LECTURE 3: Independence

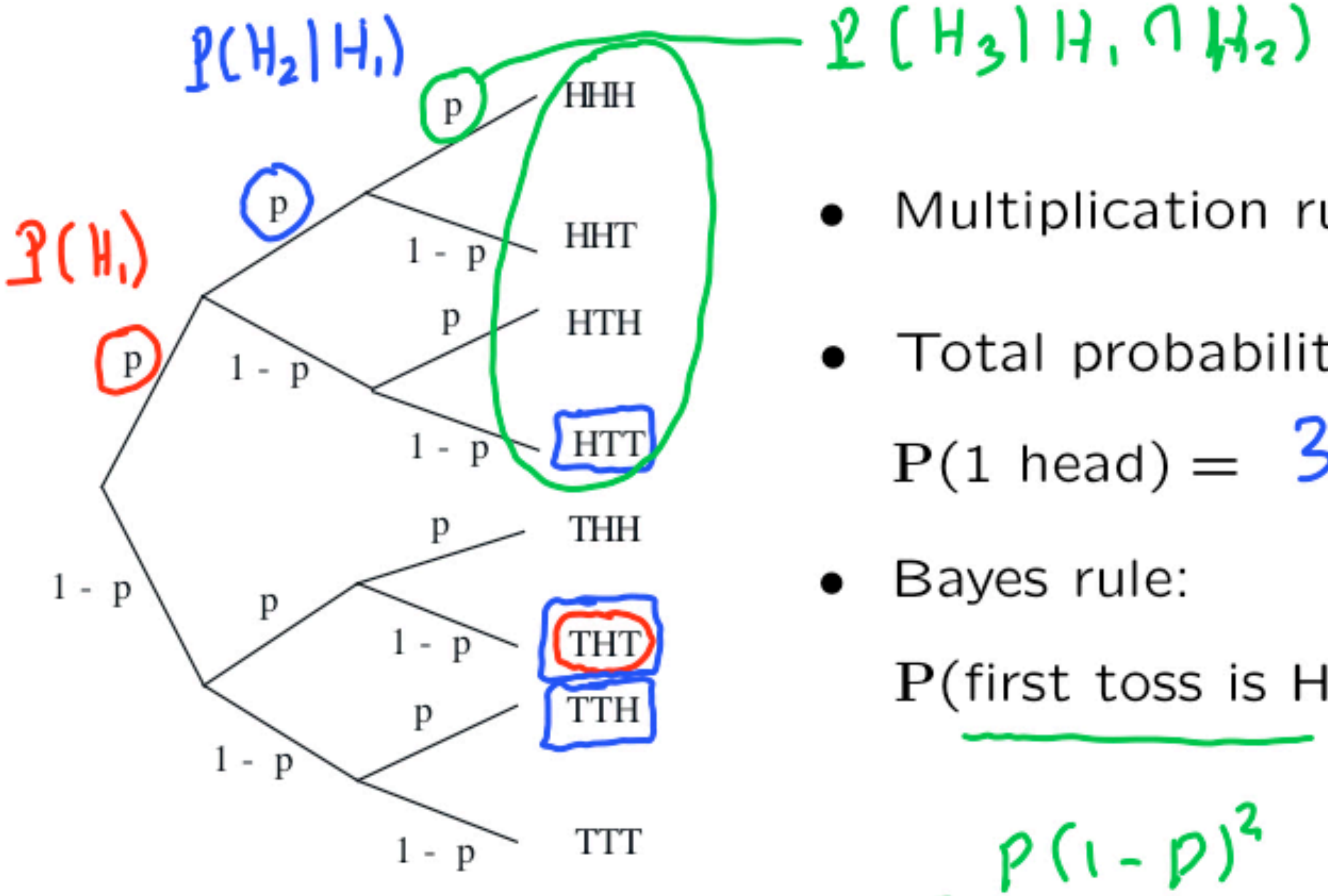
- Independence of two events
- Conditional independence
- Independence of a collection of events
- Pairwise independence
- Reliability
- The king's sibling puzzle

A model based on conditional probabilities

- 3 tosses of a biased coin: $\mathbf{P}(H) = p$, $\mathbf{P}(T) = 1 - p$

$$\mathbf{P}(H_2 \mid H_1) = p = \mathbf{P}(H_2 \mid T_1)$$

$$\mathbf{P}(H_2) = \mathbf{P}(H_1)\,\mathbf{P}(H_2 \mid H_1) + \mathbf{P}(T_1)\,\mathbf{P}(H_2 \mid T_1) = p$$



- Multiplication rule: $\mathbf{P}(THT) = (1-p)\,p\,(1-p)$

- Total probability:

  $\mathbf{P}(1 \text{ head}) = 3p(1-p)^2$

- Bayes rule:

  $\mathbf{P}(\text{first toss is H} \mid 1 \text{ head}) = \dfrac{\mathbf{P}(H_1 \cap 1 \text{ head})}{\mathbf{P}(1 \text{ head})}$

  $$= \frac{p(1-p)^2}{3p(1-p)^2} = \frac{1}{3}$$

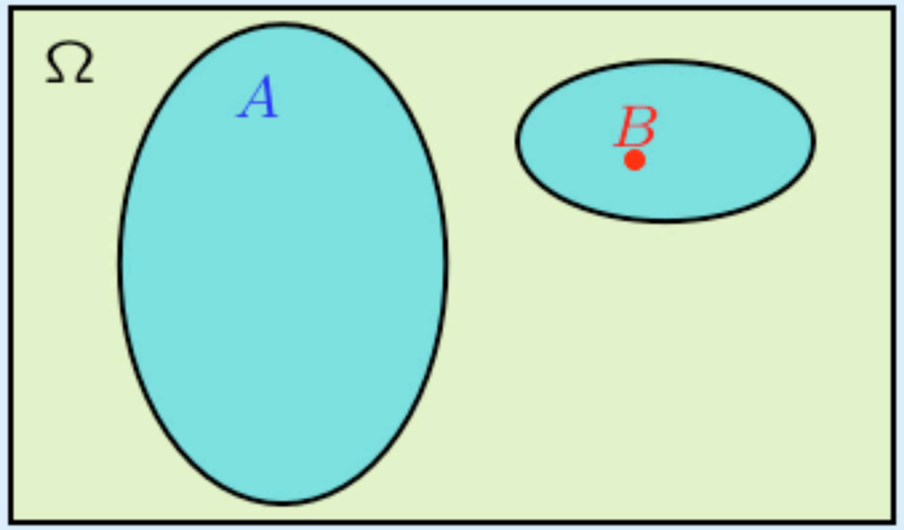
Independence of two events

- **Intuitive "definition":** $\mathbf{P}(B \mid A) = \mathbf{P}(B)$
  - occurrence of A provides no new information about B

$\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B \mid A) = \mathbf{P}(A)\,\mathbf{P}(B)$

Definition of independence: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$

- Symmetric with respect to A and B
- implies $\mathbf{P}(A \mid B) = \mathbf{P}(A)$
- applies even if $\mathbf{P}(A) = 0$



$\mathbf{P}(A) > 0$ $\mathbf{P}(B) > 0$

Independent?

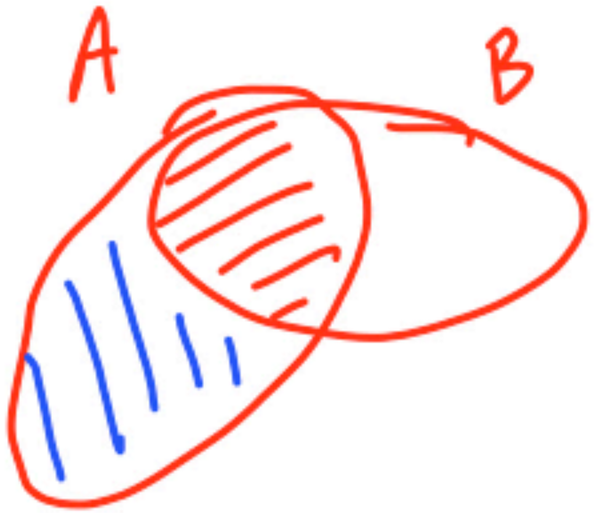
$\mathbf{P}(A \cap B) = 0$

$\mathbf{P}(A)\,\mathbf{P}(B) > 0$

Independence of event complements

Definition of independence: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$

- If A and B are independent, then A and B^c are independent.
  - Intuitive argument
  - Formal proof



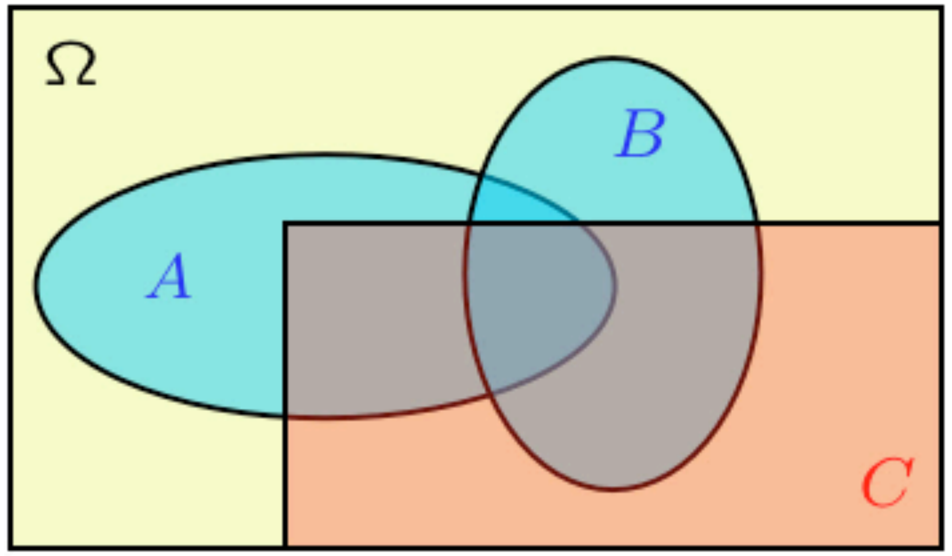
$$A = (A \cap B) \cup (A \cap B^c)$$

$$\mathbf{P}(A) = \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B^c)$$
$$= \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A \cap B^c)$$

$$\mathbf{P}(A \cap B^c) = \mathbf{P}(A) - \mathbf{P}(A)\mathbf{P}(B) = \mathbf{P}(A)(1 - \mathbf{P}(B))$$
$$= \mathbf{P}(A)\mathbf{P}(B^c)$$

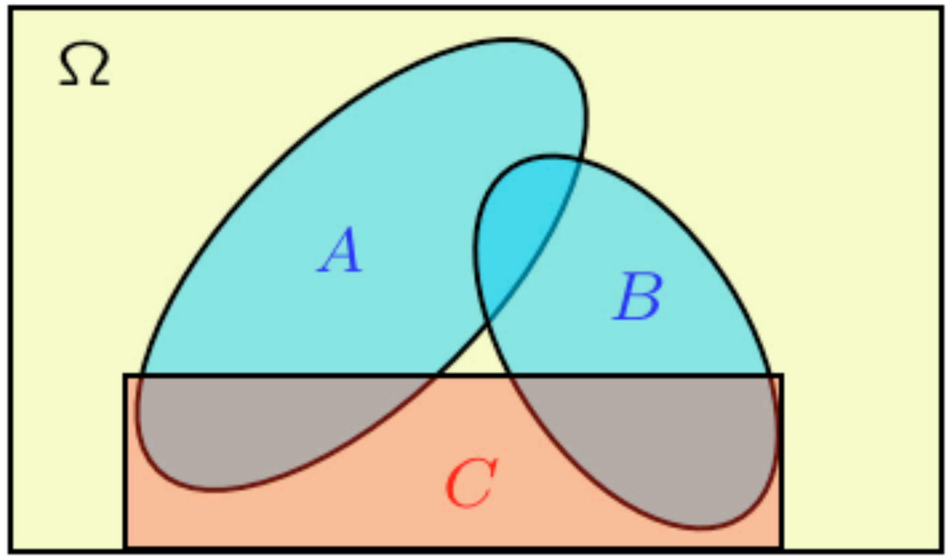
Conditional independence

- Conditional independence, given C,
  is defined as independence under the probability law $\mathbf{P}(\cdot \mid C)$



$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$

Assume A and B are independent



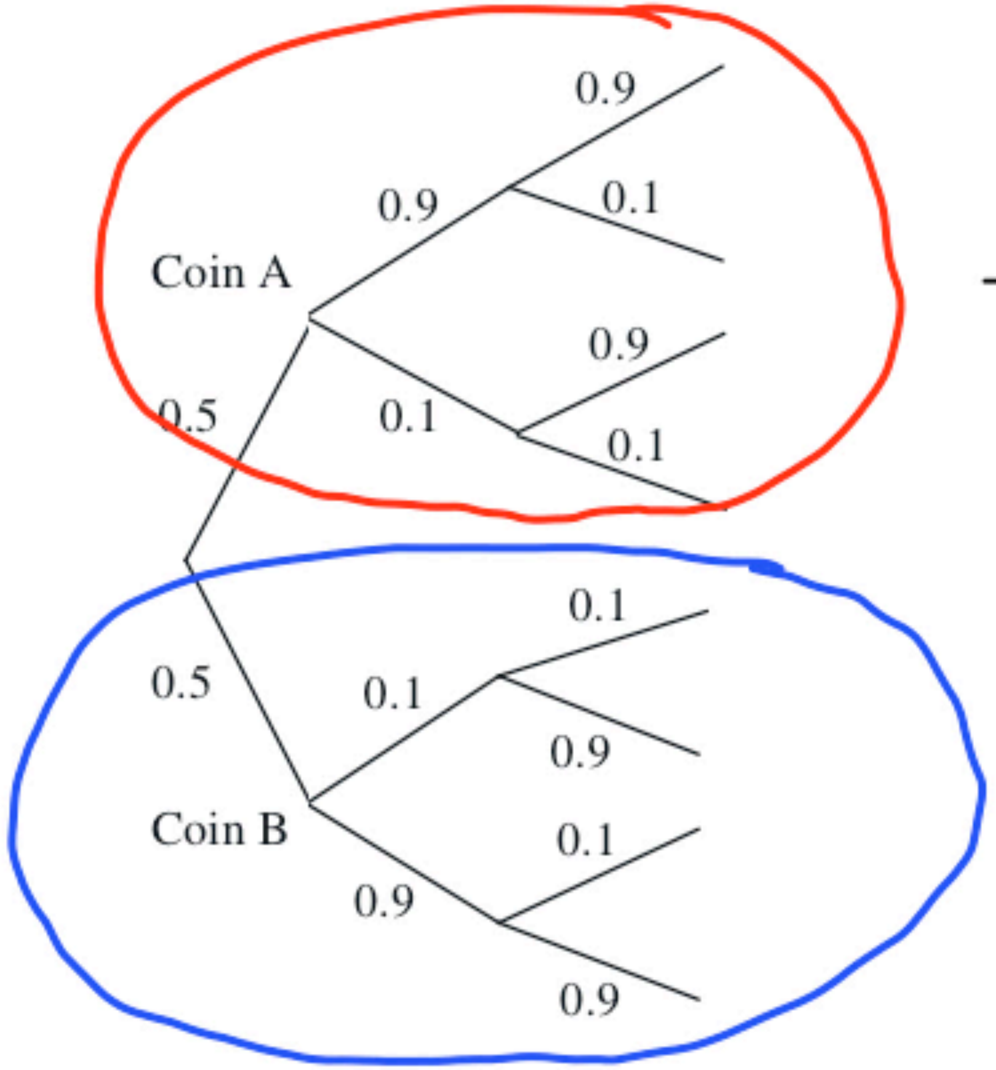
- If we are told that C occurred,
  are A and B independent? No

Conditioning may affect independence

given a coin:
independent tosses

- Two unfair coins, A and B:
  $\mathbf{P}(H \mid \text{coin } A) = 0.9$, $\mathbf{P}(H \mid \text{coin } B) = 0.1$

- choose either coin with equal probability



- Are coin tosses independent?

Nö!

  - Compare:
    $\mathbf{P}(\text{toss } 11 = H)$ $= P(A)\,P(H_{11} \mid A) + P(B)\,P(H_{11} \mid B)$
    $= 0.5 \times 0.9 + 0.5 \times 0.1 = 0.5$

    $\mathbf{P}(\text{toss } 11 = H \mid \text{first 10 tosses are heads})$

    $\approx P(H_{11} \mid A) = 0.9$

Independence of a collection of events

- **Intuitive "definition":** Information on some of the events does not change probabilities related to the remaining events

$A_1, A_2, \ldots$ indep $\Rightarrow P(A_3 \cap A_4^c) = P(A_3 \cap A_4^c \mid A_1 \cup (A_2 \cap A_5^c))$

$P(A_3) = P(A_3 \mid A_1 \cap A_2) = P(A_3 \mid A_1 \cap A_2^c) = P(A_3 \mid A_1^c \cap A_2)$

Definition: Events $A_1, A_2, \ldots, A_n$ are called **independent** if:

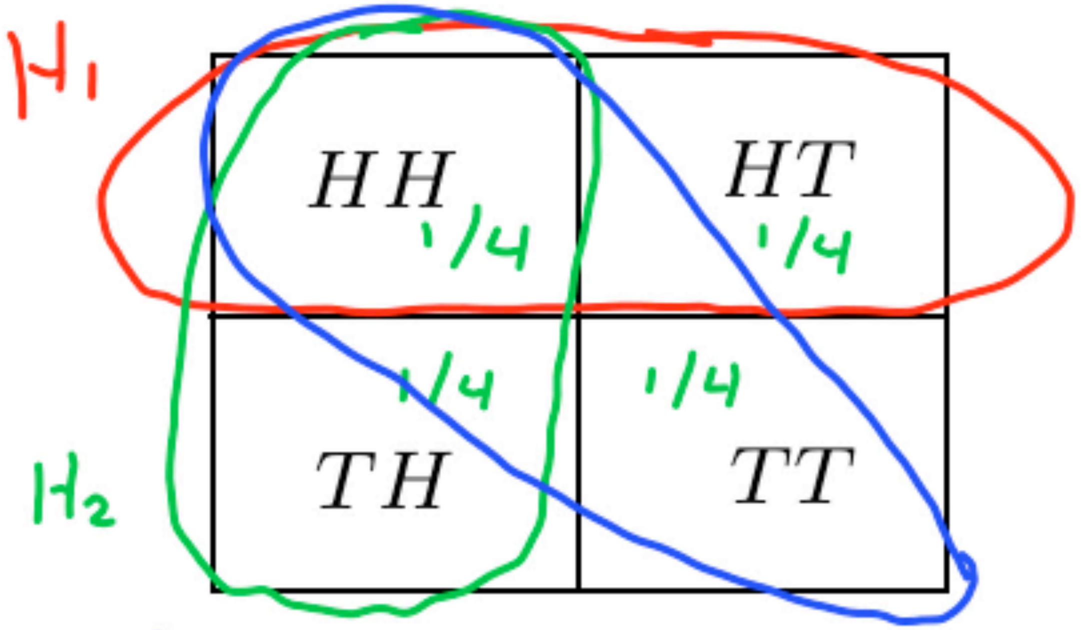
$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_m) = \mathbf{P}(A_i)\mathbf{P}(A_j) \cdots \mathbf{P}(A_m)$ for any distinct indices $i, j, \ldots, m$

$n = 3$:

$$\left.\begin{aligned} \mathbf{P}(A_1 \cap A_2) &= \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) \\ \mathbf{P}(A_1 \cap A_3) &= \mathbf{P}(A_1) \cdot \mathbf{P}(A_3) \\ \mathbf{P}(A_2 \cap A_3) &= \mathbf{P}(A_2) \cdot \mathbf{P}(A_3) \end{aligned}\right\} \text{pairwise independence}$$

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) \cdot \mathbf{P}(A_3)$$

Independence vs. pairwise independence



- Two independent fair coin tosses
  - H_1: First toss is H
  - H_2: Second toss is H

  $\mathbf{P}(H_1) = \mathbf{P}(H_2) = 1/2$

- C: the two tosses had the same result $= \{HH, TT\}$

$P(H_1 \cap C) = P(H_1 \cap H_2) = 1/4$ $\quad P(H_1)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\quad H_1, C$: indep.

H_2, C: indep.

$P(H_1 \cap H_2 \cap C) = P(HH) = 1/4$

$P(H_1)P(H_2)P(C) = 1/8$

diff.

$P(C \mid H_1) = P(H_2 \mid H_1) = P(H_2) = 1/2 = P(C)$

$P(C \mid H_1 \cap H_2) = 1 \neq P(C) = 1/2$

H_1, H_2, and C are pairwise independent, but not independent

Reliability

p_i: probability that unit i is "up"

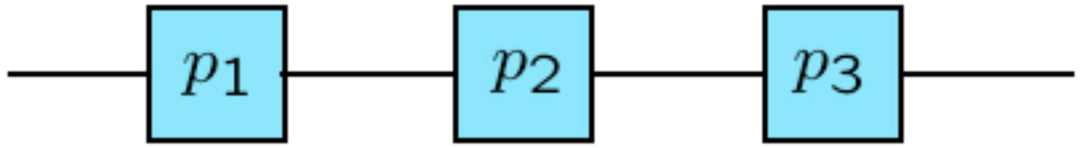
independent units

U_i : ith unit up

$U_1, U_2, \ldots, U_n$ independent

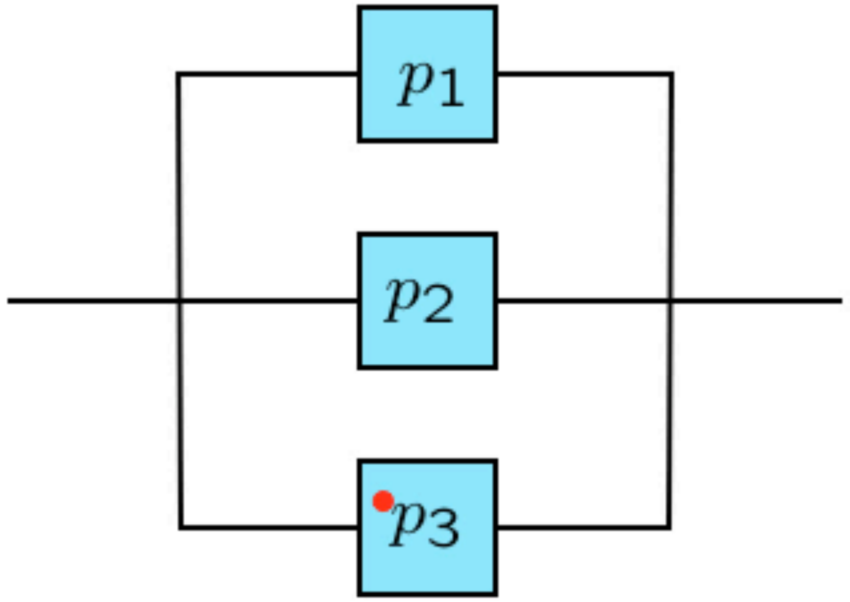
F_i : ith unit down

$\Rightarrow F_i$ independent



probability that system is "up"?

$$\mathbf{P}(\text{system up}) = \mathbf{P}(U_1 \cap U_2 \cap U_3)$$
$$= \mathbf{P}(U_1)\,\mathbf{P}(U_2)\,\mathbf{P}(U_3) = p_1 p_2 p_3$$



$$\mathbf{P}(\text{system is up}) = \mathbf{P}(U_1 \cup U_2 \cup U_3)$$
$$= 1 - \mathbf{P}(F_1 \cap F_2 \cap F_3)$$
$$= 1 - \mathbf{P}(F_1)\,\mathbf{P}(F_2)\,\mathbf{P}(F_3)$$
$$= 1 - (1-p_1)(1-p_2)(1-p_3)$$

The king's sibling

- The king comes from a family of two children. What is the probability that his sibling is female?

~~1/2~~ ?

boys have precedence

$P(\text{boy}) = P(\text{girl}) = 1/2$

independent

BB $1/4$	BG $1/4$
GB $1/4$	GG $1/4$

$2/3$

- till 1 boy $\Rightarrow P(G) = 1$
- till 2 boys $\Rightarrow P(G) \doteq 0$