Proposition : The average number of compares $C(N)$ needed to apply quicksort to arrays of size N (with distinct keys) is $\sim 2N \cdot \ln N$. natural log

Proof: Let us define a recurrence $C(N)$

$$C(0) = 0 \qquad C(1) = 0$$

for $N \geq 2$

$$C(N) = (N+1) + C(\text{size of right array}) + C(\text{size left})$$

\# of compares for partitioning

merge sort $2 \cdot C(N/2)$

quicksort: $\boxed{\leq v \; v \; \geq v}$

Left	partition	Right
0	1	N-1
1	1	N-2
2	1	N-3
3	1	N-4
	...	
N-3	1	2
N-2	1	1
N-1	1	0

total = N

$1/N$

$$N \cdot C(N) = N \cdot \underbrace{(N+1)}_{\text{partitioning}} + \cancel{N}\frac{C(0)+C(N-1)}{\cancel{N}} + \cancel{N}\frac{C(1)+C(N-2)}{\cancel{N}} + \cancel{N}\frac{C(2)+C(N-3)}{\cancel{N}}$$

$$\cdots + \cancel{N}\frac{C(N-2)+C(1)}{\cancel{N}} + \cancel{N}\frac{C(N-1)+C(0)}{\cancel{N}}$$

$$N \cdot C(N) = N \cdot (N+1) + \left\{ \begin{array}{l} C(0) + C(N-1) + C(1) + C(N-2) + \\ C(N-2) + C(1) + C(N-1) + C(0) \end{array} \right\}$$

$\ldots$

$$N \cdot C(N) = N^2 + N + 2\left(C(0) + C(1) + \ldots C(N-1)\right)$$

$$\begin{array}{l} N \cdot C(N) \\ -(N-1)C(N-1) \end{array} = (N-1)(N) - 2\left(C(0) + C(1) \ldots C(N-2)\right)$$

$$= 2N + 2\left(C(N-1)\right).$$

Rearrange terms and divide by $N \cdot (N+1)$

$$\frac{C(N)}{N+1} = \frac{C(N-1)}{N} + \frac{2}{N+1}$$

$$\frac{C(N)}{N+1} = \frac{C(N-1)}{N} + \frac{2}{N+1}$$

Apply telescoping

$$= \frac{C(N-2)}{N-1} + \frac{2}{N} + \frac{2}{N+1}$$

$$= \frac{C(N-3)}{N-2} + \frac{2}{N-1} + \frac{2}{N} + \frac{2}{N+1}$$

$$= \frac{0}{1} + \frac{0}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \ldots \frac{2}{N+1}$$

$C(1)$ $C(2)$

Multiply by $N+1$ pull out the 2

$$C(N) = (N+1) \cdot 2\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots \frac{1}{N+1}\right)$$

$$\sim 2(N+1)\int_3^{N+1} \frac{1}{x}\,dx \sim 2(N+1) \cdot \ln N$$

$$\approx 1.3\, N \cdot \log_e N$$