

LEVERAGING TEMPORAL FEATURES OF THE DIVERGENCE MEASURE TO DETECT CHAOS IN CONSERVATIVE SYSTEMS

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ABSTRACT. The recurrence-based divergence quantifier (*DIV*), traditionally applied to dissipative systems, is shown here to be an effective finite-time chaos indicator for conservative dynamics. We benchmark its performances against the well-established fast Lyapunov indicator (FLI), focusing on the standard map, a canonical model of Hamiltonian chaos. Through extensive numerical simulations on moderately long orbits, we find strong agreement between *DIV* and FLI, supporting the reported correlation between the divergence of recurrences and positive Lyapunov exponents. Additionally, our study sheds more light into asymptotic time properties of *DIV* by revealing distinct power laws for regular and chaotic trajectories, both in the original and reconstructed phase spaces. In particular, for regular dynamics, *DIV* decays in average with the time N as $1/N$, mirroring the decay rate of the maximal Lyapunov exponent. For chaotic trajectories, *DIV* decreases at a much slower rate in average, close to $1/\sqrt{N}$. This scaling insight opens new avenues for characterizing chaos from time series. Our numerical results thus demonstrate *DIV* to be a computationally viable and theoretically rich tool for chaos detection in conservative systems.

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1. INTRODUCTION

The story of recurrence plot (RP) and the beginning of its methodological developments is often traced back to the seminal paper of Eckmann et al. (1987). Let be γ a finite orbit, $\gamma =$

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$\{z_0, z_1, z_2, \dots, z_n\}$ of points in \mathbb{R}^d , $d \geq 1$, of some dynamical systems. The $\{z_i\}_i$ might be the results of mapping iterations, flow discretisation or observed states (measurements). The 0–1 recurrence matrix $R_\gamma = (r_{i,j})_{0 \leq i,j \leq n}$ is constructed from the recurrence coefficient

$$(1) \quad r_{i,j} = \Theta(\|z_i - z_j\| - \epsilon),$$

where Θ is the Heaviside function, ϵ is a real cutoff parameter, and $\|\bullet\|$ is a norm in \mathbb{R}^d . When $r_{i,j} = 1$, z_i and z_j are recurrence points, which means that they are ϵ close. The matrix R_γ always contains the line of identity as trivial recurrence points. A RP, as indicated by the name, is in its root a graphical tool. It consists of visualising the binary matrix R_γ with recurrence points encoded as dark pixels. Interestingly enough, dynamical properties of the orbit (such as its oscillatory nature, the presence of drifting components, the presence of extreme or rare events, etc.) are encoded in different structures and textures of the RP (Eckmann et al. (1987); Webber Jr and Zbilut (1994)). (Several typologies obtained on the standard map, the model on which we base our analysis, will be exemplified later in Sect. 2.) In addition to qualitative graphical assessments of RP, which can be difficult to interpret or intuit visually, several recurrence variables have been introduced by Webber Jr and Zbilut (1994) and developed in greater depth over the years (Marwan et al., 2007), a field termed recurrence quantification analysis (RQA). Such variables include the recurrence rate RR (percentage of recurrence points), the determinism DET (proportion of recurring points forming diagonal lines) or lengths of vertical lines, to name but a few. In this contribution, we focus specifically on the RQA variable related to the length of the longest diagonal lines ℓ_{\max} in a RP (leaving aside the line of identity and its vicinity), and more precisely its inverse — another RQA variable — the divergence DIV (Webber Jr and Zbilut, 1994), defined as

$$(2) \quad DIV = 1/\ell_{\max}.$$

A diagonal line of ℓ length units emanating from the times (i, j) in a RP corresponds to a sequence of ℓ successive recurrence points,

$$(3) \quad \begin{cases} r_{i-1,j-1} = 0, \\ r_{i,j} = 1, \\ r_{i+1,j+1} = 1, \\ \dots \\ r_{i+\ell-1,j+\ell-1} = 1, \\ r_{i+\ell,j+\ell} = 0. \end{cases}$$

It has long been reported ℓ_{\max} (or DIV) to be in direct proportion to the largest Lyapunov exponent (Eckmann et al., 1987; Zbilut and Webber, 1992; Webber Jr and Zbilut, 1994; Trulla et al., 1996), as heuristically understood here. Assume the trajectory to be embedded into some \mathbb{R}^d and that $r_{i,j}$ is a recurrence point. It follows that z_j is ϵ close to z_i . If we interpret $\delta_{i,j} = z_i - z_j$ as a deviation vector and follow its time evolution, in case of chaotic dynamics, we expect the quantity $\{\|\delta_{i+k,j+k}\|\}_k$ to grow exponentially fast in average with k at a rate dictated by the largest Lyapunov exponent. Thus, the probability of $z_{i+k,j+k}$ still being an ϵ neighbour of $z_{i,j}$ (a recurrent point) decreases with time k very fast. In other words, the length of the diagonal that stems from the observed times (i, j) is expected to be small. A more rigorous link between diagonal lines and chaos is made apparent when looking at the distribution of diagonal lines of length ℓ and the second-order Rényi entropy (also called correlation entropy), K_2 (Grassberger and Procaccia, 1983). For purely deterministic systems $K_2 = 0$ and for stochastic systems $K_2 \rightarrow +\infty$. Deterministic chaos is characterised by finite values, $K_2 > 0$. It turns out that K_2 can be estimated directly from RPs as presented in Thiel et al. (2003). An algorithm to allow its numerical estimation is detailed by Asghari et al. (2004) and Marwan et al. (2007). The procedure involves essentially three major steps. Firstly, one need to compute the cumulative distribution of the diagonal lines for several choice of ϵ . Secondly, one need to identify automatically a scaling region and clusters with ϵ . Finally, a last step involving a linear regression

leads to the estimation of K_2 . Although some contributions have applied the RP and RQA frameworks to conservative dynamics (see in particular [Zou et al. \(2007a,b, 2016\)](#) or [Asghari et al. \(2004\)](#); [Kovács \(2019, 2020\)](#) in the context of gravitational n -bodies like dynamics), their developments have been mostly driven by dissipative dynamics. This might be partly due to the important application of RPs and RQAs to the field of time series analysis (see [Marwan \(2023\)](#) for a bibliographical view of RPs and RQAs), the raise of chaos in low-dimensional dynamical systems, and the desire to analyse complex systems from the nonlinear time series perspective (intimately connected with phase space reconstruction and delay embedding theorem, see, *e.g.*, [Abarbanel et al. \(1993\)](#)). This contribution has two main objectives: first, to further apply the methodology and potential of RPs to conservative problems by focusing on the standard map model, paradigm of Hamiltonian chaos; and second, to propose a methodology for using the specific DIV metric as an indicator of chaos. Indeed, our work provides numerical evidences that the more heuristic divergence DIV furnishes a simple and robust workaround of the more cumbersome automatisation of the estimation of K_2 , and might be reliably used to distinguish between regular and chaotic motions¹.

The remainder of the paper is structured as follows. In Sec. 2, we present the model and computational tools that form the basis of our analysis. In Sec. 3, we explore numerically the asymptotic time behavior of DIV and reveal distinct power laws of the average $\langle DIV \rangle$ depending on the nature of the orbit, being regular or chaotic. Our results follow from an extensive parametric investigation based on the standard map model. The power laws we reveal are obtained not only in the original two dimensional phase space but also in reconstructed phase spaces, following a nonlinear time series perspective (*i.e.*, by starting from the knowledge of observables). In Sec. 4, we demonstrate that DIV can effectively be used as chaos indicator. We assess the performances of the DIV measure in the original 2 dimensional phase space against the fast Lyapunov indicator, a variational method based on the tangent map. Finally, Sec. 5 summaries and closes the paper.

2. MODEL AND COMPUTATIONAL TOOLS

2.1. The standard map. The standard map is a 2-dimensional area preserving map paradigmatic of Hamiltonian chaos. The map is obtained as the Poincaré map of the periodically kicked rotator model ([Chirikov, 1979](#)). Given a point (p_0, θ_0) on the two dimensional torus $\mathbb{T} = [0, 2\pi]^2$, the dynamics for $n \geq 0$ reads

$$(4) \quad \begin{cases} p_{n+1} = p_n + K \sin \theta_n \pmod{2\pi}, \\ \theta_{n+1} = \theta_n + p_{n+1} \pmod{2\pi}, \end{cases}$$

where K is the nonlinearity parameter. When $K = 0$, the dynamics is easily described. The “action” p is constant and the angle θ evolves linearly with time. When $K \neq 0$, resonances grow in size and chaos manifests ([Meiss, 2008](#)).

2.2. Computation of the divergence DIV . The divergences $DIVs$ follow from the maximal line length in the diagonal direction of the RPs. The RPs and the RQAs measures are computed using the Julia language and specific packages ([Bezanson et al., 2017](#); [Datseris, 2018](#)). Several parameters are involved during the computation of the RPs matrices, provided here for the sake of reproducibility.

¹The landscape of chaos indicators and complexity measures developed over the last decades is rich, and still active. Among the various existing methods, it is rather customary to distinguish between variational methods, relying on the Jacobian associated to the dynamics (or, an estimation of it), and orbit based diagnostics. The divergence DIV has the advantage to belong to the later family; yet, its complexity is $\mathcal{O}(N^2)$, N being the length of the orbit. A non-exhaustive list of indicators include the Lyapunov exponent ([Benettin et al., 1976](#)), the fast Lyapunov indicator and variations of it ([Froeschlé et al., 1997](#); [Fouchard et al., 2002](#)), the mean-exponential growth of nearby orbits (MEGNO, [Cincotta and Simó \(2000\)](#)), the smaller alignment index and its generalisation (SALI and GALI respectively, see [Skokos \(2001\)](#); [Skokos et al. \(2007\)](#)), frequency analysis ([Laskar, 1993](#)), topological braid chaos methods ([Thiffeault, 2005](#)), the 0 – 1 test ([Gottwald and Melbourne, 2004](#)), Lagrangian descriptors methods based on lengths of orbits ([Daquin et al., 2022](#); [Hillebrand et al., 2022](#)) (requiring the trajectory and the knowledge of nearby trajectories), Birkhoff averages ([Sander and Meiss, 2020](#)).

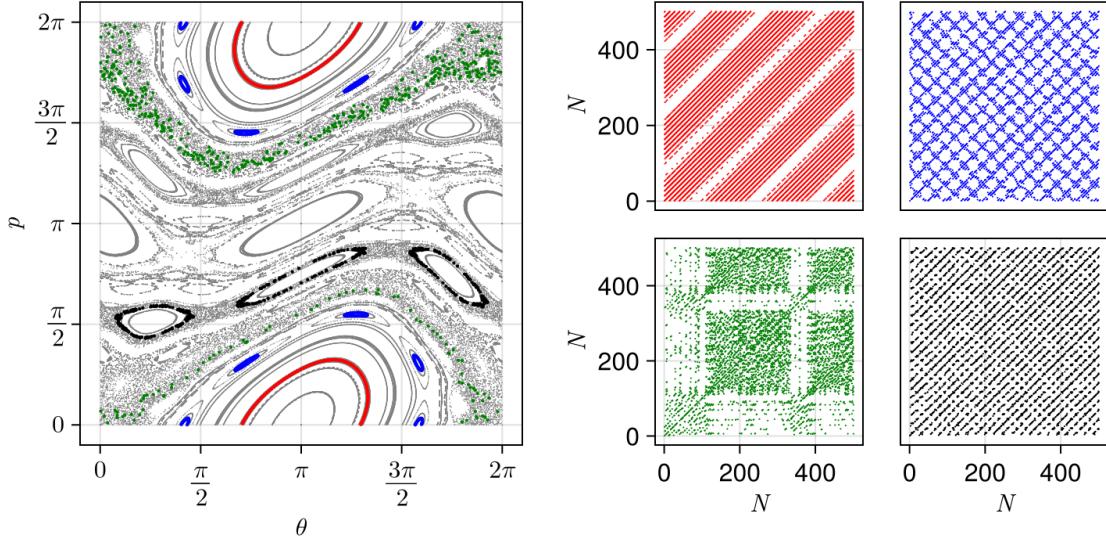


FIGURE 1. (Left) The phase portrait of the standard map at $K = 1$ for $N = 500$ iterations. (Right) Recurrence plots associated to four representative trajectories shown in color in the phase space; red - quasi-periodic orbit $(\theta_0, p_0) = (3.5, 1.0)$, blue - orbit trapped in a secondary resonance $(\theta_0, p_0) = (3.85, 1.7)$, green - large scale chaotic orbit $(\theta_0, p_0) = (3.3, 1.8)$, black - sticky orbit $(\theta_0, p_0) = (3.14, 2.215)$. The different textures in the RPs serve as a basis for recurrence quantification analysis. In particular, we focus here on the divergence metric DIV , associated to the non-trivial longest diagonal line in a given RP.

Unless otherwise stated, the norm used in Eq. (1) is the Euclidean norm. To minimise arbitrariness, the parameter ϵ follows from a normalised value, a fixed recurrence rate (RR),

$$(5) \quad RR = \frac{1}{N^2} \sum_{i,j} r_{i,j},$$

here set to $RR = 5\%$. For RPs computed up to time N , the longest diagonal line is the line of identity (trivial diagonal of recurrences). This line is excluded from the RPs, together with its vicinity (we set the Theiler window to 2). The smallest diagonal line considered in RPs is $\ell_{\min} = 2$. The fig. 1 is an illustrative composite plot showing the phase space of the standard map at $K = 1$ together with several RPs for four orbits with distinct properties: an orbit immersed within the main resonant island, an orbit in a secondary resonance, an orbit experiencing stickiness and finally one orbit experiencing large scale chaos.

2.3. The fast Lyapunov indicator. A strictly positive largest finite time Lyapunov exponent (FTLE) is usually considered as signature of deterministic chaos (see Skokos (2009) for a review), *i.e.*, nearby trajectories diverge exponentially fast in average. Here, to assess the presence of chaos, we instead compute a closely related quantity, namely the fast Lyapunov indicator, hereafter denoted FLI (Froeschlé et al., 1997). The FLI is a well established variational chaos indicator, valid for discrete and continuous systems. Contrarily to the FTLE, the FLI does not average the growths of the tangent vector over time, an average that impedes its fast convergence. Given a smooth mapping $x_{n+1} = M(x_n)$, $n \in \mathbb{N}$, let us denote by v_0 a unitary deviation vector. The tangent map dynamics

reads

$$(6) \quad \begin{cases} x_{n+1} = M(x_n), \\ v_{n+1} = DM(x_n)v_n. \end{cases}$$

There are several definitions of the FLI literature, and in the following we follow the one of Lega and Froeschlé (2001) that suppresses oscillations,

$$(7) \quad \text{FLI}(x_0, v_0; N) = \sup_{n \leq N} \log \|v_n\|.$$

For regular orbits, FLI grows as $\mathcal{O}(\log N)$ with time N . For chaotic orbits, the norm of the deviation vector grows exponentially fast, and thus the FLI tends to grow as $\mathcal{O}(N)$. Those two distinct time evolutions allow the detection of chaos in a short time (Guzzo and Lega, 2023).

3. TIME BEHAVIOR OF DIV

Aiming to harness the potential of the *DIV* measure as chaos indicator, we first need to ensure that the metric reacts differently on regular and chaotic trajectories. We have investigated time asymptotic properties of *DIV* on three scenarios, detailed subsequently, by probing parametrically the dynamics of the standard map for many different initial conditions. Our parametric study focuses on properties in the original 2 dimensional phase space, but also in reconstructed phase spaces when considering generated time series. Quite interestingly, our numerical results highlight clear distinct power laws underpinning the time behavior of the spatial average $\langle \text{DIV} \rangle$ for regular or chaotic trajectories. In the following, the final time N is set to $N = 25,000$, considered as here as an “asymptotic time.”

3.1. Time behavior of $\langle \text{DIV} \rangle$ in the original phase space. We computed the *DIV* for a set of 200 initial conditions distributed randomly into $[0, 2\pi]^2$ for parameters K randomly chosen in the range $[0.6, 4]$. By using the FLI computed at the final time N , each initial condition is assigned to a specific label (regular or chaotic). The Fig. 2 shows the time evolution of the divergences *DIVs* for this set of initial conditions. Initial conditions that are FLI regular are color coded in blue, while FLI chaotic trajectories appear in red. We observe that the *DIVs* tend to form two distinct clusters, indicating that the *DIV* metric effectively captures the regular or chaotic nature of the orbit through its distinct time evolution. We note the spreading of the *DIVs* values to be less pronounced in the case of regular orbits. The fits of the ensemble averages $\langle \text{DIV} \rangle$ of the regular and chaotic cluster lead to power laws (dashed black curves) with clearly distinct exponents. Regular orbits are characterised by

$$(8) \quad \langle \text{DIV}(N) \rangle \propto N^\gamma, \gamma \sim -1,$$

(numerically, we find $\gamma_{\text{reg}} = 1.045$), whilst chaotic orbits experience a much slower decay rate (numerically, we estimate $\gamma_{\text{cht}} = -0.48$). It should be noted that the asymptotic decay of $\langle \text{DIV} \rangle$ in the regular case aligns with the asymptotic decay of the maximal Lyapunov characteristic exponent (Benettin et al., 1976; Contopoulos et al., 1978).

We further confirmed our observations by repeating the previous experiment for 4 frozen values of K , namely $K \in \{0.6, 1.1, 2.6, 4\}$. The chosen values are representative of phase spaces dominated by stability ($K = 0.6$) or chaoticity ($K = 4$). The intermediate value of $K = 1.1$ reflects a mixed phase space regime, where chaotic and regular structures cohabit in a balanced proportion. Our previous observations fully extrapolate to these 4 cases. In particular, regular trajectories have their *DIVs* following closely the curve $N^{\gamma_{\text{reg}}}$, with $\gamma_{\text{reg}} \leq -1$, whilst chaotic trajectories have in average a slower decay rate, close to $-1/2$, as shown in Fig. 3. A finer analysis was performed as a function of the nonlinearity parameter K , although this required reducing the final time to $N = 500$. We found $\gamma_{\text{cht}} \sim -1/2$ to be characteristic of the decay rate of $\langle \text{DIV} \rangle$ in the chaotic regime, see Fig. 4. (The power-laws with pronounced exponents seem to be characteristic of deterministic systems. For a stochastic ARMA process, the exponent is much smaller, reflecting the quasi absence of decay rate of *DIV*, see appendix A). In appendix B, we present numerical evidences that our results are robust

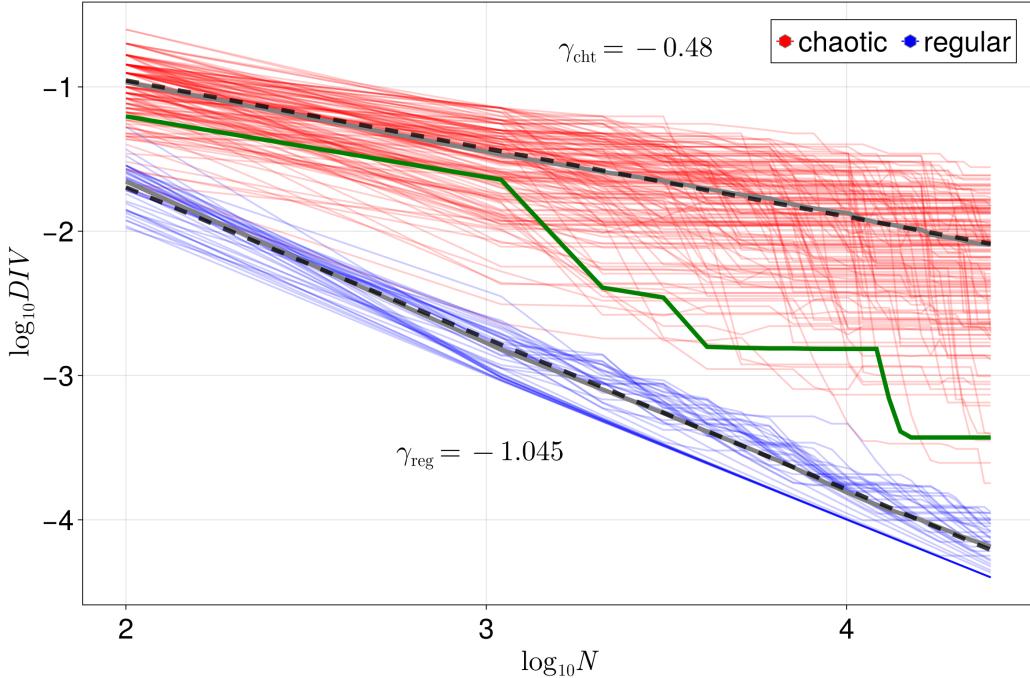


FIGURE 2. The measure DIV as a function of the time series length, N . The ensemble of 200 trajectories is separated according to the threshold values into regular (blue) and chaotic (red) parts, 152 and 48 members, respectively. The fit (dashed lines) to the ensemble averages (gray solids) shows different power-law exponents for the two kinds of motion. The green curve will be discussed thereafter.

to the parameters of the RPs, that is, when RR varies in a realistic manner or when the norm to gauge the recurrence is changed. The appendix C extends our conclusions to two resonant archetypal 1 degree-of-freedom non-autonomous Hamiltonian models where the divergence DIV , computed from the 2-dimensional discretisation of the dynamics via the stroboscopic map, also obeys the laws just described. Thus, there is a body of results that supports these numerical findings.

A close inspection of specific curves in the set of chaotic trajectories of Fig. 2 reveals that some orbits have a final DIV comparable to regular orbits. The green orbit of Fig. 2 is one of such orbit. We identified initial conditions and parameters of these orbits for a more detailed analysis and found that those orbits exhibit stickiness during their evolution. The sharp and successive drops in DIV are correlated to time intervals during which the orbit is temporarily trapped in specific regions of the phase space. During this time, the decay of DIV tends to align to the decay rate of regular trajectories, $\gamma_{\text{reg.}} = -1$. When the orbit leaves the sticky zone and experiences large excursion, the exponent is closer to $\gamma_{\text{cht.}} = -1/2$. Altogether, it produces the apparent staircase pattern. This dynamical behavior is also observed and confirmed through the FLI analysis, where the FLI behaves as plateau during the sticky events (no hyperbolicity contributing to the growth of the norm of the tangent vector). The panel of Fig. 5 illustrates this phenomenology on a representative case. Thus, we find that the DIV metric is sufficiently sensitive to detect the presence of sticky dynamics, which manifest as significant fluctuations in DIV over specific time windows corresponding to temporary captures. (Note that this observation provides another RQA measure able to capture stickiness, besides the recurrence rate RR , see Palmero et al. (2022).)

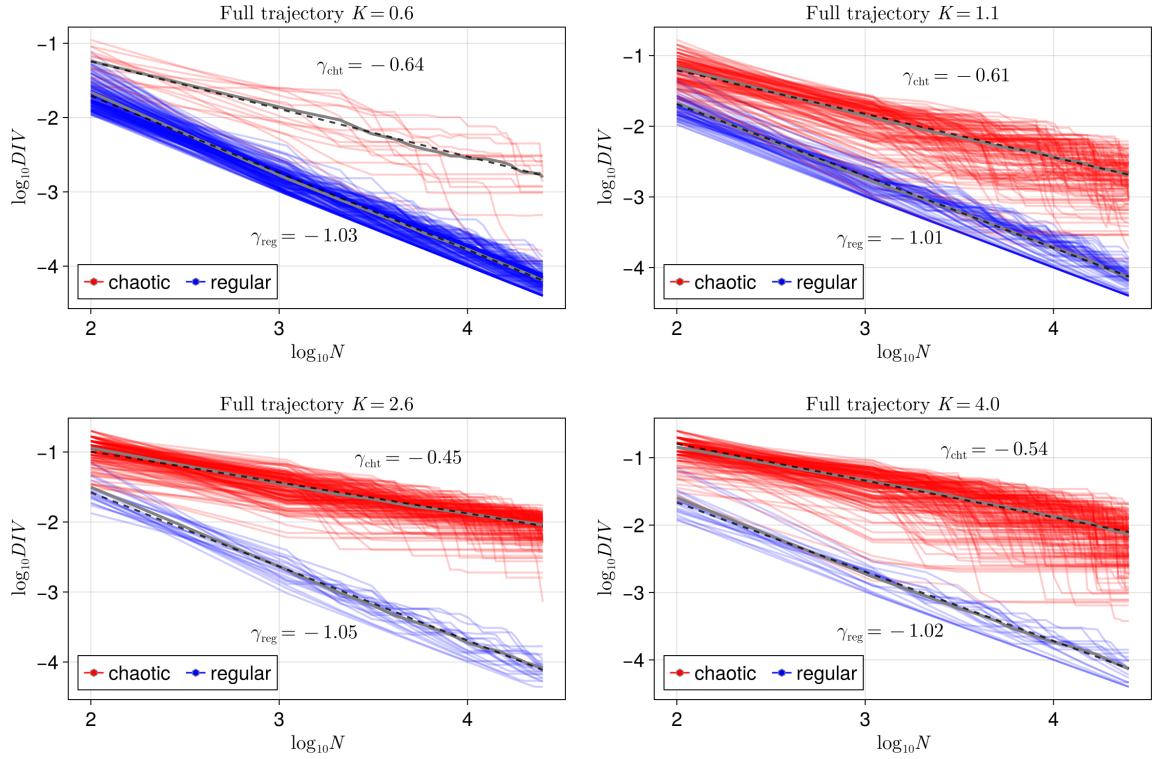


FIGURE 3. Same as in Fig. 2 for varying nonlinearity parameters K ranging dynamical regimes dominated by stability ($K = 0.6$) and chaoticity ($K = 4$).

3.2. Time behavior of $\langle \text{DIV} \rangle$ starting from an observable, no embedding. The distinct power laws just revealed are based on the computation of the RP from the 2-dimensional trajectory. Adopting a nonlinear time series perspective, we now conduct similar numerical experiments starting from the knowledge of a specific univariate observable $z = g(p, \theta)$ over time, $\{z_0, z_1, \dots, z_N\}$, g being an observable function. The method of delay-coordinates often follows to reconstruct the phase-space. This is the approach taken subsection 3.3. For now, we compute the RP plot with no embedding, *i.e.*, without any phase space reconstruction. Given the low-dimensionality of the original system, according to Iwanski and Bradley (1998), it is expected the RP plot to be almost independent from the reconstruction process. As the DIV follows directly from RPs, we thus expect as byproduct the DIV to be also almost independent from the latter. Under this setting, Eq. (1) becomes now

$$(9) \quad r_{i,j} = \Theta(|z_i - z_j| - \epsilon).$$

Note that $\|\bullet\|$ of Eq. (1) has been replaced by the absolute value. We repeated the steps of subsection 3.1 for the following choice of observables: $g(p, \theta) = p$ and $g(p, \theta) = \theta$. The results are shown in Fig. 6.

3.3. Time behavior of $\langle \text{DIV} \rangle$ starting from an observable, including embedding. Contrarily to the approach taken in subsection 3.2, starting from the univariate time series $\{z_0, z_1, \dots, z_N\}$, we now reconstruct a phase space in \mathbb{R}^d , for some estimated d , via the delay-coordinates method with lag-time τ (Abarbanel et al., 1993). More specifically, let be $m = N - (d-1)\tau$, then the reconstructed trajectory reads $\{y_i\}_{i=0}^m$ with

$$(10) \quad y_i = (z_i, z_{i+\tau}, \dots, z_{i+(d-1)\tau}).$$

There are several methods to determine the embedding parameters d and delay time τ . Here, we fix the time delay to $\tau = 1$ and determine d using the false nearest neighbors method (Kantz and

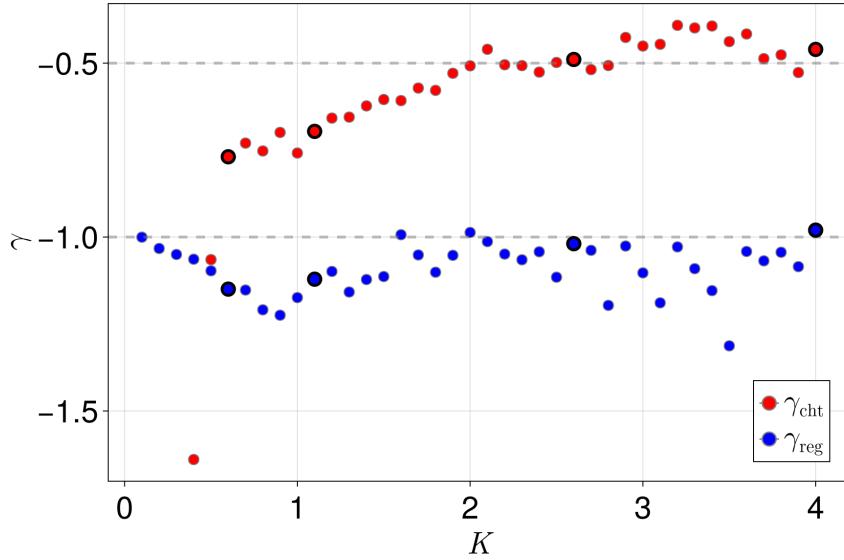


FIGURE 4. Evolution of the power laws exponent γ in regular or chaotic cases as a function of K computed at $N = 500$. The separation between regular and chaotic orbits is well reflected by the exponent. For regular components, the exponent is always smaller than $\gamma_{\text{reg}} \leq 1$, whilst for chaotic ensembles, $\gamma_{\text{cht.}}$ is larger and tends to align to $\gamma_{\text{cht.}} = -1/2$ for very chaotic components.

(Schreiber, 2003; Small, 2005). Once the phase space is reconstructed, the divergence is computed from the RP matrix

$$(11) \quad r_{i,j} = \Theta(\|y_i - y_j\| - \epsilon),$$

where $\|\bullet\|$ denotes the Euclidean norm in \mathbb{R}^d . Fig. 7 presents the analogue of Figs. 3 and 6 starting from the two observables $z(p, \theta) = \theta$ and $z(p, \theta) = p$ and extrapolates our previous observations. In appendix B, we present evidences that the results of the whole section do not chiefly depend on the choices made in computing the RPs: norm and threshold ϵ .

4. PERFORMANCE ASSESSMENT OF DIV AS CHAOS INDICATOR IN THE ORIGINAL PHASE SPACE

We now qualitatively and quantitatively assess the sensitivity of the DIV measure as finite time chaos indicator. Given the similarities of the power laws revealed in Sec. 3 in the original or reconstructed phase spaces, we focus solely on the performances in the original phase space, *i.e.*, using the 2 dimensional original trajectory. We benchmark the performance against the FLI, considered here as “ground-truth.” Our assessment is performed at $N = 500$, thus working with data lengths that are relevant to real-world scenarios.

Given an initial condition x_0 and a final time N , it is first desirable to define a threshold α to establish a binary classification of the orbit as regular or chaotic from the value $DIV(x_0; N)$. More precisely, we are looking to a criteria

$$(12) \quad \begin{cases} DIV(x_0; N) \leq \alpha \Rightarrow \text{“The orbit stemming from } x_0 \text{ is regular,”} \\ DIV(x_0; N) > \alpha \Rightarrow \text{“The orbit stemming from } x_0 \text{ is chaotic.”} \end{cases}$$

To determine the threshold α , we follow here standard approaches used for variational methods relying on the shape of the histogram of the metric computed for many initial conditions (Szezech Jr et al., 2005). Fig. 8 shows the phase space of the standard map for $K = 1$ together with the landscapes of

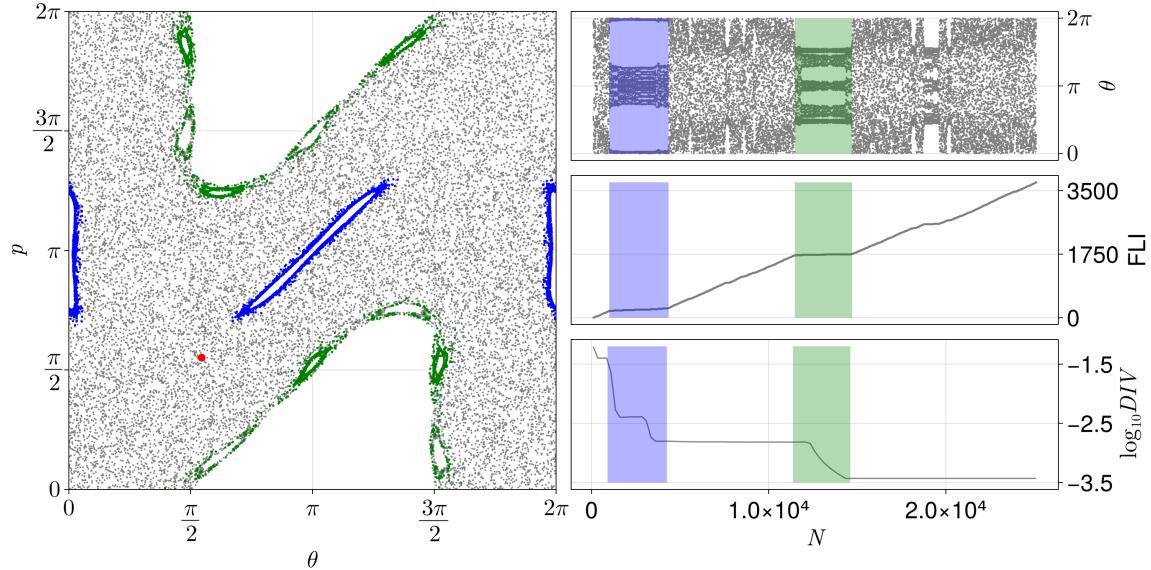


FIGURE 5. (Left) Phase space of the chaotic orbit with initial condition marked in red extracted from Fig. 2. The orbit is chaotic with a low divergence value (green curve). Two portions of the orbit are colored (in blue and green) in accordance with the 2 most prominent sticky events encountered during its evolution. (Right, from top to bottom) Time evolution of the angle θ , the FLI and the DIV . Sticky events (localised variations of θ) correspond to plateau in the FLI and sharp decay in the DIV measure, as visually materialised with the blue and green boxes serving as guides. The sharp decreases of DIV during the sticky events explain the low final value of DIV , despite being chaotic.

the FLI and DIV for 250 initial conditions spread along the dashed line joining the initial conditions $(\theta, p) = (\pi, 0)$ and $(\pi, 2\pi)$. For both indicators, we identify sharp increases, in 1 – 1 correspondence, when they cross transversely the thin chaotic layers. The bottom row of Fig. 8 shows the histograms of their final values for a resolved 400×400 Cartesian mesh of initial conditions in $[0, 2\pi]^2$. The FLI distribution is rather right skewed, with the main peak located close to $\log(N) = \log(500) \sim 2.69$. This value characterizes the background of regular orbits. In this case, setting the threshold value α larger than this furnishes a reliable threshold to separate chaotic from regular trajectories, here set to $\alpha = 4$. The distribution of the DIV s is rather bimodal, where each mode reflects the typical values taken on regular and chaotic components. In this case, the threshold α is set as the minimal value between the two modes, leading here to $\alpha = 2.15$. In the following, we systematically rely on histogram inspection to determine ad hoc α values (for regimes dominated by regularity, the histogram is skew right, for a regime dominated by chaoticity, it becomes skew left).

Fig. 9 portrays a series of dynamical maps corresponding to a scan of $[0, 2\pi]^2$ (top row) and a magnification of it (bottom row) materialised by the red square appearing near the $2 : 1$ periodic orbit. Each domain is meshed with a regular 400×400 grid of initial conditions. Initial conditions leading to chaotic motions (either with the FLI or DIV) are marked in white, whilst regular orbits appear in black. Qualitatively, the dynamical maps of the first two columns of Fig. 9 are similar, meaning that the DIV metric is indeed sound as chaos indicator. By plotting the initial conditions for which the labels are in disagreement, we are able to reveal the mismatch set (third column of Fig. 9). The two-scale analysis of the mismatch sets highlights that mismatch points are predominantly concentrated

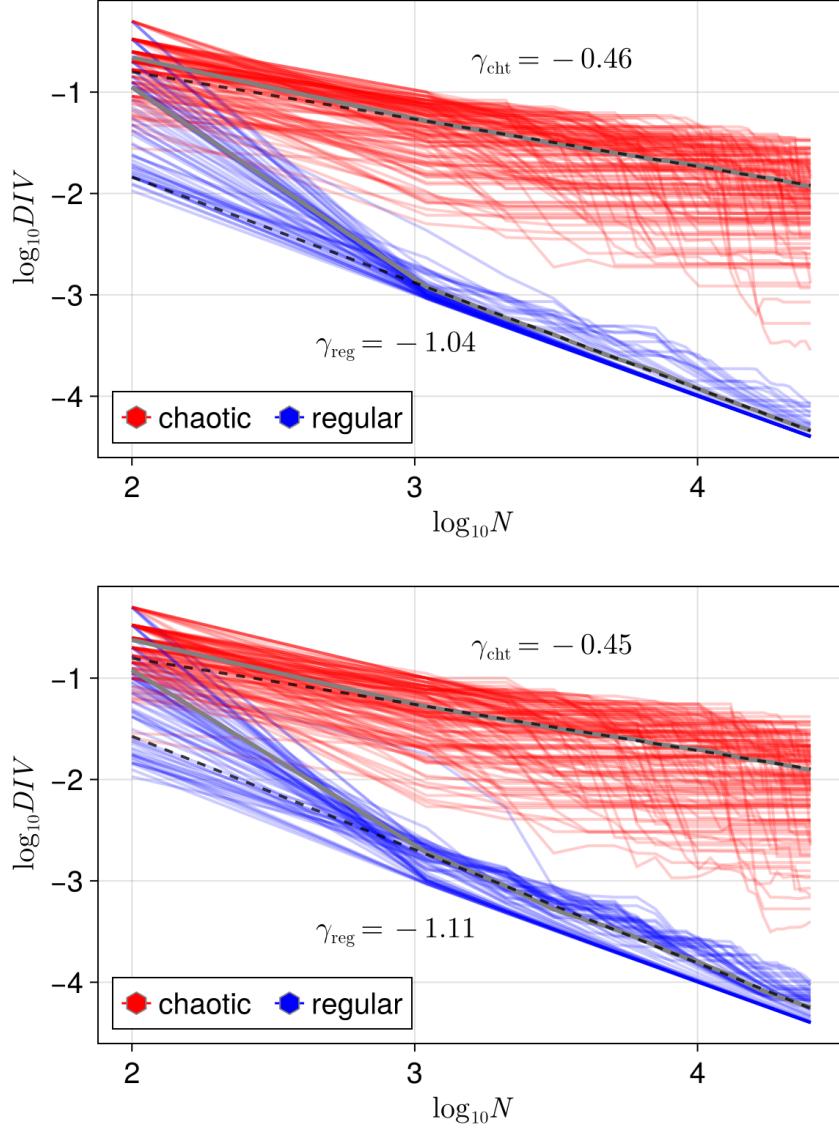


FIGURE 6. Time evolution of DIV as a function of time based on the two observables corresponding to θ (left) and momentum p (right). No reconstruction of the phase space is performed.

along the edges of the dynamical structures.

In order to compare the dynamical maps more quantitatively, we compute a simple metric with a straightforward interpretation: the probability of agreement P_A between the labels produced (chaotic or regular) with either method over a specific domain D (discretised as a regular Cartesian mesh of initial conditions.) The metric reads

$$(13) \quad P(D) = \frac{\#\text{Labels in agreement}}{\#\text{Initial conditions in } D}.$$

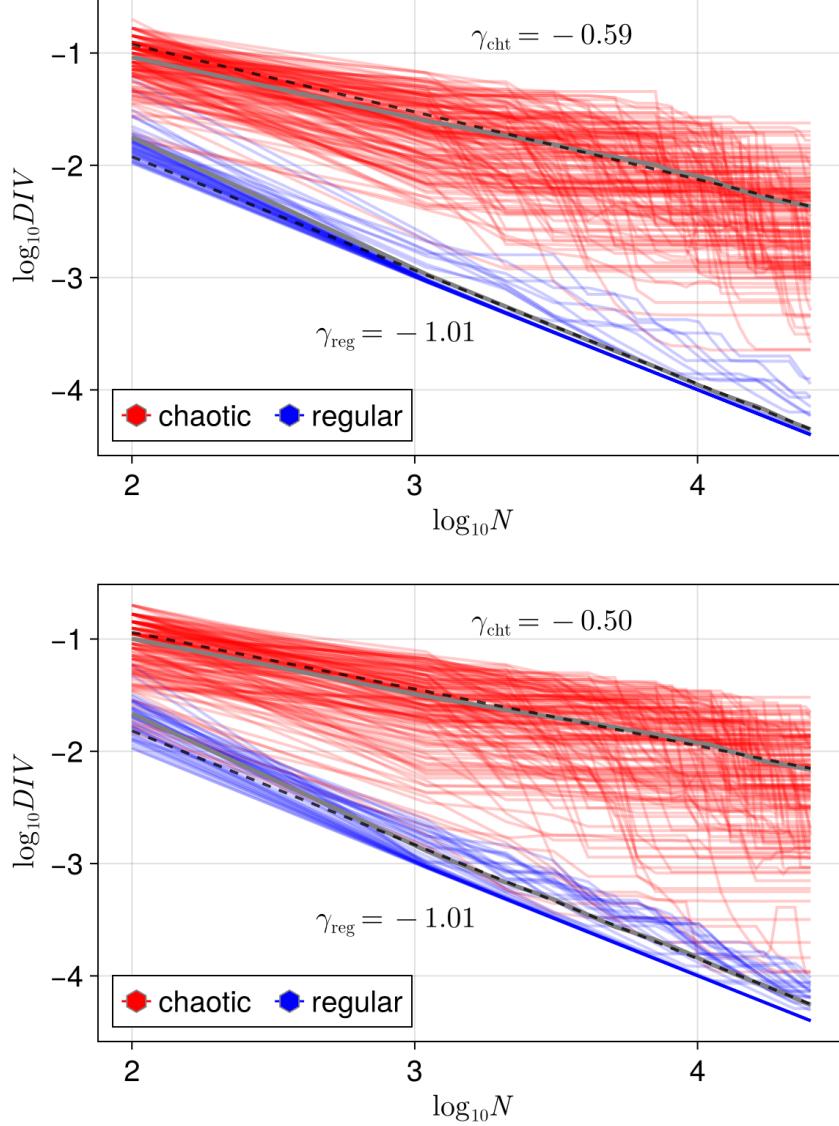


FIGURE 7. Time evolution of DIV as a function of time based on the two observables corresponding to θ (left) and momentum p (right) in the reconstructed phase space.

Note that $P(D)$ concatenates the outputs of resolved dynamical maps. For $N = 500$, we found the DIV method to perform overall very well against the FLI, with agreement over 95% over the range of nonlinearity parameters $[0.6, 4]$, as illustrated in the left panel of Fig. 10. The range of values of $K \in [1, 2.2]$ corresponds to a dip in $P(D)$. These dynamical regimes are characterized by regular and chaotic structures that cohabit in a rather balanced manner (mixed phase space regime). Given that mixed phase spaces contain many thin secondary structures, it is not surprising that the indicator is challenged. Nevertheless, the agreements are still excellent, above 95%. Increasing the time to $N = 1,000$ doesn't alter much this picture but improve very slightly the agreement. The right panel of Fig. 10 complements the former computation and shows, as a function of the nonlinearity parameter K , the volume of chaotic trajectories estimated either with the FLI or the DIV . The computation closely follows the one of Sander and Meiss (2020). Throughout the range of K values, we observe

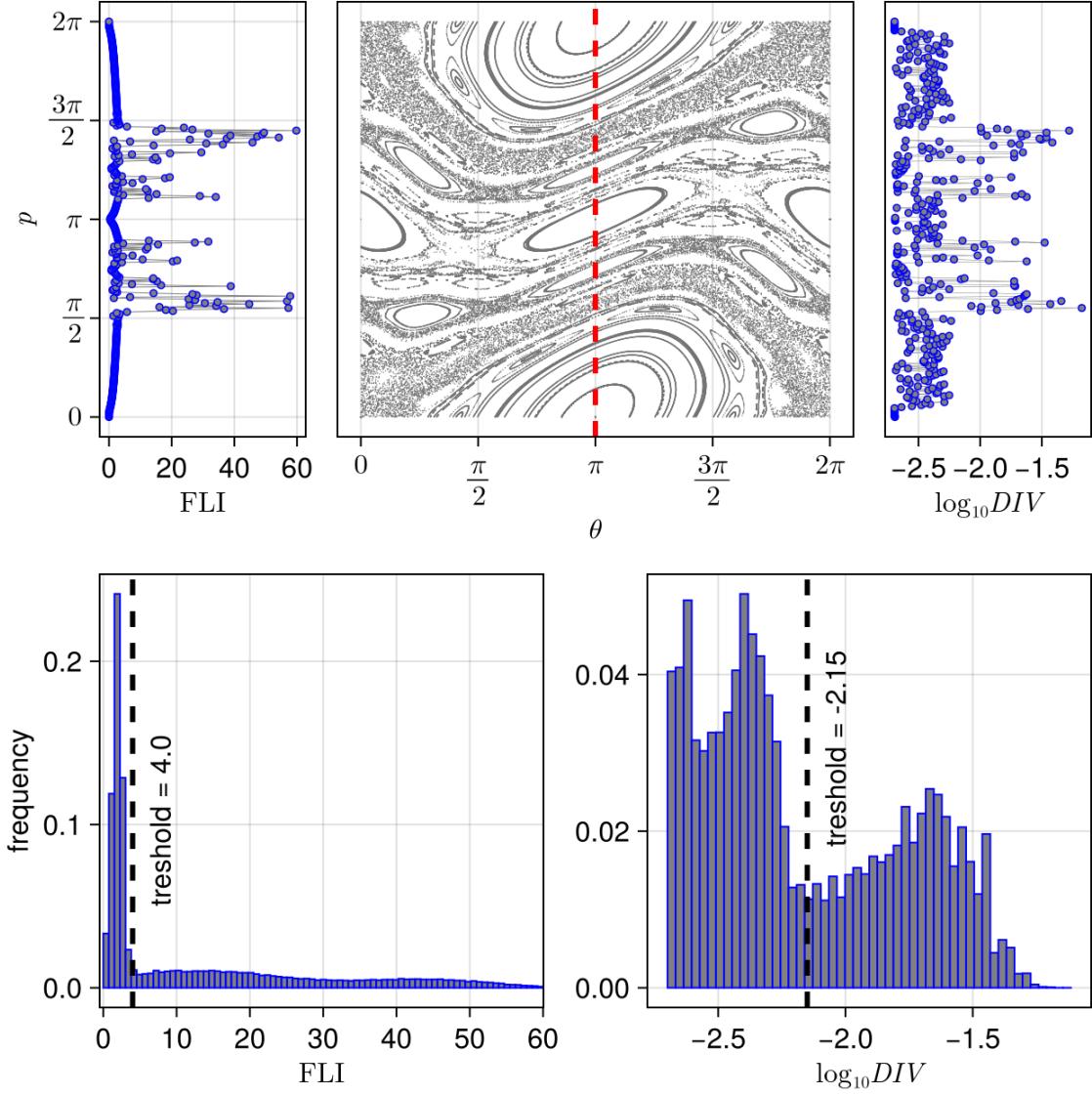


FIGURE 8. (Top row) Phase portrait of the standard map at $K = 1$ along with FLI and DIV landscapes computed at $N = 500$ along the dashed vertical red line of initial conditions. (Bottom row) Distributions of the FLI and DIV for 400×400 uniformly distributed initial conditions in $[0, 2\pi]^2$. Inspection of the histograms allow to set a threshold α to binarily classify orbits as regular or chaotic. See text for details.

that the two curves closely follow one another, further demonstrating the effectiveness of the DIV method in serving as a reliable chaos indicator.

5. CONCLUSIONS

Although RPs and RQAs have been predominantly applied to dissipative systems, our study demonstrates the effectiveness and potential of the DIV metric to serve as a simple diagnostic to detect chaos

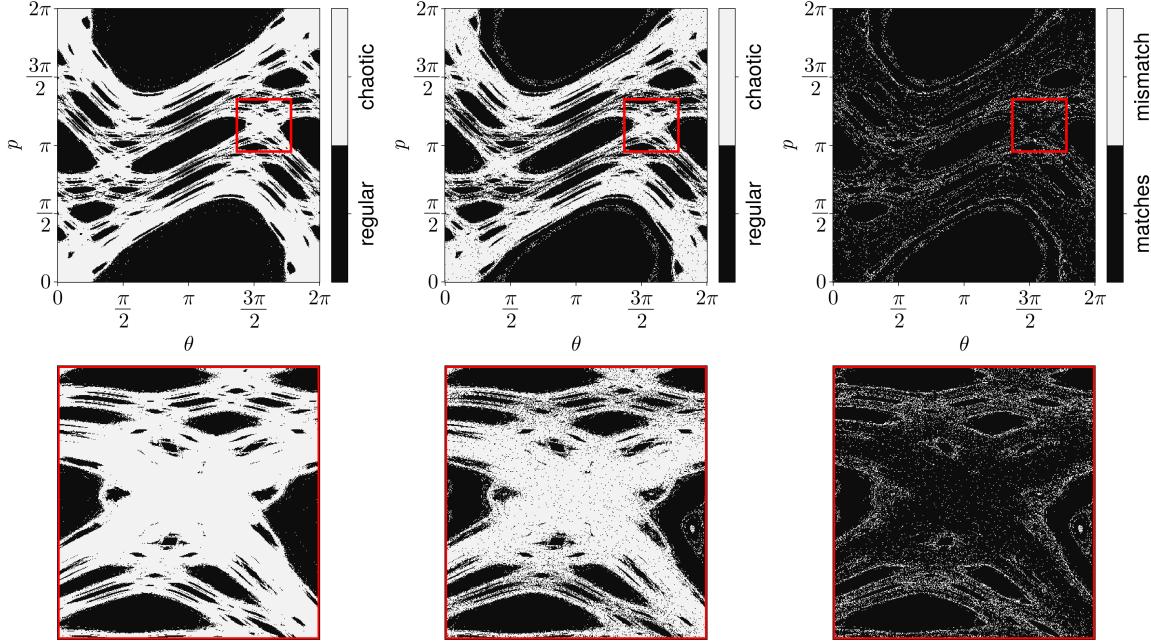


FIGURE 9. (Top row) Stability maps encoding the regularity (black) or chaoticity (white) obtained with the FLI and the DIV , respectively. The last figure shows the mismatch points, *i.e.*, the points for which the classification disagree (white). (Bottom row) Same analysis at a smaller scale delineated by the red box in the top row. Most of the mismatch points are located in the vicinity of the edges of the dynamical structures.

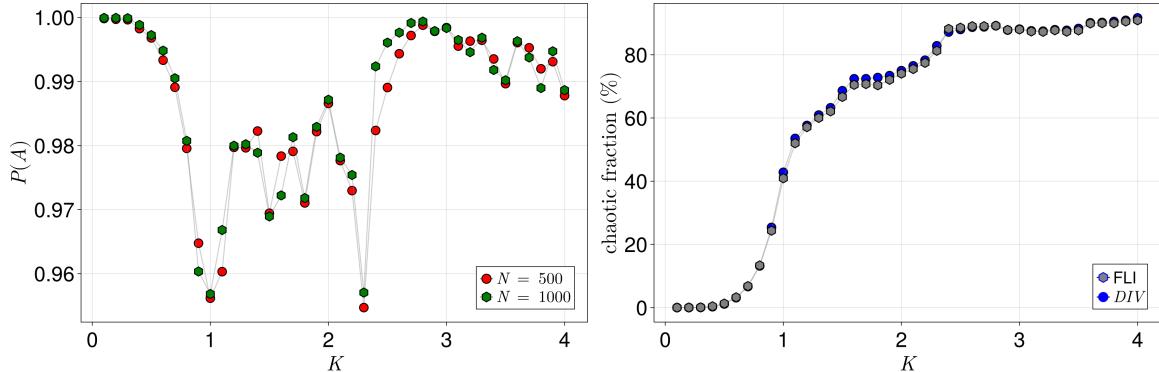


FIGURE 10. (Left) Evolution of the probability of agreement P_A (confer Eq. (13)) versus the nonlinearity parameter K . P_A is computed over the domain $D = [0, 2\pi]^2$. (Right) The size of the chaotic region in the phase space for different K computed either with the FLI or the DIV . The two curves closely follow each other, strengthening further the use of DIV as chaos indicator. Both panels are based on a mesh of 400×400 initial conditions computed at $N = 500$.

	$\gamma_{\text{reg.}}$	$\gamma_{\text{cht.}}$
L_2 -norm	-1.04	-0.48
L_1 -norm	-1.04	-0.46
L_∞ -norm	-1.03	-0.47

TABLE 1. Sensitivity of the exponents with respect to the choice of the norm of Eq. (1).

in conservative systems. We have compared the performance of this indicator against the fast Lyapunov indicator, a well established variational chaos detection method. We focused on the standard map, a paradigmatic discrete model of Hamiltonian chaos. From our extensive numerical simulations, we have quantitatively assessed the overall good agreement between the methods, further strengthening, but this time in the conservative regime, the strong correlation between the divergence measure and the presence of chaos in the dynamics. Our comparison has focused on moderately long orbits with 500 data points, stepping towards real-world applicability of the method. The distinct power-laws of DIV we have revealed in average depending on the regularity or chaoticity of the trajectories, on a much longer timescale, valid in the original and reconstructed phase space, shed also more light on asymptotic properties of the DIV metric. In particular, we have observed a decay of DIV as $1/N$ for regular orbits, which is interestingly the same rate as the maximal Lyapunov characteristic exponent. These distinct power laws could also be particularly valuable when dealing with a limited number of orbits, a context in which setting the threshold α from the histogram becomes challenging. Those properties suggest a possible new approach for chaos detection in time series, based on analyzing the slope of the decay rate of DIV computed over windowed segments of the original series. Our current efforts are focused on leveraging this property, besides the noise free case, alongside providing analytical insights to support our findings.

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DATA AVAILABILITY STATEMENT

The data presented in this study can be reproduced based on the models and list of parameters described in the text.

APPENDIX A. DIV VERSUS N FOR ARMA PROCESS

Figure 11 reports the time evolution of an autoregressive AR(1) model. We observe a quasi absence of decay of DIV versus the time. Note that the exponent found is one order of magnitude smaller than characteristic exponents found in the deterministic case.

APPENDIX B. ROBUSTNESS WITH RESPECT TO CHOICES

The power-laws and probability of agreements presented in the manuscript depend, a priori, on the choice of the norm and the neighboring condition defined by ϵ (fixed through a fixed RR .) Tables 1 and 2 show that the exponents we found are unaffected when the parameters vary. The boldfaced fonts repeat the exponents found in the manuscript's corpus, presented here again for the sake of completeness. Only the case of the initial 2D trajectory has been explored.

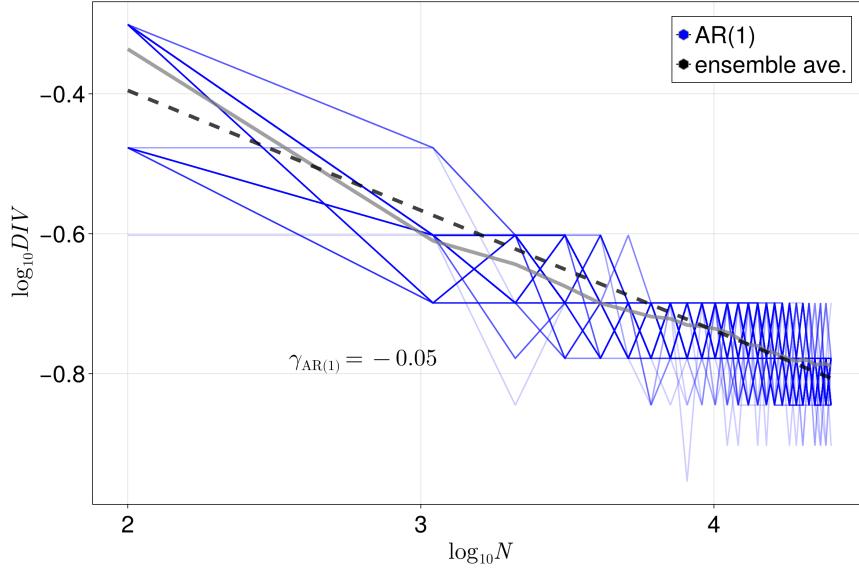


FIGURE 11. Time evolution of DIV for a AR(1) process together with the fit of it's time evolution.

	$\gamma_{\text{reg.}}$	$\gamma_{\text{cht.}}$
$RR = 0.05$	-1.048	-0.48
$RR = 0.025$	-1.07	-0.49
$RR = 0.1$	-1.00	-0.47

TABLE 2. Sensitivity of the exponents with respect to ϵ in Eq. (1) fixed through various recurrence rates RR .

APPENDIX C. APPLICATIONS TO TWO STROBOSCOPIC MAPS COMPUTED FROM RESONANT HAMILTONIANS

We report the temporal laws of DIV obtained for two stroboscopic maps of two Hamiltonian models. Both models are time-periodic 1 degree-of-freedom (DoF) models that contain resonant harmonics. In both cases, the stroboscopic map P is 2 dimensional and has a cylindrical topology, *i.e.*, its variables $\mathbf{x} = (x, y) \in \Sigma = S^1 \times I$, $I \subset \mathbb{R}$ and S^1 is the circle. Let be

$$(14) \quad \Sigma_T = \{(\mathbf{x}, t) \in \Sigma \times \mathbb{T}_T\}, \mathbb{T}_T = \mathbb{R}/T\mathbb{Z}.$$

For an initial condition $z = (\mathbf{x}, t) \in \Sigma_T$, we consider the T -time map

$$(15) \quad \begin{aligned} \Phi^T : \Sigma_T &\rightarrow \Sigma_T \\ z &\mapsto z' = (\mathbf{x}', t') = \Phi^T(z). \end{aligned}$$

The stroboscopic map then reads

$$(16) \quad \begin{aligned} P : \Sigma &\rightarrow \Sigma \\ \mathbf{x} &\mapsto P(\mathbf{x}) = \mathbf{x}'. \end{aligned}$$

Presentation of the results follow closely Sec. 3.1. In particular, the computation of DIV is performed in the original 2-dimensional phase space of the map P (no reconstruction whatsoever).

	$\gamma_{\text{reg.}}$	$\gamma_{\text{cht.}}$
Original 2D phase space	-1.06	-0.40
Observable I , reconstruction	-1.08	-0.40
Observable ϕ , reconstruction	-1.06	-0.36
Observable I , no reconstruction	-1.15	-0.41
Observable ϕ , no reconstruction	-1.06	-0.43

TABLE 3. Exponents for the power-laws of $\langle \text{DIV} \rangle$ found for the Poincaré map associated to the continuous model \mathcal{J} of Eq. (17) under different dynamical settings. The denomination ‘‘observable I ’’ and ‘‘observable ϕ ’’ is an abuse of notation and refer to the discrete variables of the Poincaré mapping associated to the original momentum I and angle ϕ . The parameters according to time-delay phase space reconstruction Takens (1981) are $d = 5$ and $\tau = 1$.

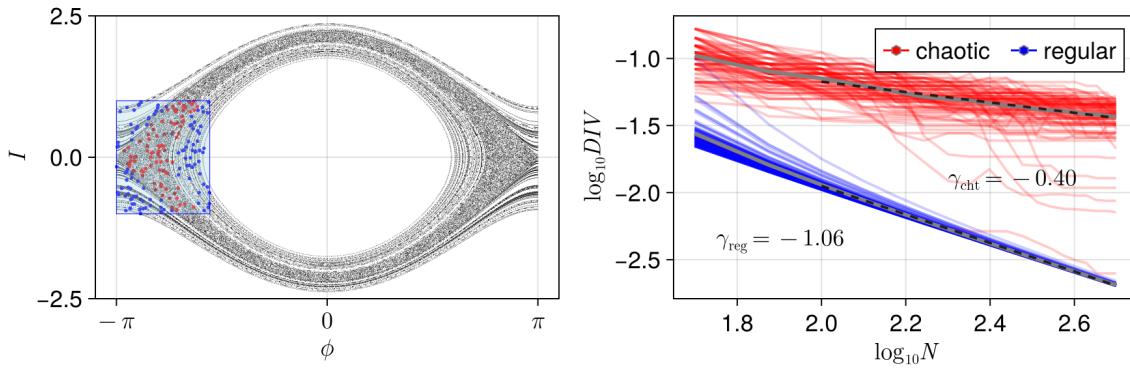


FIGURE 12. (Left) Phase portrait of the modulated pendulum. The light blue box depicts the initial conditions of 200 members of the ensemble used to estimate the power-law of DIV vs N . The uniform distribution in the highlighted domain provides roughly the same number of regular and chaotic orbits. (Right) Time dependence of measure DIV for chaotic (red) and regular (blue) parts of the ensemble. Initial conditions of trajectories are shown in left panel.

C.1. **Results for a modulated pendulum.** We consider the model

$$(17) \quad \mathcal{J}(I, \phi, t) = \frac{I^2}{2} - (1 + \alpha \cos \epsilon t) \cos \phi,$$

corresponding to a pendulum model with variable length. For this model, $T = 2\pi/\epsilon$ which is large when $\epsilon \ll 1$. We selected $\alpha = 0.25$ and $\epsilon = 0.1$. The phase space of P together with the temporal laws of DIV on regular and chaotic components are shown in Fig. 12. We found $\gamma_{\text{reg.}} = -1.06$ and $\gamma_{\text{cht.}} = -0.40$

C.2. **Results for two resonances that overlap.**

$$(18) \quad \mathcal{K}(I, \phi, t) = \frac{I^2}{2} - (\alpha_1 \cos(\phi - t) + \alpha_2 \cos(\phi + t)),$$

containing two cat-eye resonances centered at $I = -1$ and $I = 1$. For this model, $T = 2\pi$. We choose $\alpha_1 = 0.2$ and $\alpha_2 = 0.2$ (both resonances have equal dynamical weights). The phase space of P together with the temporal laws of DIV on regular and chaotic components are shown in Fig. 13. We found $\gamma_{\text{reg.}} = -1.02$ and $\gamma_{\text{cht.}} = -0.56$

	$\gamma_{\text{reg.}}$	$\gamma_{\text{cht.}}$
Original 2D phase space	-1.02	-0.52
Observable I , reconstruction	-1.06	-0.62
Observable ϕ , reconstruction	-1.03	-0.62
Observable I , no reconstruction	-1.16	-0.59
Observable ϕ , no reconstruction	-1.06	-0.59

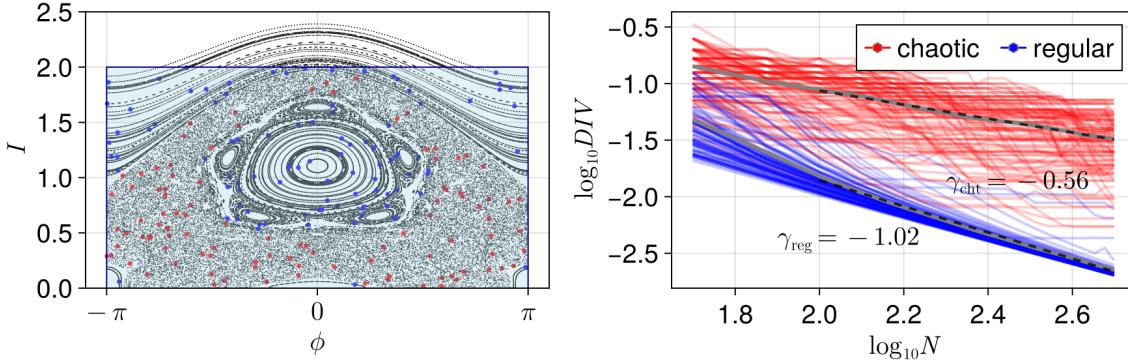
TABLE 4. Same as in Tab. 3 for the continuous model \mathcal{K} of Eq. (18)

FIGURE 13. (Left) Phase portrait for the resonance overlap Hamiltonian (18). (Right) Similar to Fig. 12.

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