Mathematical background

- Sets and De Morgan's laws
- Sequences and ther limits
- Infinite series
  - The geometric series
- Sums with multiple indices
- Countable and uncountable sets

Sets

- A collection of distinct elements

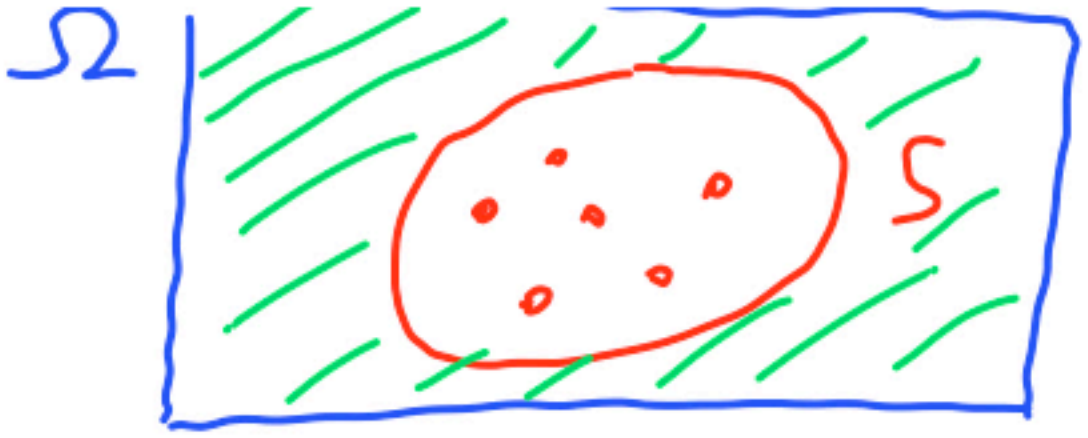
$\{a, b, c, d\}$ finite

$\mathbb{R}$: real numbers infinite

$\{x \in \mathbb{R} : \cos(x) > 1/2\}$

Ω : universal set

ϕ : empty set $\quad \Omega^c = \phi$



$x \in S \qquad x \notin S$

S^c

$x \in S^c$ if $x \in \Omega$,
$x \notin S$

$(S^c)^c = S$

$S \subset T : x \in S \Rightarrow x \in T$

$\subseteq$

Unions and intersections

$S \cup T$ $\quad x \in S \cup T \iff x \in S$ or $x \in T$

$S \cap T$ $\quad x \in S \cap T \iff x \in S$ and $x \in T$

$S_n \quad n = 1, 2, \ldots$

$x \in \bigcup_n S_n$ iff $x \in S_n$, for some n

$x \in \bigcap_n S_n$ iff $x \in S_n$, for all n

Set properties

$S \cup T \cup U$

$$S \cup T = T \cup S, \qquad S \cup (T \cup U) = (S \cup T) \cup U,$$
$$S \cap (T \cup U) = (S \cap T) \cup (S \cap U), \qquad S \cup (T \cap U) = (S \cup T) \cap (S \cup U),$$
$$(S^c)^c = S, \qquad S \cap S^c = \emptyset,$$
$$S \cup \Omega = \Omega, \qquad S \cap \Omega = S.$$

$S \cap (T \cap U) = (S \cap T) \cap U$

$$\left.\begin{array}{l} S \subset T \\ T \subset S \end{array}\right\} \Rightarrow S = T$$

De Morgan's laws

$$\boxed{(S\cap T)^c = S^c \cup T^c}$$

$S \to S^c \quad T \to T^c$

$S^c \to S \quad T^c \to T$

$$(S^c \cap T^c)^c = S \cup T$$

$$\boxed{S^c \cap T^c = (S \cup T)^c}$$

$$\Big(\bigcap_n S_n\Big)^c = \bigcup_n S_n^c$$

$$\Big(\bigcup_n S_n\Big)^c = \bigcap_n S_n^c$$

$$x \in (S\cap T)^c \Leftrightarrow x \notin S\cap T \Leftrightarrow \left\{\begin{array}{c} x\notin S \\ \text{or} \\ x \notin T \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} x\in S^c \\ \text{or} \\ x \in T^c \end{array}\right\} \Leftrightarrow x \in S^c \cup T^c$$

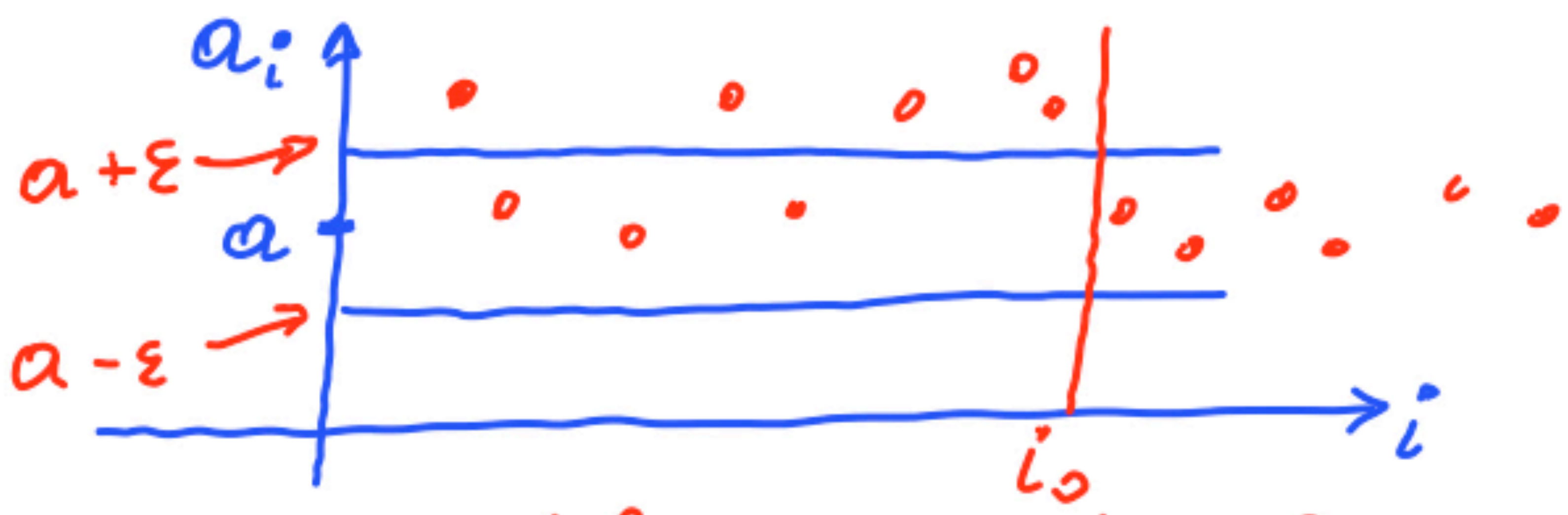
Mathematical background: Sequences and their limits

$a_1, a_2, a_3, \dots$ $\qquad i \in \mathbb{N} = \{1, 2, 3, \dots\}$

sequence a_i , $\{a_i\}$ $\qquad a_i \in S \qquad S = \mathbb{R} \qquad \mathbb{R}^n$

function $f: \mathbb{N} \to S$

$f(i) = a_i$



$$\left.\begin{array}{l} a_i \xrightarrow[i\to\infty]{} a \\ \\ \lim\limits_{i\to\infty} a_i = a \end{array}\right\}$$

For any $\varepsilon > 0$, there exists i_0, such that if $i \geq i_0$, then $|a_i - a| < \varepsilon$

$a_i^2 \to a^2$

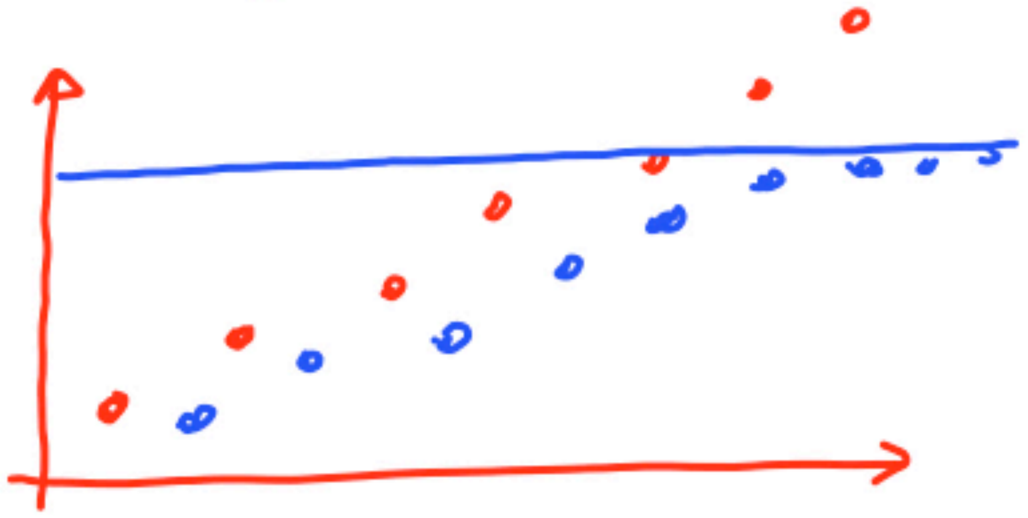
$$\left.\begin{array}{l} a_i \to a \\ b_i \to b \end{array}\right\} \Rightarrow \begin{array}{l} a_i + b_i \to a + b \\ a_i b_i \to a b \end{array}$$

g: continuous

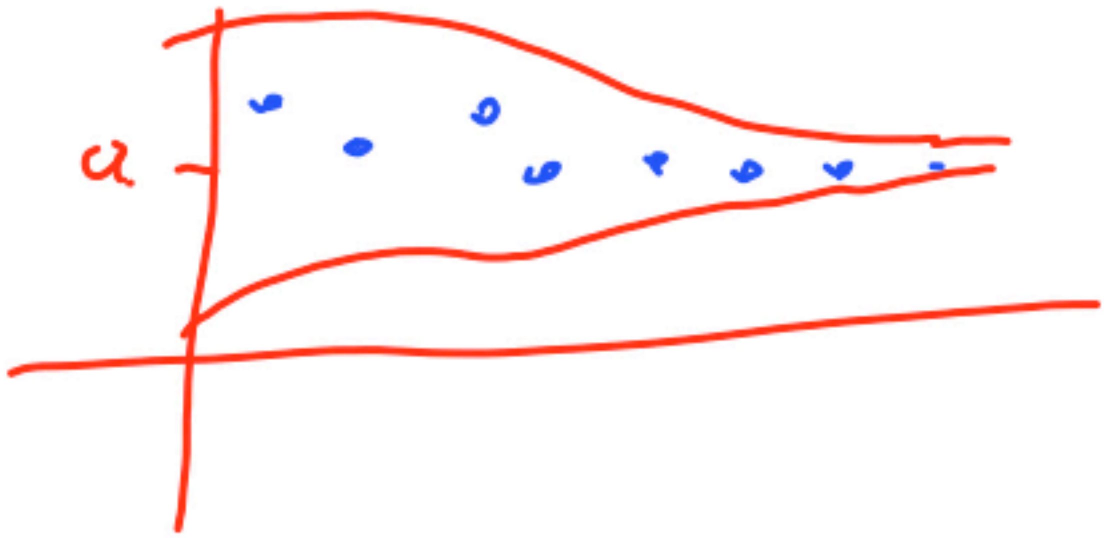
$\Rightarrow g(a_i) \to g(a)$

Mathematical background: When does a sequence converge?

- If $a_i \leq a_{i+1}$, for all i, then either:
  - the sequence "converges to ∞"
  - the sequence converges to some real number a



- If $|a_i - a| \leq b_i$, for all i, and $b_i \to 0$, then $a_i \to a$



Mathematical background: Infinite series

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i .$$ provided limit exists

- If $a_i \geq 0$: limit exists ←

- if terms a_i do not all have the same sign:
  - limit need not exist
  - limit may exist but be different if we sum in a different order
  - **Fact:** limit exists and independent of order of summation if $\sum_{i=1}^{\infty} |a_i| < \infty$

Mathematical background: Geometric series

$$S = \boxed{\sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \cdots \quad = \frac{1}{1-\alpha} \qquad |\alpha| < 1}$$

$$(1-\alpha)(1+\alpha+\cdots+\alpha^n) = 1 - \alpha^{n+1}$$

$n \to \infty$

$$(1-\alpha) S = 1$$

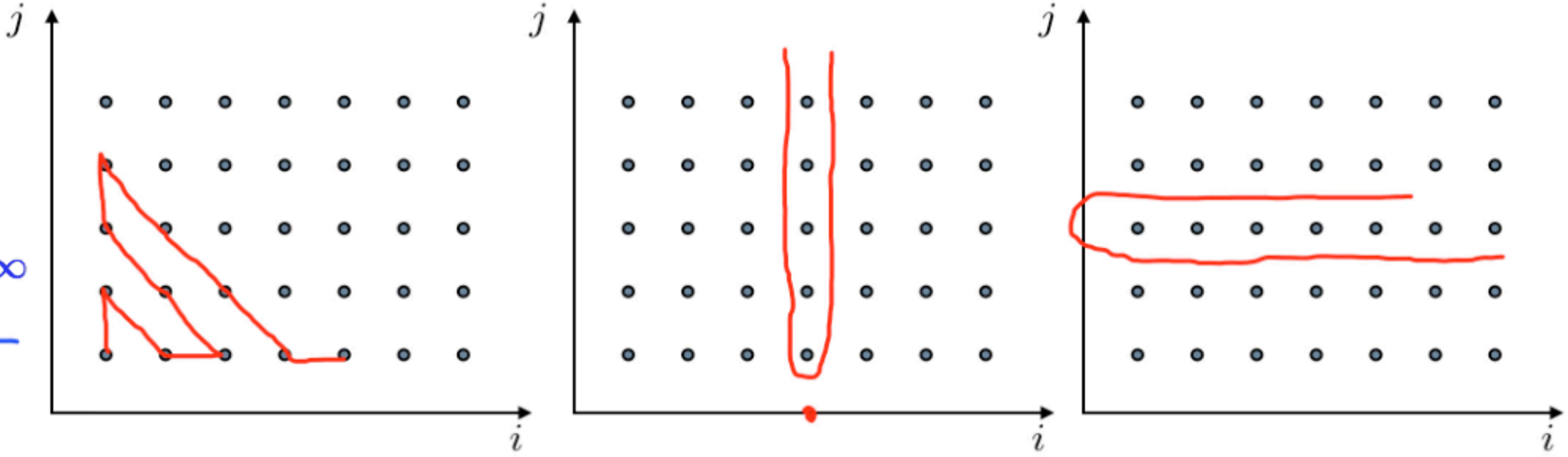
$$S = 1 + \sum_{i=1}^{\infty} \alpha^i = 1 + \alpha \sum_{i=0}^{\infty} \alpha^i = 1 + \alpha S \Rightarrow S(1-\alpha) = 1$$

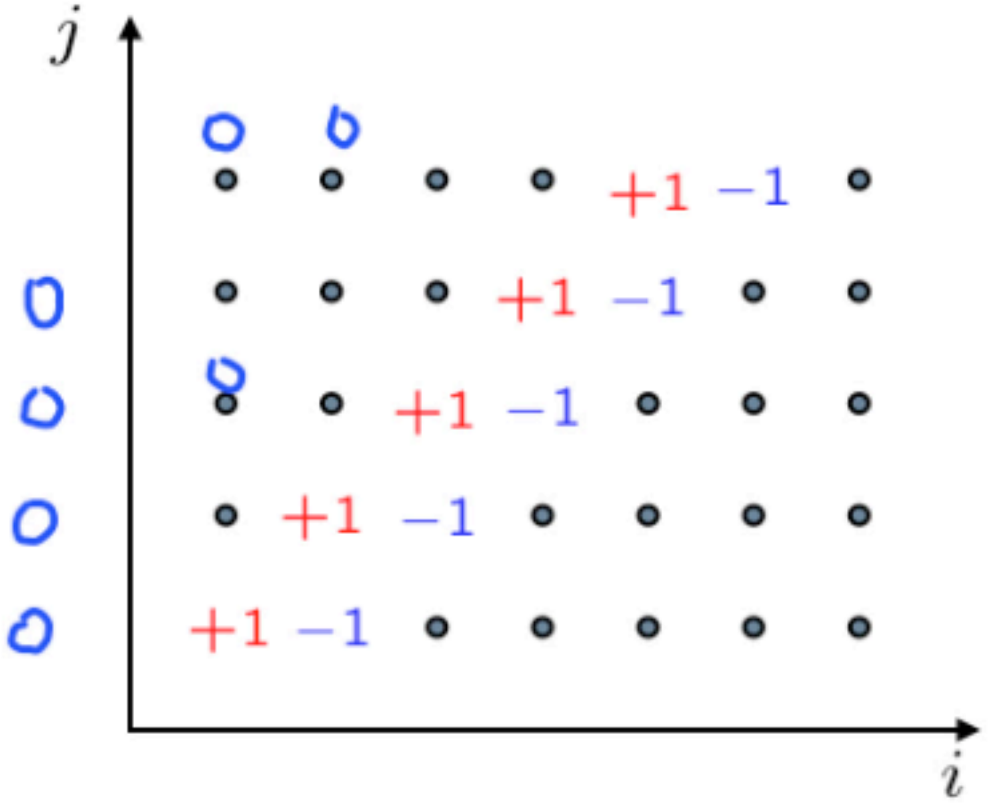
$S < \infty$ taken for granted

About the order of summation in series with multiple indices

$$\sum_{i\geq 1, j\geq 1} a_{ij}$$

$$\sum |a_{ij}| < \infty$$





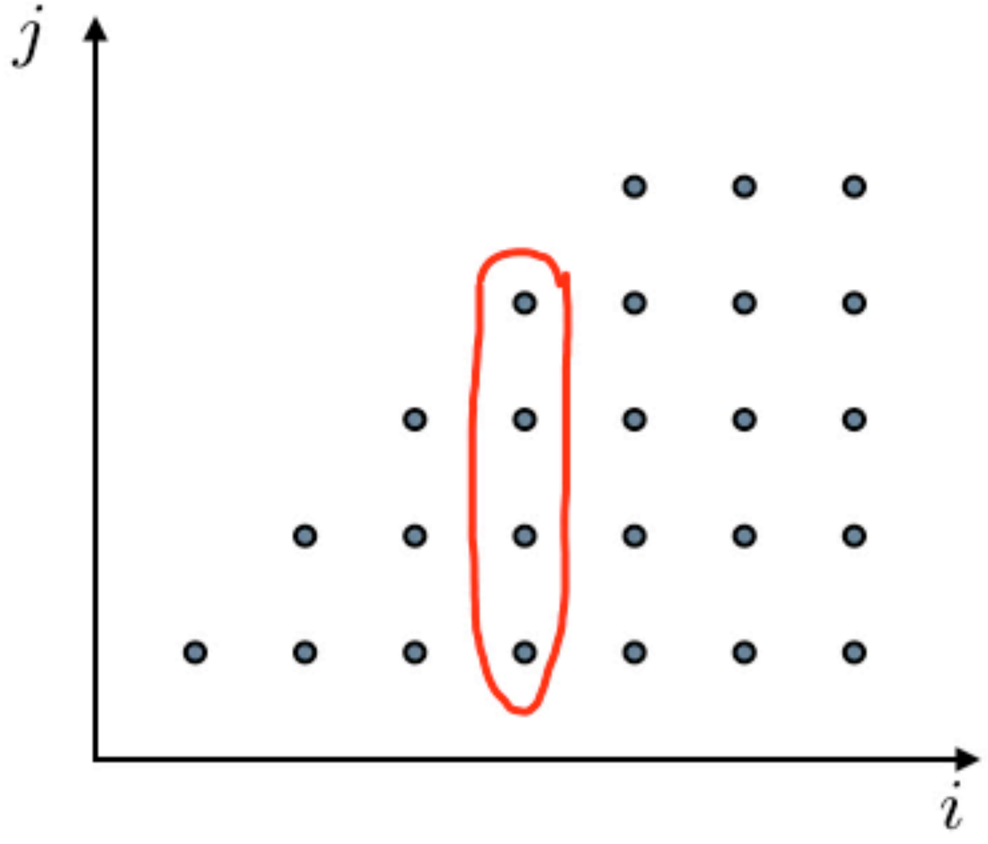
$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

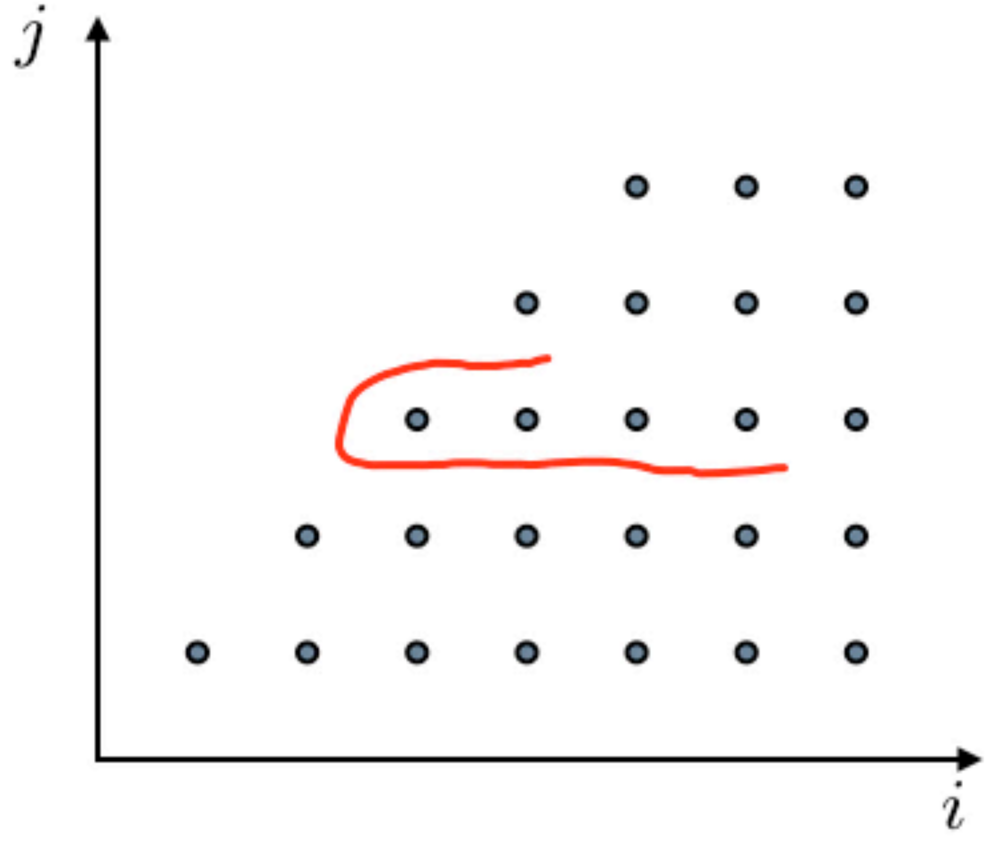
$1 + 0 + 0 \cdots = 1$ $\qquad\qquad$ $\sum 0 = 0$

About the order of summation in series with multiple indices

if

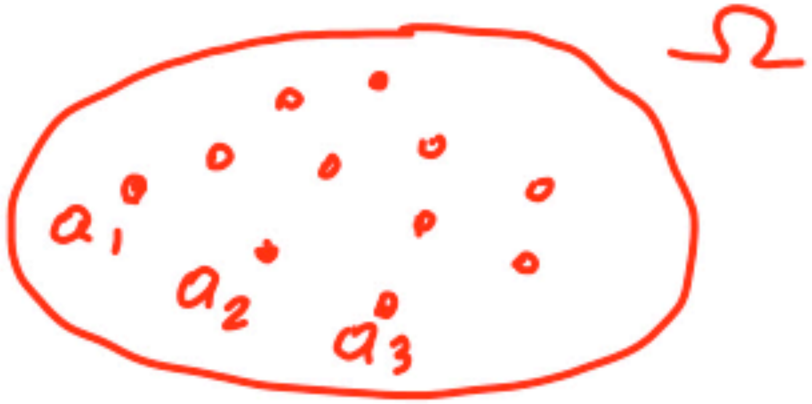
$$\sum_{(i,j):\ j\le i} |a_{ij}| < \infty$$





$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} a_{ij}$$

Countable versus uncountable infinite sets

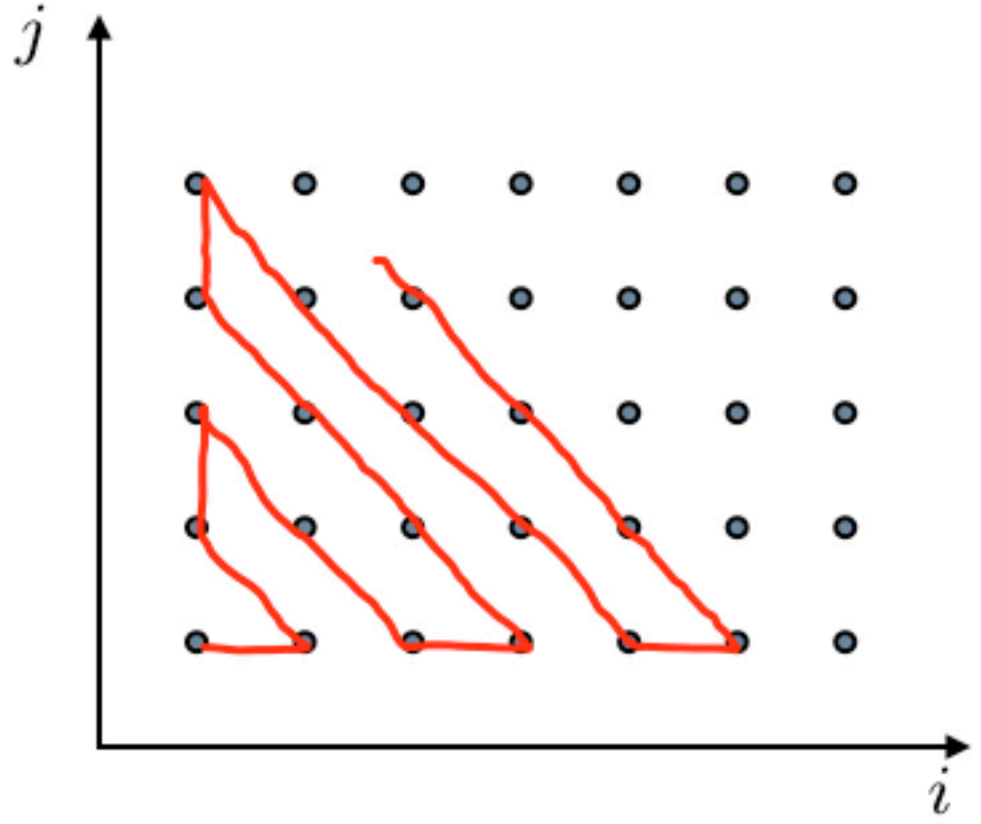


- Countable: can be put in 1-1 correspondence with positive integers
  - positive integers $1, 2, 3, \ldots$
  - integers $0, 1, -1, 2, -2, 3, -3, \ldots$

$\{a_1, a_2, a_3, \ldots\} = \Omega$

  - pairs of positive integers
  - rational numbers q, with $0 < q < 1$

$\underline{1/2}, 1/3, 2/3, 1/4, \cancel{2/4}, 3/4, 1/5, 2/5 \ldots$



- Uncountable: not countable
  - the interval $[0, 1]$
  - the reals, the plane, ...

The reals are uncountable

- Cantor's diagonalization argument

$\rightarrow \{x \in (0,1) : \text{decimal expansion only has } 3,4\}$

If countable "$\{x_1, x_2, x_3, \ldots\}$

x_1: $.3434433\ldots$

x_2: $.4443443$

x_3: $.3343444$

$.433\ldots = x$

$\neq x_i$ for all i