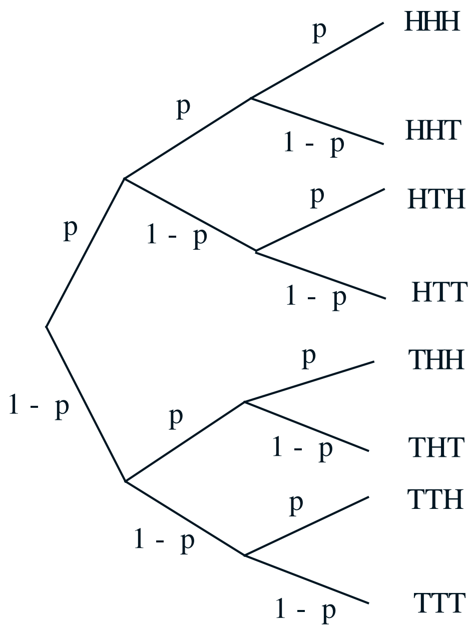
LECTURE 3: Independence

- Independence of two events
- Conditional independence
- Independence of a collection of events
- Pairwise independence
- Reliability
- The king's sibling puzzle

A model based on conditional probabilities

- 3 tosses of a biased coin: $\mathbf{P}(H) = p$, $\mathbf{P}(T) = 1 - p$



- Multiplication rule: $\mathbf{P}(THT) =$

- Total probability:

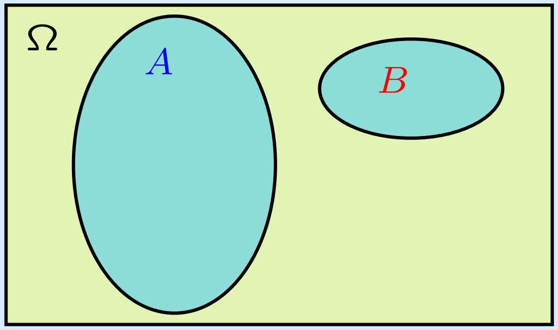
  $\mathbf{P}(\text{1 head}) =$

- Bayes rule:

  $\mathbf{P}(\text{first toss is H} \mid \text{1 head}) =$

Independence of two events

- **Intuitive "definition":** $\mathbf{P}(B \mid A) = \mathbf{P}(B)$
  - occurrence of A provides no new information about B



$\mathbf{P}(A) > 0$ $\mathbf{P}(B) > 0$

Independent?

Definition of independence: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$

- Symmetric with respect to A and B
- implies $\mathbf{P}(A \mid B) = \mathbf{P}(A)$
- applies even if $\mathbf{P}(A) = 0$

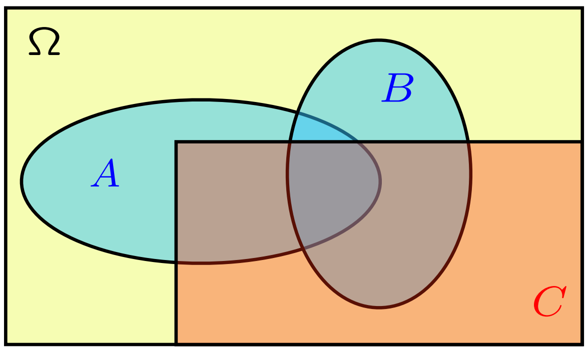
Independence of event complements

Definition of independence: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$

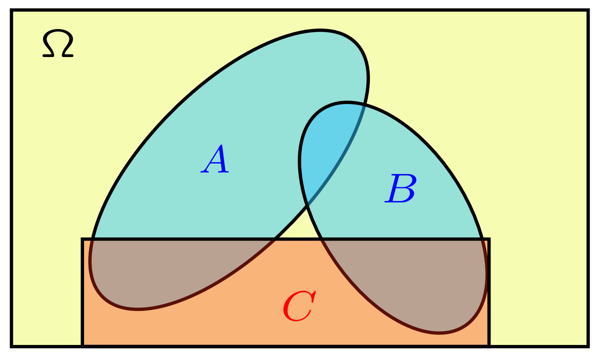
- If A and B are independent, then A and B^c are independent.
  - Intuitive argument
  - Formal proof

Conditional independence

- Conditional independence, given C, is defined as independence under the probability law $\mathbf{P}(\cdot \mid C)$



Assume A and B are independent



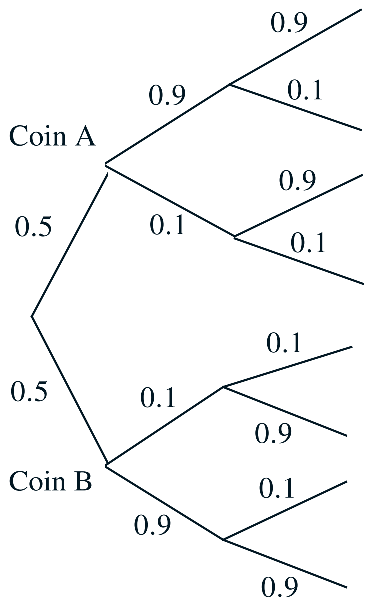
- If we are told that C occurred, are A and B independent?

Conditioning may affect independence

- Two unfair coins, A and B:
  $\mathbf{P}(H \mid \text{coin } A) = 0.9$, $\mathbf{P}(H \mid \text{coin } B) = 0.1$

- choose either coin with equal probability

- Are coin tosses independent?



- Compare:
  $\mathbf{P}(\text{toss } 11 = H)$

  $\mathbf{P}(\text{toss } 11 = H \mid \text{first 10 tosses are heads})$

Independence of a collection of events

- **Intuitive "definition":** Information on some of the events does not change probabilities related to the remaining events

Definition: Events $A_1, A_2, \ldots, A_n$ are called **independent** if:

$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_m) = \mathbf{P}(A_i)\mathbf{P}(A_j) \cdots \mathbf{P}(A_m)$ for any distinct indices $i, j, \ldots, m$

$n = 3$:

$$\left.\begin{aligned} \mathbf{P}(A_1 \cap A_2) &= \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) \\ \mathbf{P}(A_1 \cap A_3) &= \mathbf{P}(A_1) \cdot \mathbf{P}(A_3) \\ \mathbf{P}(A_2 \cap A_3) &= \mathbf{P}(A_2) \cdot \mathbf{P}(A_3) \end{aligned}\right\} \quad \text{pairwise independence}$$

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) \cdot \mathbf{P}(A_3)$$

Independence vs. pairwise independence

- Two independent fair coin tosses
  - H_1: First toss is H
  - H_2: Second toss is H

  $$\mathbf{P}(H_1) = \mathbf{P}(H_2) = 1/2$$

HH	HT
TH	TT

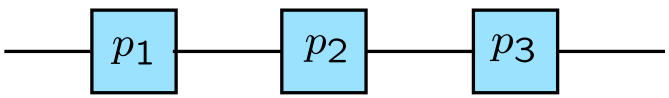
- C: the two tosses had the same result

H_1, H_2, and C are pairwise independent, but not independent

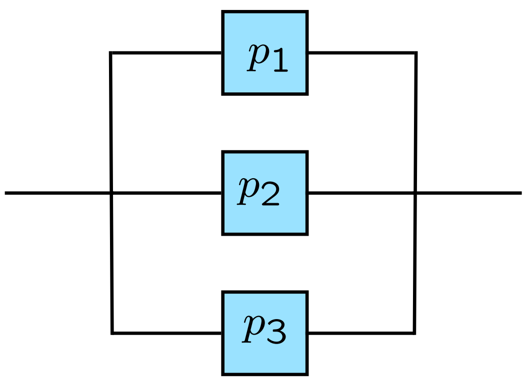
Reliability

p_i: probability that unit i is "up"

independent units



probability that system is "up"?



The king's sibling

- The king comes from a family of two children.
  What is the probability that his sibling is female?