Social Choice Distortion Analysis Using Optimization Techniques

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1 Introduction

At the intersection of theoretical computer science and economics, Social Choice Theory is a research area that has seen a plethora of activity in recent years. As suggested by the name, the field is focused on collectivizing individual preferences into an optimal decision for the group. A common setting of interest in the field is the single-winner election, where a set of voters (agents) submits preferences over a set of candidates (alternatives). In this setting and others, agents' preferences are often captured using utility functions, to which voters provide numerical values expressing the relative intensity of their preferences amongst the candidates. Utility functions can take on several forms, a common one being the normalized form, in which each agents' preference values amongst all possible alternatives are non-negative and sum to 1. As an example, someone who prefers two candidates equally would associate each with a value of $\frac{1}{2}$. Another area of interest is metric social choice, where, as opposed to the sometimes-arbitrary values supplied by voters in normalized social choice, agents' preferences are embedded in a possibly high-dimensional metric space reflecting their "ideological" distance from the alternatives. The axes of this metric space might denote, for example, leanings on social issues or economic policies. Then, the closer a candidate is to a voter, the more ideologically aligned the two individuals are.

In general, aggregating the utility functions of the agents gives rise to a notion of total social utility or social cost of an alternative, which is to be maximized or minimized respectively. Specifically, in the case of normalized social utility, a goal would be to select the alternative that maximizes the sum of utilities indicated by each agent. In the case of metric social choice, a goal would be to select the alternative that minimizes the sum of Euclidean distances from each agent to the selected alternative. While these are both valid goals, for the rest of the paper, we will focus on this latter case since it has received the most research attention in recent years.

Now, if the agents and alternatives are embedded in a metric space and the distance between every agent and candidate is known, then a simple linear program would easily find the optimal candidate. However, in any practical setting, it is unlikely that an agent is able to give a cardinal measures of preferences, as they may not even be known by the agents themselves, if they exist at all. It is more realistic to assume that agents are able to provide ordinal information on their candidate preferences (i.e., provide a total order over the candidates corresponding to their preferences). In this scenario, the agents and alternatives may be embedded in any metric space that is consistent with the agents' preferences (i.e., the candidate nearest to them in the space is their first-ranked choice, etc.). So, under this limited information model, a natural question would be whether there exists a mechanism (voting rule) that always selects the optimal candidate, when given access to only ordinal information and not the underlying metric space.

The answer is no. A simple example from [1] illustrates this. Suppose there are two candidates, c_1 and c_2, and n voters embedded on $\mathbb{R}$ (see image below).

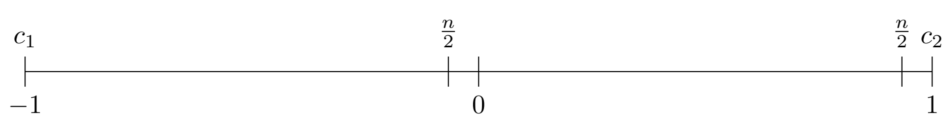

Figure 1: Example that can lead to sub-optimal candidate selection.

Half of the n agents have a slight preference for c_1 and the other half have an extremely strong preference towards c_2. A voting rule using strictly ordinal information may choose either c_1 or c_2 arbitrarily, since from its perspective both candidates are equally liked by voters. However, c_2 clearly minimizes social cost (the sum of voter-candidate distances) compared to c_1. This simple example illustrates that using ordinal information rather than cardinal information can result in voting rules selecting a non-optimal candidate as the winner of an election.

1.1 Distortion Preliminaries

The previous phenomenon motivates the concept of the distortion of a voting rule, first proposed by Procaccia and Rosenschein [2]. Loosely speaking, distortion of a voting rule f is defined as the worst-case ratio between the social cost of the winning candidate chosen by f and the social cost of the optimal candidate.

We will now define this more precisely, borrowing notation from [3]. Let V denote a set of n voters and C denote a set of m candidates. Additionally, let $\mathcal{P}$ denote the collective preferences of the voters over the candidates. We assume that the voters and candidates are embedded in a metric space endowed with a distance function d, such that $d \backsim \mathcal{P}$. The latter notation simply means that d is compatible with the voter's preferences (i.e. if voter i prefers candidate c_1 over candidate c_2 then $d(i, c_1) \leq d(i, c_2)$). The social cost of a candidate c is then $SC(c) = \sum_{i \in V} d(i, c)$. The optimal candidate $x^* = argmin_{x \in C} SC(x)$. A voting rule $f(V, C, \mathcal{P}) \rightarrow w$ is then a deterministic mechanism which outputs a candidate when given a set of voters, their candidate preferences, and the set of candidates. Having established these preliminaries, we can now define the distortion of a voting rule f, namely

$$\text{distortion}(f) = \max_{\mathcal{P}} \sup_{d: d \backsim P} \frac{SC(f(V, C, \mathcal{P}))}{SC(x^*)}.$$

The distortion of a voting rule is in a sense a marker for how well that voting rule can approximately optimize the cardinal objective when given access to only ordinal information. Using this definition, Anshelevich et al. [4] proved the distortion of many well-known voting rules with regard to both the standard sum objective and the median objective (a less common but also robust measure). These bounds are summarized in the table below. Note, across all voting rules, there is a distortion lowerbound of 3 for the sum objective (this is actually captured by our previous example), and 5 for the median objective.

Voting Rule Distortions				
Voting Rule	**Upperbound - Sum**	**Lowerbound - Sum**	**Upperbound - Median**	**Lowerbound - Median**
Plurality	$2m - 1$	$2m - 1$	∞	∞
Borda	$2m - 1$	$2m - 1$	∞	∞
k-Approval, $k > 1$	∞	∞	∞	∞
Veto	∞	∞	∞	∞
Harmonic Rule	$O(\frac{m}{\sqrt{\ln m}})$	$\Omega(\frac{m}{\ln m})$	∞	∞
STV	$O(\ln m)$	$\Omega(\sqrt{\ln m})$	?	$\Omega(\sqrt{\ln m})$
Uncovered set	5	5	5	5
Copeland	5	5	5	5
Scoring rules	∞	$1 + 2\sqrt{\ln m - 1}$	∞	∞
Deterministic voting rules	∞	3	∞	5

1.2 Roadmap

In this paper, we will first lay out the framework proposed by Kempe [3] which utilizes linear programming and duality to analyze the distortion of voting rules. The benefit of this framework is that it provides an interesting network flow based interpretation of distortion bounds. From here, we examine the work of

Kizilkaya and Kempe [5], which builds off of the previous paper, to develop a voting rule which achieves the optimal distortion of 3. Finally, we look beyond single-winner elections to the setting of multiwinner (committee) elections. We will show a novel formulation for social cost for a chosen committee, which is inspired by the work of Caragiannis et al. [6]. We end by exhibiting a lowerbound for our committee cost mechanism on $\mathbb{R}$.

2 The LP Duality Framework

In [3], Kempe introduces a new framework utilizing linear programming to analyze the distortion of voting rules. We will lay out this framework and focus on using it as a tool for proving the distortion bounds of voting rules.

To frame voting problems as optimization problems, the information can be embedded in a metric space, as described in the introduction. Recall that in this space, there is an implicit understanding that the distance between a voter and candidate represents their ideological differences. We will refer to the fact that voters have smaller distances to the candidates they prefer more as consistency constraints.

Now, in [3], Kempe constructs the following optimization problem: given m candidates, n voters each with a total order over the candidates, a proposed winning candidate w (possibly chosen by some voting rule), and an optimal candidate x^*, find a metric space maximizing distortion. Essentially, the goal of the optimization problem is to find the worst-case metric space that is consistent with both the voters' ranked preferences and the known optimal candidate, and to use this information to derive an upperbound for distortion. The following sections explain how Kempe uses this LP formulation to provide insights into the distortion of voting rules.

2.1 The Primal

One way we can represent a metric space is by assigning a distance to every voter-candidate pair. Our goal is then to find an assignment of distances that maximizes distortion subject to that the distances form a metric space and follow the consistency constraints defined above. We can solve a normalized LP, given by [3], that sets the cost of the optimal candidate (in terms of metric distance) to 1, and maximizes the cost of the proposed winning candidate (chosen by the ordinal voting rule), thus maximizing their ratio. Note that this LP exactly captures the ordinal optimization problem we are attempting to tackle; it solves for $d_{v,x}$, which represents the distance between voter v and candidate x, for all voters and candidates. Below is the primal of the LP.

$$
\begin{array}{ll}
\text{maximize} & \sum_v d_{v,w} \\
\text{subject to} & d_{v,x} \le d_{v',x} + d_{v',y} + d_{v,y} \text{ for all candidates } x, y \text{ and all voters } v, v' \text{ (Triangle inequality)} \\
& d_{v,x} \le d_{v,y} \text{ for all candidates } x, y \text{ and all voters } v \text{ such that } v \text{ prefers } x \text{ to } y \text{ (Consistency)} \\
& \sum_v d_{v,x^*} = 1 \text{ (Set optimal candidate's cost to 1)} \\
& \sum_v d_{v,x} \ge 1 \text{ for all candidates } x \text{ (Cost of all candidates is greater than cost of the optimal)} \\
& d_{v,x} \ge 0 \text{ (Non-negative distances)}
\end{array}
$$

The triangle inequality and non-negativity constraints arise from the fact that the voters and candidates are in a metric space. Note, by normalizing the sum of the distances to the optimal candidate to be 1, we no longer have to write the objective as a ratio. By itself, this LP is not very helpful given that it has an exponential number of constraints and does not provide much additional insight into the optimization problem. This motivates the taking of its dual, which ends up having a useful interpretation.

2.2 The Dual

The primal LP above yields a rather complicated dual, where $\psi_{x,y}^{(v,v')}$ are the dual variables for the triangle inequality constraints, $\phi_{x,y}^{(v)}$ are the dual variables for the consistency constraints, and α_x are the dual variables for the normalization/optimality constraints:

$$
\begin{array}{ll}
\text{minimize} & \sum_x \alpha_x \\
\text{subject to} & \alpha_x + \sum_{y:x \succ_v y} \phi^{(v)}_{x,y} - \sum_{y:y \succ_v x} \phi^{(v)}_{y,x} + \sum_{y,v'} (\psi^{(v,v')}_{x,y} - \psi^{(v,v')}_{y,x} - \psi^{(v',v)}_{x,y} - \psi^{(v',v)}_{y,x}) \\
& \geq \begin{cases} 1, & \text{if } x = w \\ 0, & \text{if } x \neq w \end{cases} \qquad \text{for all } v, x \\
& \psi^{(v,v')}_{x,y} \geq 0 \quad \text{for all } v, v', x, y \\
& \phi^{(v)}_{x,y} \geq 0 \quad \text{for all } v, x, y \\
& \alpha_x \leq 0 \quad \text{for all } x \neq x^*
\end{array}
$$

The advantage of looking at the dual over the primal is that it makes no reference to the metric space in which the voters and candidates are embedded. As a result, to find an upper bound on distortion, one does not need to look at all the possible metric spaces that satisfy the voter's preferences, but rather they simply need to find a feasible solution to the dual. By weak duality, this provides an upper bound on the value of the primal. As we we observed earlier, this means it exactly upper bounds distortion. The dual seems tricky to understand; however, [3] shows that we can interpret it as a flow problem on a specific graph.

2.3 Using Flows

Section 3.3 of [3] describes a graph $H = (U, E)$ with vertex set $U = V \times C$ where V is the set of voters and C is the set of candidates. The edges are as follows:

- When voter v prefers candidate x over candidate y, there is a directed edge from (v, x) to (v, y) called a preference edge.
- For every candidate x and voter pairs v, v' such that $v \neq v'$, there is directed edge from (v, x) to (v', x) called a sideways edge.

Having established the graph, we can now frame the dual LP as solving a minimum cost flow problem on this graph. Let (v, w) for all voters $v \in V$ be source nodes with exactly 1 unit of flow, and let (v, x^*) for all voters $v \in V$ be sink nodes that can terminate flow. In other words, flow starts at (voter - proposed winner) nodes and terminates at (voter - optimal candidate) nodes. Given a flow g on H, the cost of g for a voter v is

$$\gamma^{(g)}_v = \sum_{e \text{ into } (v,x^*)} g_e + \sum_{x \neq x^*} \sum_{v' \neq v} (g_{(v',x) \to (v,x)} + g_{(v,x) \to (v',x)}).$$

It is proven in [3] that $max_v \gamma^{(g)}_v$ (the maximum flow cost of a voter) upperbounds distortion, because there is a feasible dual solution with this value. While we will not repeat the proof in this paper, we will provide some intuition about why the flow cost function upper bounds distortion.

First, the dual LP can be interpreted as trying to route all the flow in the graph from the source nodes to the sink nodes as cheaply as possible. For some fixed $v \in V$ and flow g, the cost of g to v can be broken into two parts. By the first sum, v is charged a cost exactly equal to the amount of flow it drains at the node (v, x^*). By the second double summation, v is also charged a cost exactly equal to the amount of flow it sends and receives by sideways edges. Notice, voters do not incur a cost for using preference edges. Intuitively, this should make sense. If a flow needs to use a substantial amount of sideways edges in the graph, it indicates that a substantial number of voters preferred the optimal candidate over the winning candidate. On the otherhand, if a large number of voters support the winning candidate over the optimal candidate, then flow should be able to be cheaply routed.

Using this new interpretation, Kempe [3] was able to prove that two voting rules, Ranked Pairs and Schulze, have distortion $\Theta(\sqrt{n})$. Moreover, his framework spurred additional research in the field culminating recently with Kizilkaya and Kempe's [5] work, which we describe later in this paper.

2.4 A Basic Example of Routing Flow

In this section, we will present a simple example of how flows can be used to prove the distortion of voting rules. Specifically, we will show that every deterministic voter rule has a distortion of at least 3. This recovers lowerbound shown in [1] and mentioned in the introduction.

For our example, let $V = \{v_1, v_2\}$ and $C = \{c_1, c_2\}$. The voter preferences are the following: $v_1 : c_1 \succ c_2$ and $v_2 : c_2 \succ c_1$. Without loss of generality, suppose that c_1 is the winning candidate chosen by a voting rule f and c_2 is the optimal candidate. Below is the resulting graph induced by these voters and their candidate preferences.

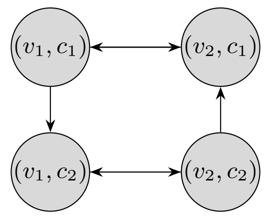

Figure 2: Example of routing flow on voter-candidate graph.

The sideways edges correspond to the edges between rows and the preference edges correspond to edges between columns. Each node in the first row is seeded with 1 unit of flow. Thus, in total, there are 2 units of flow that must be drained in the second row using the preference and sideways edges.

Lemma 1. *Let f be any deterministic voting rule. Then, distortion$(f) \geq 3$.*

Proof. In this graph, there is only one way to route the flow. Namely, the node (v_2, c_1) must shift its unit of flow to (v_1, c_1) using a sideways edge. Then, the node (v_1, c_1) must shift 2 units of flow to (v_1, c_2), where it is drained. The total cost of routing this flow is then 3 since v_1 and v_2 both get charged 1 using a sideways edge and then v_1 gets charged 2 for draining two units of flow. Consequently, the maximum cost incurred by a voter is 3, which implies that the *distortion*$(f) \geq 3$. □

Notice, in our proof, we made no mention of a specific voting rule. Hence, regardless of the deterministic voting rule, its distortion is lowerbounded by 3. While this is a simple example, it demonstrates how flows can simplify distortion proofs and provide new intuition for them. With this intuition in hand, the framework can be used to prove more complicated properties as seen in the next section.

3 Generalization of the LP Duality Framework for Randomized Voting Rules

This section reviews two prominent voting rules: PluralityVeto and k-RoundPluralityVeto. The former belongs to the family of deterministic voting rules and is used as a stepping stone to understanding the latter, which incorporates randomization in the voting rule. We illustrate each of the voting rules and explain how the idea of network flow and winner distribution is used to prove an upper bound for their distortion.

To start with, recall that there exist n candidates and m voters with their corresponding sets denoted by C and V, respectively. Each voter has a preference list over the candidates indicating how the individual has ranked the candidates. The winner of the election or so-called optimal candidate is the one that minimizes the total cost in the given metric space.

The first voting rule that we elaborate on is called PluralityVeto. In this voting rule, each candidate c receives a plurality score indicating the number of voters whose top choice has been c. Then, the winner of the election is selected following a greedy approach in which voters are processed one by one. For all voters $v \in V$, the algorithm looks at the least preferred candidate of the voter (c), and decrements the plurality score of candidate c by one. When the plurality score of a candidate reaches zero, the candidate is eliminated. The elimination process continues until only a single candidate remains. This candidate is announced as the winner of the election. It is worth noting that the algorithm does not require information about the whole ranking of voters, but only their top and bottom choices. As shown in the following theorem, the algorithm has a distortion of 3.

Theorem 2. *PluralityVeto has the optimal metric distortion of* 3*.*

Proof. To prove the theorem, one needs to use so-called domination graphs to model the problem and apply Lemma 3 to find an upper bound for the distortion. A domination graph for a candidate $c \in C$ is bipartite graph (V, V, E_c), where the E_c represents the set of edges. In this setup, two nodes v and v' are connected if the voter v ranks candidate c higher or equal to the top choice of voter v'.

Lemma 3. *Given a ranked choice profile of voters and f being the voting rule, f has a distortion of* 3 *if its domination graph has a perfect matching.*

Armed with this knowledge, it is possible to show that PluralityVeto has a distortion of 3 as its domination graph has a perfect matching. The existence of perfect matching can be explained as follows. In the beginning of the greedy algorithm, the plurality score of candidates adds up to m because each voter has a single top choice. This indicates that for each candidate c and its plurality score p_i, there exist p_i distinct voters that have c as their top choice. Let v_i be one such a voter and let A denote the set of remaining candidates. For each such a voter v_i, one can select a unique voter v_i' in a way that the top choice of v_i' is the bottom choice of v_i in the subset A. In simple words, it indicates that v_i ranks any candidate in A weakly higher than the top choice of v_i'. By selecting (v_i, v_i') as an edge and continuing the process, we can construct a perfect matching. Thus based on Lemma 3, the distortion of PluralityVeto is 3. □

Next, we explain a robust voting rule referred to as k-RoundPluralityVeto, which belongs to a family of randomized voting rules and is a generalization of PluralityVeto. The principal idea is to stop the greedy algorithm in the k-th iteration and randomly select the winning candidate over the remaining candidates' scores. The base case of k-RoundPluralityVeto in which the $k = 0$, indeed is the well-known voting rule RandomDictatorship. This voting rule would simply select the winner of the election uniformly at random among the top choices of the candidates. The other extreme of the algorithm happens when $k = m - 1$ for which the algorithm is the same as PluralityVeto. Therefore, we already know that for two k values k-RoundPluralityVeto leads to a distortion of 3. The following shows that k-RoundPluralityVeto achieves the distortion of at most 3 for all values of k.

Formally, a randomized voting rule f returns a distribution over candidates for given ranking preferences of the voters. Let us denote the probability distribution over candidates referred to as winner distribution by $\boldsymbol{w}$ and the probability of candidate c being selected by w_c. Given a metric space d, the expected cost of the winner can be calculated by

$$\text{cost}(\boldsymbol{w}, d) = \sum_{c \in \mathcal{C}} w_c \times \text{cost}(c, d) \tag{1}$$

As explained in the previous subsection, the current existing upper bound for the distortion of randomized rules is $3 - 2/n$ which is slightly better than 3. The current best lower bound is 2.1126 [7]. The following theorem proves that the k-RoundPluralityVeto has a distortion of 3.

Theorem 4. *For a given $k \in [0, n)$, the distortion of k-RoundPluralityVeto is at most* 3*.*

Proof. The key concept used to prove the theorem is entailed in the network flow idea explained in section 2. For a given election, consider a directed graph $H = (V \times C, E)$ for which the edges are generated as described in section 2.3. Having the basic setup of the graph in mind, the idea of flow is established on the graph. Given the winner distribution $\boldsymbol{w}$ on candidates, and a particular candidate c^*, a $(\boldsymbol{w}, c^*)$-flow on the graph is a circulation g abiding to the following two rules:

- Every node (v, c) of the graph originates w_c units of flow.
- The generated flow is only absorbed at nodes (v, c^*).

Notice, this is a generalization of the flow f on H described in section 2.3. The primary difference is that we are now seeding the nodes with an initial flow based on the distribution $\boldsymbol{w}$, rather than in a static manner as was previously done. As a result, we have an equivalent function describing the cost of circulation g for a voter v (where we replace x^* with the given c^*):

$$\text{cost}_v(g) = \sum_{e \text{ into } (v,c^*)} g_e + \sum_{c \neq c^*} \sum_{v' \neq v} (g_{(v',c) \to (v,c)} + g_{(v,c) \to (v',c)}) \tag{2}$$

The cost of circulation is the maximum cost for all $v \in V$,

$$\text{cost}(g) = \max_{v \in V} \text{cost}_v(g) \tag{3}$$

Having defined the cost function, the following lemma can be used to prove the upper-bound 3 for the distortion.

Lemma 5. *Let f be randomized voting rule. For an arbitrary candidate $c^* \in C$, suppose that there exists a $(\boldsymbol{w}, c^*)$-flow g on the flow network such that cost$(g) \leq \lambda$. Then, distortion of f is less or equal to λ.*

The proof of k-RoundPluralityVeto having a distortion of at most 3 can be derived from the above lemma. The principle idea is to describe $(\boldsymbol{w}, c^*)$-flow g for any candidate c^* and prove that its cost is at most 3. Then, we can conclude that distortion of the randomized voting rule is at most 3. □

4 Beyond Single Winner Elections

While the previously discussed papers [4][3][5] focus analyzing single winner elections, one can imagine a more general scenario where an election should have multiple winners. For example, workers at a company (voters) might wish to elect a committee of size k to represent their interests in contract negotiations. Recall that in a single winner election, the goal is to select a candidate who minimizes the total distance from the voters. Now, in a multiwinner election, a natural goal would be to select a committee of size k which minimizes the sum of the distances between every voter and every member of the committee.

While this is a straightforward generalization, this notion of cost can suffer from the tyranny of the majority. In particular, a low cost committee might consist entirely of candidates liked by the majority and no candidates liked by minority groups. The intuition behind this is the fact that a single-winner voting rule can effectively be repeatedly applied to elect a committee without degrading the overall distortion of the voting rule under this cost metric [8].

Another reason to move away from this way of delegating costs in the committee setting is that it assumes that voters care equally about all members of the committee, when in many practical applications voters might be satisfied as long as a fraction of the committee sufficiently embodies their views. These two concerns have motivated the development of new notions of committee cost in the context of multiwinner elections. For example, in recent work, Caragiannis et al. [6] developed a framework in which a voter's cost to the committee is their distance to the qth closest candidate in the committee for some fixed $q \leq k$. The total cost of the committee is then the sum of the q-costs for each voter. For different ranges of q values, Caragiannis et al. [6] were able to show different upper and lower bounds on distortion.

4.1 Our Contribution

In this section, we add to this trend and provide a new conception of committee cost, which attempts to incorporate aspects of fairness. Roughly speaking, our framework attempts to prioritize that every sufficiently large subset of voters is represented by a proportional number of members in the committee. We accomplish this by introducing an adaptive parameter q, based on the size of a voter subset, such that a subset of voters' cost to a committee is their distance to the qth closest candidate in that committee. In this way, our work diverges from that of [4]'s, which only considers individual voters and their distance to the qth closest candidate on the committee for some fixed $q \leq k$.

We can now be more precise with our definitions. Let d be some metric distance function consistent with candidate preferences of the voters. We call any $S \subseteq V$ a "block" of voters, where $|S| \geq \frac{|V|}{k}$. A committee

$K \subseteq C$ is a set of candidates, whose size is k. For a block of voters S and a committee K, let $c_{S,q}(K|d)$ denote the qth closet member of the committee to the voter block according to the metric d, where $q = \left\lfloor k \cdot \frac{|S|}{|V|} \right\rfloor$. Hence, q is proportional to the size of S. The cost of the committee K to a specific voter block S under the metric d is then defined to be $SC_{S,K,d} = \sum_{i \in S} d(i, c_{S,q}(K|d))$. We denote the normalized cost of the committee K to a specific voter block S under the metric d as

$$NSC_{S,K,d} = \frac{SC_{S,K,d}}{min_{A \subseteq C, |A|=k} SC_{S,A,d}}.$$

We have normalized $SC_{S,K,d}$ by the optimal committee for the voter block under our cost model. Then, the cost of a committee K with respect to all voter blocks and a fixed d is the maximum $NSC_{S,K,d}$ over all $S \subseteq V$ with $|S| \geq \frac{|V|}{k}$. In particular,

$$cost(K) = \sup_{S \subseteq V, |S| \geq \frac{|V|}{k}} \frac{SC_{S,K,d}}{min_{A \subseteq C, |A|=k} SC_{S,A,d}}.$$

Note, when $k = 1$, we exactly recover the notion of cost defined in the single-winner election case. That is, the top of the previous fraction transforms into distance of every voter to a single candidate and the bottom is the distance of every voter to optimal candidate (the one which minimizes the sum of the distance to every voter). We could now use this new committee cost mechanism to formulate a corresponding definition of the distortion of a multiwinner voting rule. However, for simplicity, we leave this for future work, and instead focus on understanding the cost mechanism.

4.2 A Simple Example

Having established this new mechanism for committee cost, we provide a non-trivial, concrete example of it in action and that show in this example no deterministic multiwinner voting rule can select a committee with cost less than ϕ, the golden ratio.

For our example, let $V = \{v_1, v_2, v_3, v_4\}$ and $C = \{c_1, c_2, c_3\}$. Additionally, we set $k = 2$; recall this is the size of our committee. We implicitly define a metric space by embedding these voters and candidates on $\mathbb{R}$ (pictured below). Note, while technically in a metric space two distinct points cannot occupy the same location, we permit this in our example for notational simplicity with the understanding that our results still hold if we simply perturbed the voters an infinitesimal amount.

Now, the voter blocks in this example are all subsets $S \subseteq V$ such that $|S| \geq 2$. There are three possible committees which can be formed from the set of candidates: $K_1 = \{c_1, c_2\}$, $K_2 = \{c_2, c_3\}$, and $K_3 = \{c_1, c_3\}$. Our goal (optimization problem) is then to select the committee $K \in \{K_1, K_2, K_3\}$ such that $cost(K)$ is minimized. We would like to show for this example that no committee exists with a cost less than ϕ, where ϕ is the golden ratio. This implies in turn that no deterministic multiwinner voting rule exists which can select a a committee with cost less than ϕ.

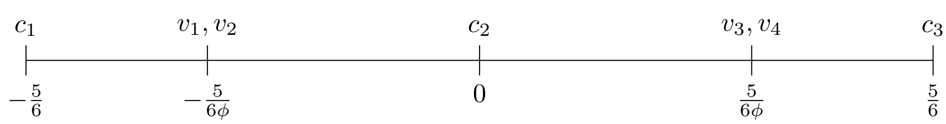

Figure 3: Example of our committee cost mechanism on $\mathbb{R}$.

Lemma 6. *For $K \in \{K_1, K_2, K_3\}$, $cost(K) \geq \phi$.*

Proof. First, recognize that $cost(K_1) = cost(K_2)$ by symmetry. Hence, we only need to consider the costs of K_1 and K_3. So, to prove that no committee has cost less then ϕ, we simply need to exhibit two voter blocks S_1, S_2 with the property that regardless of whether we select K_1 or K_2, one of these voter blocks will force the committee to have a cost of at least ϕ.

Let $S_1 = \{v_2, v_3\}$ and let $S_2 = \{v_3, v_4\}$. Since $|S_1| = |S_2| = 2$, this implies that $q = 1$. So, we determine the cost of a committee to these voter blocks by looking at the closest candidate in the committee to the voter block. Using this information, a simple calculation shows that the optimal committee for S_1 is K_1 with a cost of $\frac{10}{6\phi}$ and the optimal committee for S_2 is K_2 with a cost of $\frac{10}{6}(1-\frac{1}{\phi})$.

Now, suppose, we select K_1 as our winning committee, we claim that $NSC_{S_2,K_1} = \phi$. By visual inspection, the closest candidate in K_1 to S_2 is c_2, while the closest candidate in the optimal committee K_2 is c_3. Using this information, we find that

$$\begin{aligned}
NSC_{S_2,K_1} &= \frac{\sum_{i\in S_1} d(i,c_2)}{\sum_{i\in S_1} d(i,c_3)} \\
&= \frac{\frac{10}{6\phi}}{\frac{10}{6}(1-\frac{1}{\phi})} \\
&= \frac{1}{\phi-1} \\
&= \phi.
\end{aligned}$$

On the otherhand, suppose we decide to select K_2 as our winning committee. We claim that $NSC_{S_1,K_2} = \phi$. By visual inspection, the closest candidate in K_2 to S_1 is c_3, while the closest candidate in the optimal committee K_1 is c_2. We can use this information to find that

$$\begin{aligned}
NSC_{S_1,K_2} &= \frac{\sum_{i\in S_2} d(i,c_3)}{\sum_{i\in S_2} d(i,c_2)} \\
&= \frac{\frac{10}{6\phi}+\frac{10}{6}(1-\frac{1}{\phi})}{\frac{10}{6\phi}} \\
&= \frac{1}{\frac{1}{\phi}} \\
&= \phi.
\end{aligned}$$

Consequently, we have shown that regardless of the committee $K \in \{K_1, K_2, K_3\}$ that is selected, $cost(K) \geq \phi$. □

This example produces our best known lowerbound for committee cost on $\mathbb{R}$; it was generated by manipulating the distances between the voters and candidates to balance the worst-case cost ratios of each committee. In some sense, this explains the curious appearance of the golden ratio in our lowerbound. An interesting question would be whether this represents the worst case committee cost when candidates and voters are embedded on $\mathbb{R}$. While this is a relatively simple example, it provides a concrete example of how our cost function operates.

5 Conclusion and Future Work

In this paper, we introduced Social Choice Theory and the metric distortion-based analysis of it. We followed this by presenting many of the foundational results originally laid out by Anshelevich et al. [4]. From there, we presented several newer papers [3][5], which built on this work by using an equivalent LP formulation of the optimization problem. The techniques shown in these papers have proven extremely useful in simplifying and tightening upper and lower bounds for the distortion of many popular voting rules. Finally, we ended by introducing the concept of committee elections and presented a novel formulation of committee cost.

For future work, our new mechanism for committee cost presents a variety of directions which could be explored. The most prominent direction would be to use this notion of cost to develop a corresponding definition of the distortion of a multiwinner voting rule. With this definition in hand, a natural goal would be to prove distortion bounds for many of the common multiwinner voting rules (Plurality, Single Transferable Vote, etc). It also seems plausible that there exists an LP formulation for this new objective function that completely encapsulates the optimization problem, in a similar manner to Kempe's [3] construction for single-winner elections. Constructing such a LP could prove useful for analyzing the distortion bounds of multiwinner voting rules under our cost mechanism.

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