LECTURE 6: Variance; Conditioning on an event; Multiple random variables

- Variance and its properties
  - Variance of the Bernoulli and uniform PMFs
- Conditioning a r.v. on an event
  - Conditional PMF, mean, variance
  - Total expectation theorem
- Geometric PMF
  - Memorylessness
  - Mean value
- Multiple random variables
  - Joint and marginal PMFs
  - Expected value rule
  - Linearity of expectations
- The mean of the binomial PMF

Variance — a measure of the spread of a PMF

- Random variable X, with mean $\mu = \mathbf{E}[X]$
- Distance from the mean: $X - \mu$
- Average distance from the mean?

- **Definition of variance:** $\text{var}(X) = \mathbf{E}[(X-\mu)^2]$

- Calculation, using the expected value rule, $\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$

$$\text{var}(X) =$$

Standard deviation: $\sigma_X = \sqrt{\text{var}(X)}$

Properties of the variance

$$\boxed{\text{var}(aX + b) = a^2\text{var}(X)}$$

- Notation: $\mu = \mathbf{E}[X]$

- Let $Y = X + b$

- Let $Y = aX$

A useful formula: $\text{var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2$

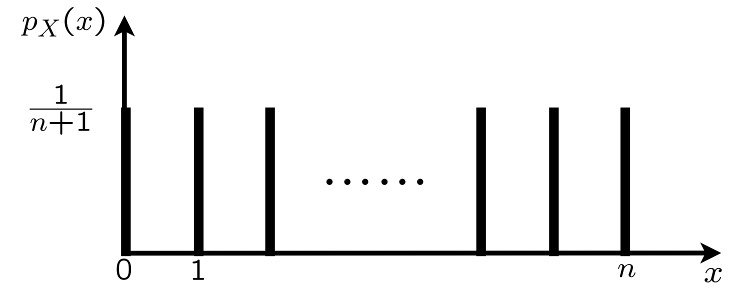
Variance of the Bernoulli

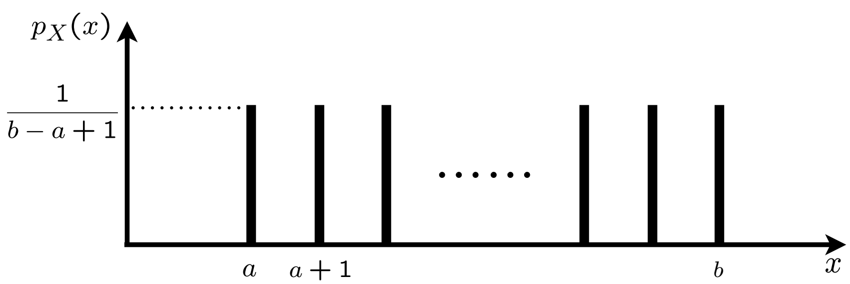
$$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases}$$

$$\text{var}(X) = \sum_x (x - \mathbf{E}[X])^2 p_X(x)$$

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Variance of the uniform





Conditional PMF and expectation, given an event

- Condition on an event $A \Rightarrow$ use conditional probabilities

$$p_X(x) = \mathbf{P}(X = x) \qquad p_{X|A}(x) = \mathbf{P}(X = x \mid A)$$

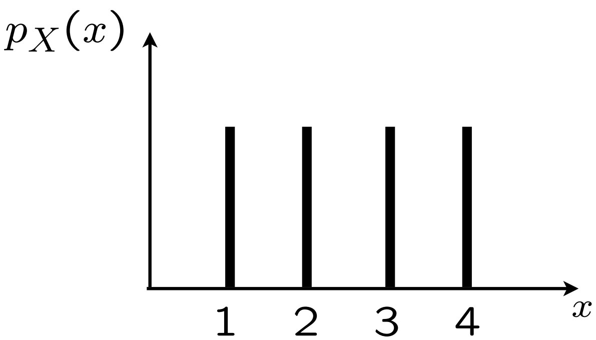
$$\sum_x p_X(x) = 1 \qquad \sum_x p_{X|A}(x) = 1$$

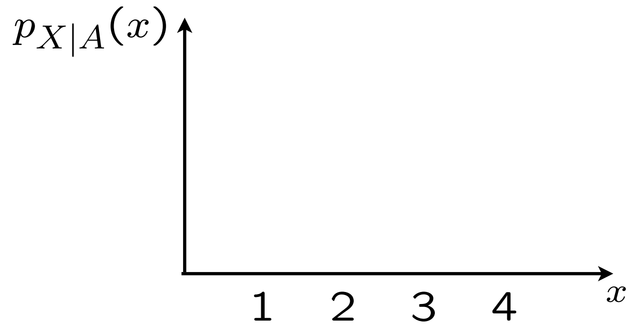
$$\mathbf{E}[X] = \sum_x x p_X(x) \qquad \mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$$

$$\mathbf{E}[g(X)] = \sum_x g(x)\, p_X(x) \qquad \mathbf{E}[g(X) \mid A] = \sum_x g(x)\, p_{X|A}(x)$$

Example of conditioning

- Let $A = \{X \geq 2\}$





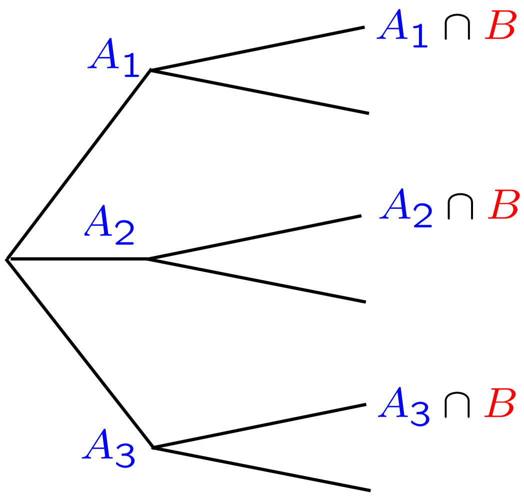
$\mathbf{E}[X] =$

$\mathbf{E}[X \mid A] =$

$\text{var}(X) =$

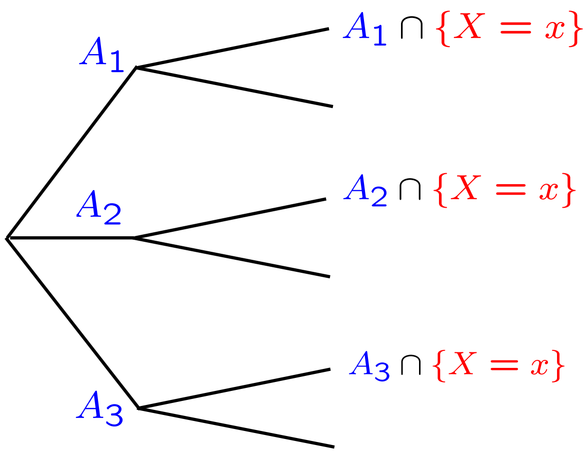
$\text{var}(X \mid A) =$

Total expectation theorem



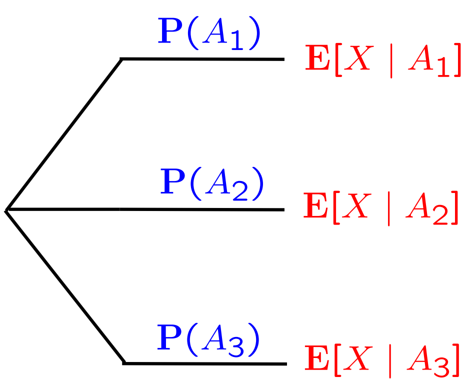
$$\mathbf{P}(B) = \mathbf{P}(A_1)\,\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\,\mathbf{P}(B \mid A_n)$$

Total expectation theorem



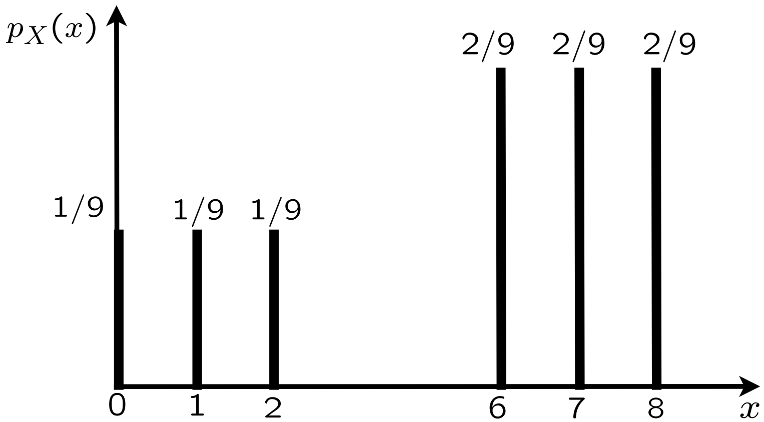
$$\mathbf{P}(B) = \mathbf{P}(A_1)\,\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\,\mathbf{P}(B \mid A_n)$$

$$p_X(x) = \mathbf{P}(A_1)\,p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\,p_{X|A_n}(x)$$



$$\boxed{\mathbf{E}[X] = \mathbf{P}(A_1)\,\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\,\mathbf{E}[X \mid A_n]}$$

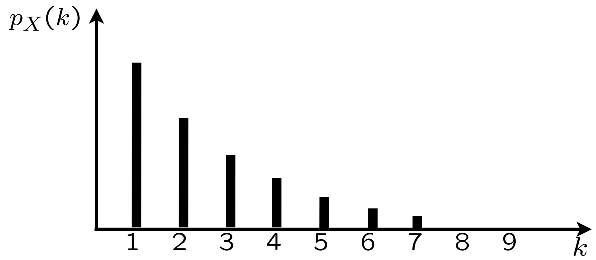
Total expectation example



Conditioning a geometric random variable

- X: number of independent coin tosses until first head; $\mathbf{P}(H) = p$

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$



Memorylessness:
Number of **remaining** coin tosses, conditioned on Tails in the first toss, is **Geometric**, with parameter p

Conditioned on $X > 1$, $X - 1$ is geometric with parameter p

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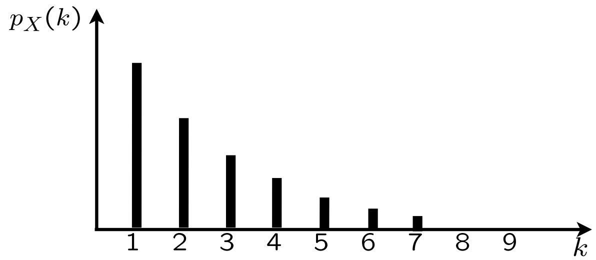
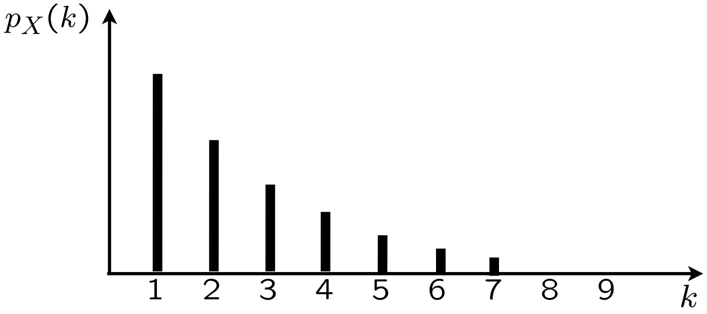


Memorylessness:
Number of **remaining** coin tosses, conditioned on Tails in the first toss, is **Geometric**, with parameter p

Conditioned on $X > n$, $X - n$ is geometric with parameter p

The mean of the geometric



$$\mathbf{E}[X] = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

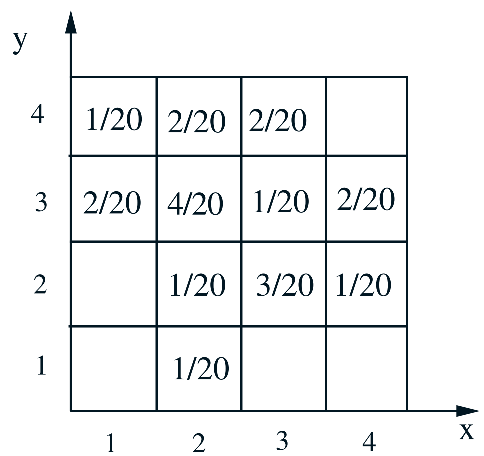
$$\boxed{\mathbf{E}[X] = \frac{1}{p}}$$

Multiple random variables and joint PMFs

$X : p_X$
$Y : p_Y$

$\mathbf{P}(X = Y) =$

Joint PMF: $p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y)$



$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

More than two random variables

$$p_{X,Y,Z}(x,y,z) = \mathbf{P}(X = x \text{ and } Y = y \text{ and } Z = z)$$

$$\sum_x \sum_y \sum_z p_{X,Y,Z}(x,y,z) = 1$$

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x,y,z)$$

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z)$$

Functions of multiple random variables

$Z = g(X, Y)$

PMF: $p_Z(z) = \mathbf{P}(Z = z) = \mathbf{P}\Big(g(X, Y) = z\Big)$

Expected value rule: $\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y)\, p_{X,Y}(x, y)$

Linearity of expectations

$$\mathbf{E}[aX+b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\boxed{\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]}$$

$$\mathbf{E}[2X + 3Y - Z] =$$

The mean of the binomial

- X: binomial with parameters n, p
  - number of successes in n independent trials

$$\mathbf{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boxed{\mathbf{E}[X] = np}$$

$X_i = 1$ if ith trial is a success;
$X_i = 0$ otherwise

(indicator variable)

$$X = X_1 + \cdots + X_n$$