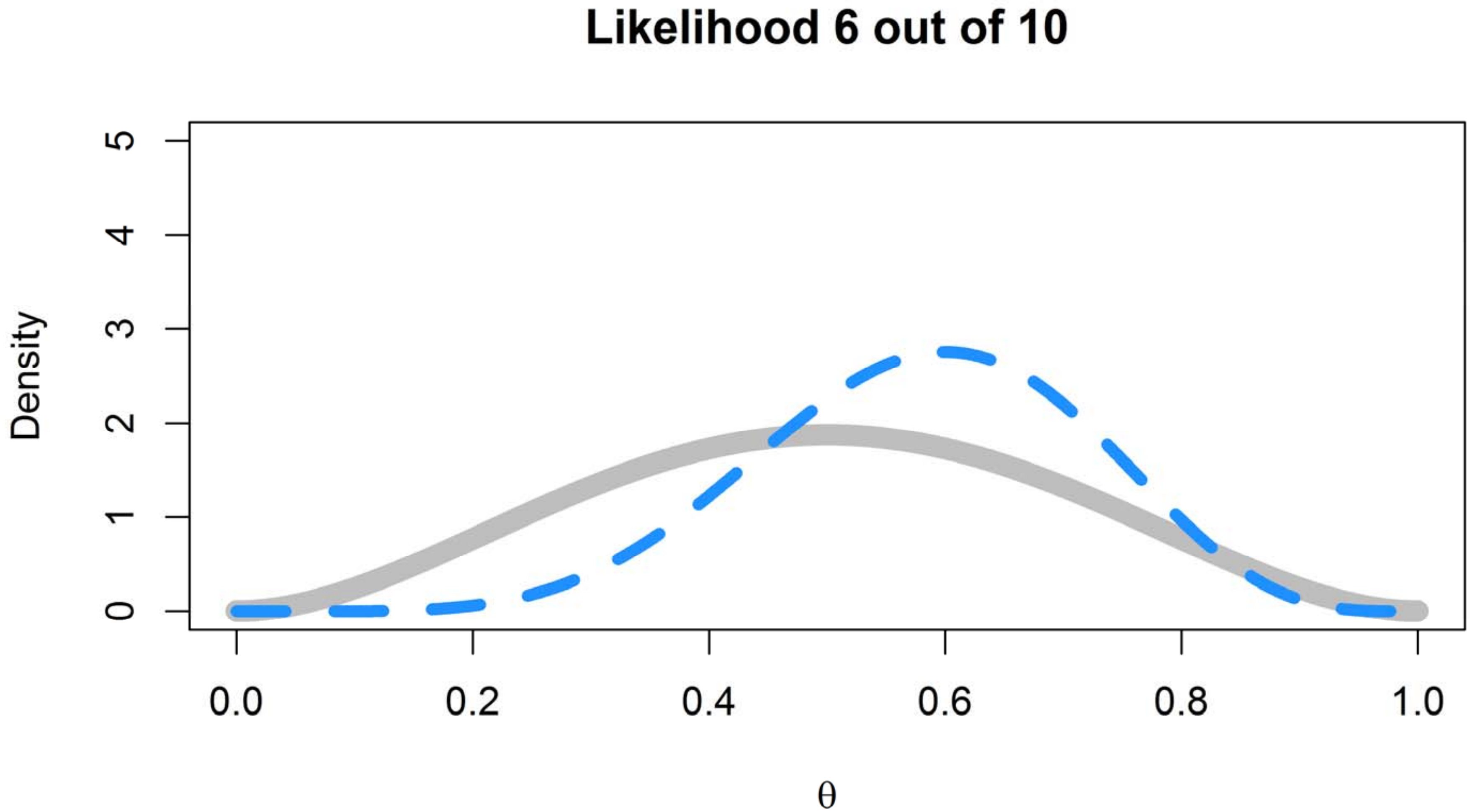
$$\beta^* = \beta_{prior} + \beta_{likelihood} - 1$$


Prior Beta(1,1)
Density
θ
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0

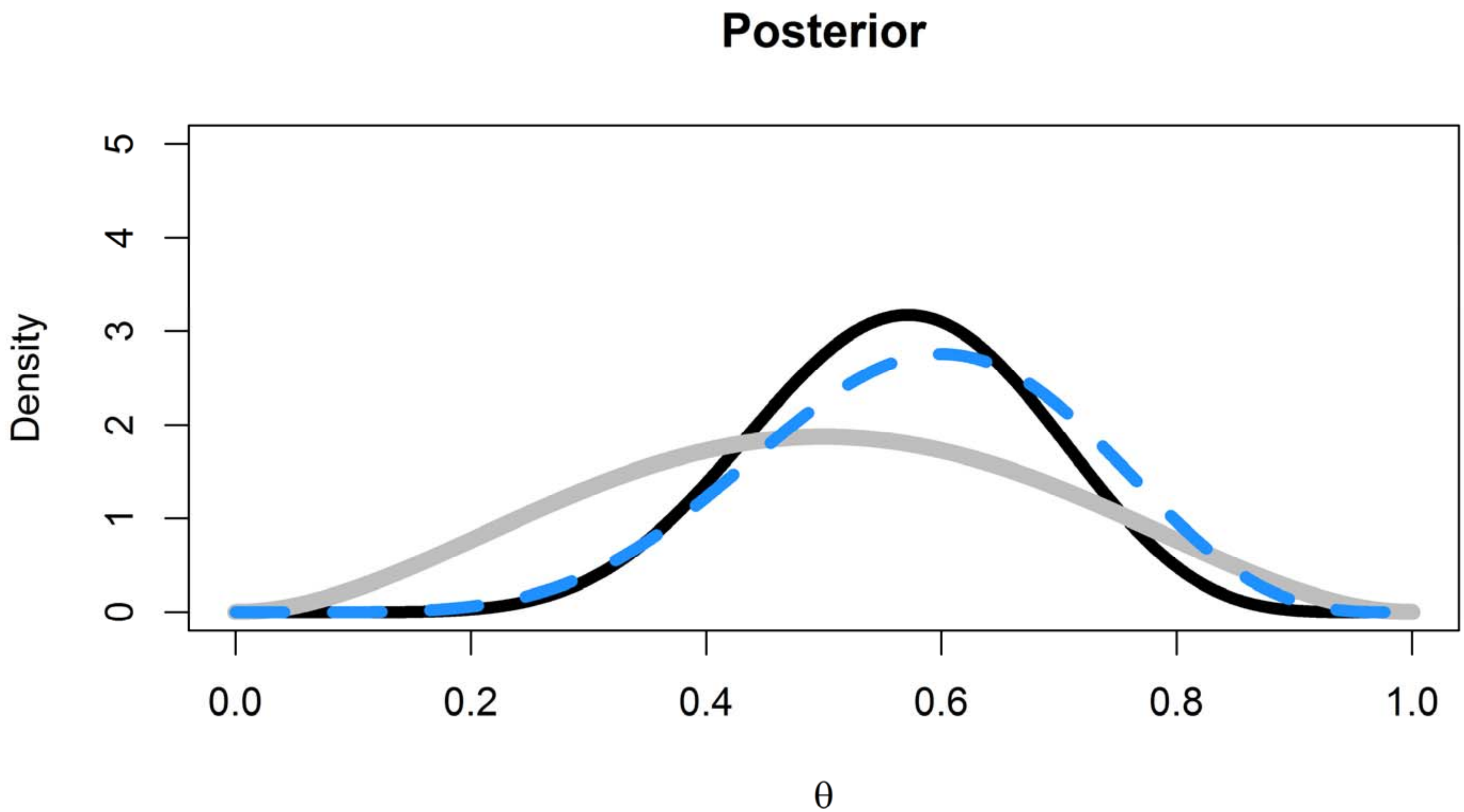

Likelihood 6 out of 10
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ


Posterior
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ


Prior Beta(3,3)
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ



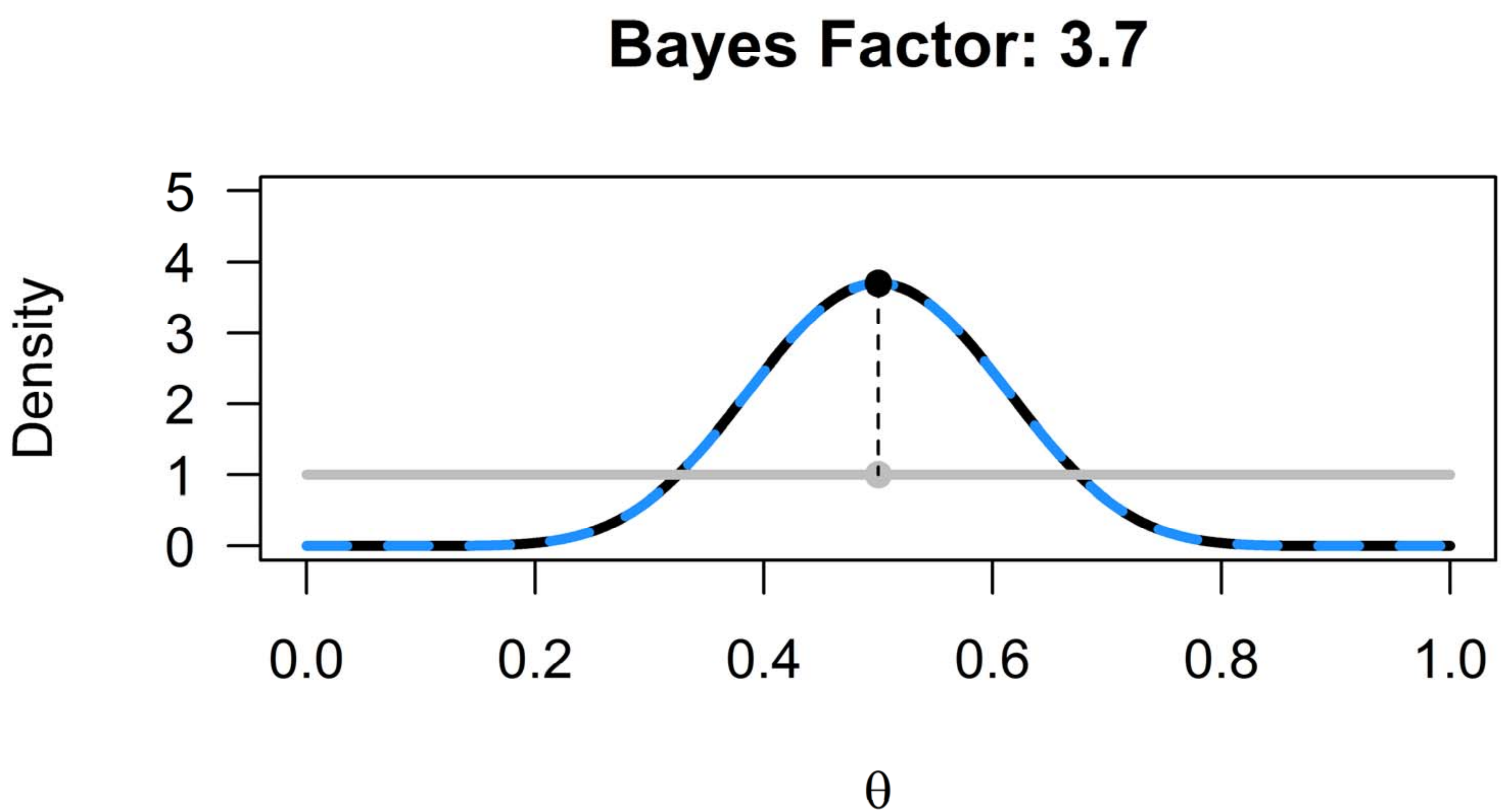
Likelihood 6 out of 10
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ

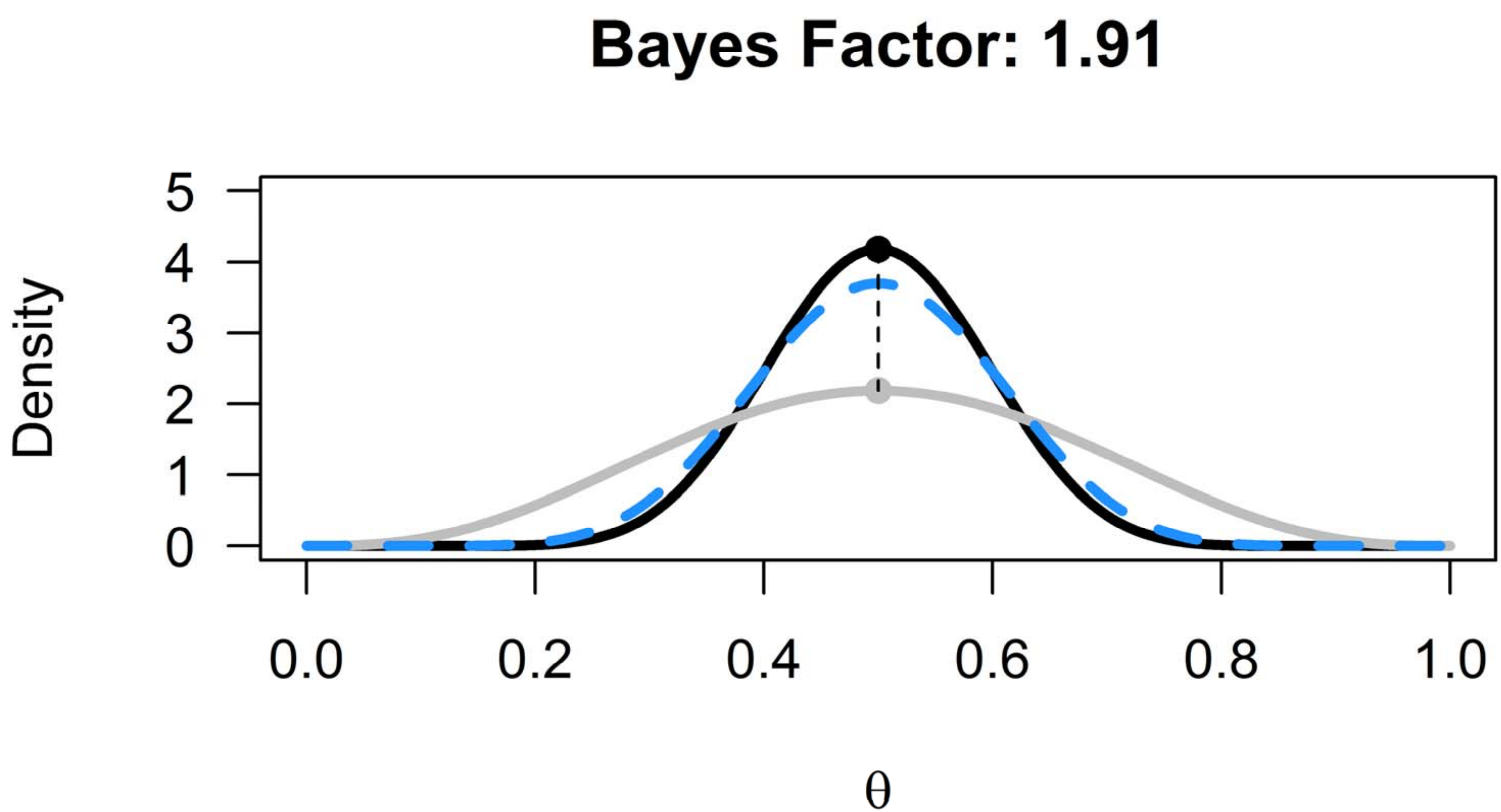


Posterior
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ

A Bayes factor is the *relative evidence* for one model compared to another.

The data from 20 coin flips gives 10 heads. Our prior is either Beta(1, 1) or Beta(4, 4)


Bayes Factor: 3.7
Density
5
4
3
2
1
0
0.0
0.2
0.4
0.6
0.8
1.0
θ

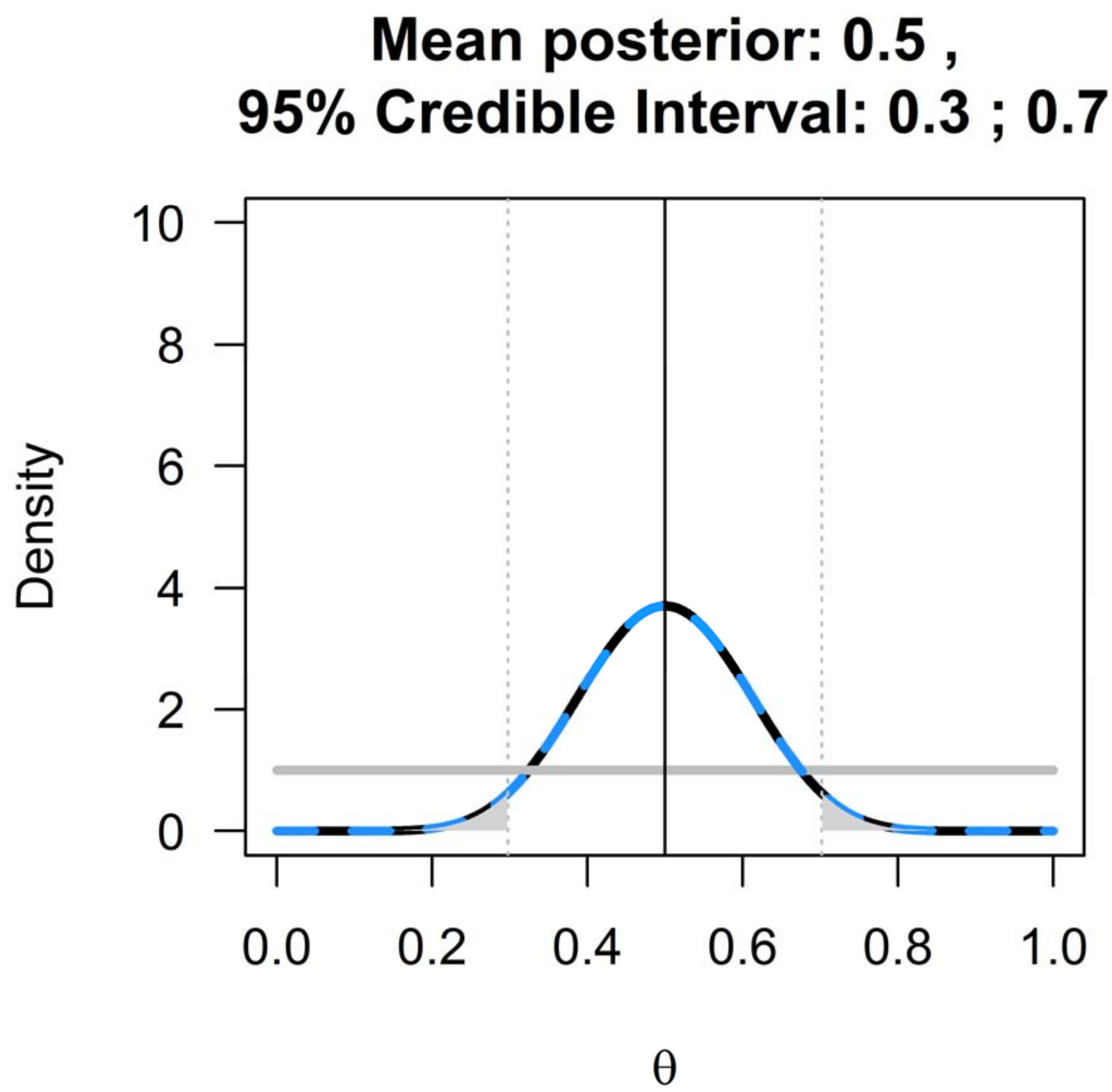


Bayes Factor: 1.91
Density
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
θ

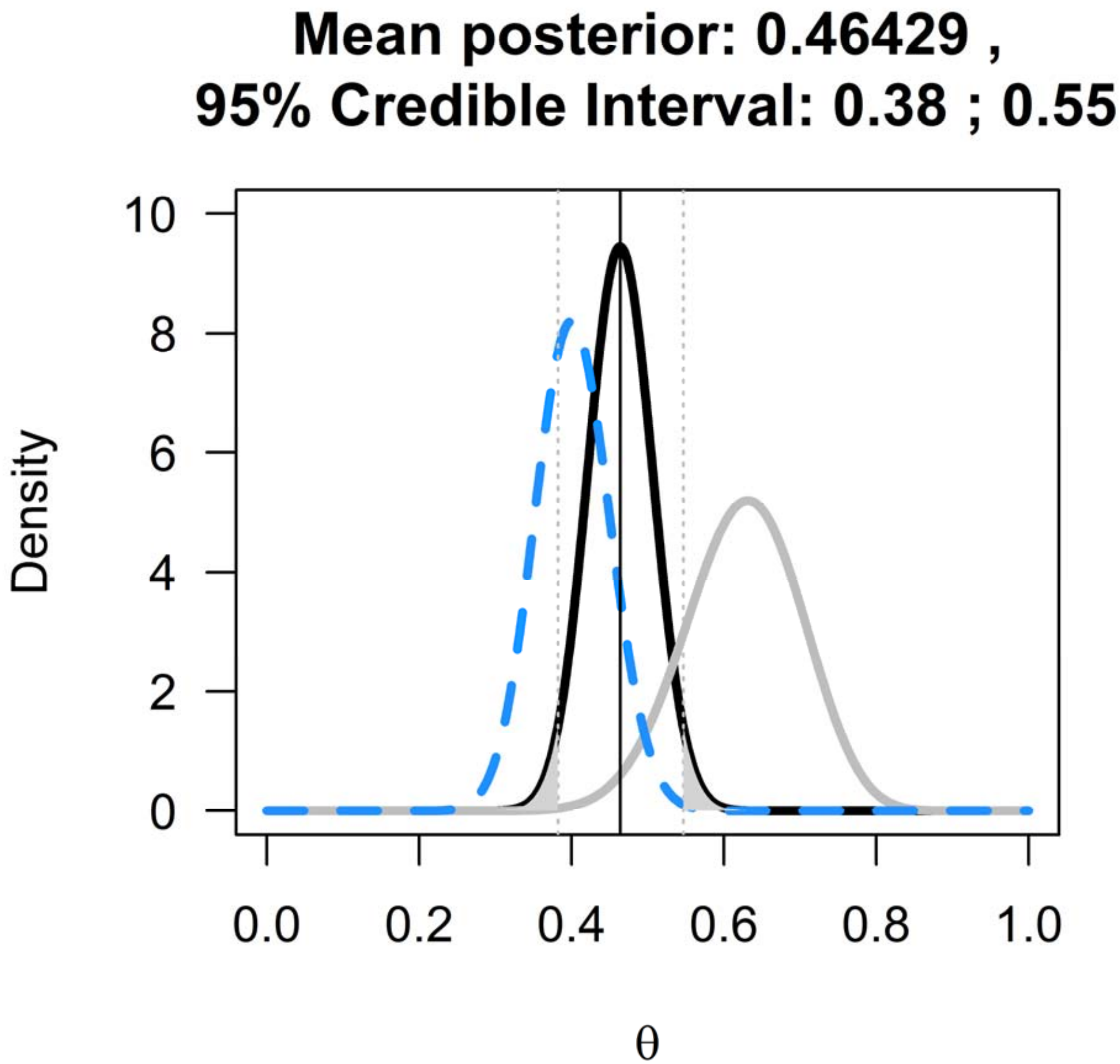
After the looking at the data, $\theta = 0.5$ has become 1.91 or 3.70 times more likely, depending on the prior

Bayesian Estimation

Instead of testing two models (prior vs. posterior) you can also use the posterior to estimate plausible values.



Mean posterior: 0.5 ,
95% Credible Interval: 0.3 ; 0.7
Density
10
8
6
4
2
0
0.0
0.2
0.4
0.6
0.8
1.0
θ


Mean posterior: 0.46429 ,
95% Credible Interval: 0.38 ; 0.55
Density
0
2
4
6
8
10
0.0
0.2
0.4
0.6
0.8
1.0
θ

A 95% credible interval contains the values you find most **plausible**.

Bayesian statistics allows us to update and quantify our beliefs.