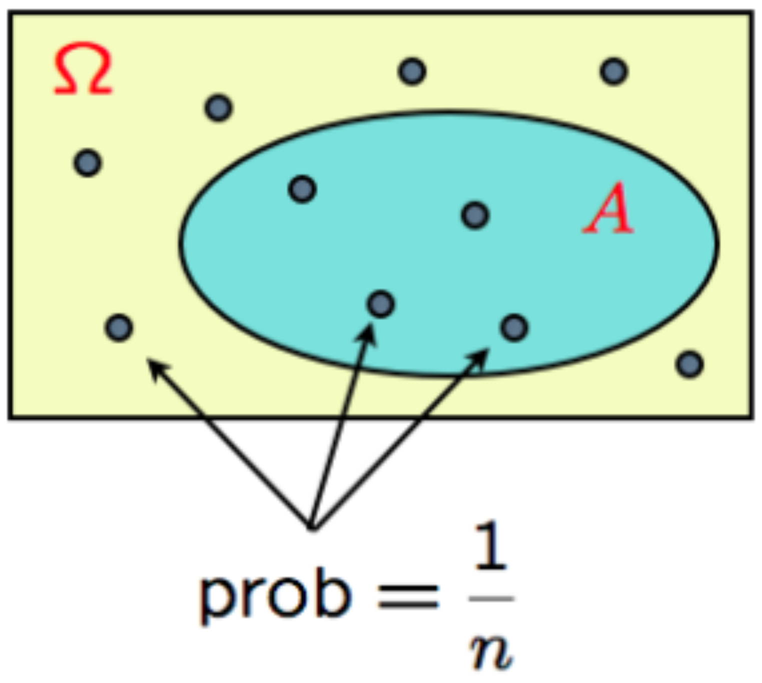
LECTURE 4: Counting

Discrete uniform law

– Assume Ω consists of n equally likely elements
– Assume A consists of k elements



Then : $$\mathrm{P}(A) = \frac{\text{number of elements of } A}{\text{number of elements of } \Omega} = \frac{k}{n}$$

- Basic counting principle

- Applications

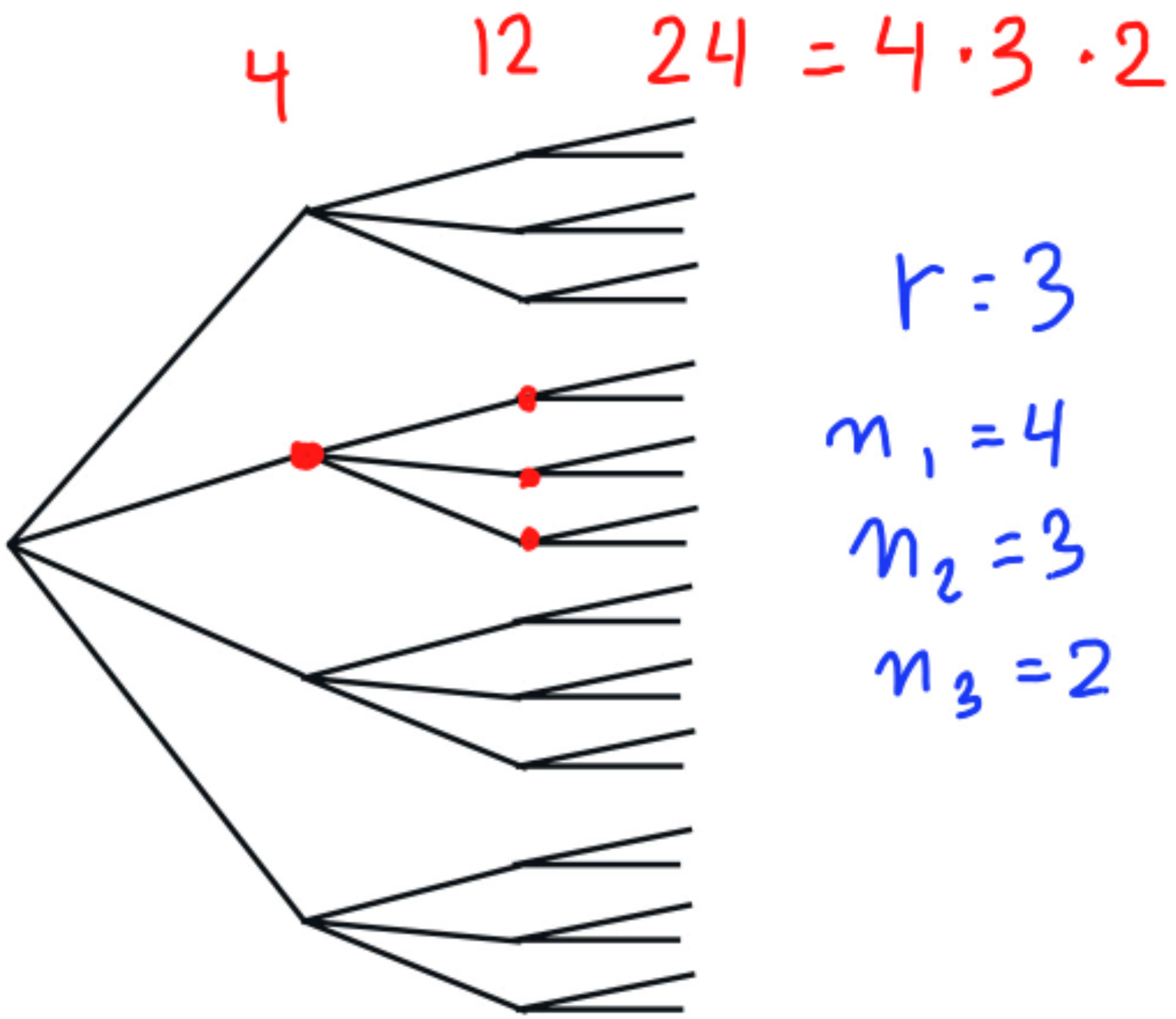
permutations
combinations
partitions

number of subsets
binomial probabilities

Basic counting principle

4 shirts
3 ties
2 jackets

Number of possible attires?



- r stages
- n_i choices at stage i

Number of choices is: $n_1 \cdot n_2 \cdots n_r$.

Basic counting principle examples

- Number of license plates with 2 letters followed by 3 digits:

  $26\cdot 26\cdot 10\cdot 10\cdot 10$

  - ... if repetition is prohibited: $26\cdot 25\cdot 10\cdot 9\cdot 8$

- **Permutations:** Number of ways of ordering n elements:

$n\cdot(n-1)(n-2)\cdots 1 = n!$

- Number of subsets of $\{1, \ldots, n\}$:

$2\cdot 2\cdots 2 = 2^n$

$n=1$ $\{1\}$ $2^1 = 2$

$\{1\}$ $\emptyset$

Example

- Find the probability that:
  $\underbrace{\text{six rolls of a (six-sided) die}}$ $\underbrace{\text{all give different numbers.}}$ $\swarrow A$

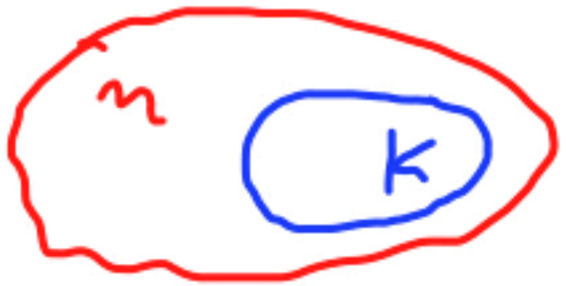
  (Assume all outcomes equally likely.)

typical outcome $P(2,3,4,3,6,2) = 1/6^6$

" element of A: $(2,3,4,1,6,5) = 6!$

$$P(A) = \frac{\#\text{ in } A}{\#\text{ possible outcomes}} = \frac{6!}{6^6}.$$

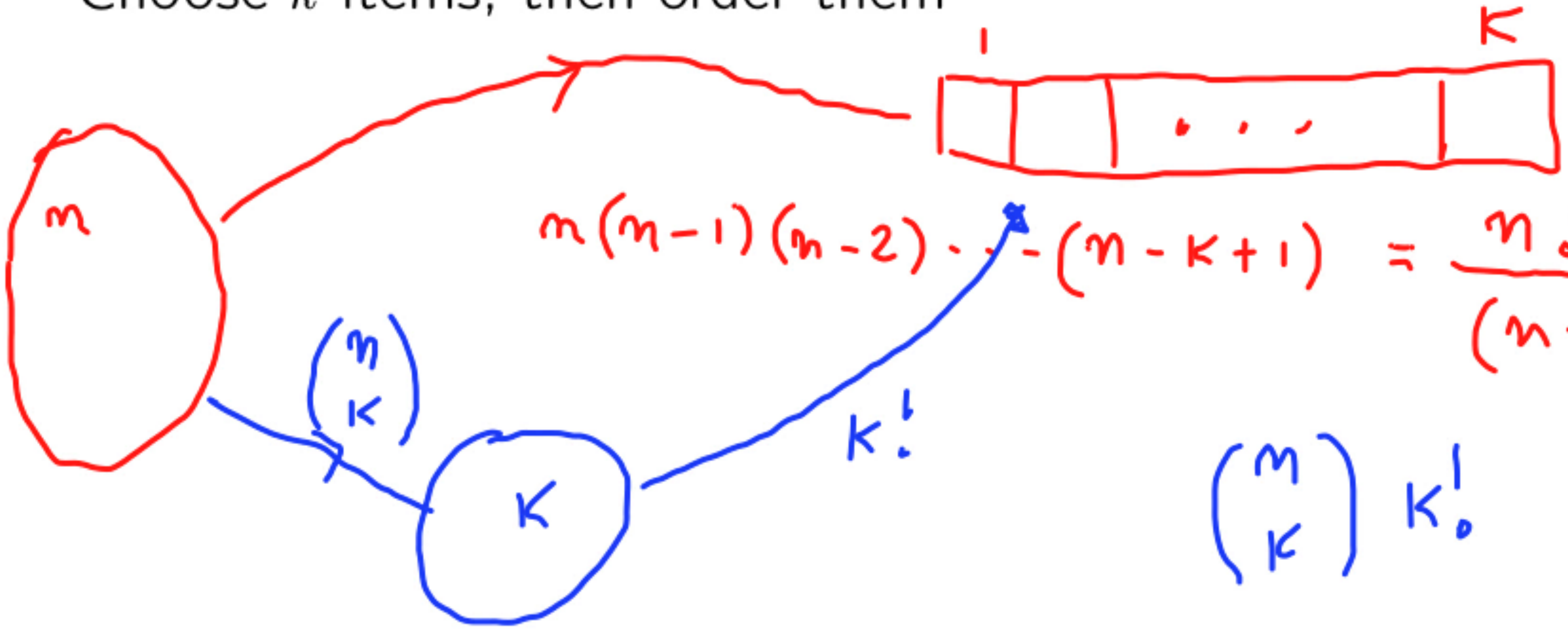
Combinations



- Definition: $\binom{n}{k}$: number of k-element subsets of a given n-element set $\boxed{= \frac{n!}{k!(n-k)!}}$

$n = 0, 1, 2, \ldots$

- Two ways of constructing an **ordered** sequence of k **distinct** items: $k = 0, 1, \ldots, n$
  - Choose the k items one at a time
  - Choose k items, then order them



$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

$\binom{n}{k} k!$ =

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \frac{n!}{n!\,0!} \qquad 0! = 1 \quad \text{convention}$$

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1 \qquad \emptyset$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = \#\,\text{all subsets} = 2^n$$

Binomial coefficient $\binom{n}{k}$ $\longrightarrow$ Binomial probabilities

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- $n \geq 1$ independent coin tosses; $\mathbf{P}(H) = p$

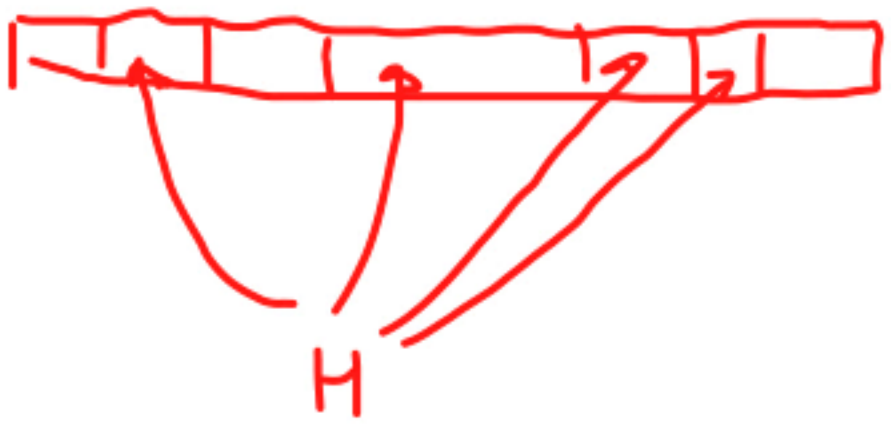
  $n = 6$

- $\mathbf{P}(HTTHHH) = p(1-p)(1-p)ppp = p^4(1-p)^2$

- $\mathbf{P}(\text{particular sequence}) = p^{\#\text{ heads}}(1-p)^{\#\text{ tails}}$

- $\mathbf{P}(\text{particular } k\text{-head sequence}) = p^k(1-p)^{n-k}$

$$\mathbf{P}(k \text{ heads}) = p^k(1-p)^{n-k} \cdot (\#\ k\text{-head sequences})$$



$$\binom{n}{k}$$

A coin tossing problem

Assumptions:

- independence
- $\mathbf{P}(H) = p$

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

- First solution:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{P(H_1 H_2 \text{ and one } H \text{ in tosses } 3,\ldots,10)}{P(B)}$$

$$= \frac{p^2 \cdot \binom{8}{1} p^1 \cdot (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{8}{\binom{10}{3}}.$$

A coin tossing problem

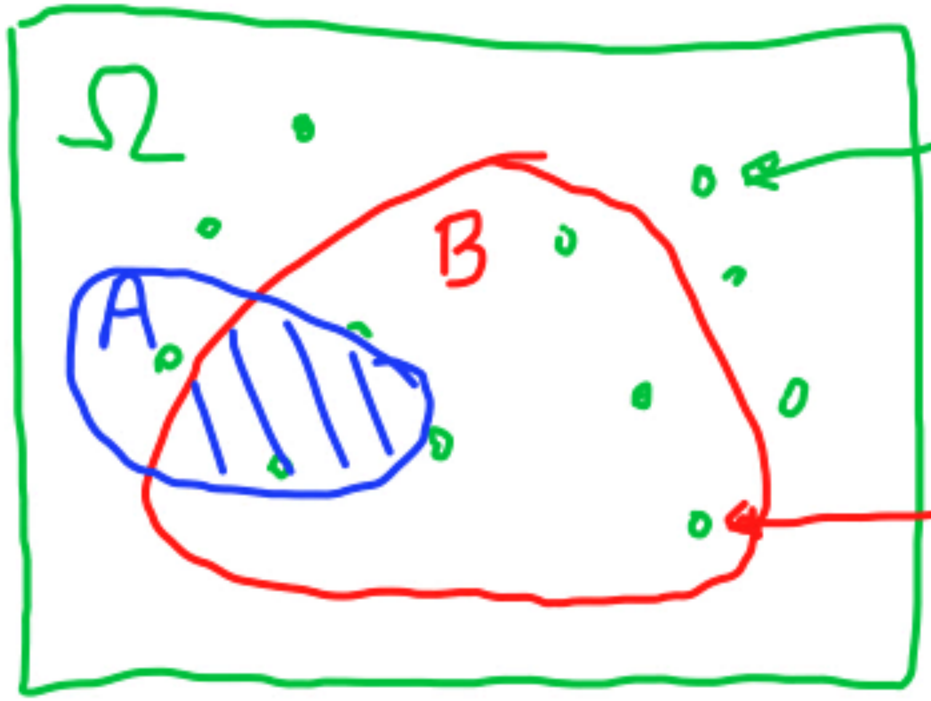
Assumptions:

- independence
- $\mathbf{P}(H) = p$

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Given that there were 3 heads in 10 tosses,
  what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

- Second solution: Conditional probability law (on B) is uniform



length 10 seq.

3-head seqs

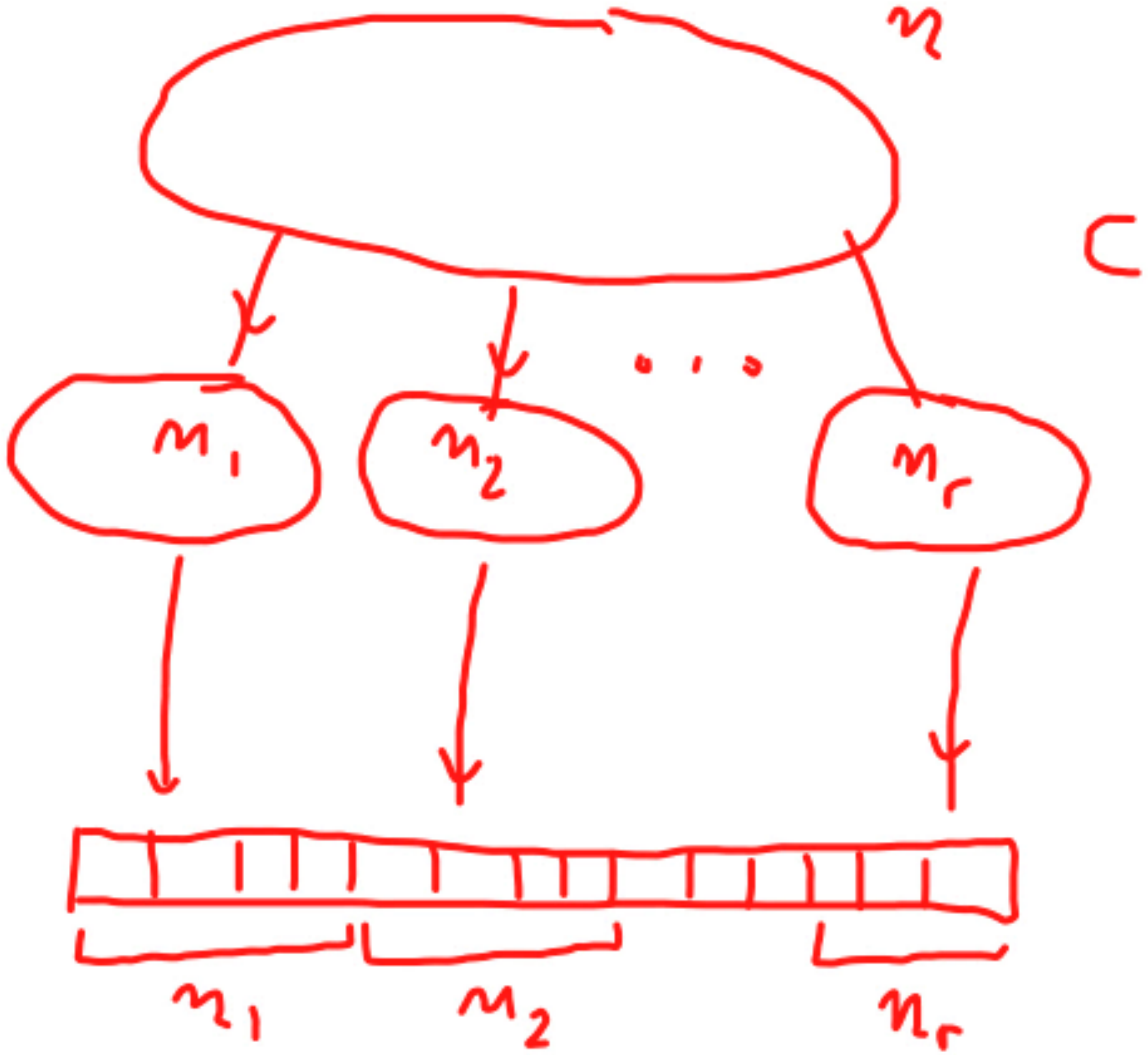
$p^3(1-p)^7$

$$\frac{\# \text{ in } (A \cap B)}{\# \text{ in } B} = \frac{8}{\binom{10}{3}}.$$

Partitions

- $n \geq 1$ distinct items; $r \geq 1$ persons
  give n_i items to person i
  - here $n_1, \ldots, n_r$ are given nonnegative integers
  - with $n_1 + \cdots + n_r = n$
- Ordering n items: $n!$
  - Deal n_i to each person i, and then order

$$c \; n_1! \, n_2! \cdots n_r! = n!$$



$r=2 \qquad n_1 = k \qquad n_2 = n-k$

$$\text{number of partitions} = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad \text{(multinomial coefficient)}$$

Example: 52-card deck, dealt (fairly) to four players.
Find $\mathbf{P}$(each player gets an ace)

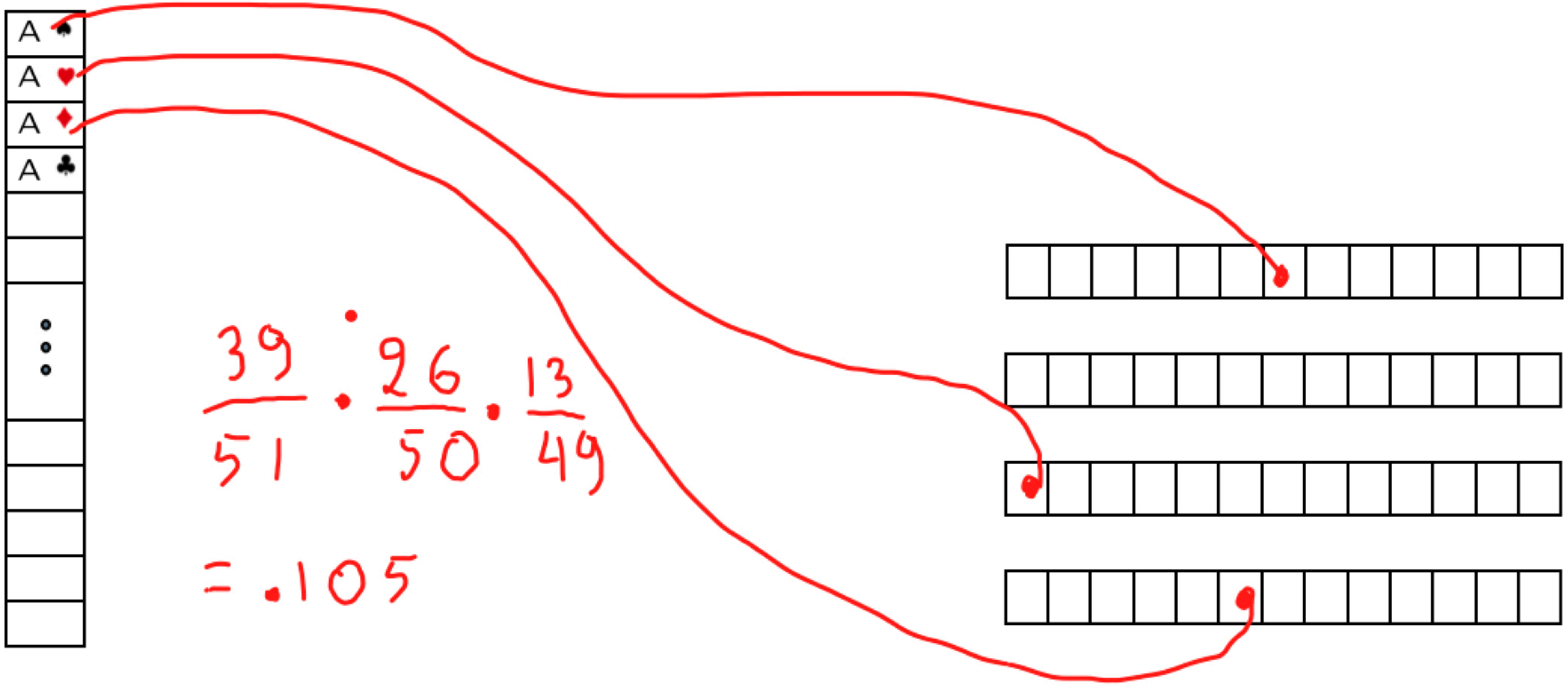
- Outcomes are: partition equally likely
  - number of outcomes: $\dfrac{52!}{13!\,13!\,13!\,13!}$
- Constructing an outcome with one ace for each person:
  - distribute the aces $4 \cdot 3 \cdot 2 \cdot 1$
  - distribute the remaining 48 cards $\dfrac{48!}{12!\,12!\,12!\,12!}$
- Answer:

$$\frac{4 \cdot 3 \cdot 2 \cdot \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$

Example: 52-card deck, dealt (fairly) to four players. Find $\mathbf{P}$(each player gets an ace)

A smart solution

Stack the deck, aces on top



$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$$

$$= .105$$