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3.1 SYMBOL TABLES

- *API*
- *elementary implementations*
- *ordered operations*

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3.1 Symbol Tables

- *API*
- *elementary implementations*
- *ordered operations*

Why are telephone books obsolete?

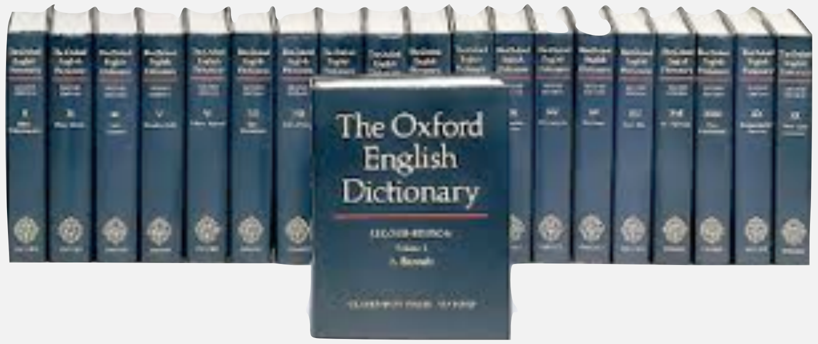
Unsupported operations.

- Change the number associated with a given name.
- Add a new name, associated with a given number.
- Remove a given name and associated number.

key = term, value = article

key = name
value = phone number

key = function name and input
value = function output



key = word, value = definition

key = time and channel
value = TV show

Symbol tables

Key-value pair abstraction.

- Insert a value with specified key.
- Given a key, search for the corresponding value.

Ex. DNS lookup.

- Insert domain name with specified IP address.
- Given domain name, find corresponding IP address.

domain name	IP address
www.cis.upenn.edu	158.130.69.163
www.upenn.edu	23.15.7.160
www.princeton.edu	140.180.223.42
www.harvard.edu	104.16.154.6
www.fivethirtyeight.com	192.0.79.32

key ↑ value ↑

Symbol table applications

application	purpose of search	key	value
dictionary	find definition	word	definition
book index	find relevant pages	term	list of page numbers
file share	find song to download	name of song	computer ID
financial account	process transactions	account number	transaction details
web search	find relevant web pages	keyword	list of page names
compiler	find properties of variables	variable name	type and value
routing table	route Internet packets	destination	best route
DNS	find IP address	domain name	IP address
reverse DNS	find domain name	IP address	domain name
genomics	find markers	DNA string	known positions
file system	find file on disk	filename	location on disk

Symbol tables: context

Also known as: maps, dictionaries, associative arrays.

Generalizes arrays. Keys need not be between 0 and $N-1$.

Language support.

- External libraries: C, VisualBasic, Standard ML, bash, ...
- Built-in libraries: Java, C#, C++, Scala, ...
- Built-in to language: Awk, Perl, PHP, Tcl, JavaScript, Python, Ruby, Lua.

every array is an associative array

every object is an associative array

table is the only "primitive" data structure

```
has_nice_syntax_for_associative_arrays["Python"] = True
has_nice_syntax_for_associative_arrays["Java"]   = False
```

legal Python code

Basic symbol table API

Associative array abstraction. Associate one value with each key.

`public class ST<Key, Value>`

	`ST()`	*create an empty symbol table*
`void`	`put(Key key, Value val)`	*put key-value pair into the table* ← `a[key] = val;`
`Value`	`get(Key key)`	*value paired with key* ← `a[key]`
`boolean`	`contains(Key key)`	*is there a value paired with key?*
`Iterable<Key>`	`keys()`	*all the keys in the table*
`void`	`delete(Key key)`	*remove key (and its value) from table*
`boolean`	`isEmpty()`	*is the table empty?*
`int`	`size()`	*number of key-value pairs in the table*

Conventions

- Values are not `null`. ← `java.util.Map` allows `null` values
- Method put() overwrites old value with new value.
- Method get() returns null if key not present.

> " *Careless use of null can cause a staggering variety of bugs. Studying the Google code base, we found that something like 95% of collections weren't supposed to have any null values in them, and having those fail fast rather than silently accept null would have been helpful to developers.* "

guava-libraries

Guava: Google Core Libraries for Java 1.6+

GUAVA

https://code.google.com/p/guava-libraries/wiki/UsingAndAvoidingNullExplained

Conventions

- Values are not `null`. ← `java.util.Map` allows `null` values
- Method `get()` returns `null` if key not present.
- Method `put()` overwrites old value with new value.

Intended consequences.

- Easy to implement `contains()`.

```
public boolean contains(Key key)
{  return get(key) != null;  }
```

- Can implement lazy version of `delete()`.

```
public void delete(Key key)
{  put(key, null);  }
```

Keys and values

Value type. Any generic type.

specify Comparable in API.

Key type: several natural assumptions.

- Assume keys are `Comparable`, use `compareTo()`.
- Assume keys are any generic type, use `equals()` to test equality.
- Assume keys are any generic type, use `equals()` to test equality; use `hashCode()` to scramble key.

built-in to Java
(stay tuned)

Best practices. Use immutable types for symbol table keys.

- Immutable in Java: `Integer`, `Double`, `String`, `java.io.File`, …
- Mutable in Java: `StringBuilder`, `java.net.URL`, arrays, ...

Equality test

All Java classes inherit a method `equals()`.

Java requirements. For any references `x`, `y` and `z`:

- Reflexive: `x.equals(x)` is `true`.
- Symmetric: `x.equals(y)` iff `y.equals(x)`.
- Transitive: if `x.equals(y)` and `y.equals(z)`, then `x.equals(z)`.
- Non-null: `x.equals(null)` is `false`.

equivalence relation

do x and y refer to the same object?

Default implementation. `(x == y)`

Customized implementations. `Integer`, `Double`, `String`, `java.io.File`, …

User-defined implementations. Some care needed.

Implementing equals for user-defined types

Seems easy.

```
public       class Date implements Comparable<Date>
{
   private final int month;
   private final int day;
   private final int year;
   ...

   public boolean equals(Date that)
   {









      if (this.day   != that.day  ) return false;
      if (this.month != that.month) return false;
      if (this.year  != that.year ) return false;
      return true;
   }
}
```

check that all significant fields are the same

Implementing equals for user-defined types

Seems easy, but requires some care.

typically unsafe to use equals() with inheritance (would violate symmetry)

```
public final class Date implements Comparable<Date>
{
   private final int month;
   private final int day;
   private final int year;
   ...

   public boolean equals(Object y)
   {
      if (y == this) return true;

      if (y == null) return false;

      if (y.getClass() != this.getClass())
         return false;

      Date that = (Date) y;
      if (this.day   != that.day  ) return false;
      if (this.month != that.month) return false;
      if (this.year  != that.year ) return false;
      return true;
   }
}
```

must be Object.
Why? Experts still debate.

optimize for true object equality

check for null

objects must be in the same class (religion: getClass() vs. instanceof)

cast is guaranteed to succeed

check that all significant fields are the same

Equals design

"Standard" recipe for user-defined types.

- Optimization for reference equality.
- Check against `null`.
- Check that two objects are of the same type; cast.
- Compare each significant field:
  - if field is a primitive type, use `==` ← but use Double.compare() with double (to deal with -0.0 and NaN)
  - if field is an object, use `equals()` ← apply rule recursively
  - if field is an array, apply to each entry ← can use `Arrays.deepEquals(a, b)` but not `a.equals(b)`

Best practices.

- No need to use calculated fields that depend on other fields.
- Compare fields mostly likely to differ first.
- Make `compareTo()` consistent with `equals()`.

`x.equals(y)` if and only if `(x.compareTo(y) == 0)`

ST test client for analysis

Frequency counter. Read a sequence of strings from standard input and print out one that occurs with highest frequency.

```
% more tinyTale.txt
it was the best of times
it was the worst of times
it was the age of wisdom
it was the age of foolishness
it was the epoch of belief
it was the epoch of incredulity
it was the season of light
it was the season of darkness
it was the spring of hope
it was the winter of despair

% java FrequencyCounter 3 < tinyTale.txt
the 10

% java FrequencyCounter 8 < tale.txt
business 122

% java FrequencyCounter 10 < leipzig1M.txt
government 24763
```

tiny example (60 words, 20 distinct)

real example (135,635 words, 10,769 distinct)

real example (21,191,455 words, 534,580 distinct)

Frequency counter implementation

```
public class FrequencyCounter
{
   public static void main(String[] args)
   {
      int minlen = Integer.parseInt(args[0]);
      ST<String, Integer> st = new ST<String, Integer>();
      while (!StdIn.isEmpty())
      {
         String word = StdIn.readString();
         if (word.length() < minlen) continue;
         if (!st.contains(word)) st.put(word, 1);
         else                    st.put(word, st.get(word) + 1);
      }
      String max = "";
      st.put(max, 0);
      for (String word : st.keys())
         if (st.get(word) > st.get(max))
            max = word;
      StdOut.println(max + " " + st.get(max));
   }
}
```

create ST

ignore short strings

read string and update frequency

print a string with max frequency

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3.1 Symbol Tables

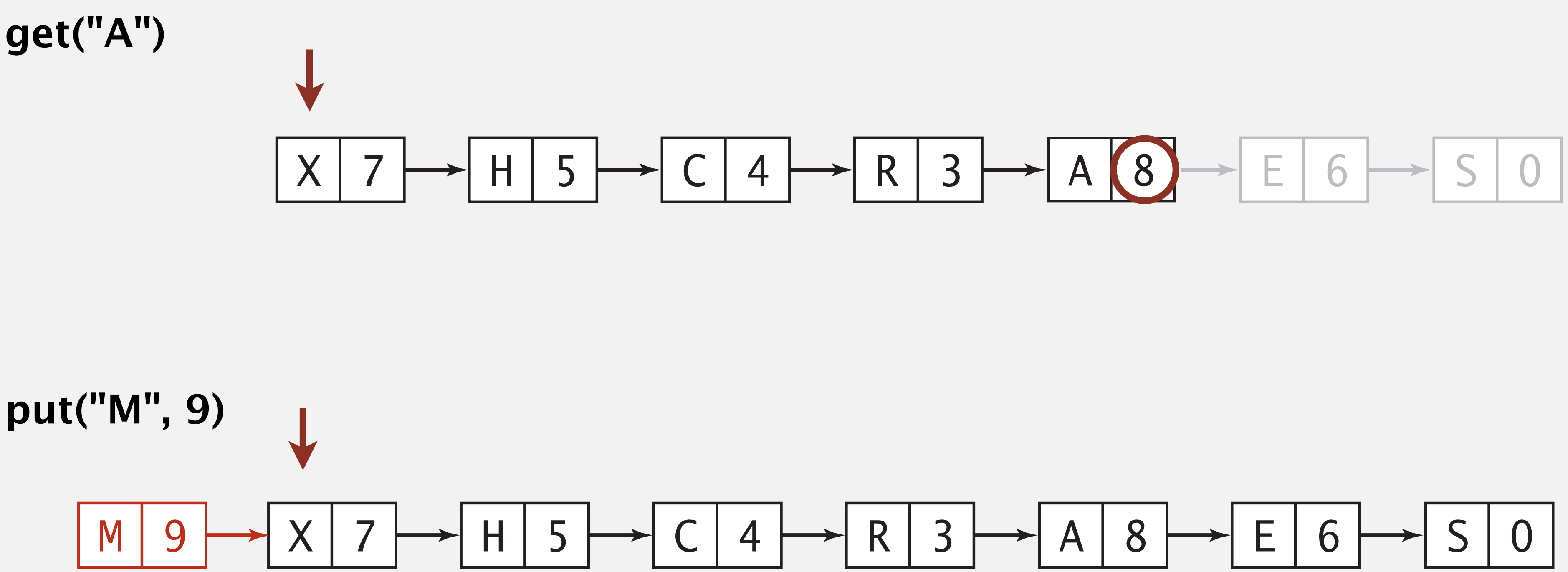
- API
- *elementary implementations*
- ordered operations

Sequential search in a linked list

Data structure. Maintain an (unordered) linked list of key-value pairs.

Search. Scan through all keys until find a match.

Insert. Scan through all keys until find a match; if no match add to front.



Elementary ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	N	N	`equals()`

Challenge. Efficient implementations of both search and insert.

Binary search in an ordered array

Data structure. Maintain parallel arrays for keys and values, sorted by keys.

Search. Use binary search to find key.

Proposition. At most $\sim \lg N$ compares to search a sorted array of length N.

keys[]

0	1	2	3	4	5	6	7	8	9
A	C	E	H	L	M	P	R	S	X

vals[]

0	1	2	3	4	5	6	7	8	9
8	4	2	5	11	9	10	3	0	7

get("P")

lo	hi	m										
0	9	4	A	C	E	H	L	M	P	R	S	X
5	9	7	A	C	E	H	L	M	P	R	S	X
5	6	5	A	C	E	H	L	M	P	R	S	X
6	6	6	A	C	E	H	L	M	P	R	S	X

entries in black are `a[lo..hi]`

entry in red is `a[m]`

`return vals[6]`

Binary search in an ordered array

Data structure. Maintain parallel arrays for keys and values, sorted by keys.

Search. Use binary search to find key.

```
public Value get(Key key)
{
   int lo = 0, hi = N-1;
   while (lo <= hi)
   {
      int mid = lo + (hi - lo) / 2;
      int cmp = key.compareTo(keys[mid]);
      if      (cmp  < 0) hi = mid - 1;
      else if (cmp  > 0) lo = mid + 1;
      else if (cmp == 0) return vals[mid];
   }
   return null;   no matching key
}
```

Binary search: insert

Data structure. Maintain an ordered array of key-value pairs.

Insert. Use binary search to find place to insert; shift all larger keys over.

Proposition. Takes linear time in the worst case.

put("P", 10)

keys[]

0	1	2	3	4	5	6	7	8	9
A	C	E	H	M	R	S	X	-	-

vals[]

0	1	2	3	4	5	6	7	8	9
8	4	6	5	9	3	0	7	-	-

Elementary ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	N	N	`equals()`
binary search (ordered array)	$\log N$	N †	$\log N$	N †	`compareTo()`

† can do with log N compares, but requires N array accesses

Challenge. Efficient implementations of both search and insert.

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3.1 Symbol Tables

- API
- elementary implementations
- ordered operations

Examples of ordered symbol table API

	keys	*values*
min()	09:00:00	Chicago
	09:00:03	Phoenix
get(09:00:13)	09:00:13	Houston
	09:00:59	Chicago
	09:01:10	Houston
floor(09:05:00)	09:03:13	Chicago
	09:10:11	Seattle
select(7)	09:10:25	Seattle
	09:14:25	Phoenix
	09:19:32	Chicago
	09:19:46	Chicago
keys(09:15:00, 09:25:00)	09:21:05	Chicago
	09:22:43	Seattle
	09:22:54	Seattle
	09:25:52	Chicago
ceiling(09:30:00)	09:35:21	Chicago
	09:36:14	Seattle
max()	09:37:44	Phoenix

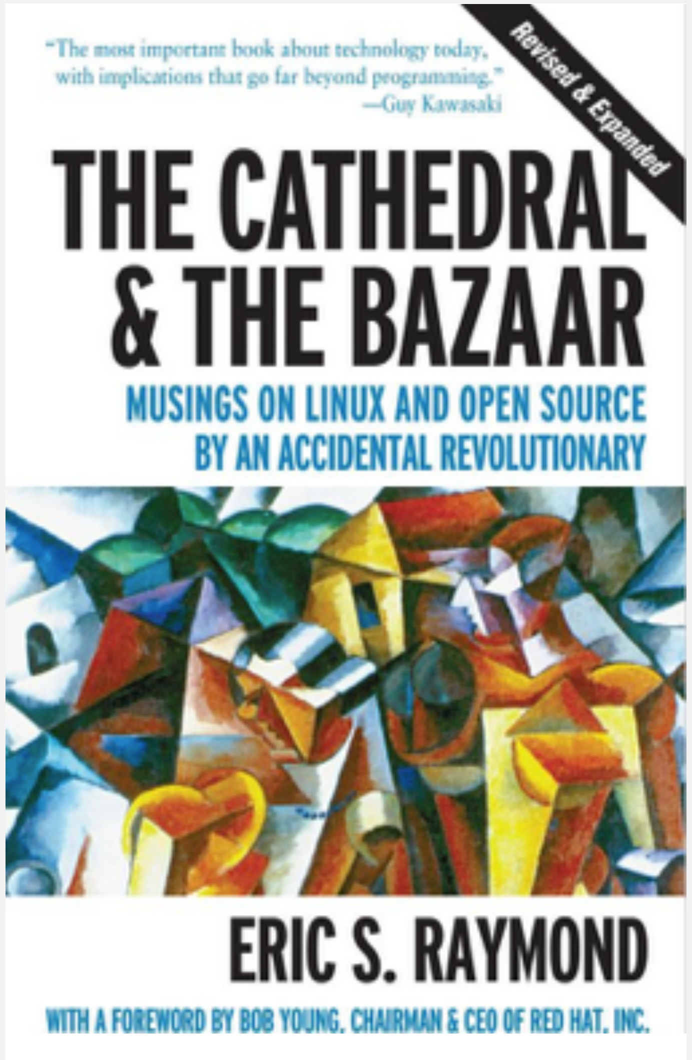
size(09:15:00, 09:25:00) *is* 5
rank(09:10:25) *is* 7

Ordered symbol table API

```
public class ST<Key extends Comparable<Key>, Value>
        ⋮
Key  min()                   smallest key
Key  max()                   largest key
Key  floor(Key key)          largest key less than or equal to key
Key  ceiling(Key key)        smallest key greater than or equal to key
int  rank(Key key)           number of keys less than key
Key  select(int k)           key of rank k
        ⋮
```

Smart data structures

" Smart data structures and dumb code works a lot better than the other way around. " – *Eric S. Raymond*



Binary search: ordered symbol table operations summary

	sequential search	binary search
search	N	$\log N$
insert	N	N
min / max	N	1
floor / ceiling	N	$\log N$
rank	N	$\log N$
select	N	1

order of growth of the running time for ordered symbol table operations

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3.2 BINARY SEARCH TREES

- *BSTs*
- *ordered operations*
- *iteration*
- *deletion (see book)*

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3.2 Binary Search Trees

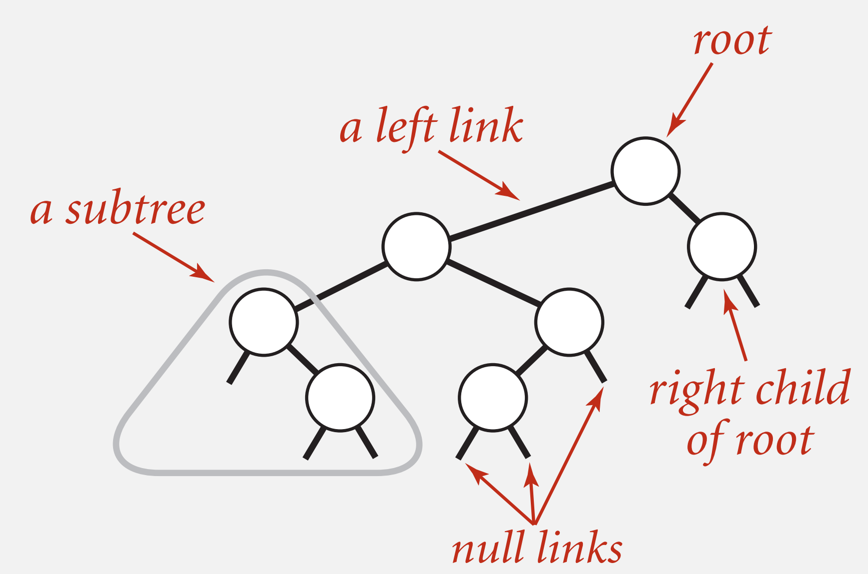
- *BSTs*
- *ordered operations*
- *iteration*
- *deletion*

Binary search trees

Definition. A BST is a binary tree in symmetric order.

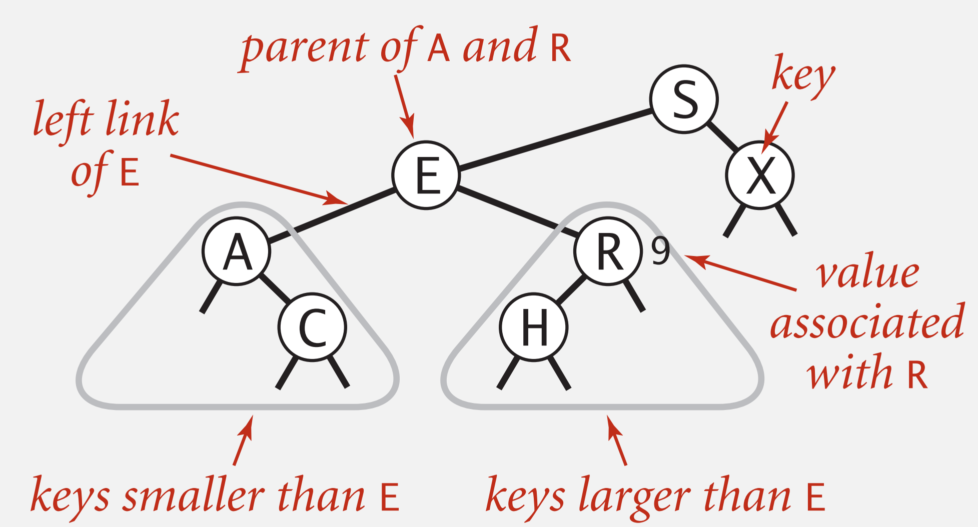
A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



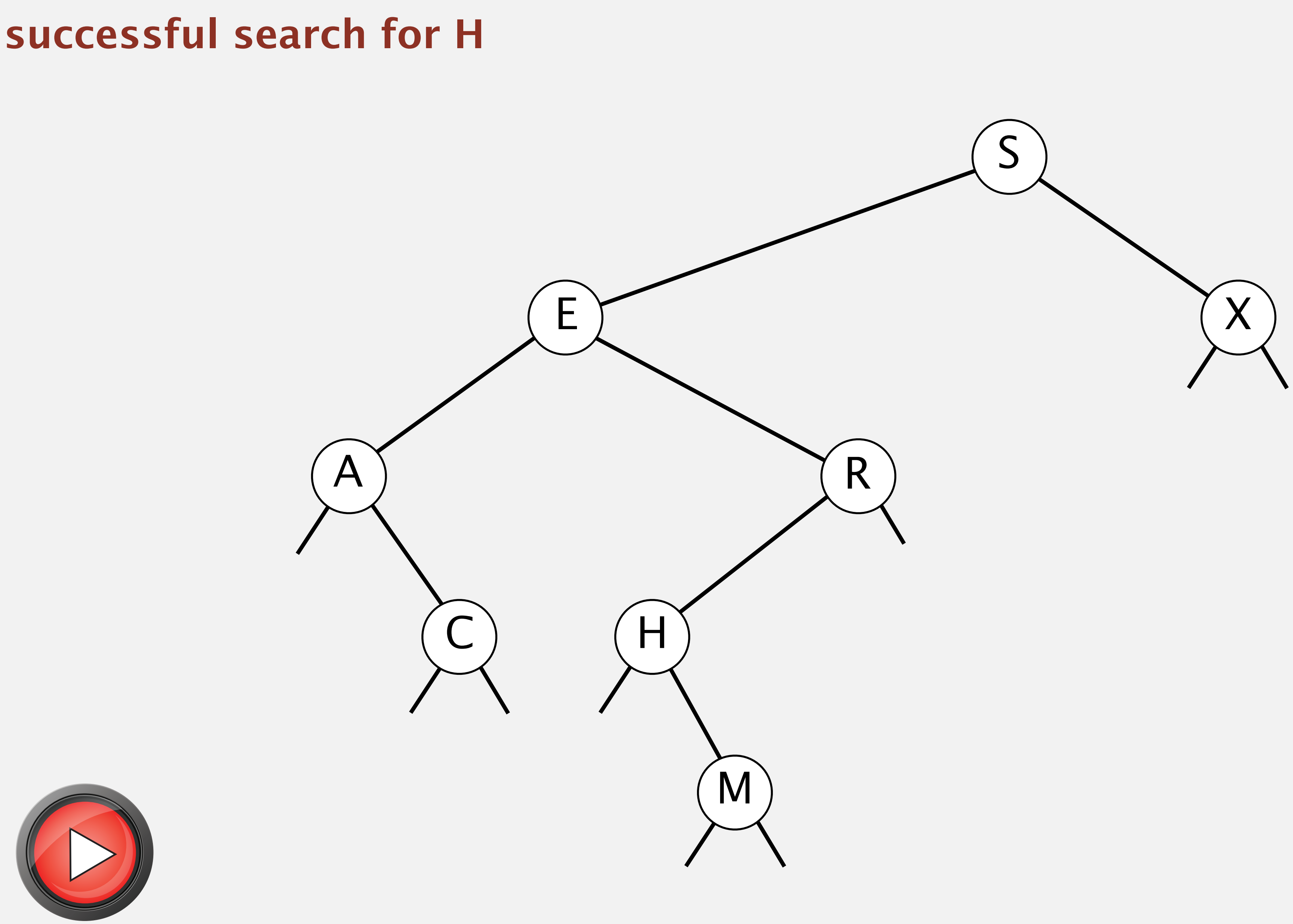
Symmetric order. Each node has a key, and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.



Binary search tree demo

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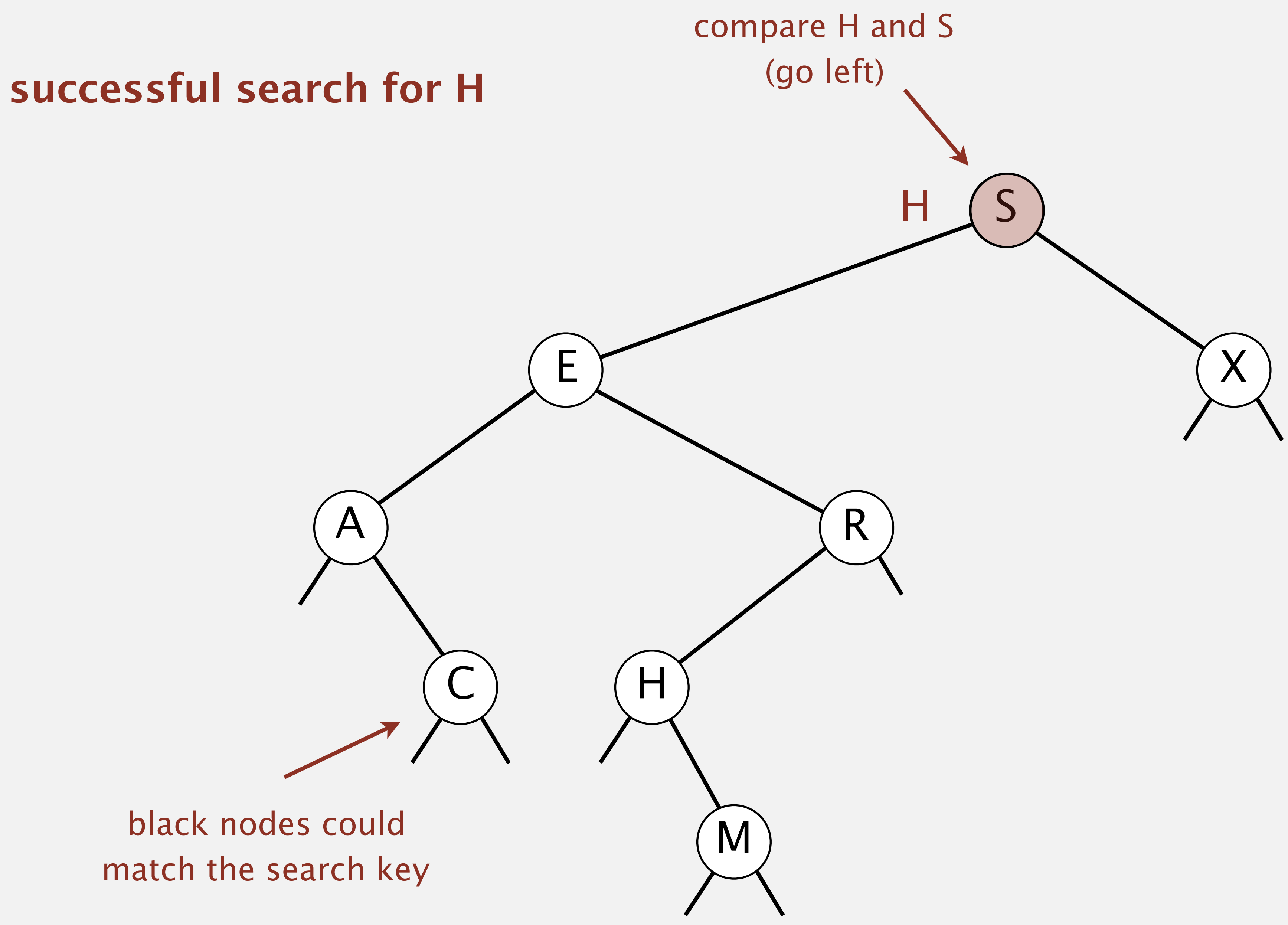

Binary search tree demo

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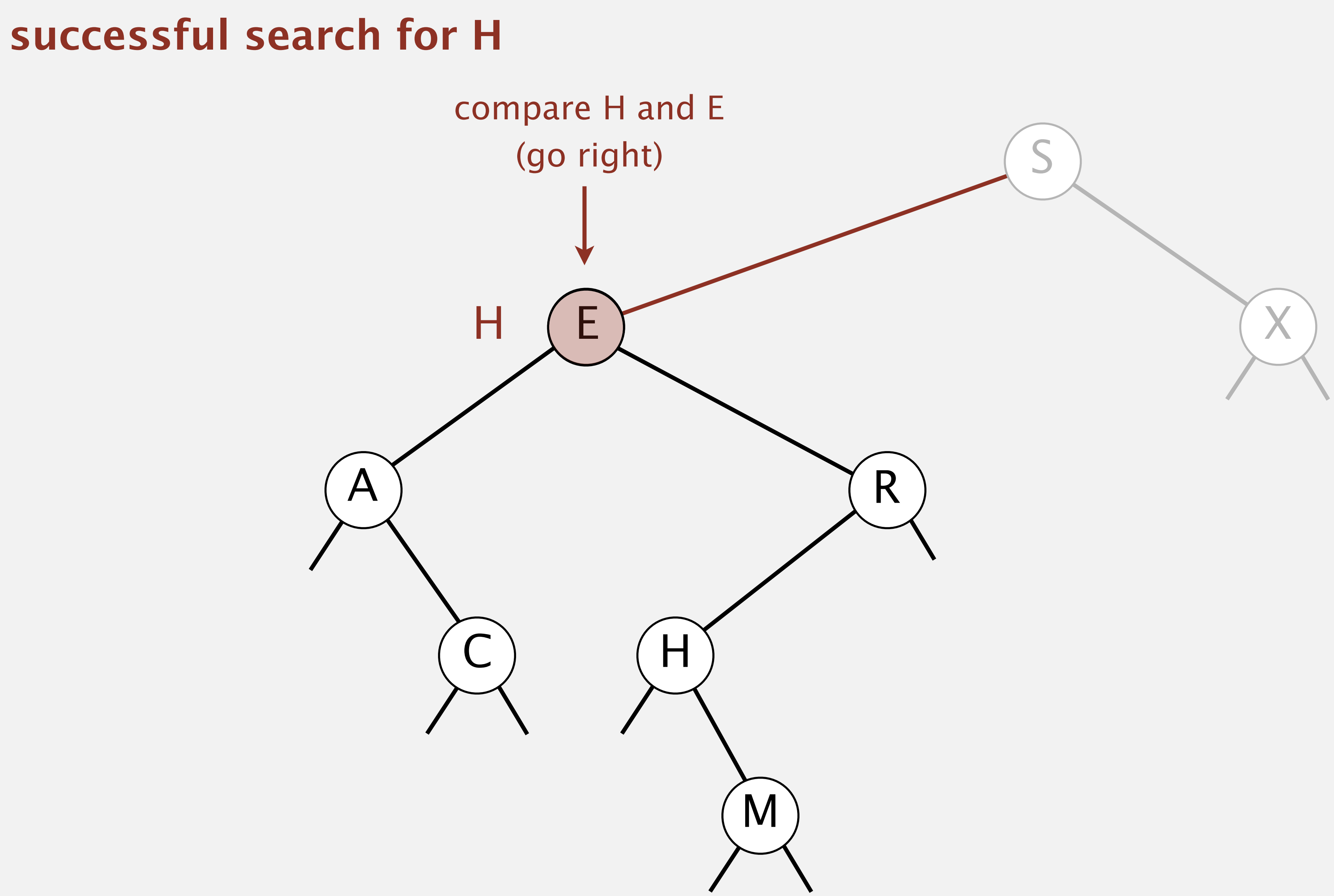
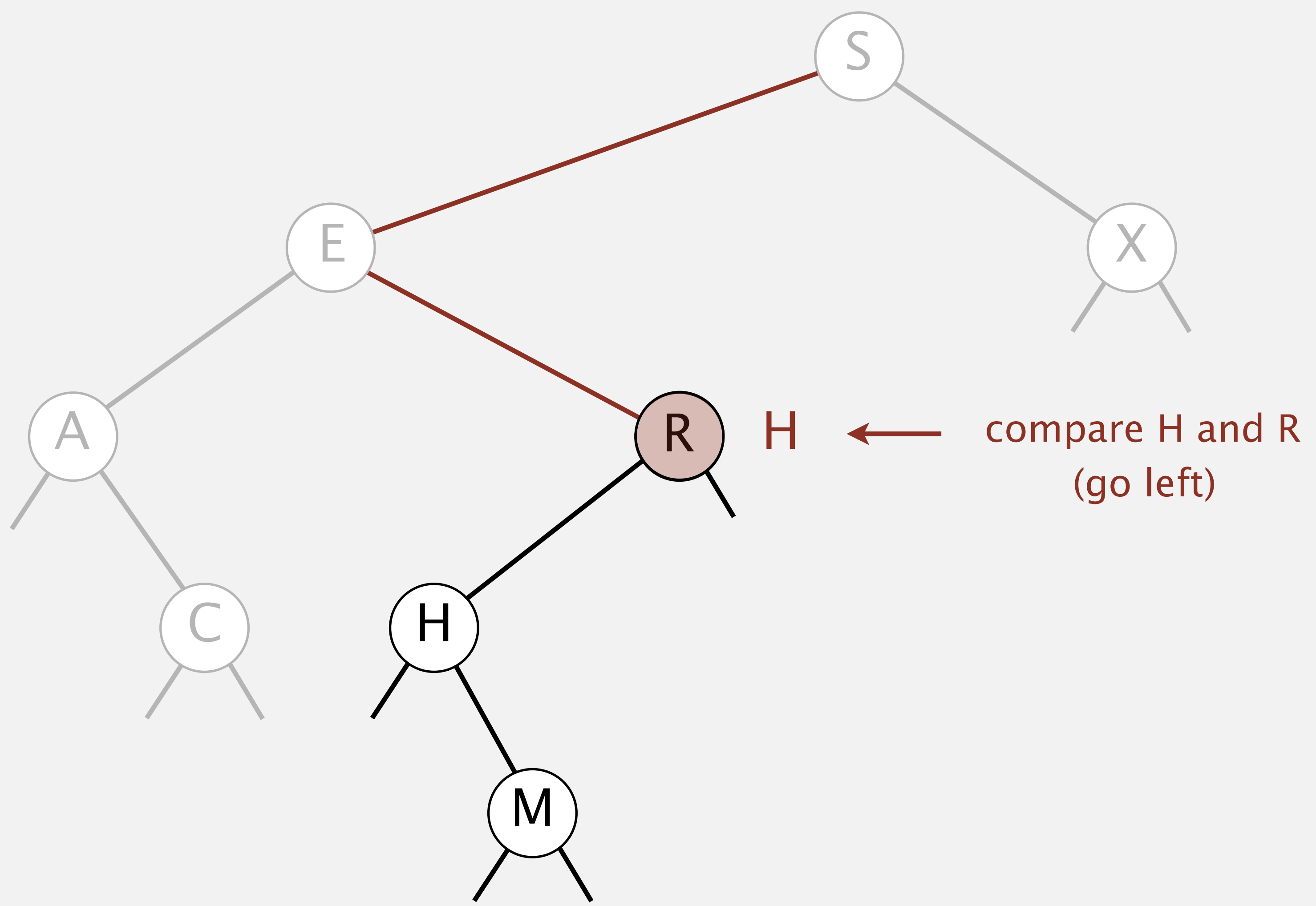


Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

successful search for H



Binary search tree demo

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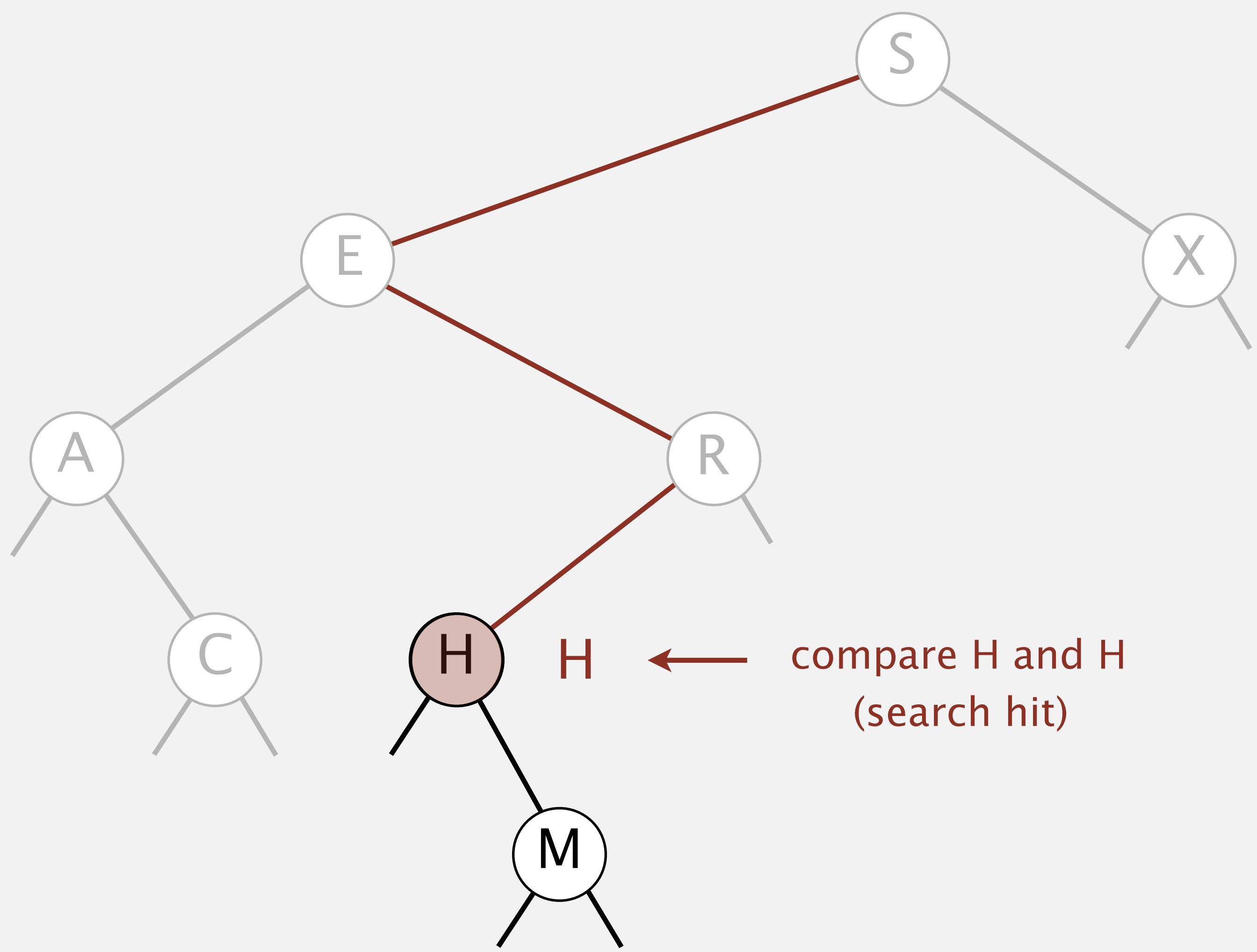
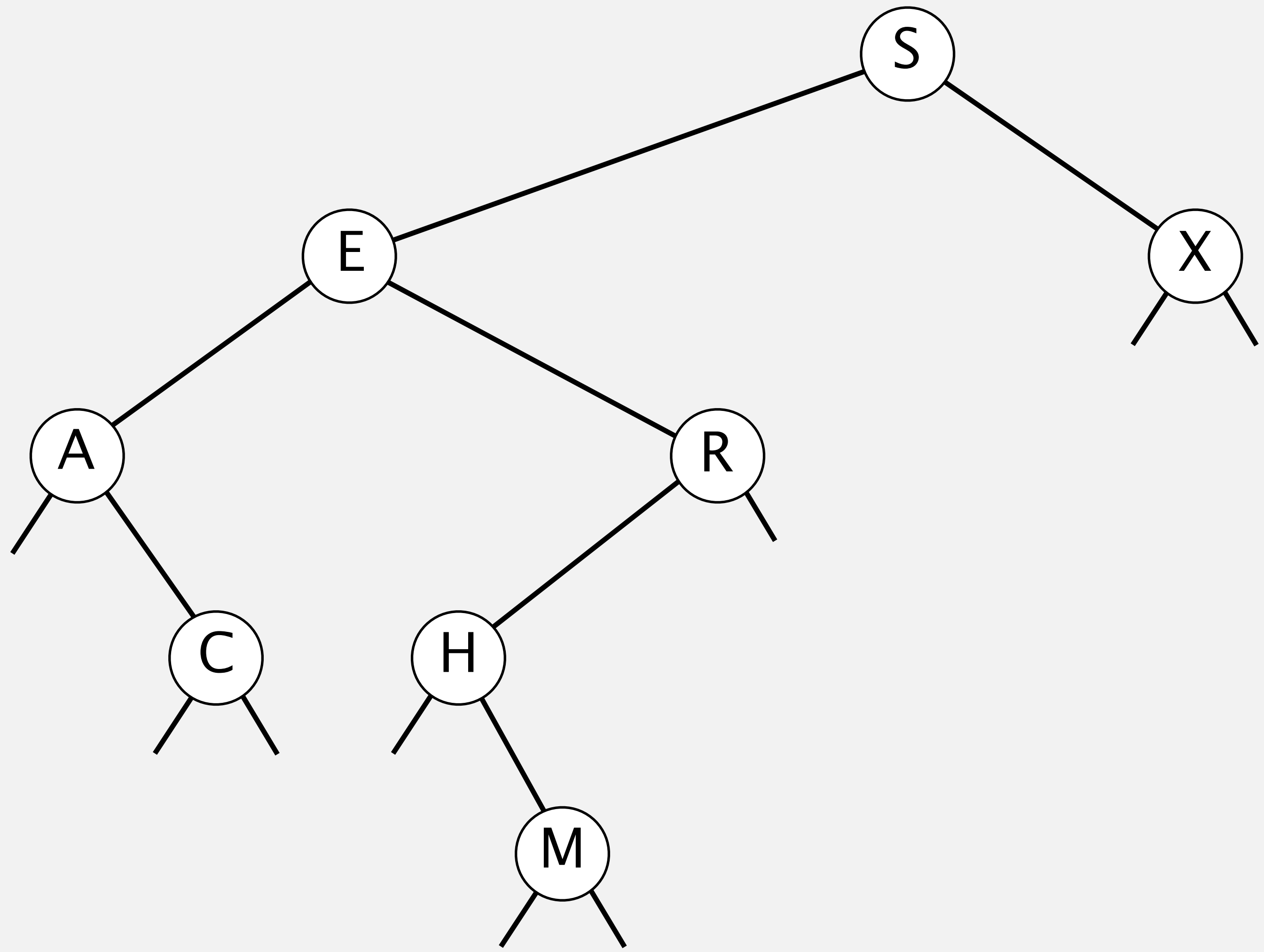


Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

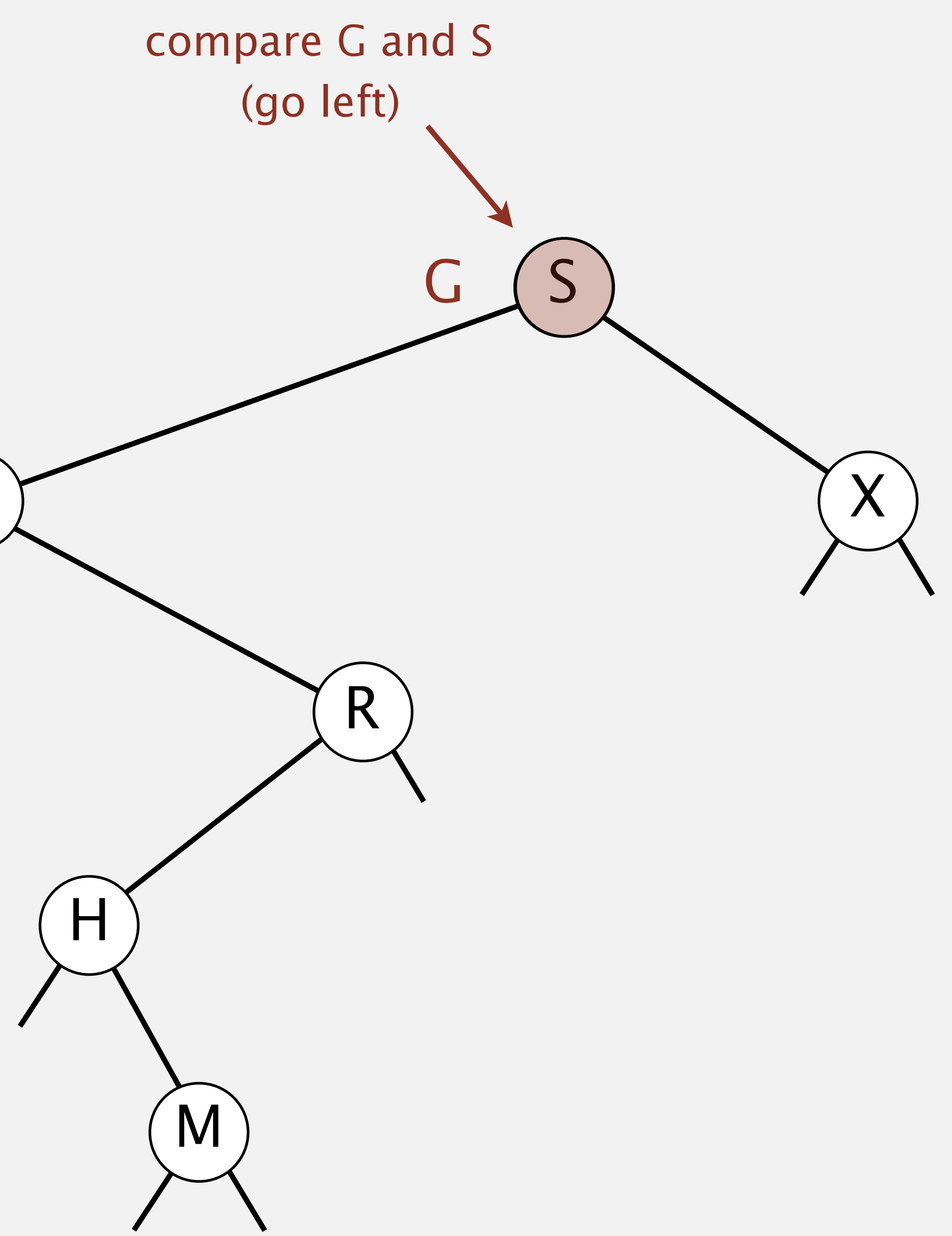
unsuccessful search for G



Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

unsuccessful search for G



Binary search tree demo

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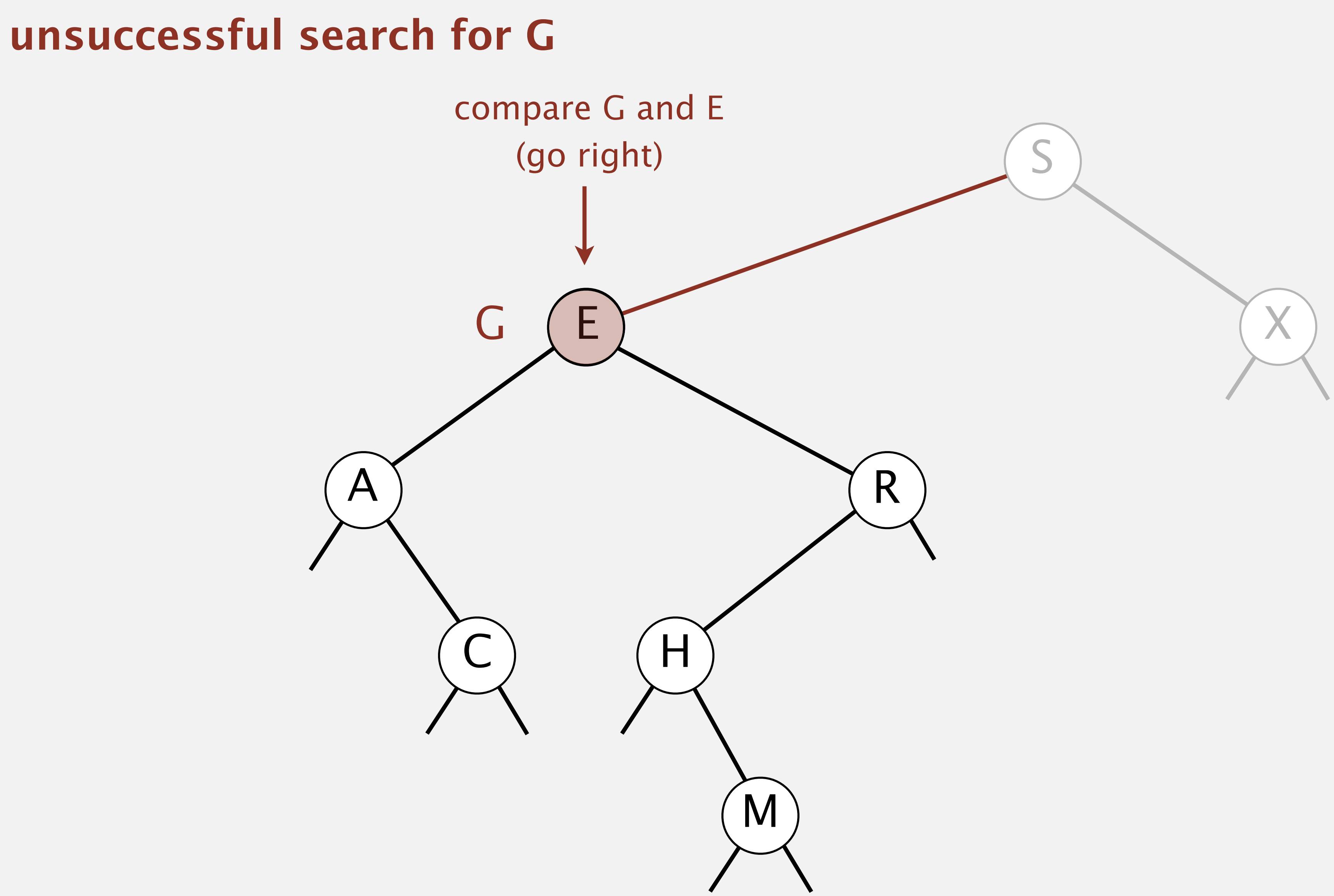

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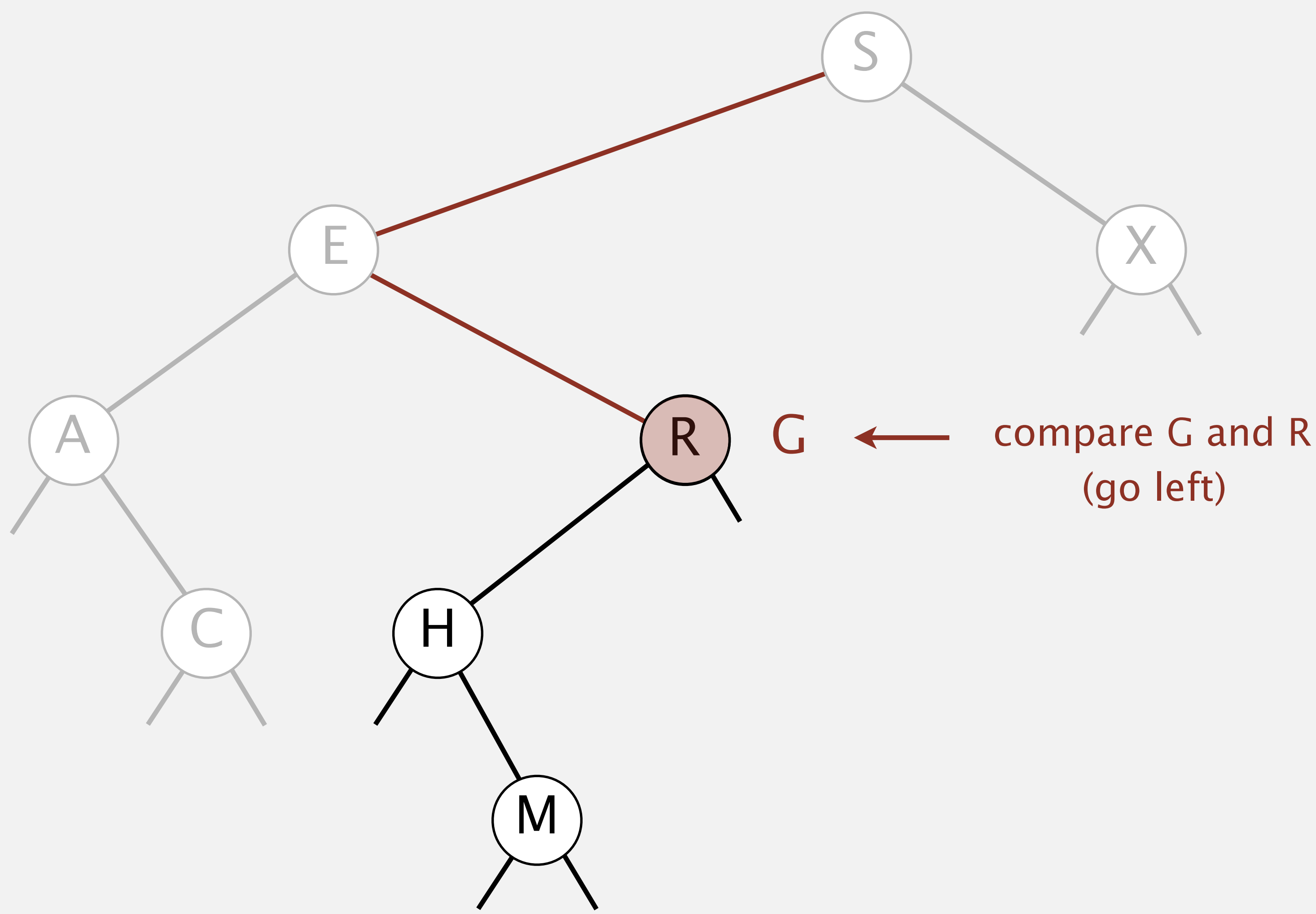

Binary search tree demo

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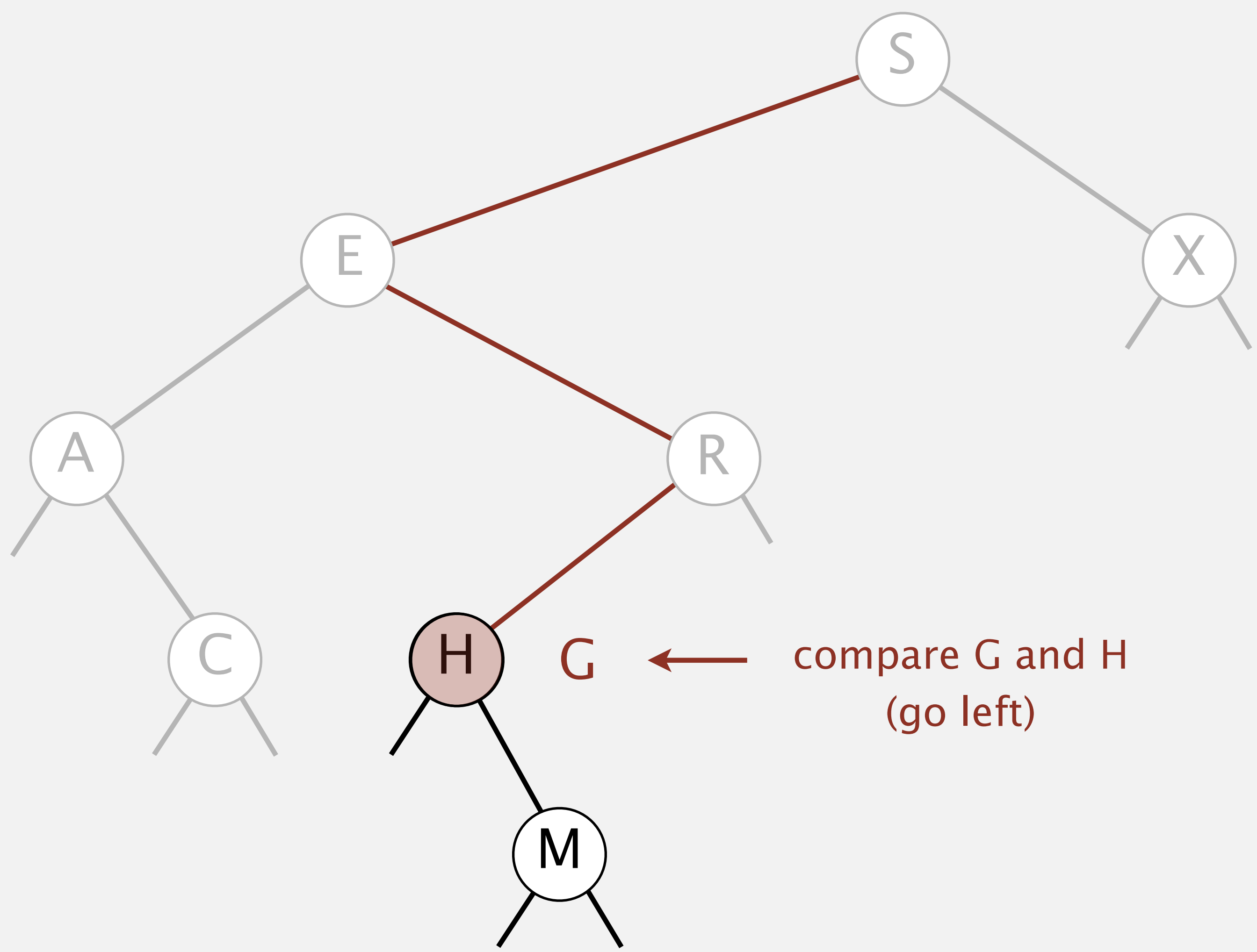
unsuccessful search for G



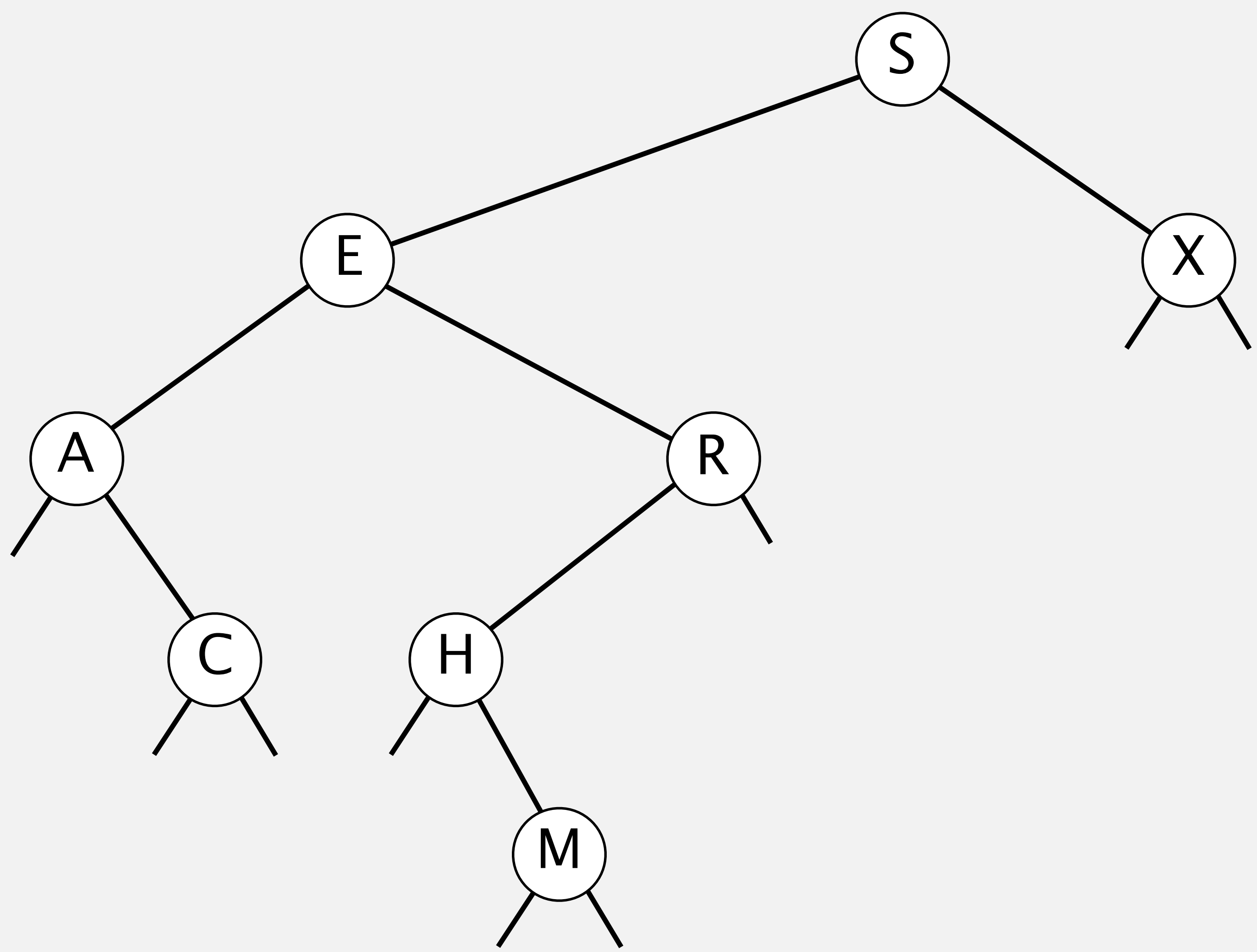
Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

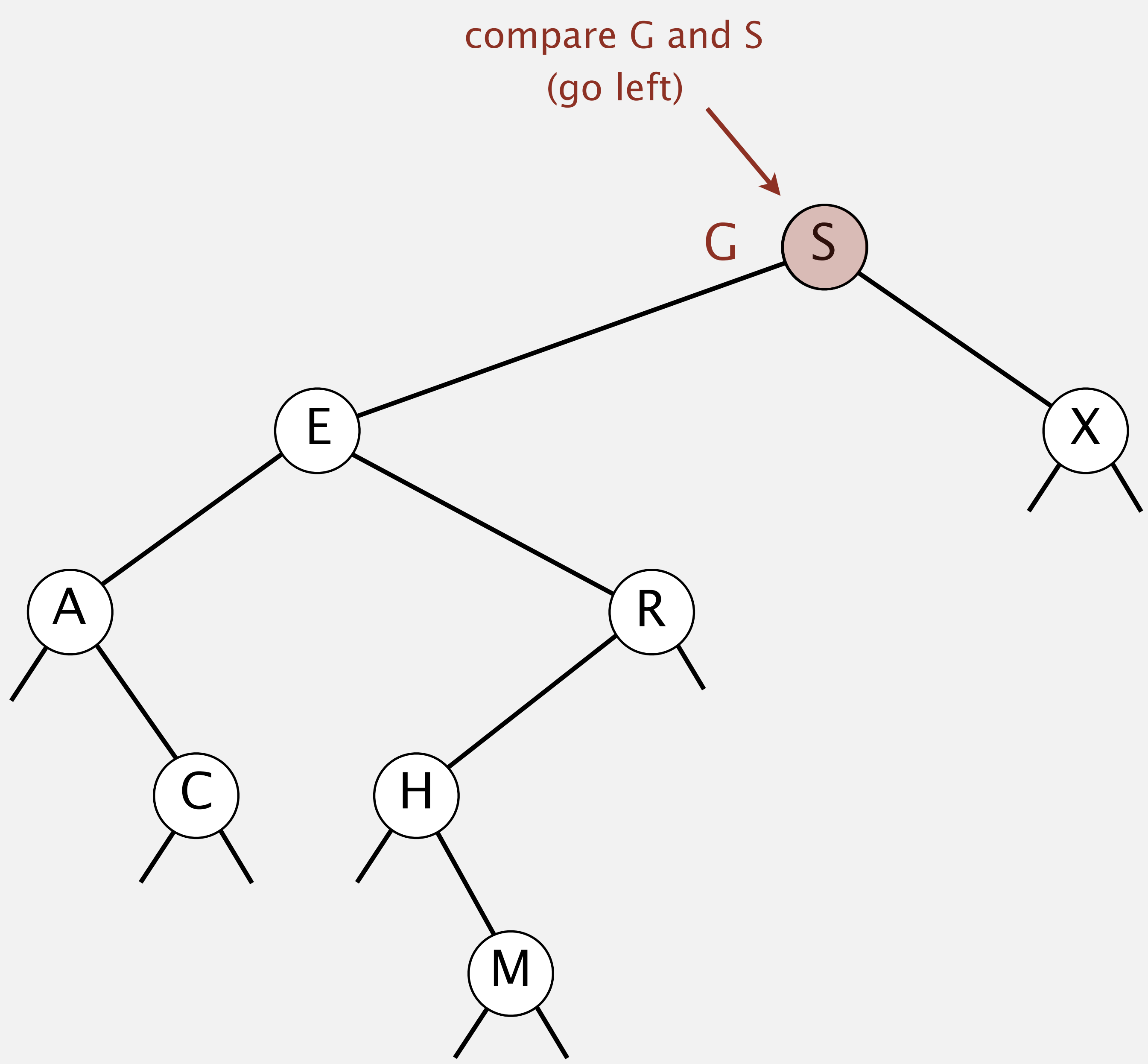
insert G



Binary search tree demo

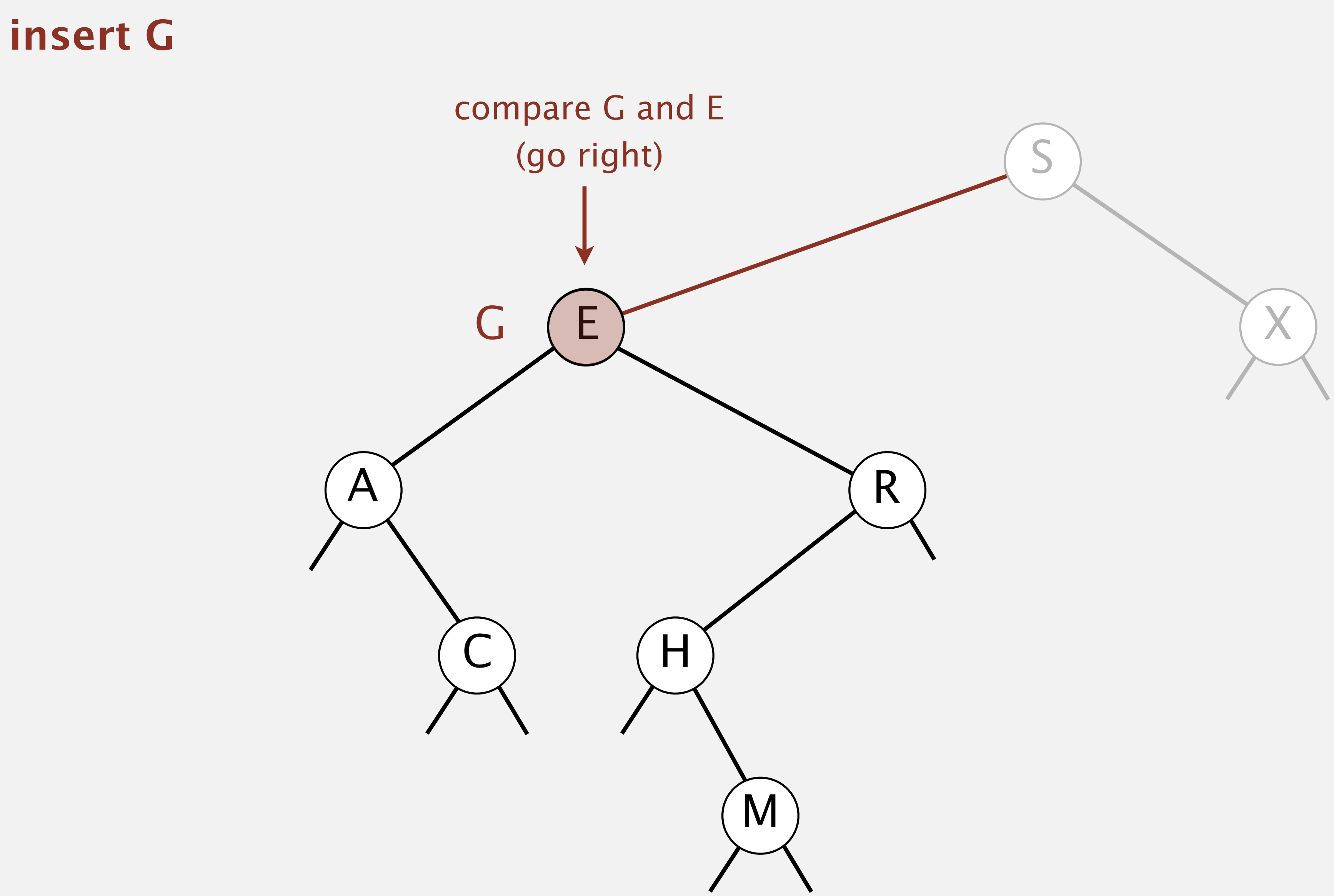
Insert. If less, go left; if greater, go right; if null, insert.

insert G



Binary search tree demo

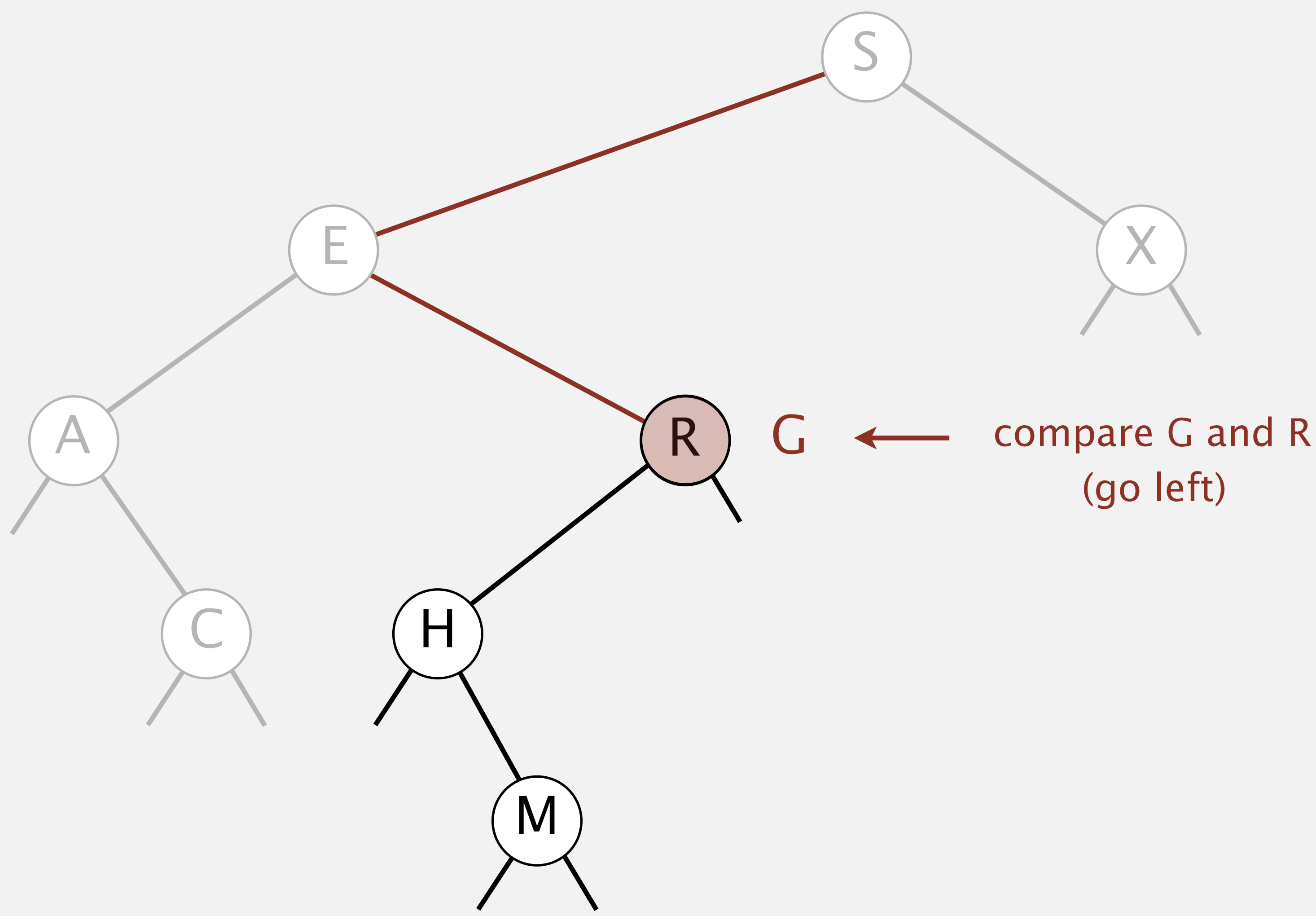
Insert. If less, go left; if greater, go right; if null, insert.



Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G



Binary search tree demo

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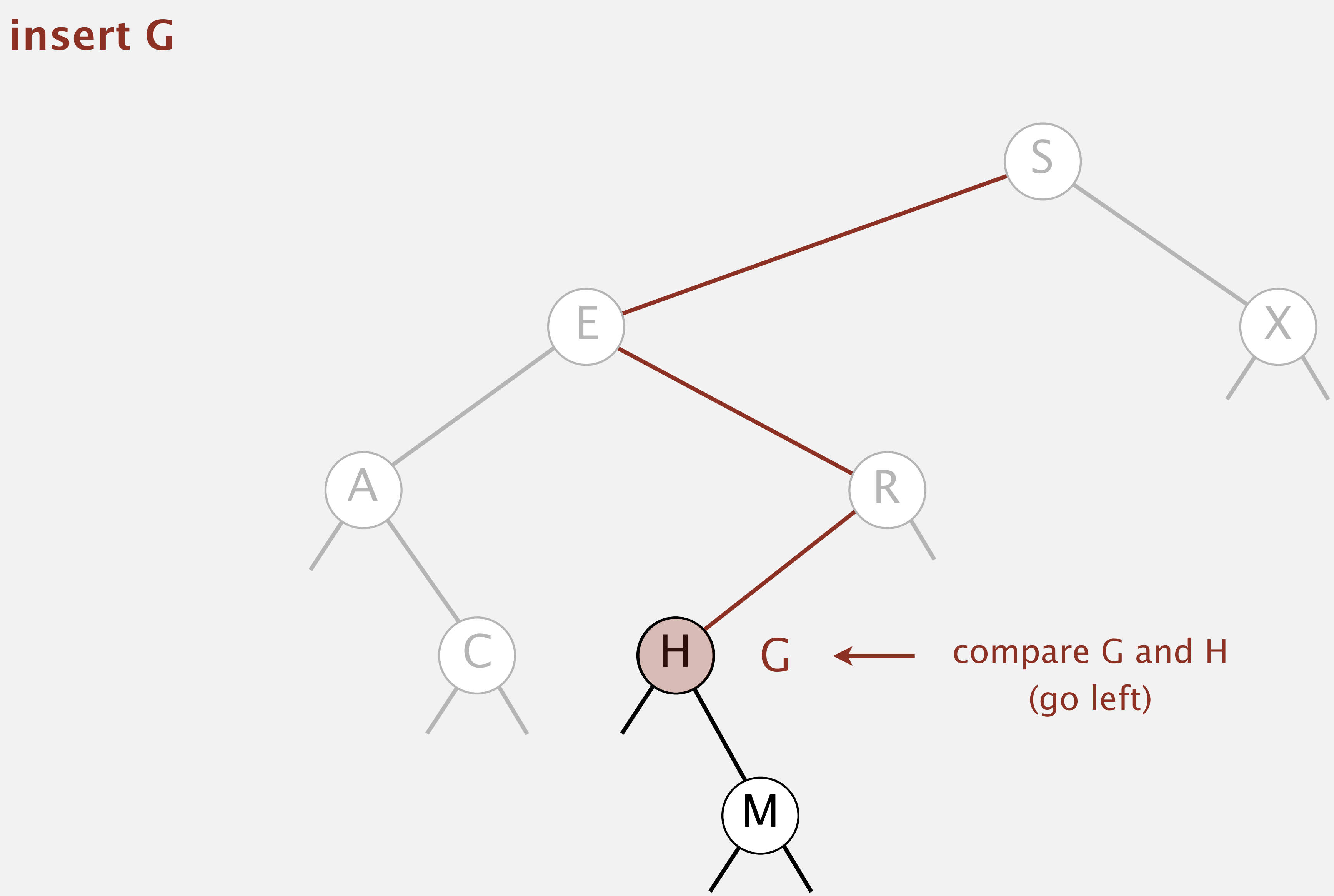

Binary search tree demo

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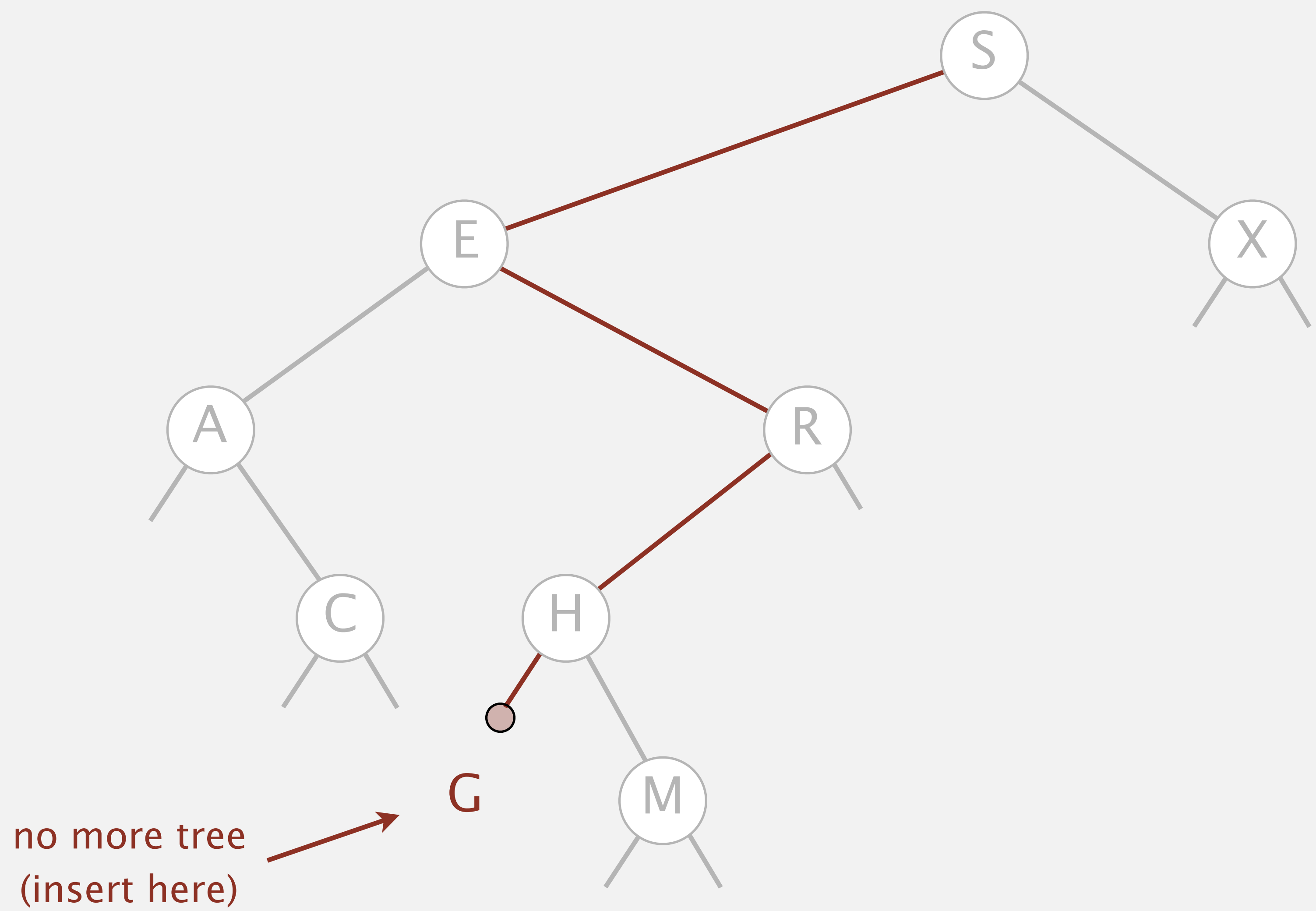

Binary search tree demo

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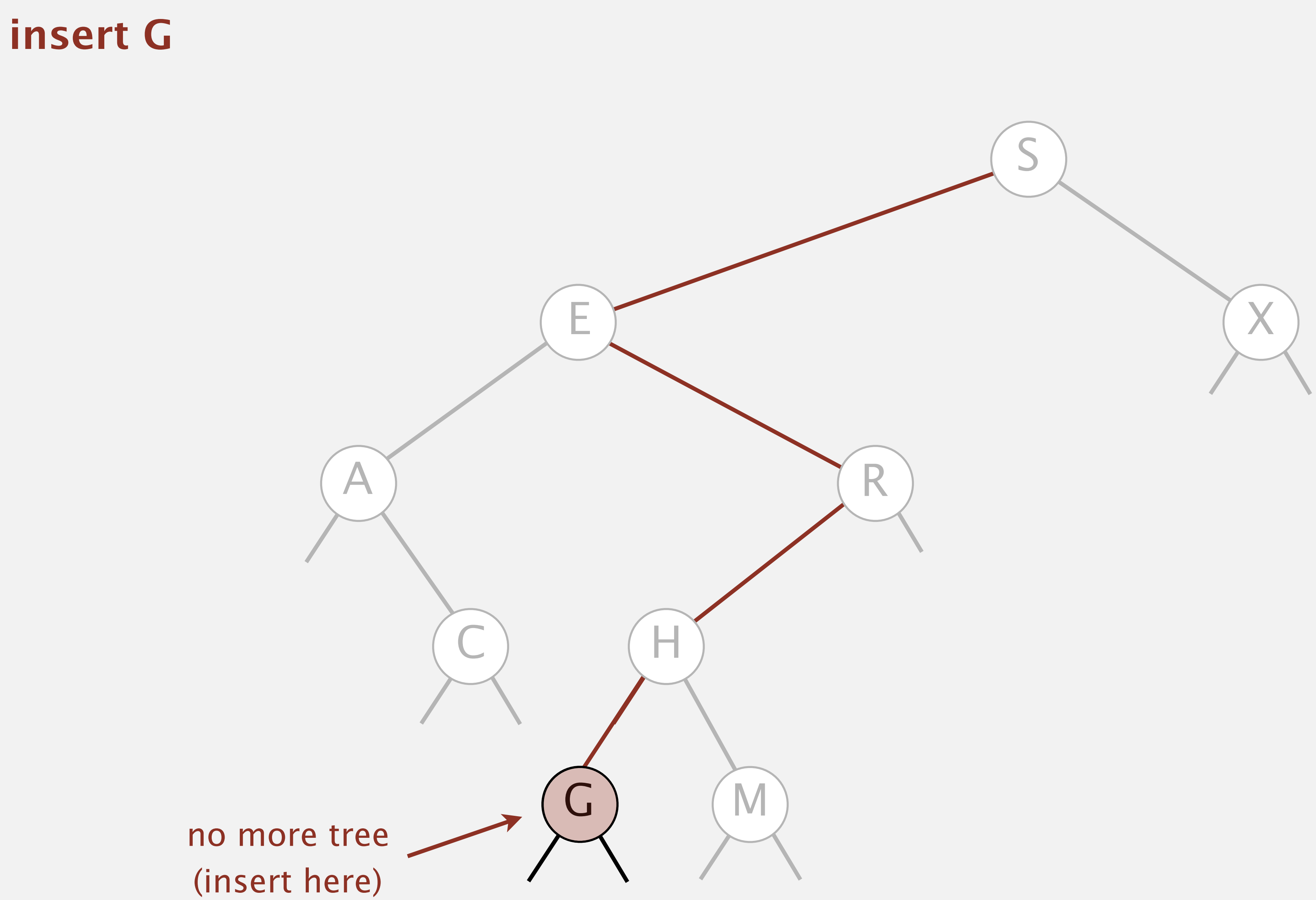

Binary search tree demo

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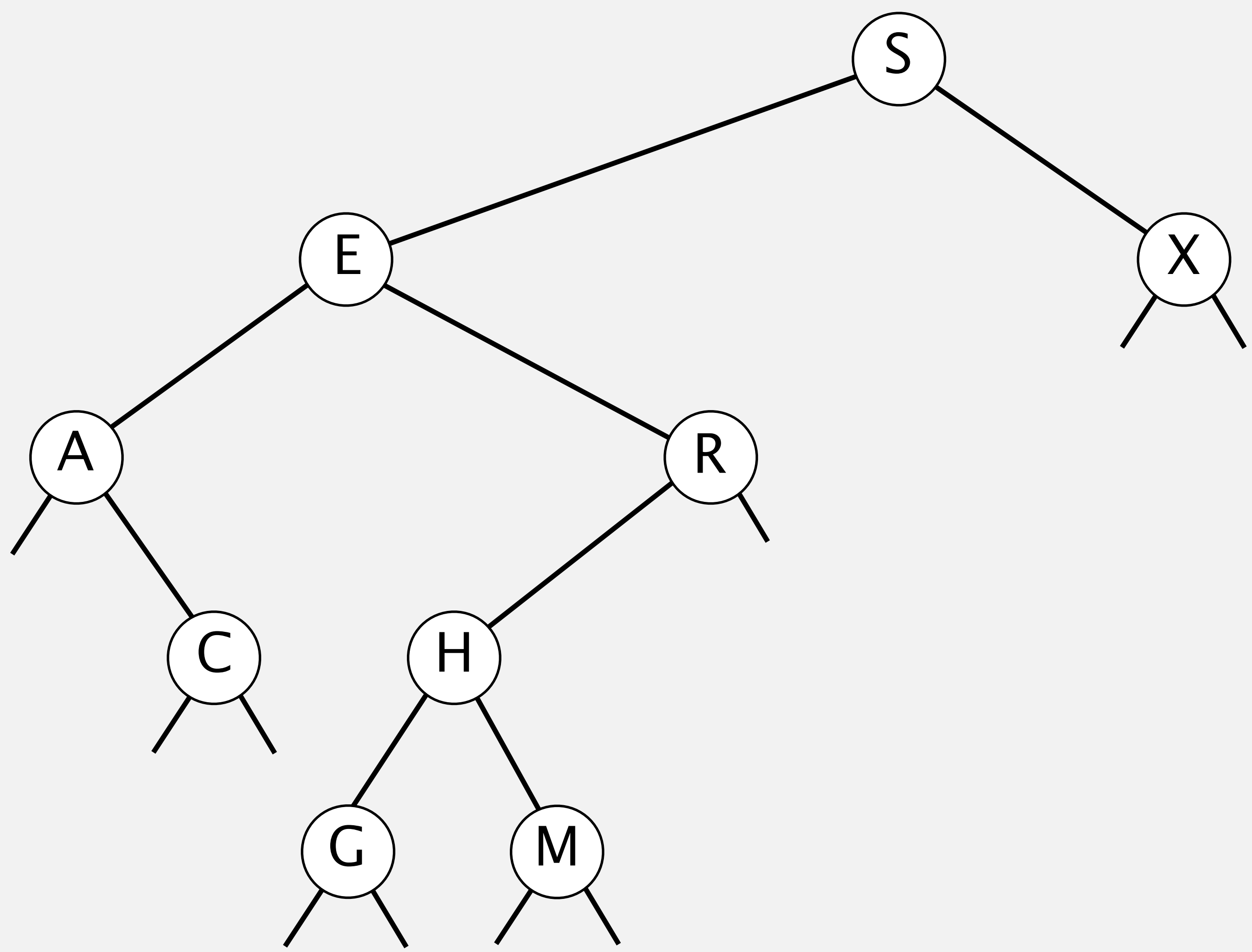
insert G



BST representation in Java

Java definition. A BST is a reference to a root `Node`.

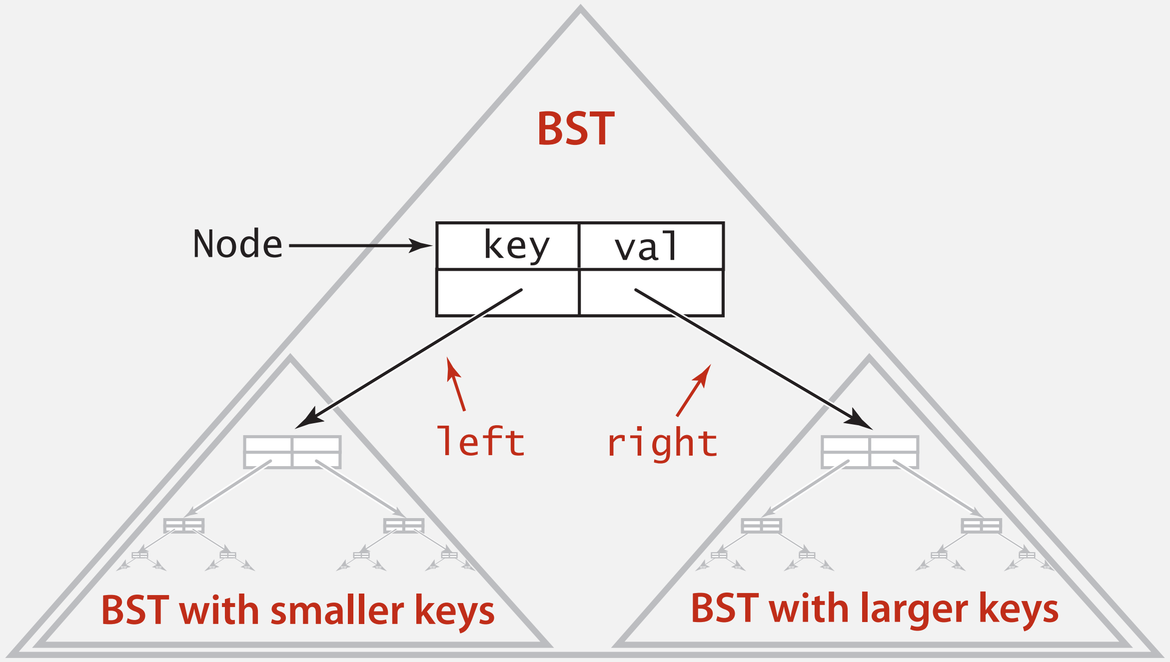
A `Node` is composed of four fields:

- A `Key` and a `Value`.
- A reference to the left and right subtree.

smaller keys   larger keys

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;
   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```

Key and Value are generic types; Key is Comparable



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;

    private class Node
    {  /* see previous slide */  }

    public void put(Key key, Value val)
    {  /* see next slides */  }

    public Value get(Key key)
    {  /* see next slides */  }

    public Iterable<Key> iterator()
    {  /* see slides in next section */  }

    public void delete(Key key)
    {  /* see textbook */  }

}
```

root of BST

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

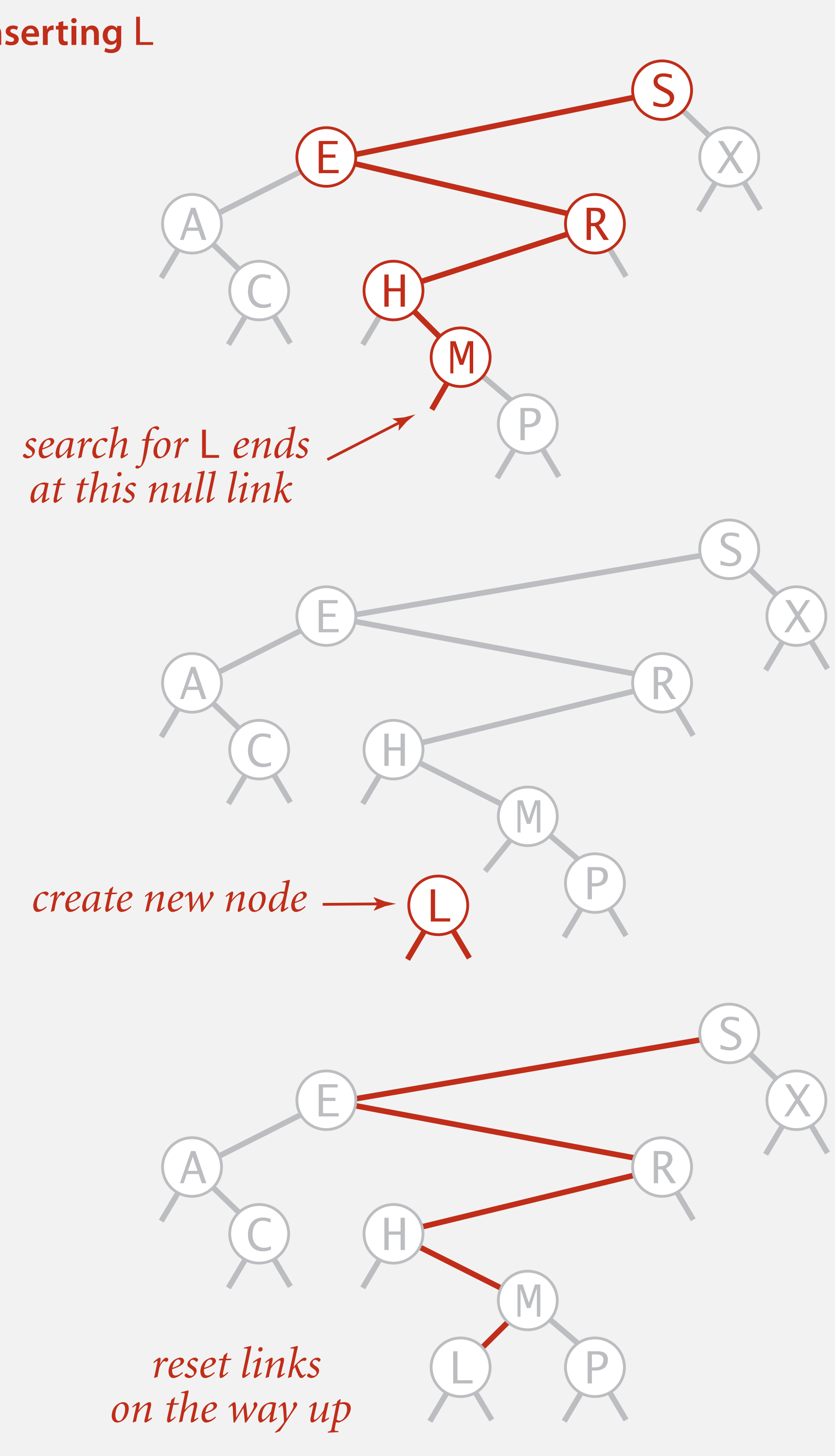
Cost. Number of compares = 1 + depth of node.

BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree $\Rightarrow$ reset value.
- Key not in tree $\Rightarrow$ add new node.



Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

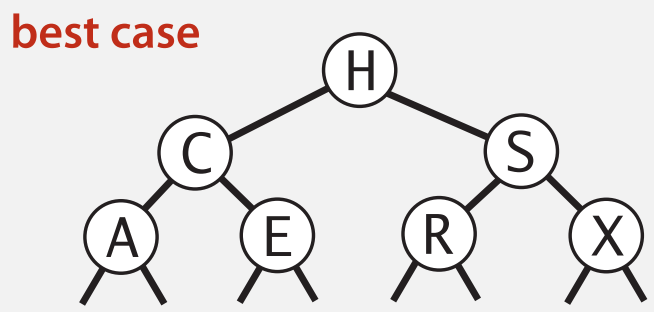
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   return x;
}
```

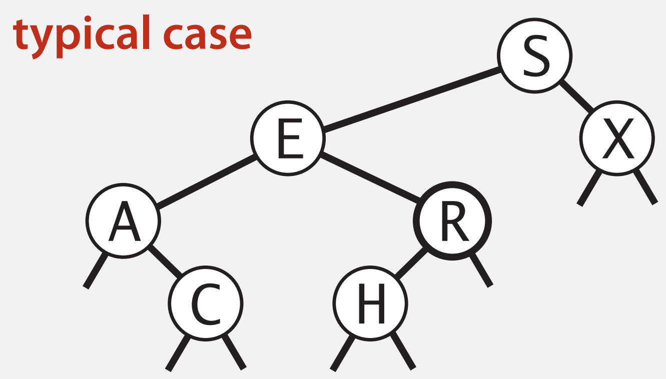
Warning: concise but tricky code; read carefully!

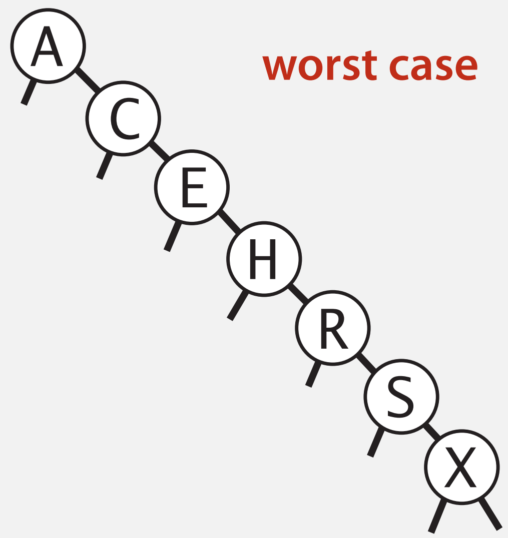
Cost. Number of compares = 1 + depth of node.

Tree shape

- Many BSTs correspond to same set of keys.
- Number of compares for search/insert = 1 + depth of node.



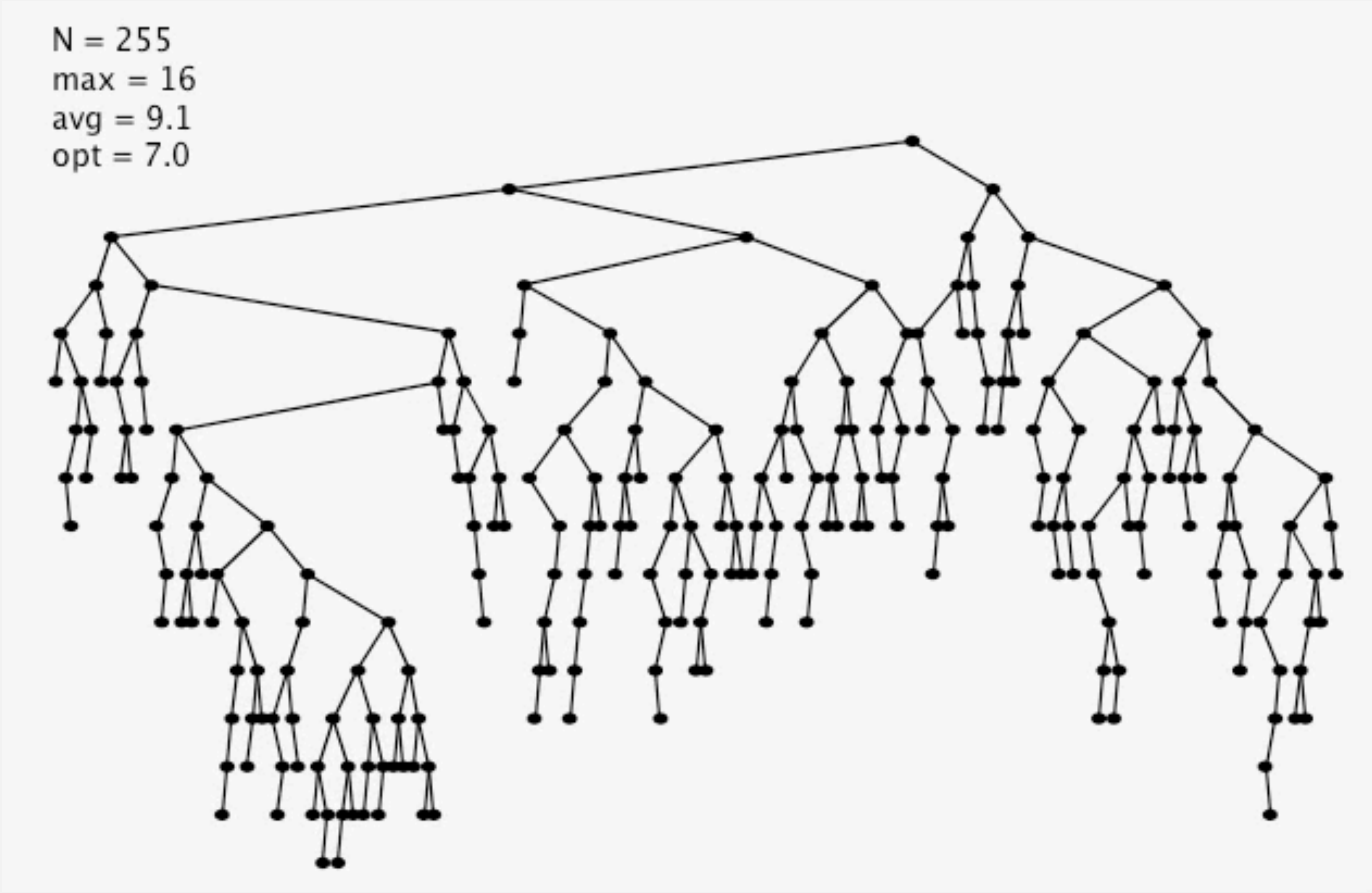




Bottom line. Tree shape depends on order of insertion.

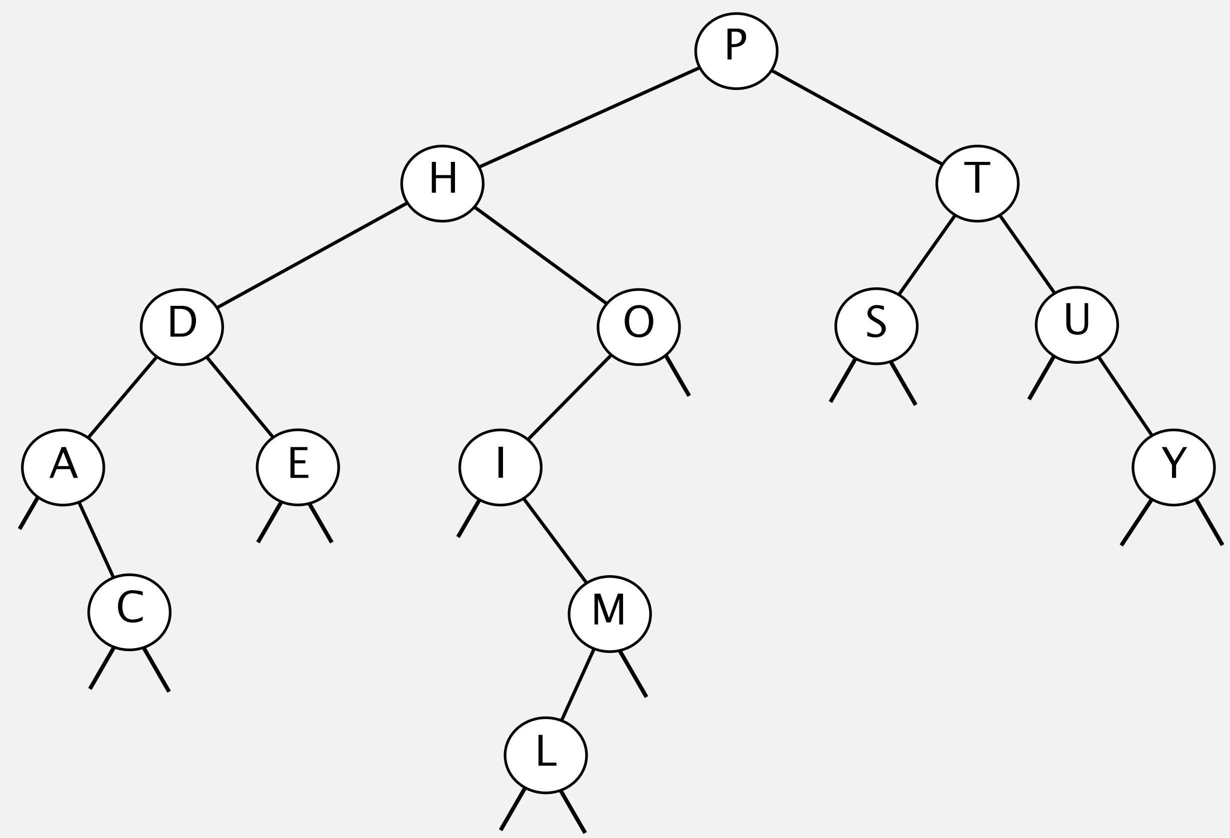
BST insertion: random order visualization

Ex. Insert keys in random order.



Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y



Remark. Correspondence is 1–1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is $\sim 2 \ln N$.

Pf. 1–1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted into a BST in random order, the expected height is $\sim 4.311 \ln N$.

expected depth of
function-call stack in quicksort

How Tall is a Tree?

Bruce Reed
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ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathrm{E}(H_n) = \alpha \log n - \beta \log\log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But... Worst-case height is $N - 1$.

[exponentially small chance when keys are inserted in random order]

ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	N	N	`equals()`
binary search (ordered array)	$\log N$	N	$\log N$	N	`compareTo()`
BST	N	N	$\log N$	$\log N$	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of log N?

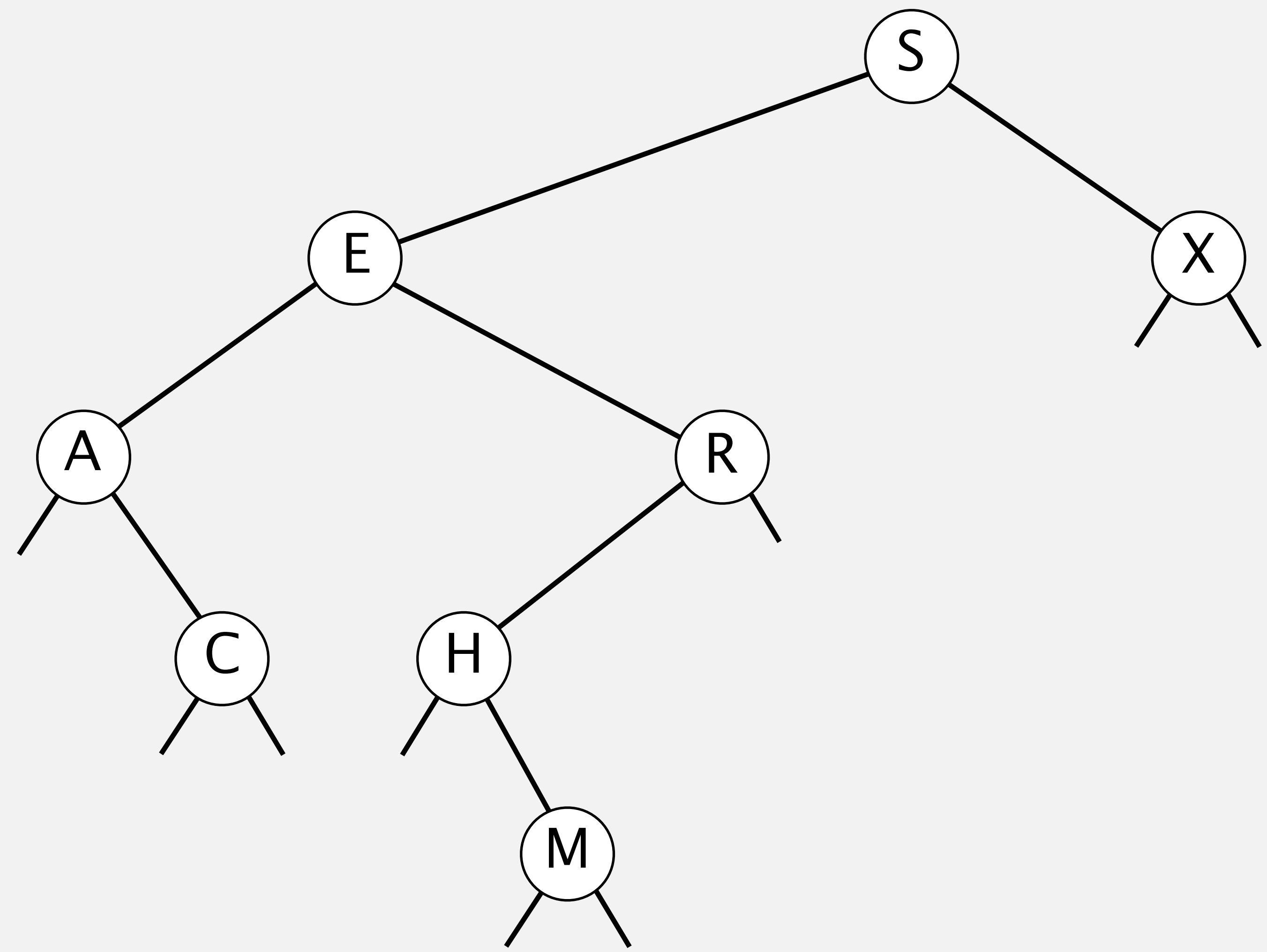
http://algs4.cs.princeton.edu

3.2 Binary Search Trees

- BSTs
- *iteration*
- ordered operations
- deletion

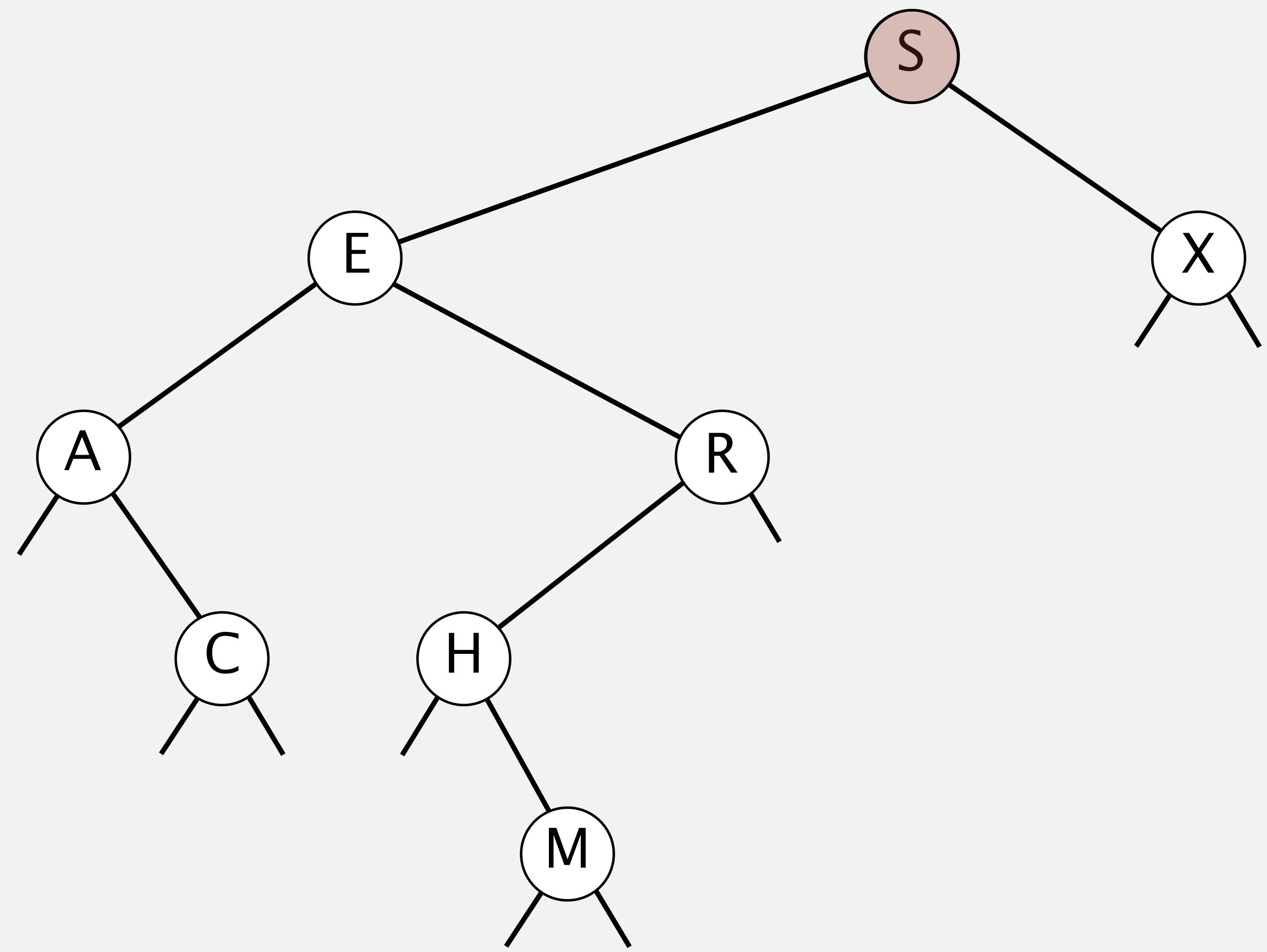
Inorder traversal demo

```
private void inorder(Node x)
{
   if (x == null) return;
   inorder(x.left);
   StdOut.println(x.key);
   inorder(x.right);
}
```



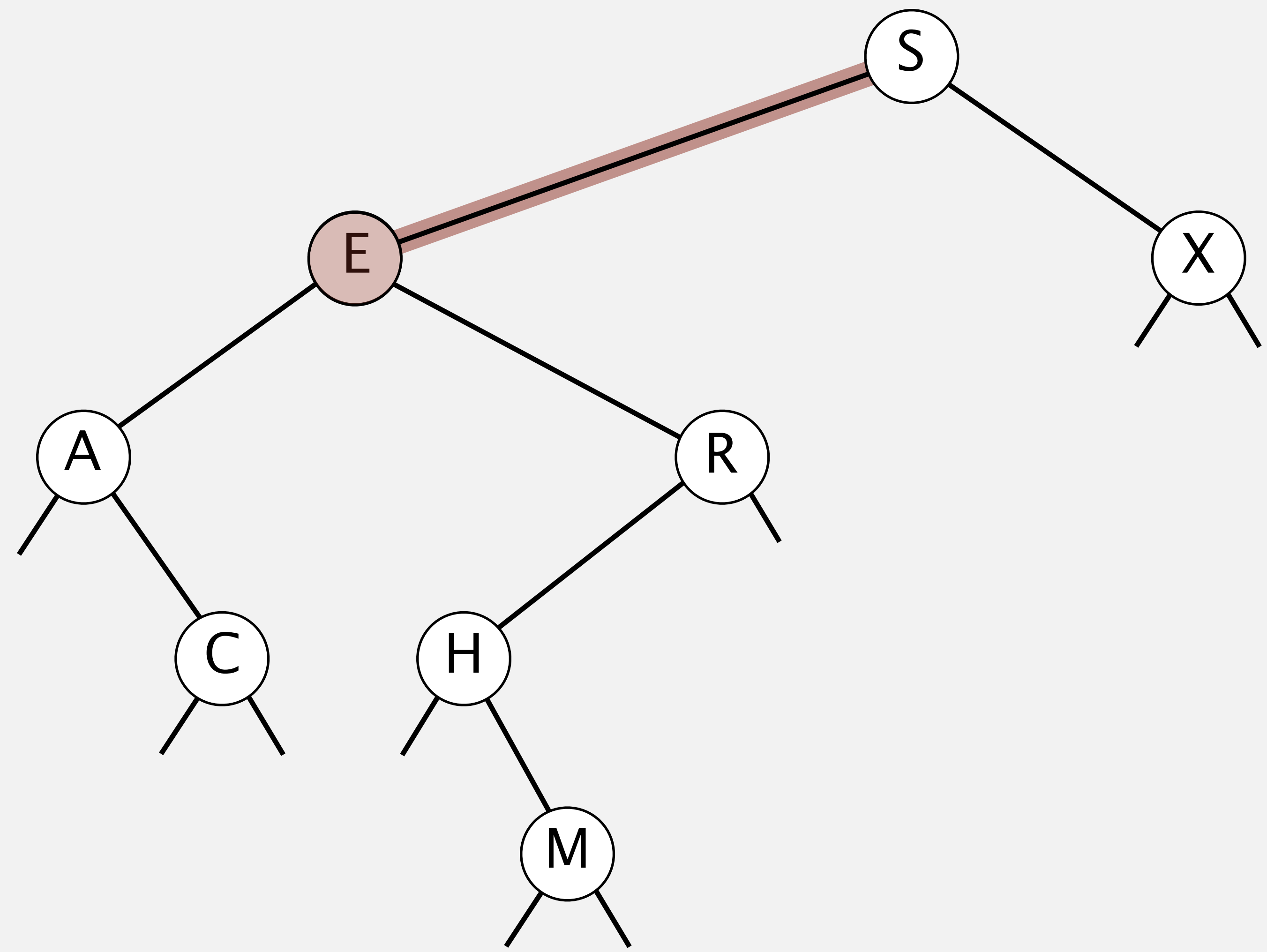
Inorder traversal demo

`inorder(S)`



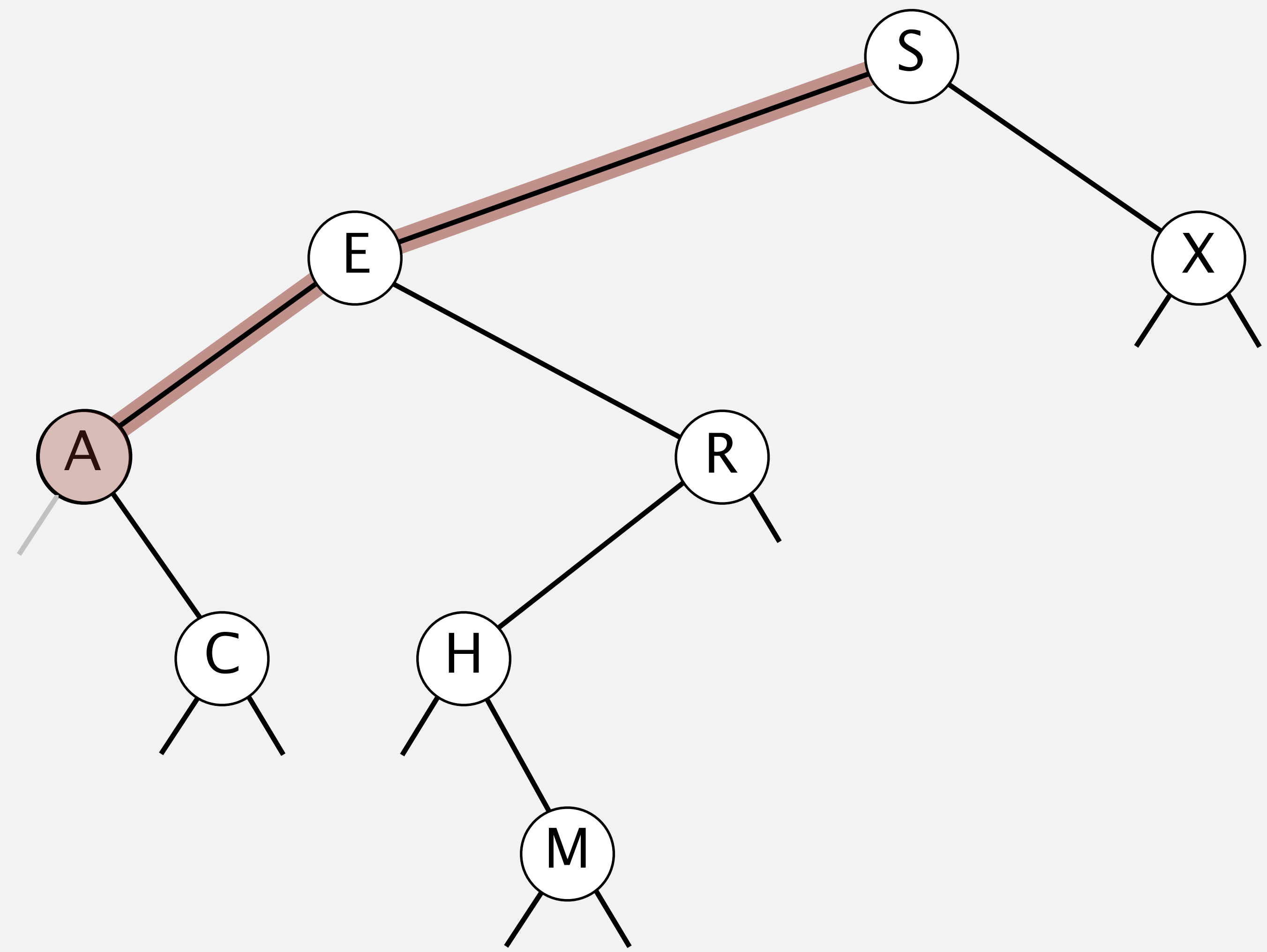
Inorder traversal demo

```
inorder(S)
   inorder(E)
```



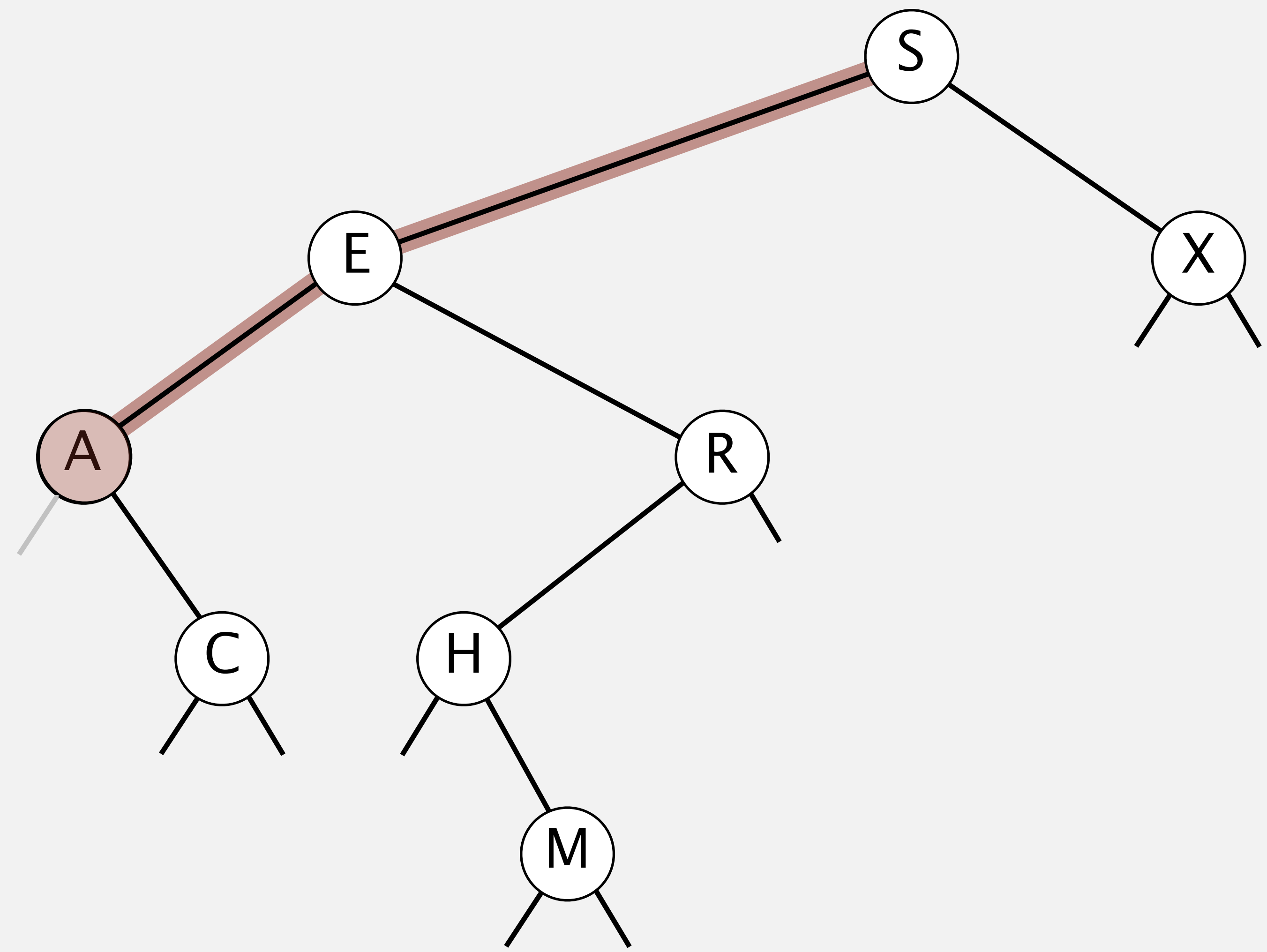
Inorder traversal demo

```
inorder(S)
   inorder(E)
      inorder(A)
```



Inorder traversal demo

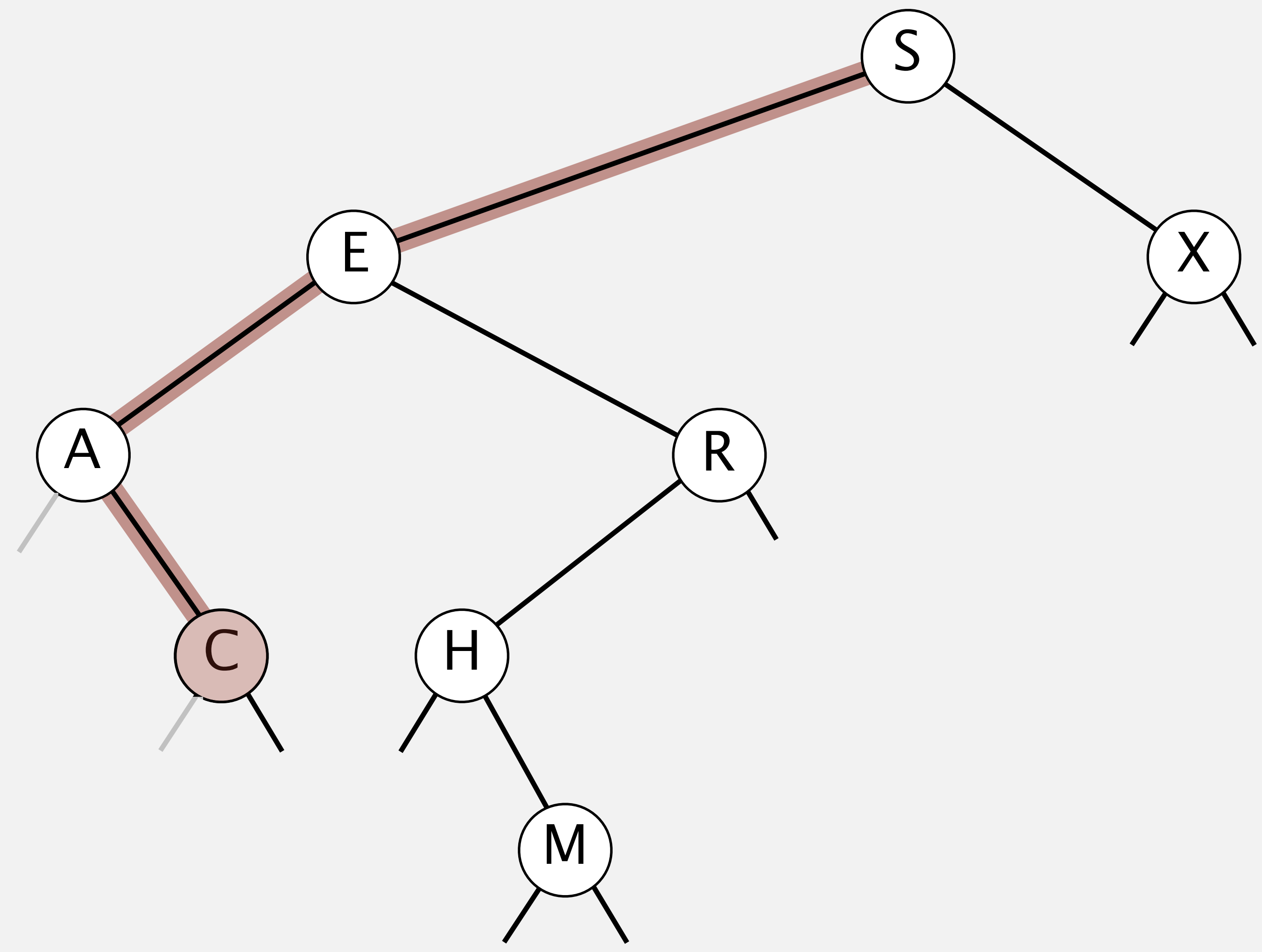
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
```



output: **A**

Inorder traversal demo

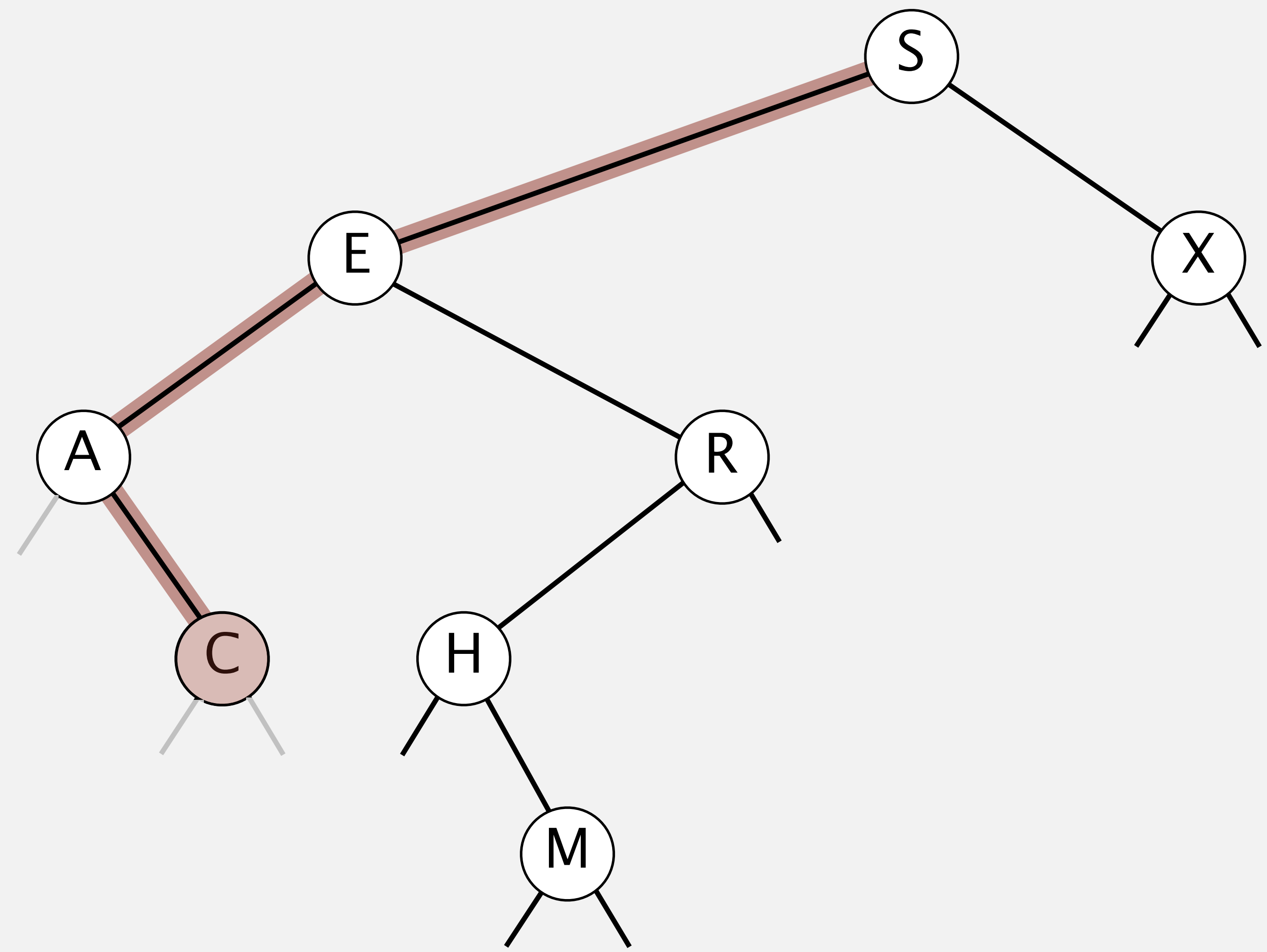
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
```



output: **A**

Inorder traversal demo

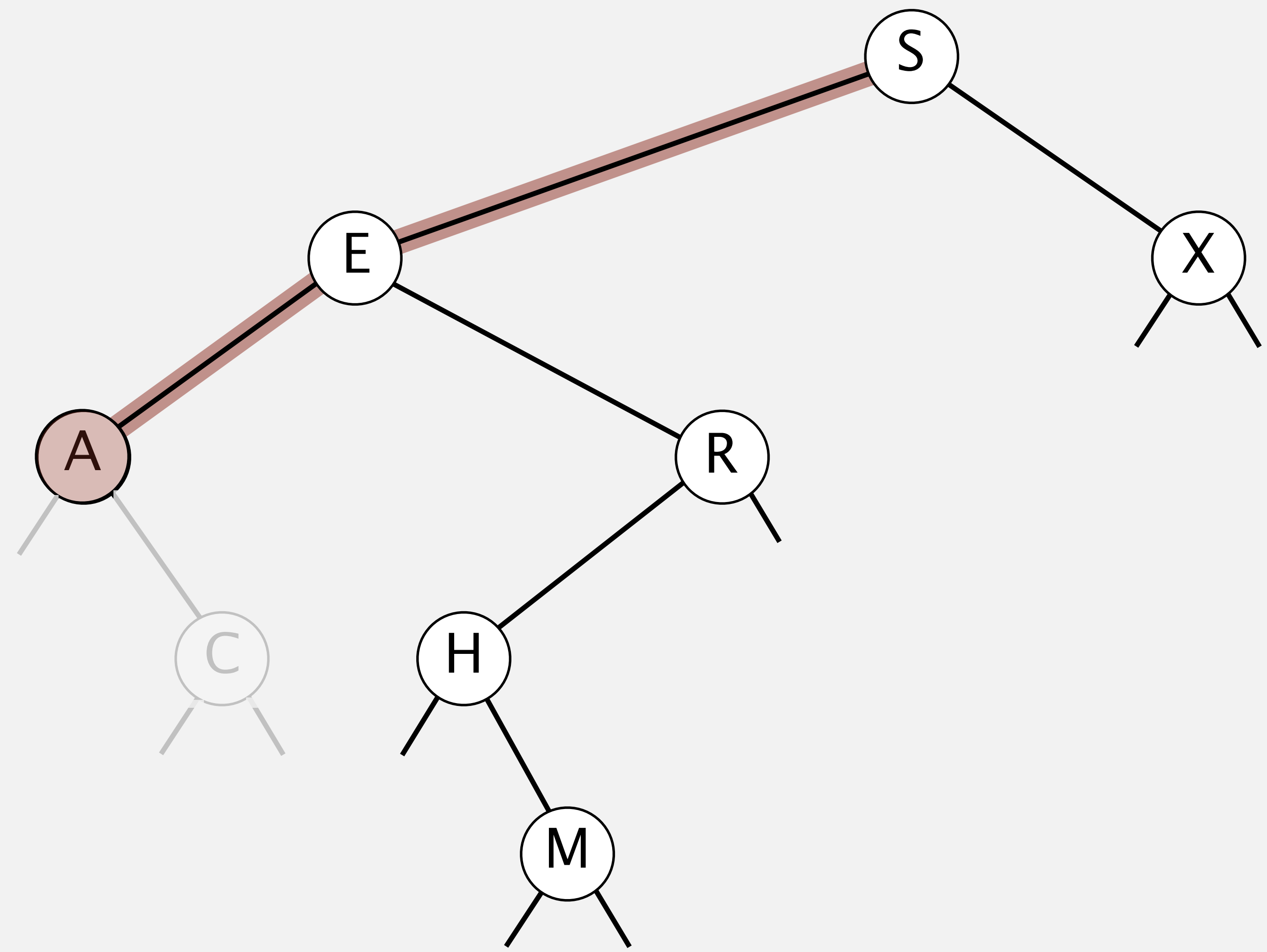
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
```



output: **A C**

Inorder traversal demo

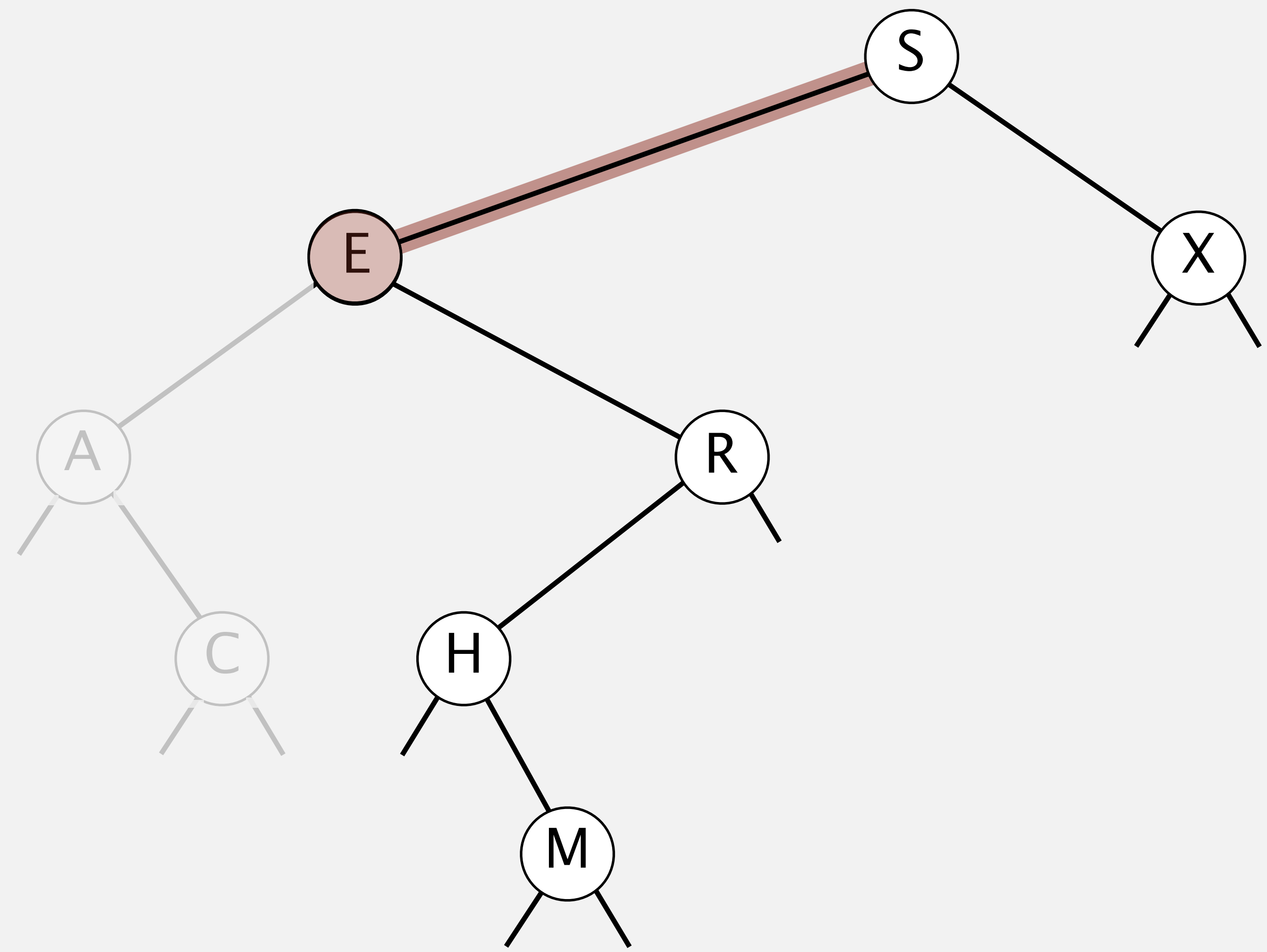
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
```



output: A C

Inorder traversal demo

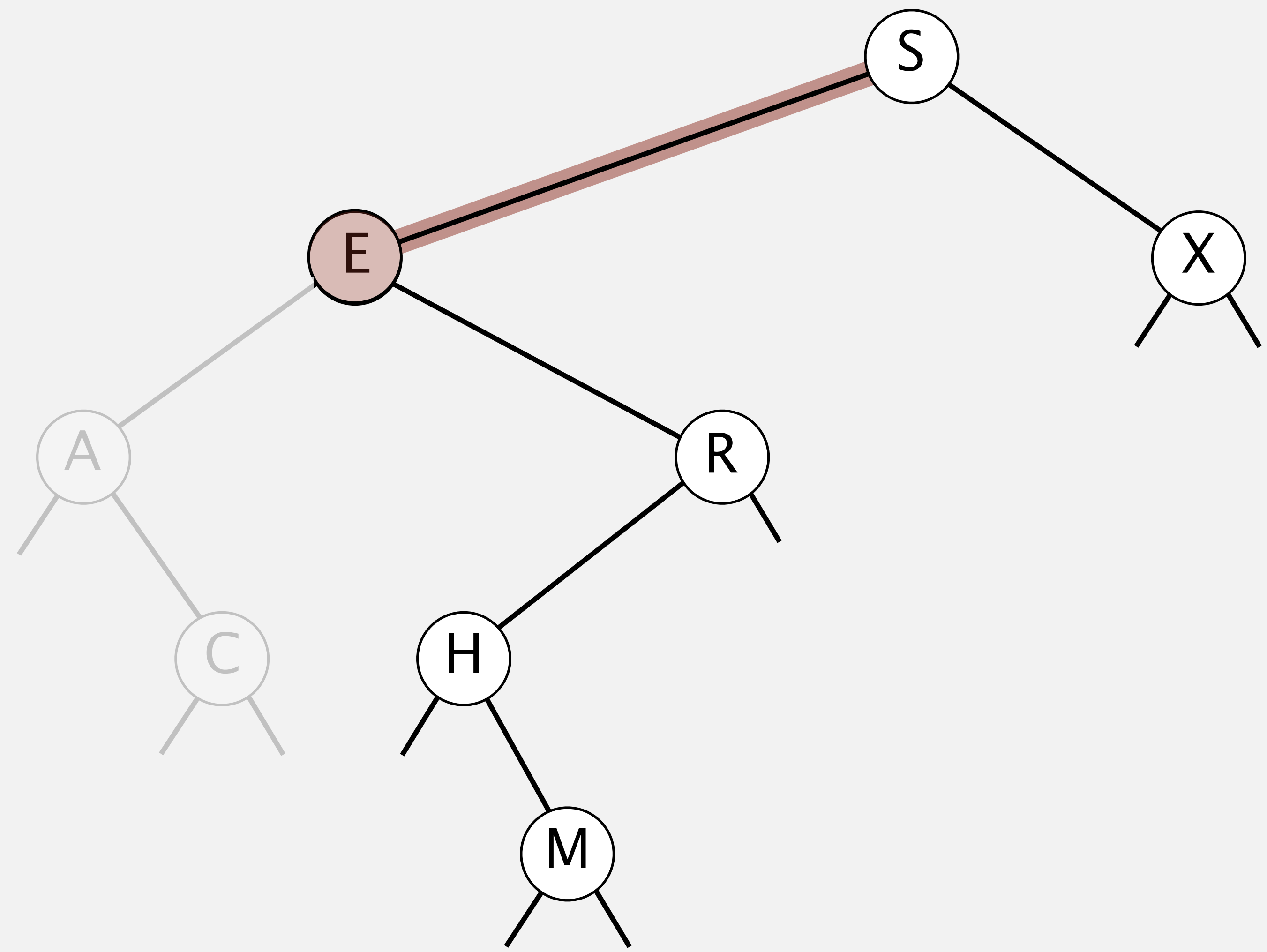
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
```



output: **A C**

Inorder traversal demo

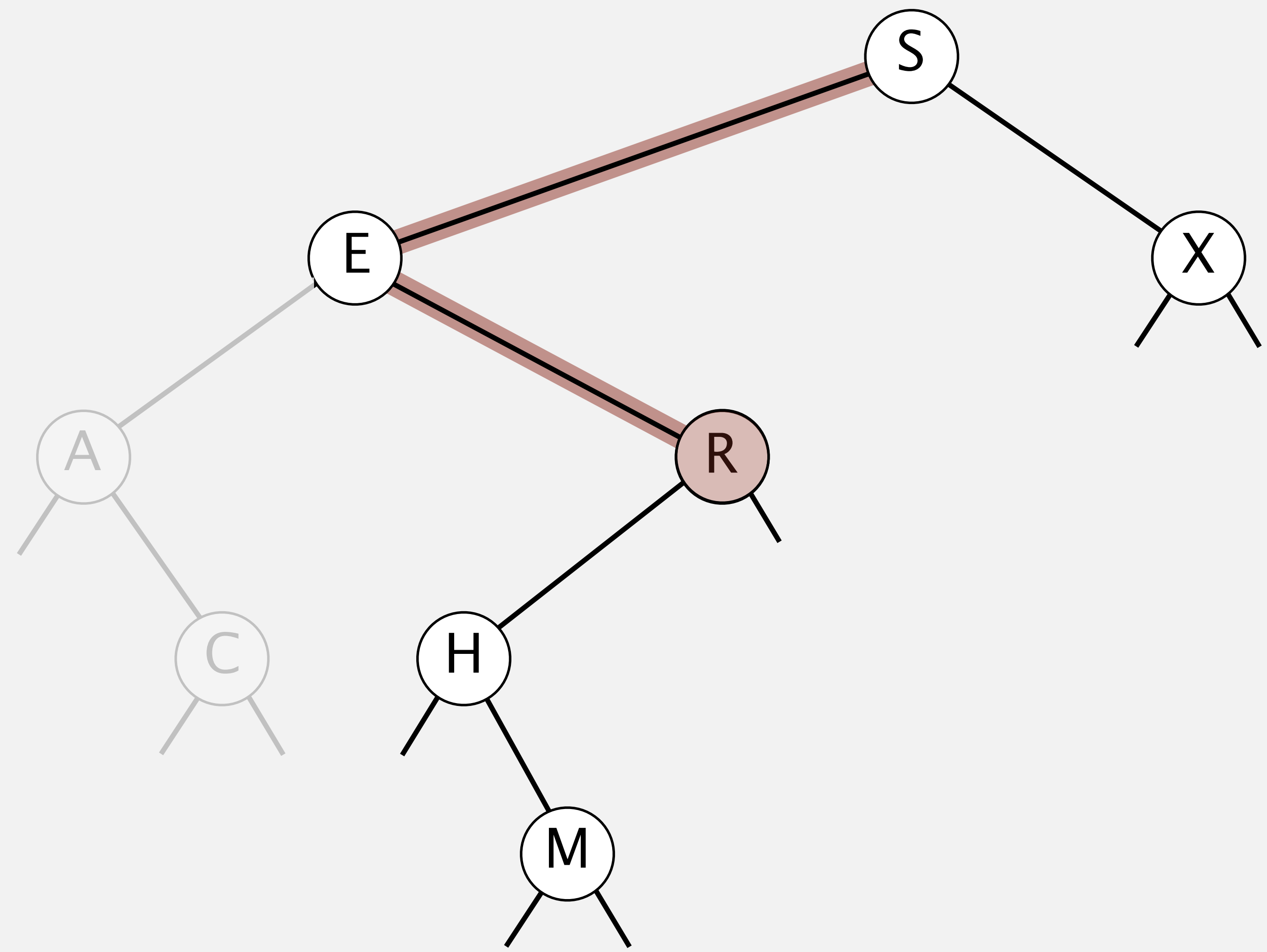
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
```



output: **A C E**

Inorder traversal demo

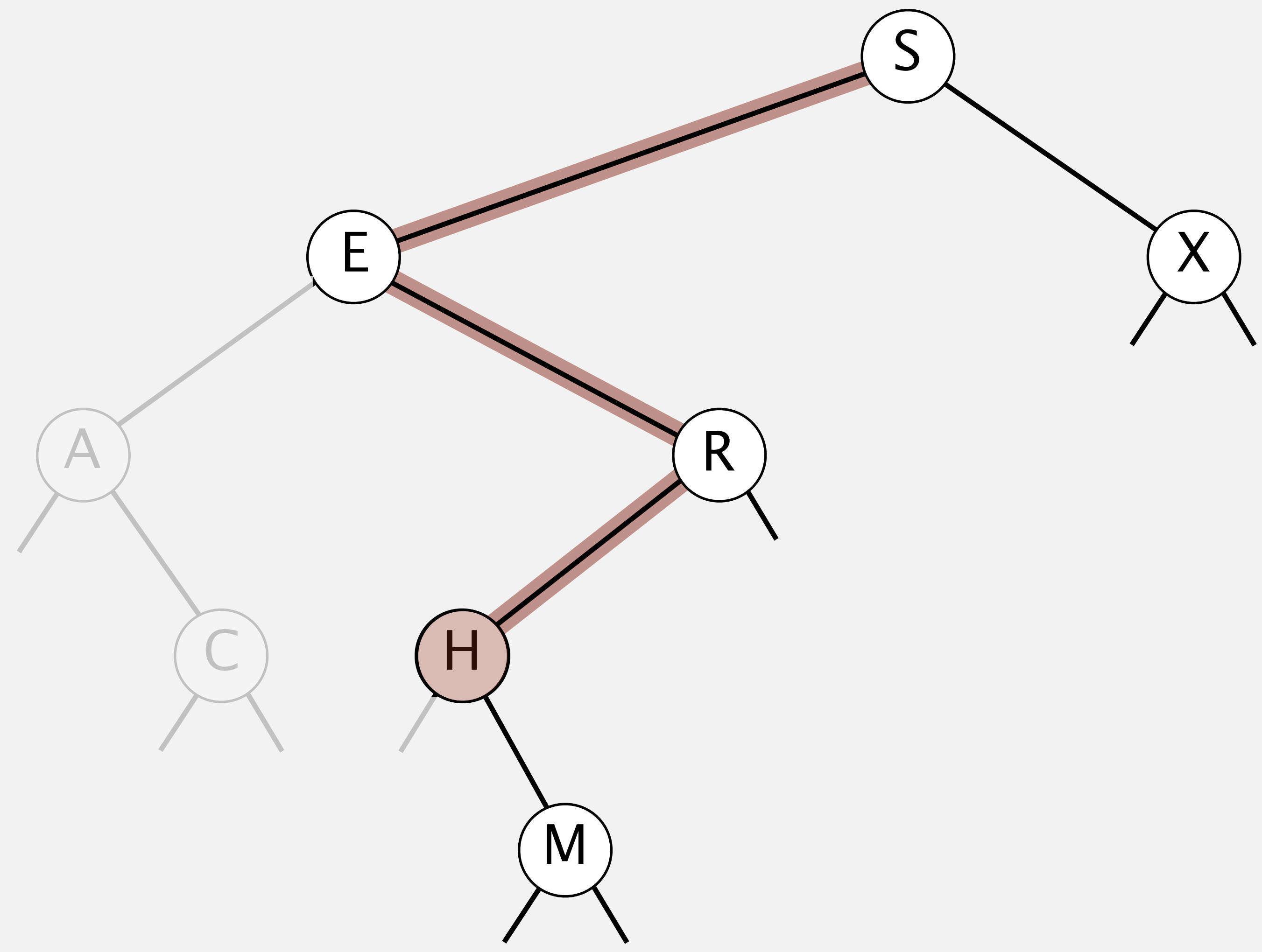
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
```



output: A C E

Inorder traversal demo

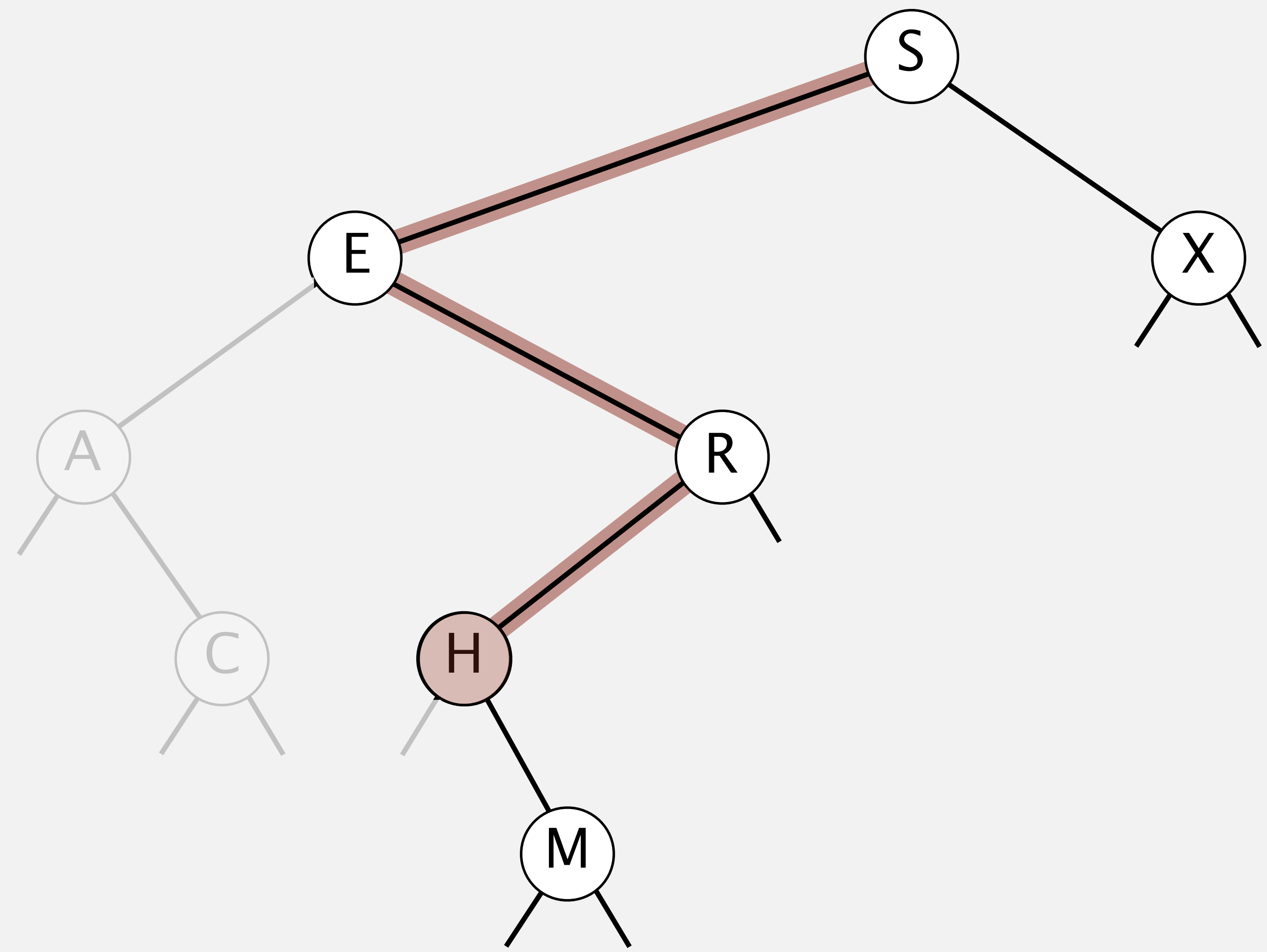
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
```



output: **A C E**

Inorder traversal demo

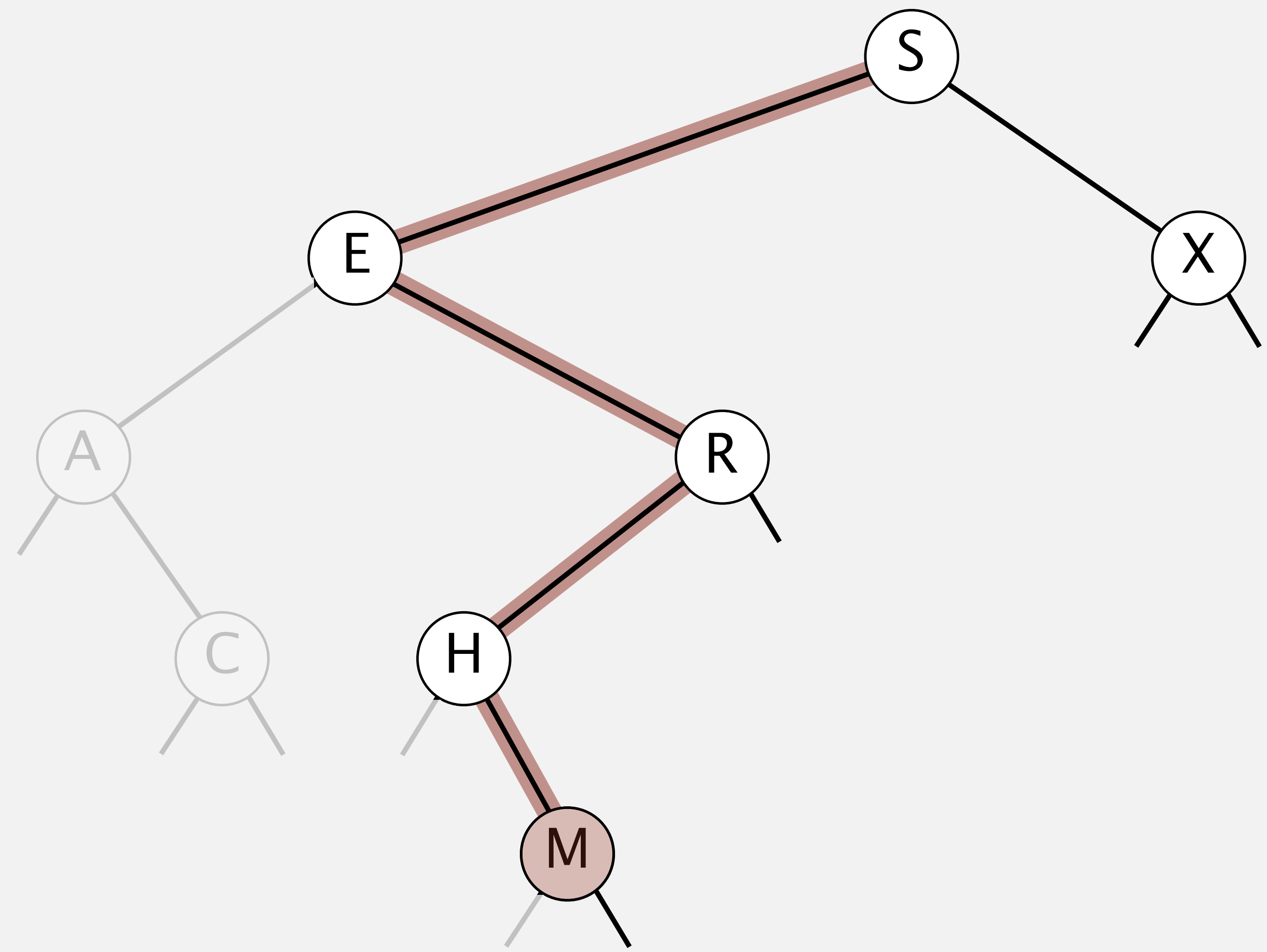
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
```



output: **A C E H**

Inorder traversal demo

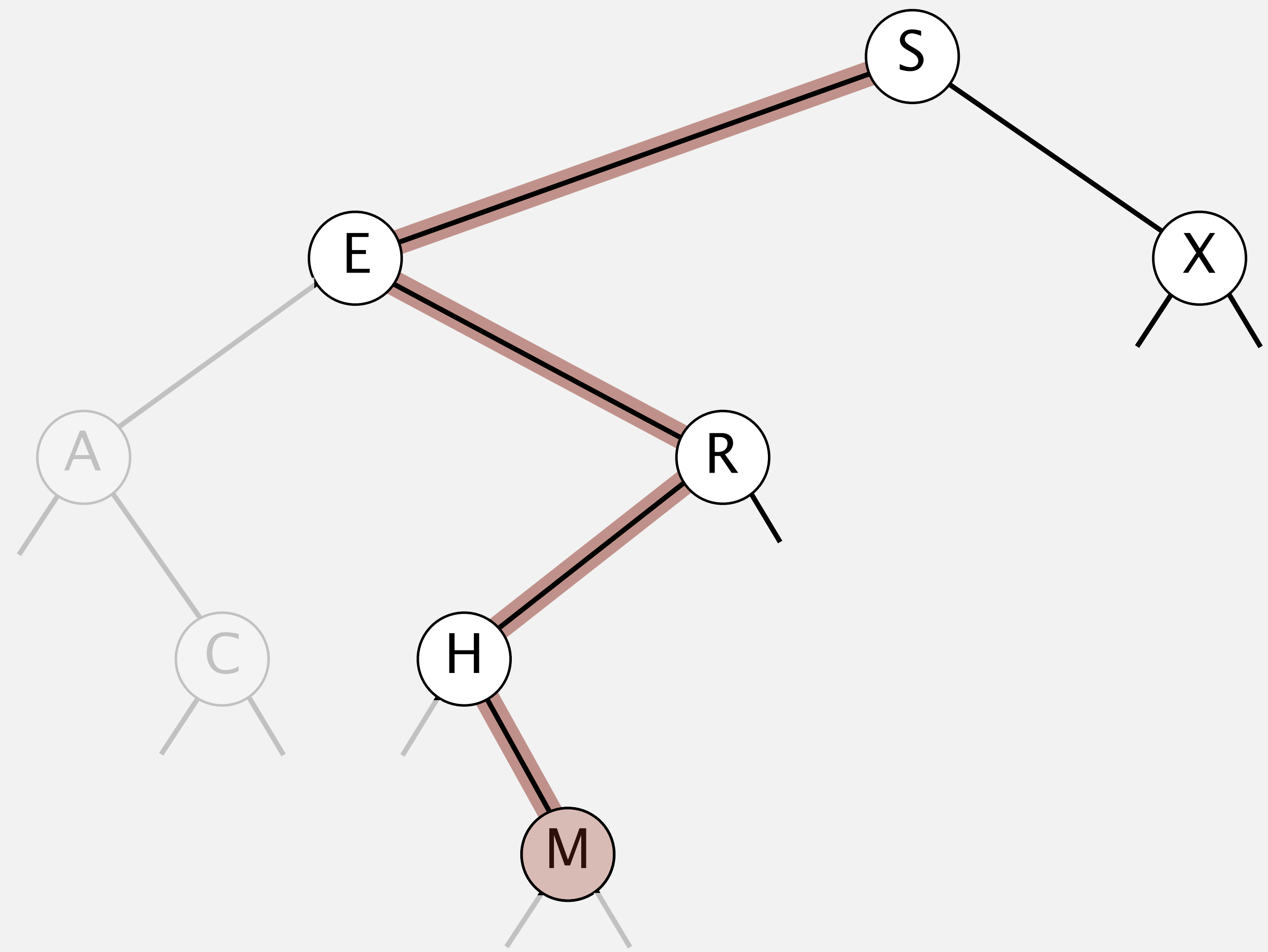
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
```



output: A C E H

Inorder traversal demo

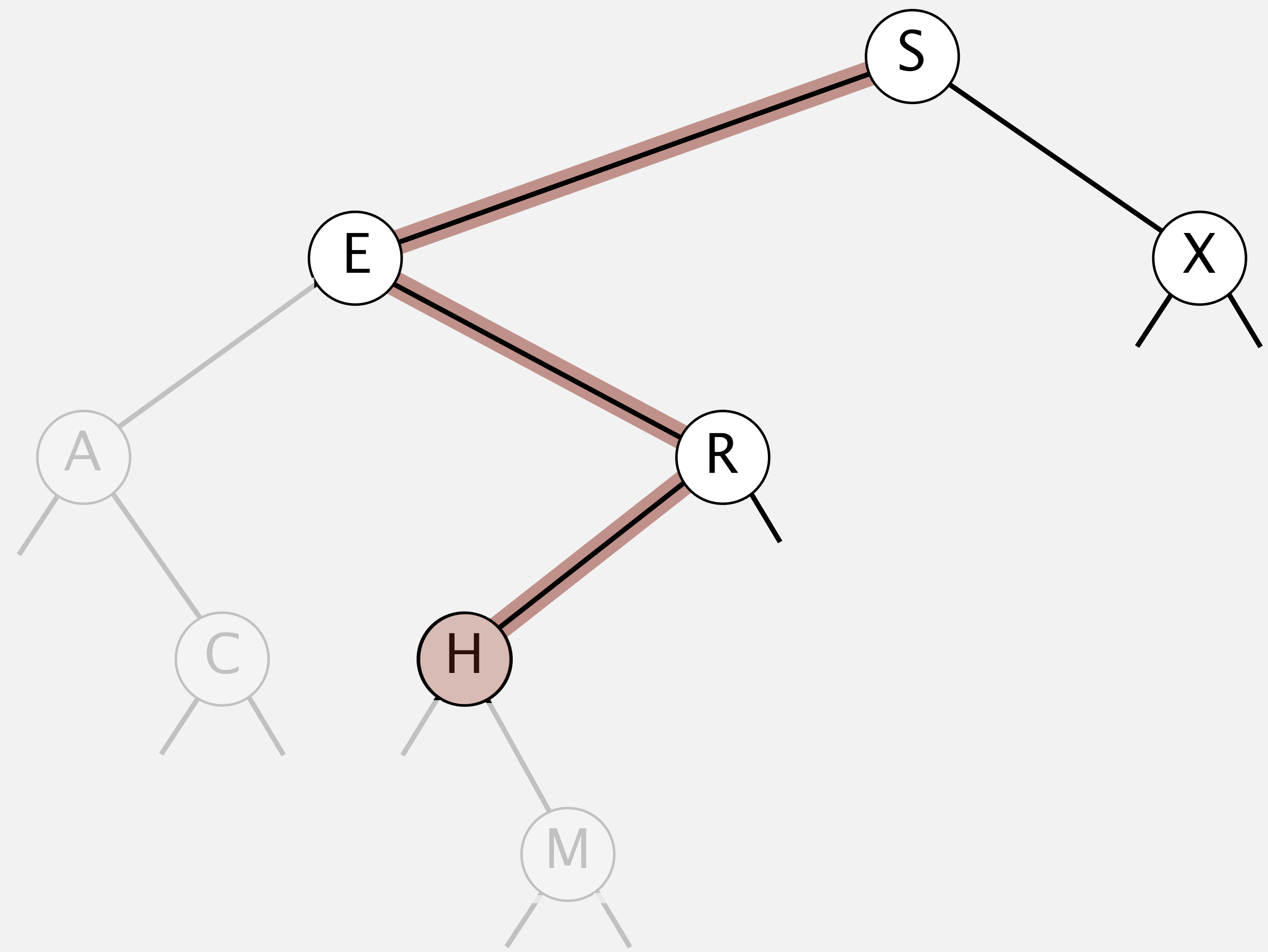
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
```



output: **A C E H M**

Inorder traversal demo

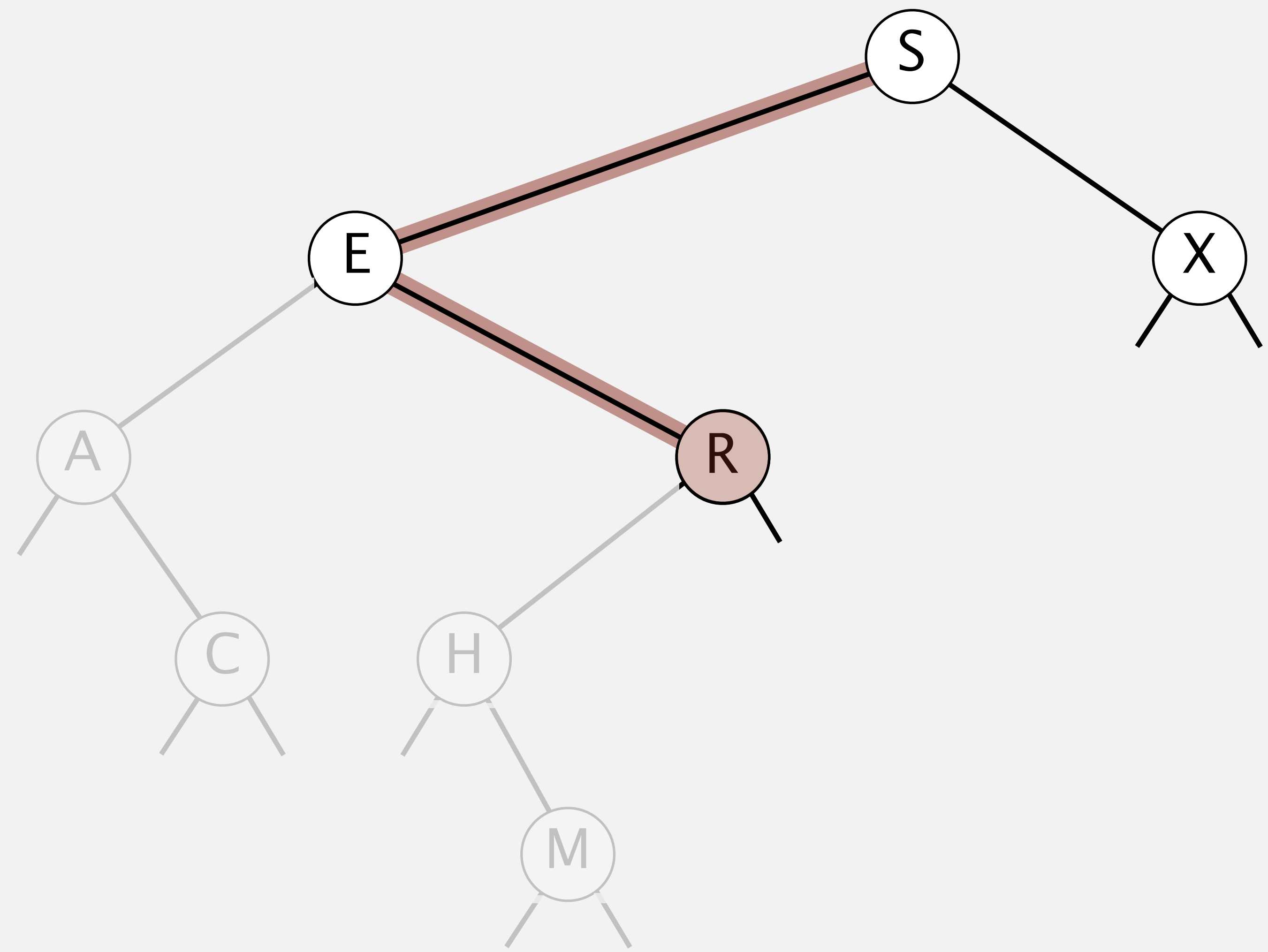
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
```



output: A C E H M

Inorder traversal demo

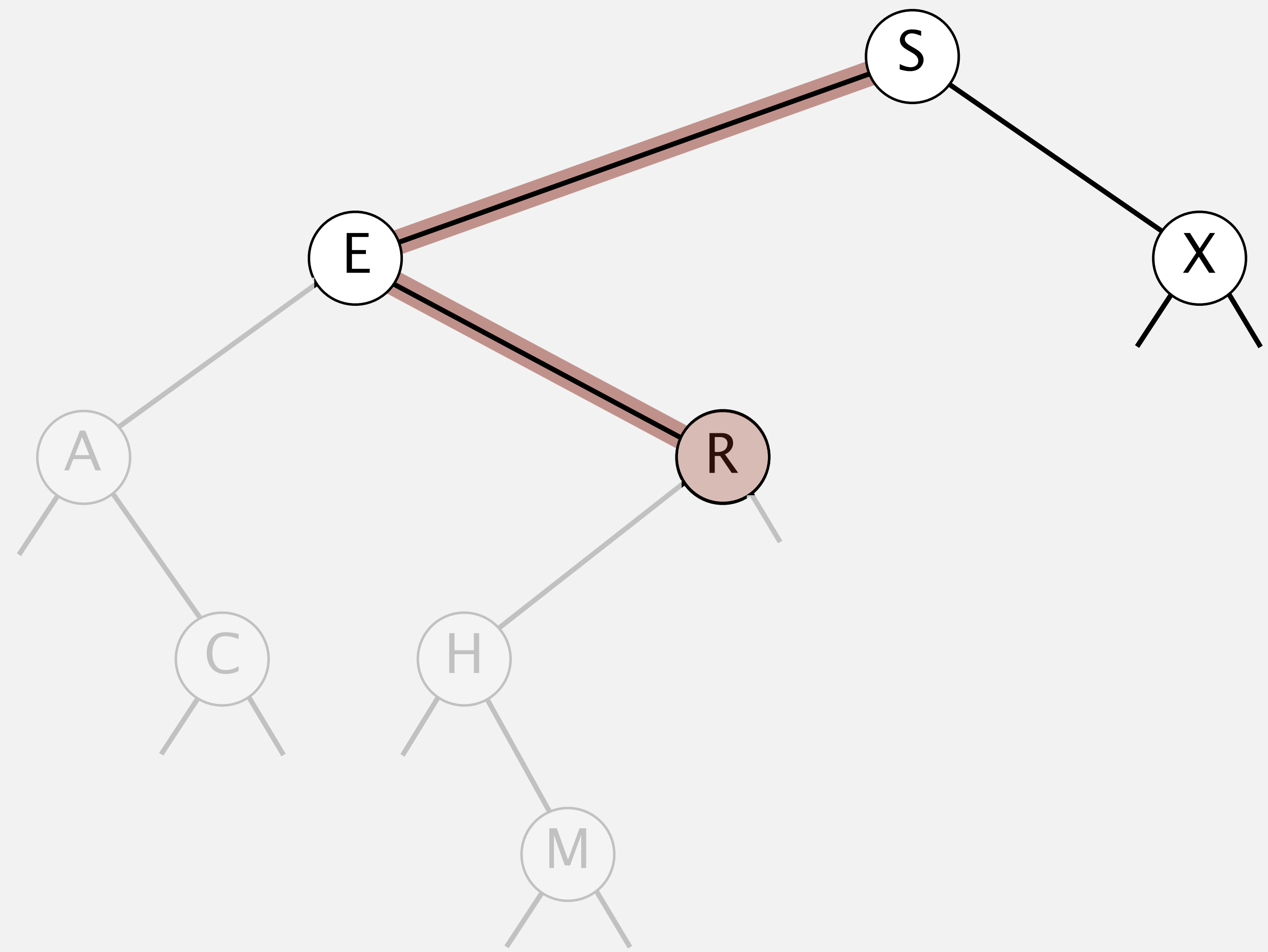
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
```



output: **A C E H M**

Inorder traversal demo

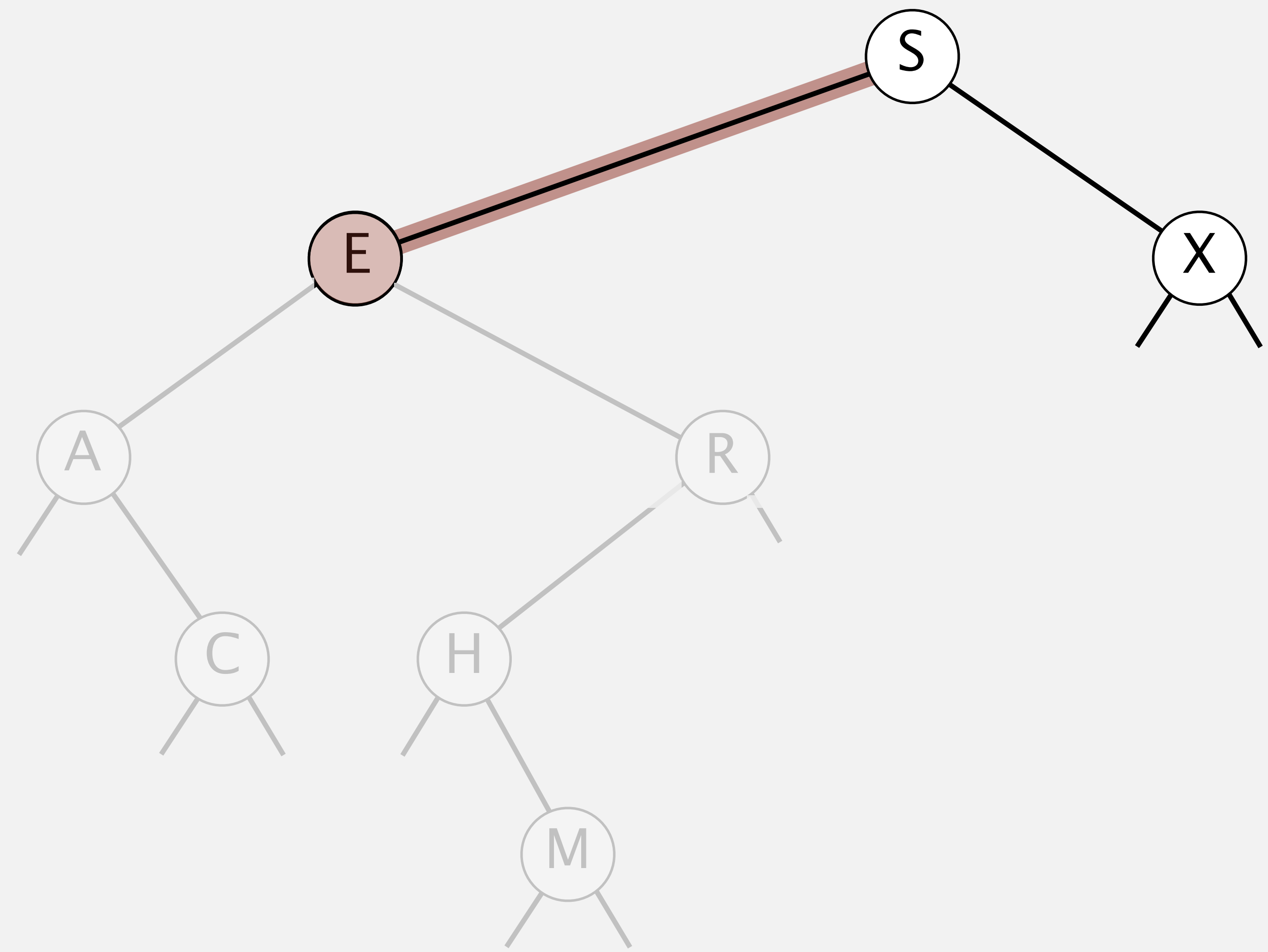
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
```



output: A C E H M R

Inorder traversal demo

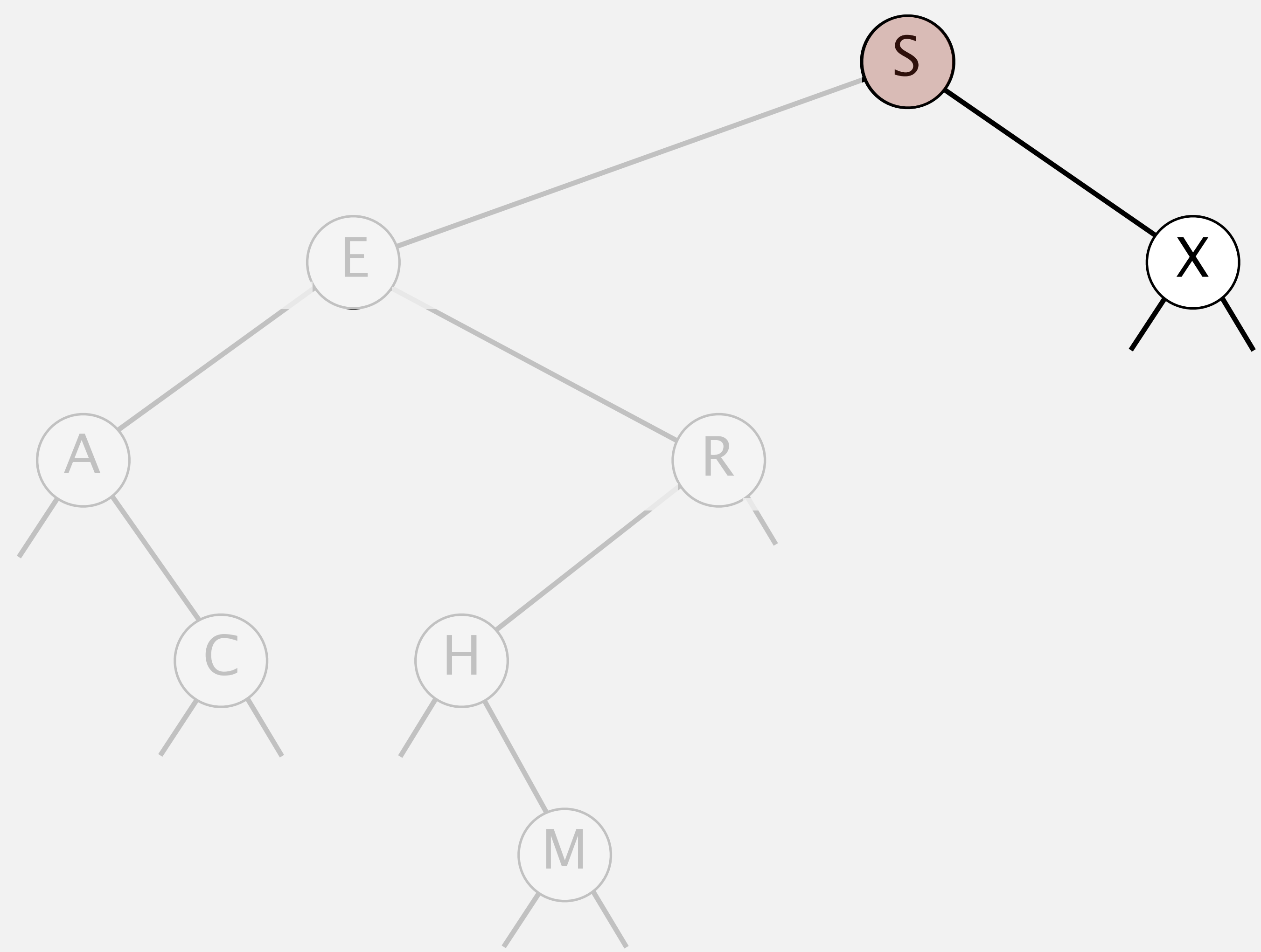
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
```



output: **A C E H M R**

Inorder traversal demo

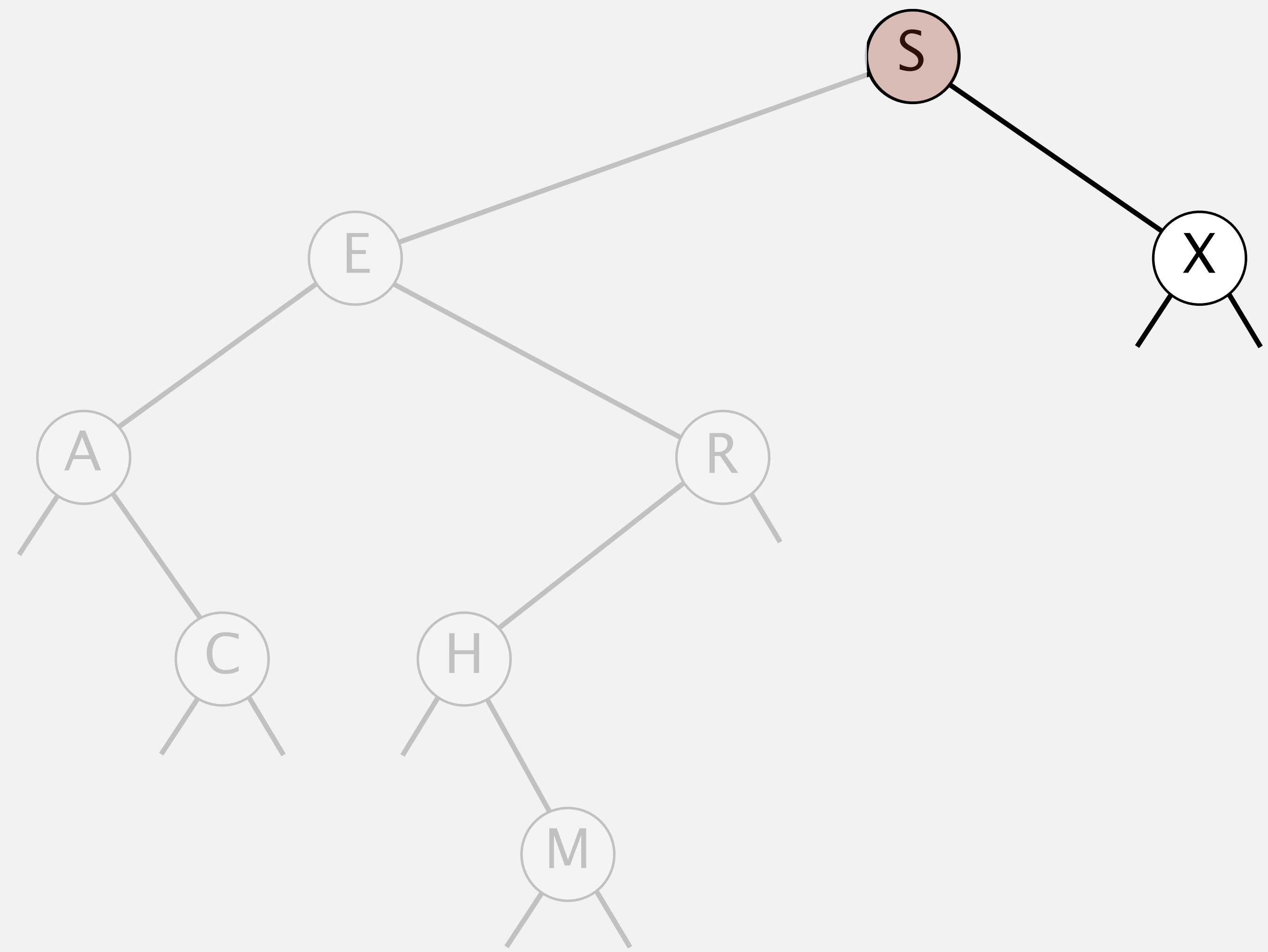
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
```



output: A C E H M R

Inorder traversal demo

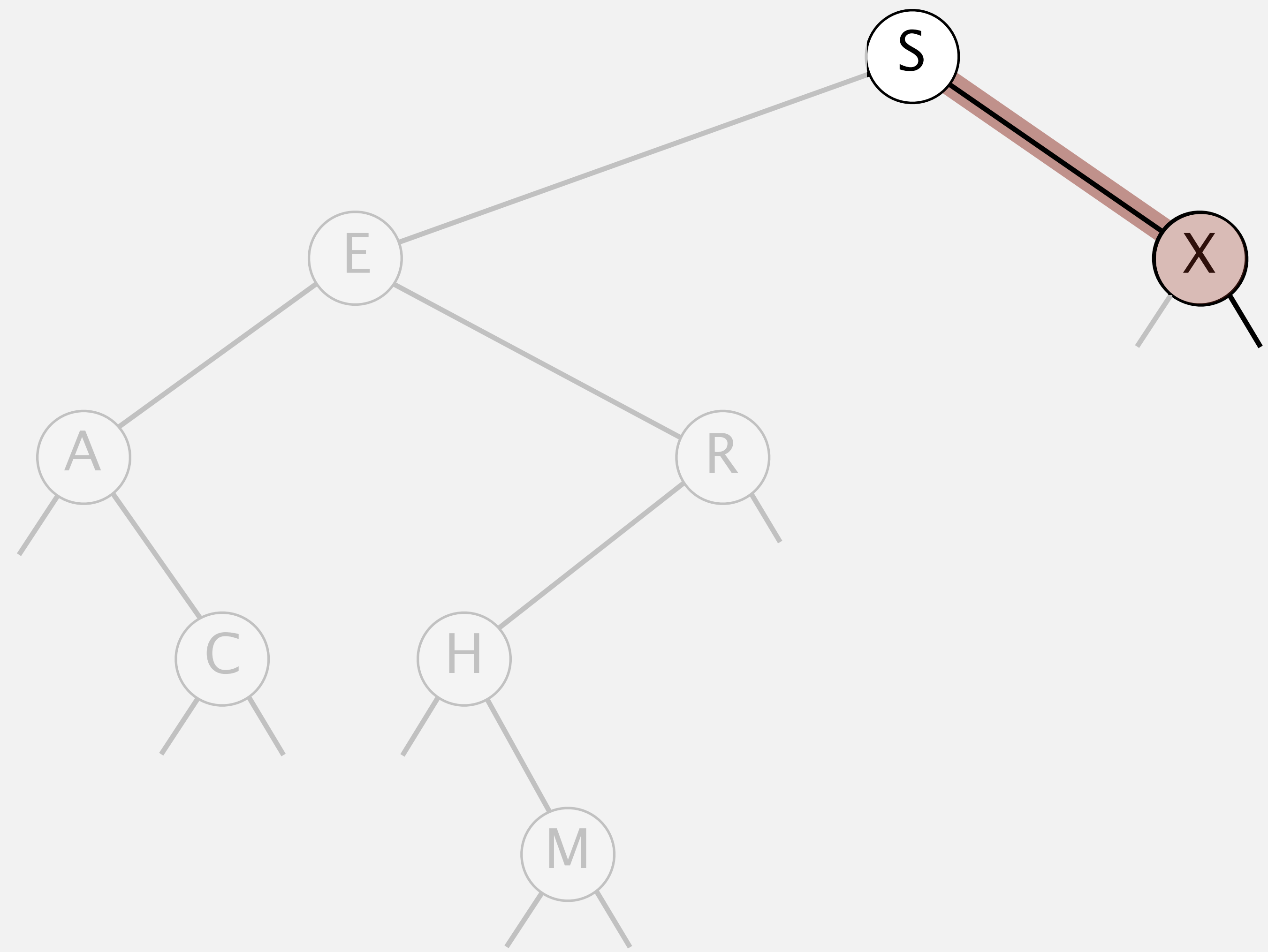
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
```



output: A C E H M R S

Inorder traversal demo

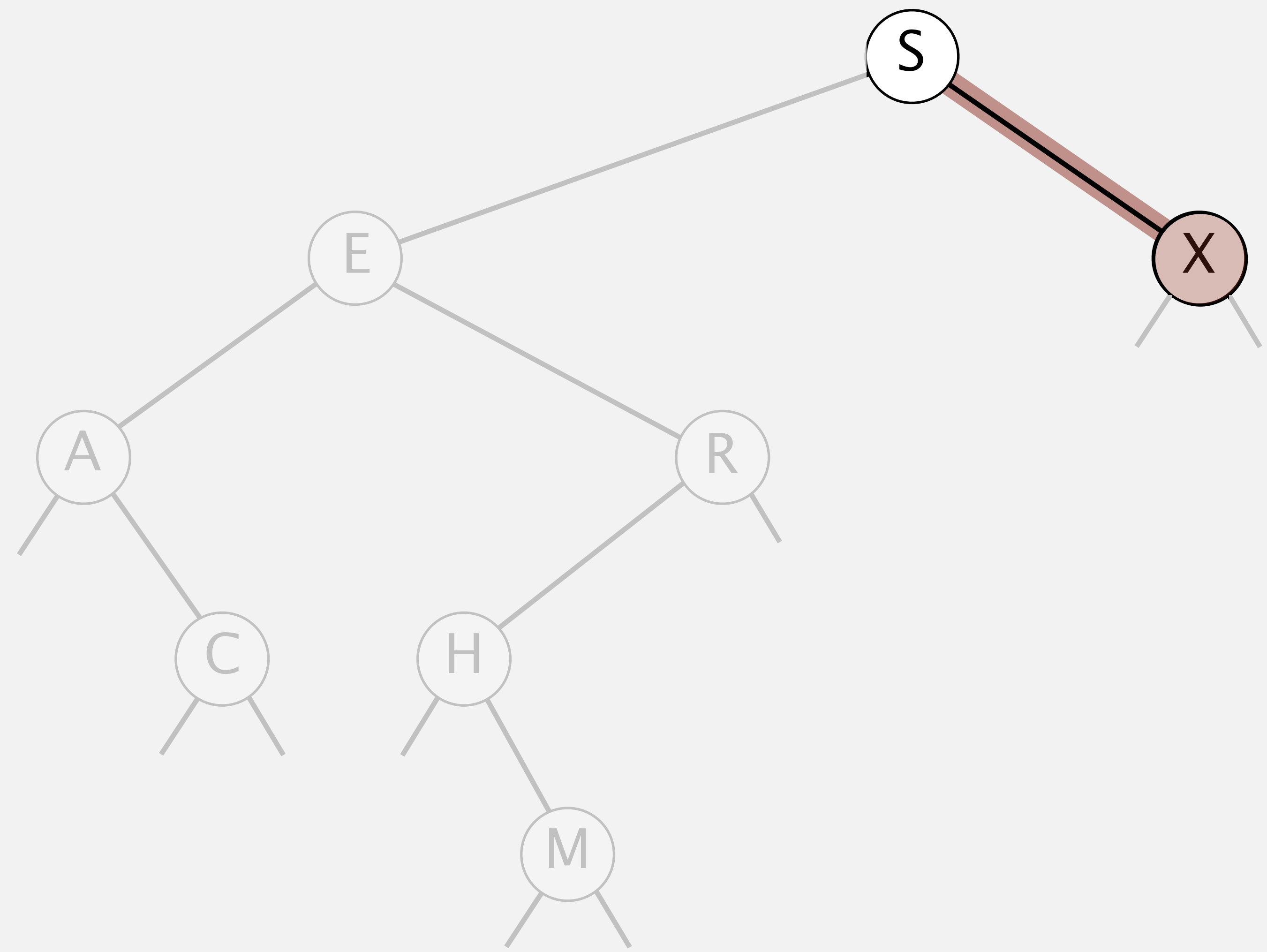
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
```



output: A C E H M R S

Inorder traversal demo

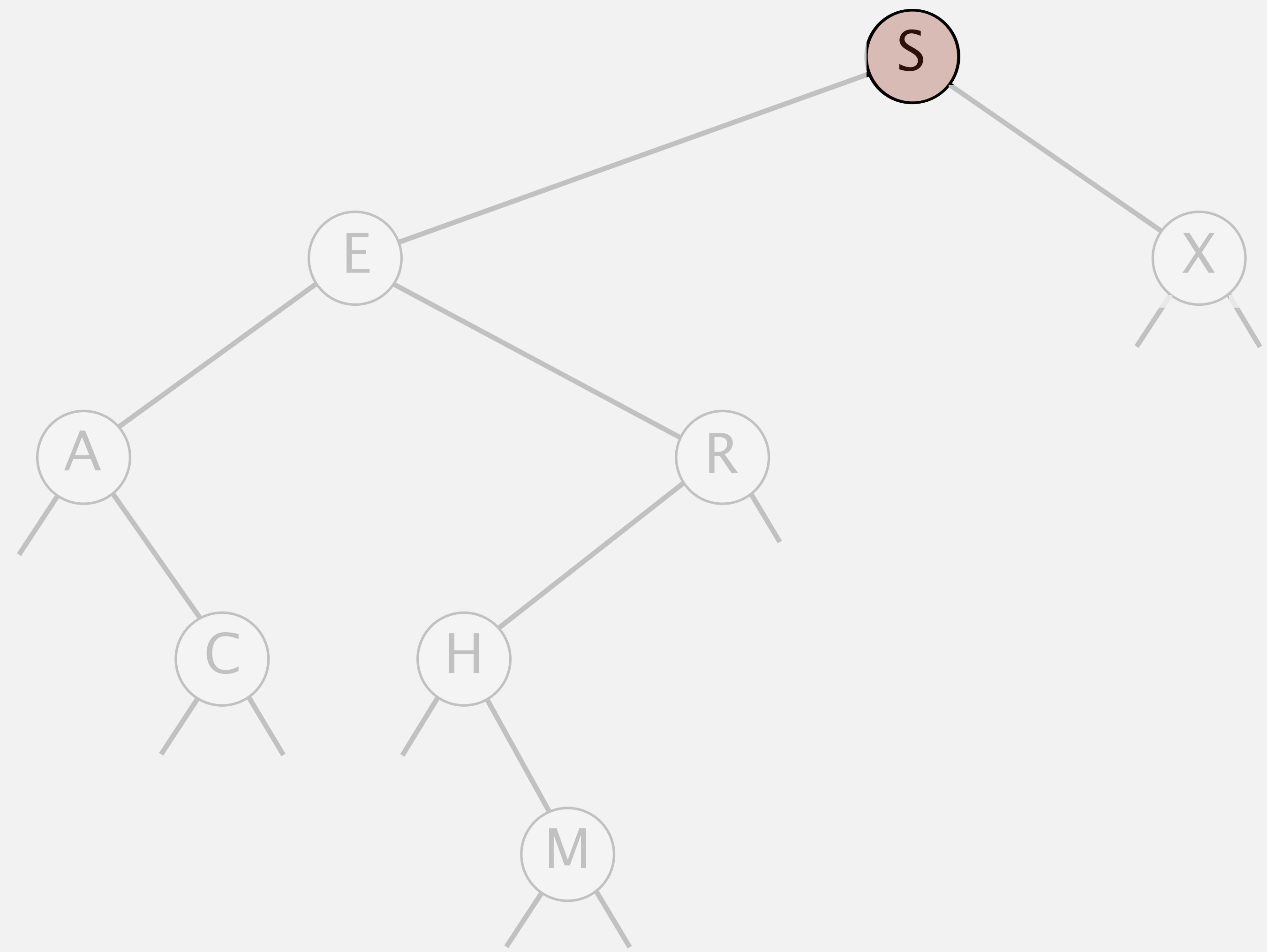
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
      print X
```



output: **A C E H M R S X**

Inorder traversal demo

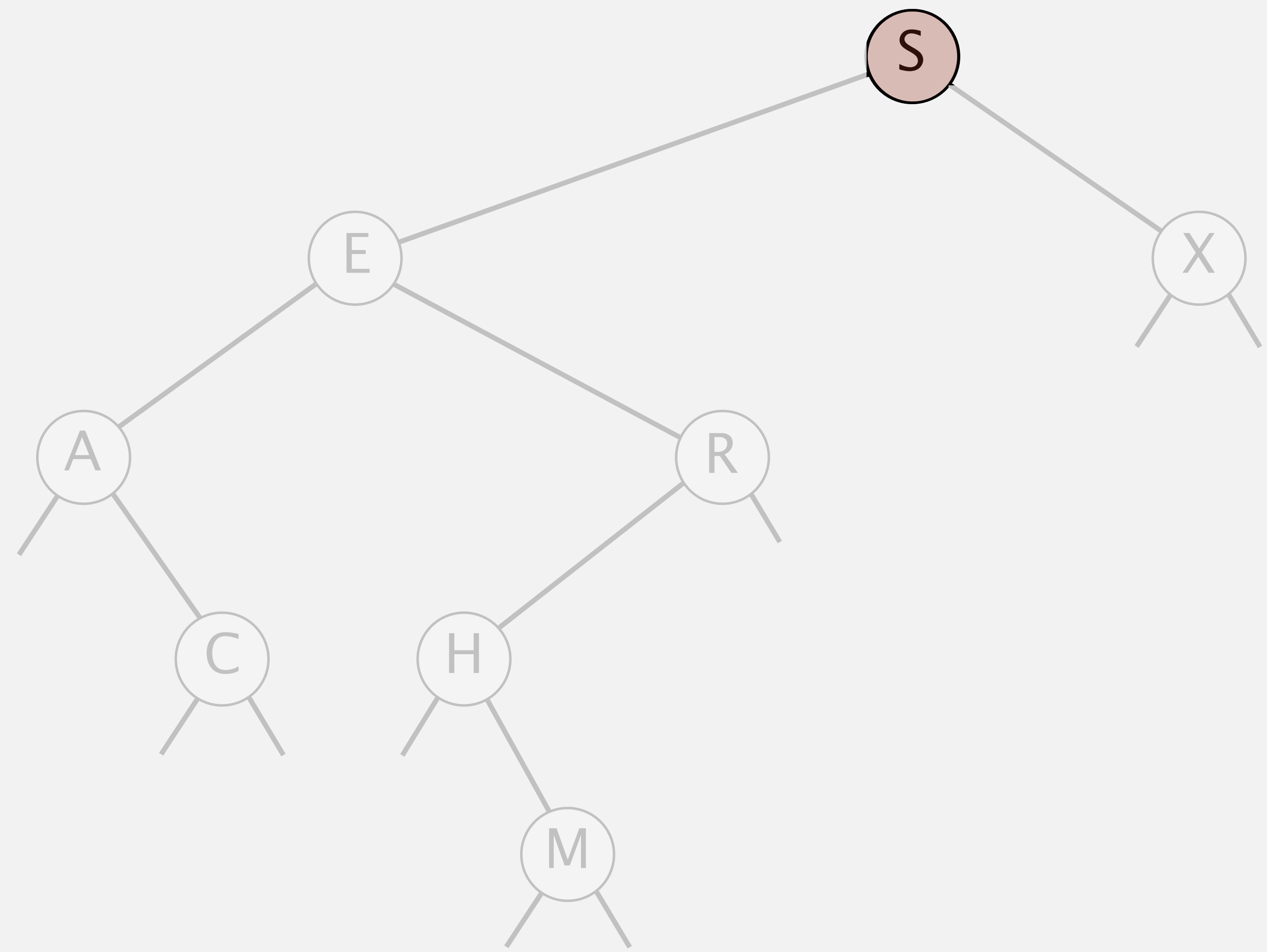
```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
      print X
      done X
```



output: A C E H M R S X

Inorder traversal demo

```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
      print X
      done X
   done S
```

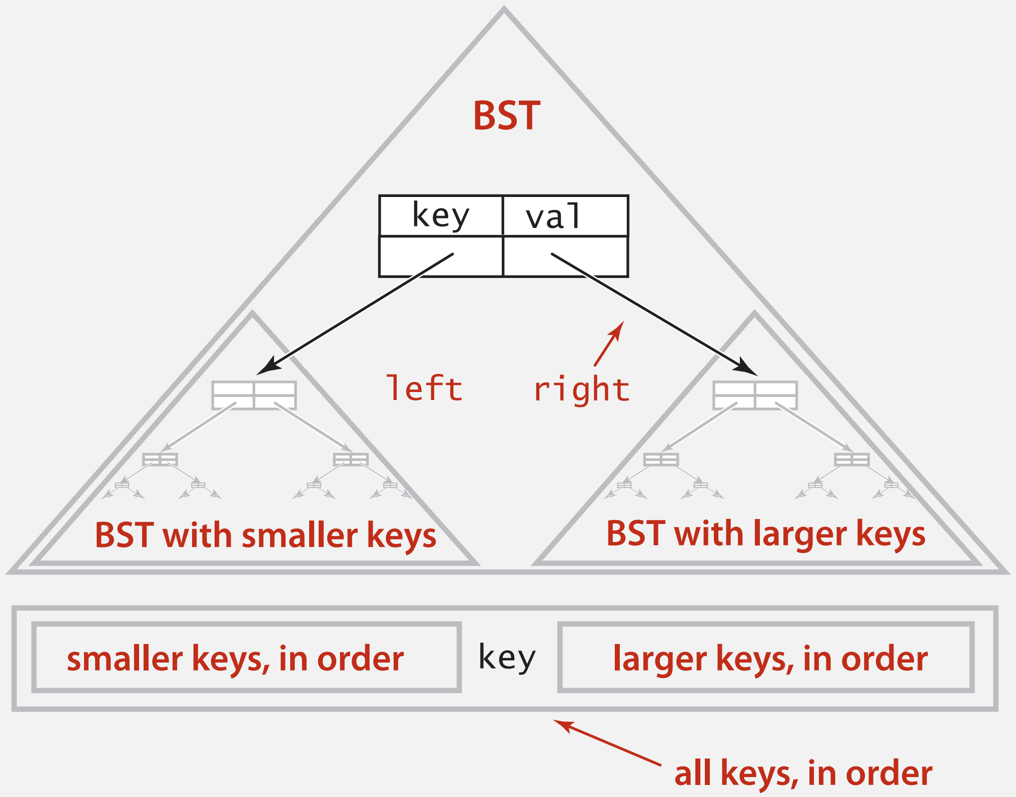


output: A C E H M R S X

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

http://algs4.cs.princeton.edu

3.2 Binary Search Trees

- *BSTs*
- *iteration*
- *ordered operations*
- *deletion*

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	$\log N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\log N$	h
rank	N	$\log N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

ST implementations: summary

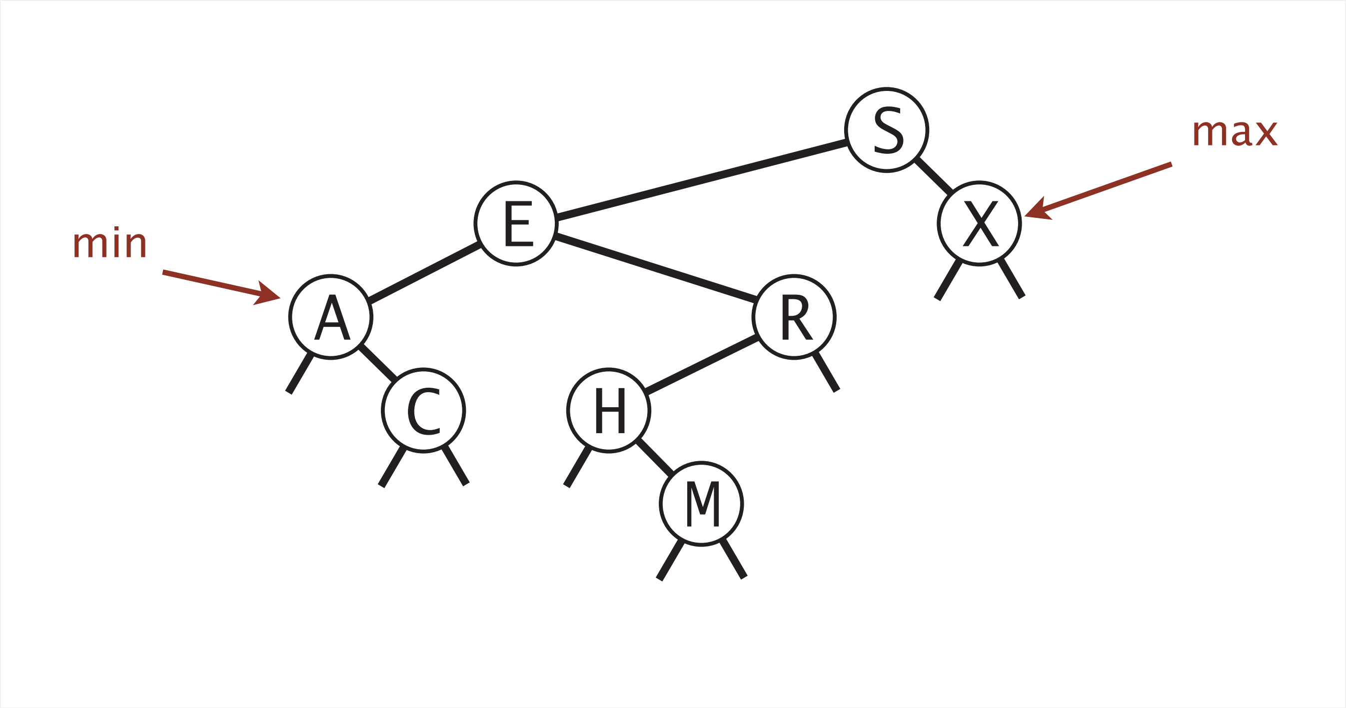
implementation	guarantee		average case		ordered ops?	key interface
	search	insert	search hit	insert		
sequential search (unordered list)	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	$\log N$	N	✔	`compareTo()`
BST	N	N	$\log N$	$\log N$	✔	`compareTo()`
red-black BST	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`

Next lecture. Guarantee logarithmic performance for all operations.

Minimum and maximum

Minimum. Smallest key in BST.

Maximum. Largest key in BST.

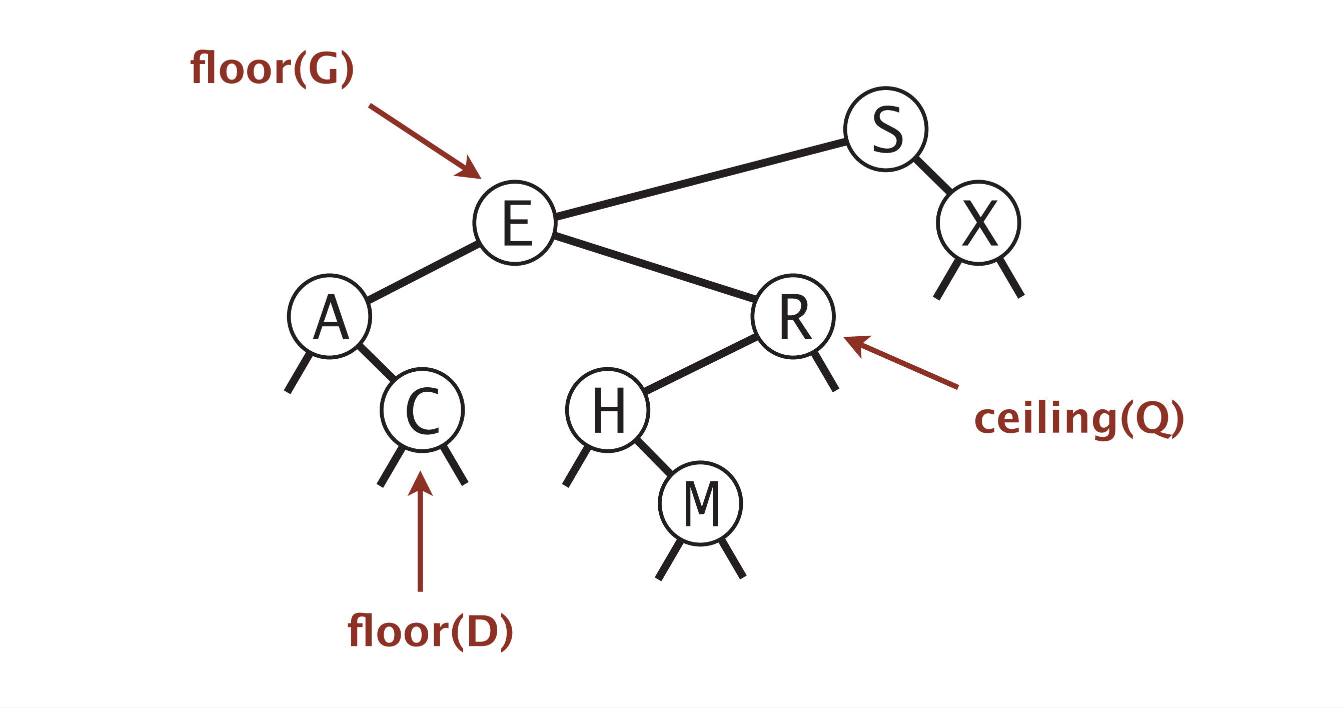


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key in BST $\leq$ query key.

Ceiling. Smallest key in BST $\geq$ query key.

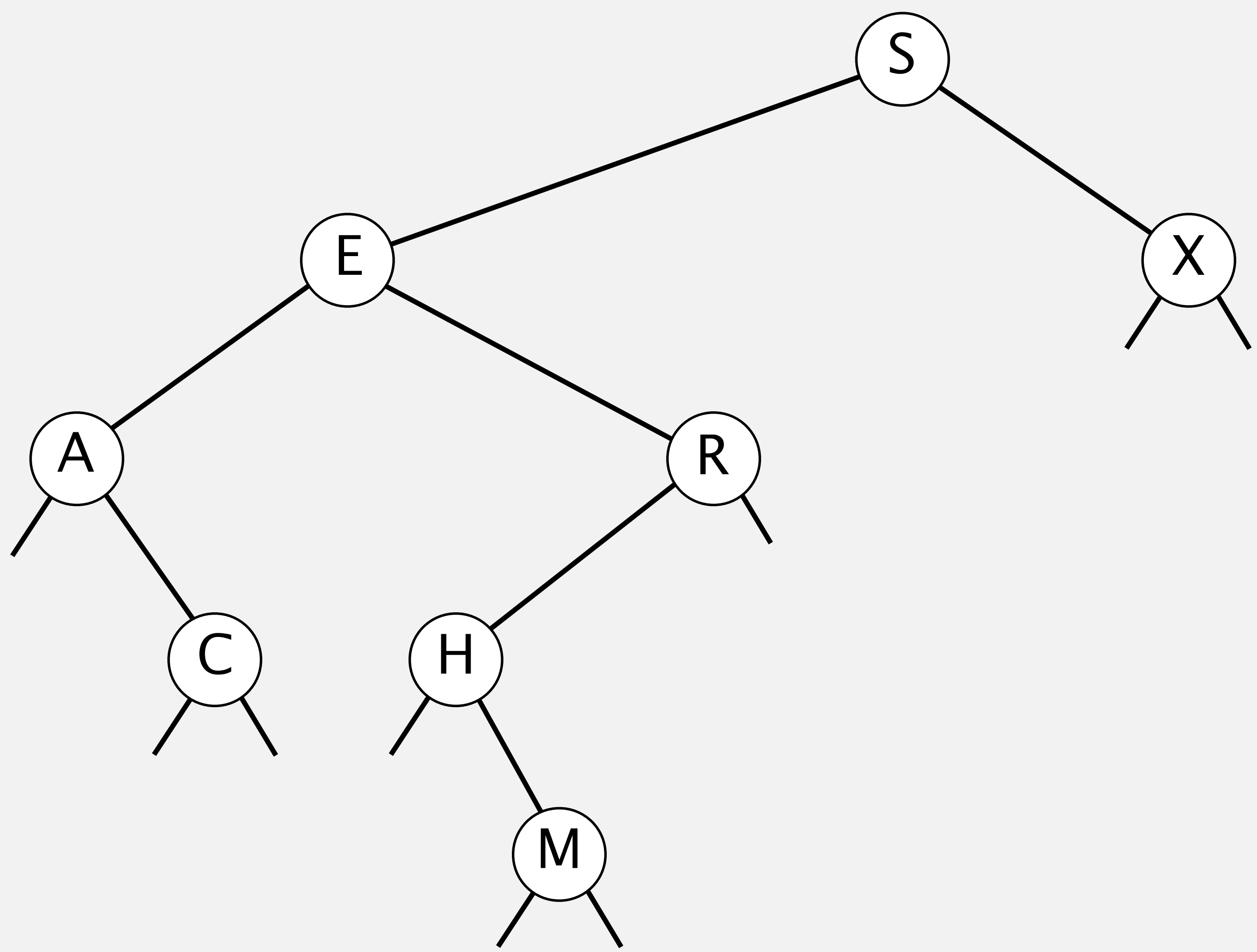


Q. How to find the floor / ceiling?

Floor in a BST demo

Floor. Find the largest key in a BST that is $\leq k$?

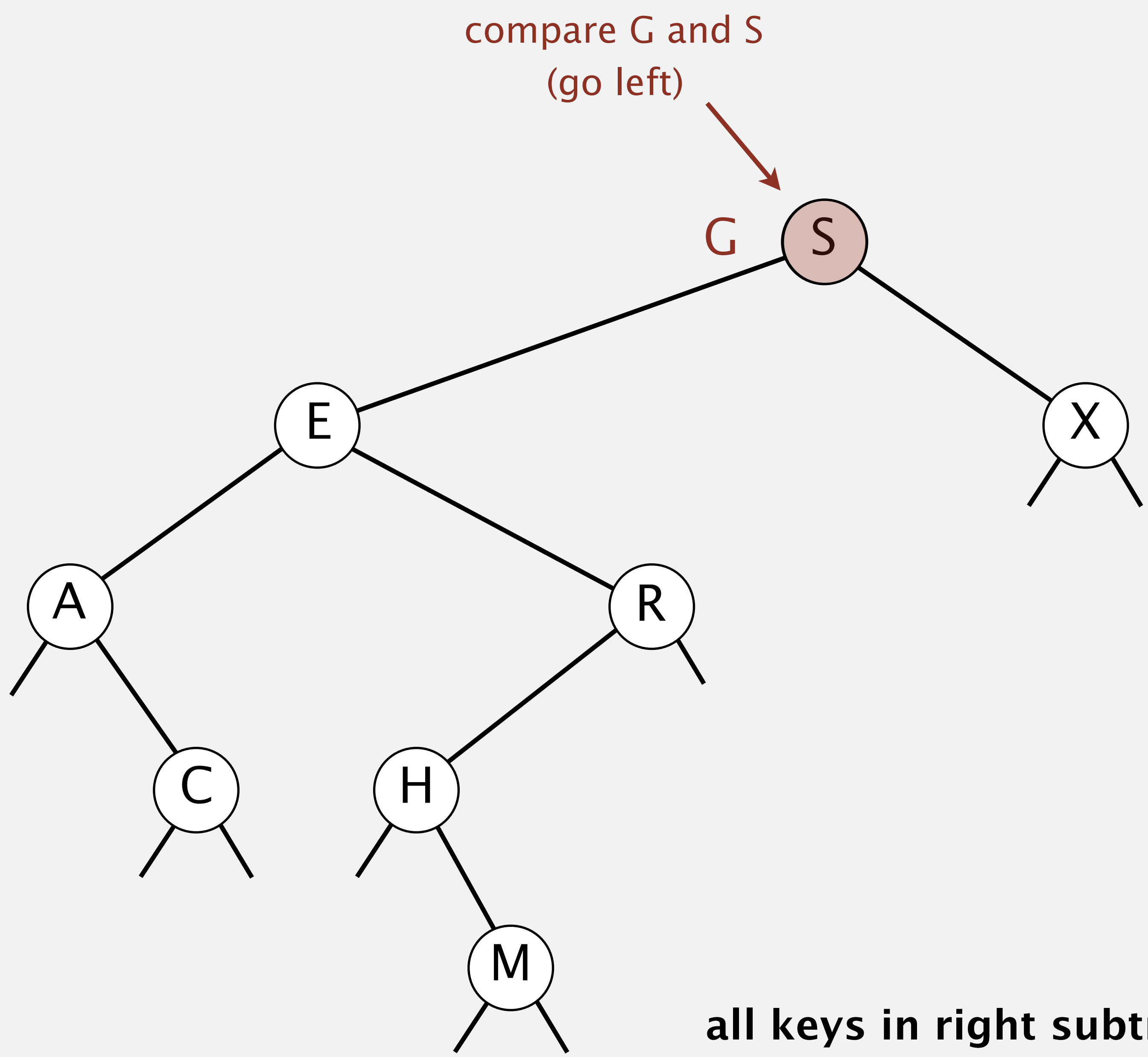
floor of G



Floor in a BST demo

Floor. Find the largest key in a BST that is ≤ k?

floor of G



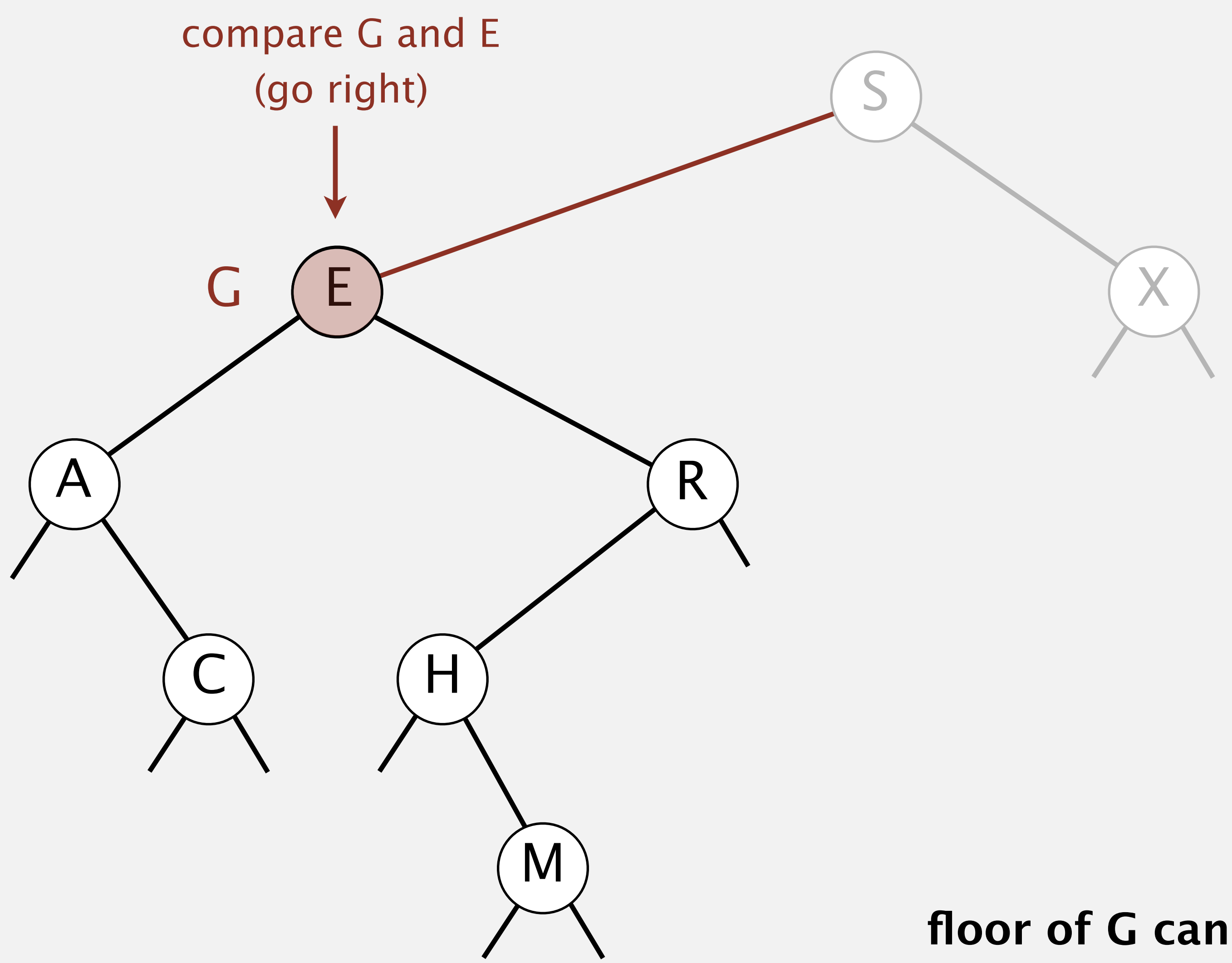
all keys in right subtree of S are greater than G
⇒ compute floor of G in left subtree

Floor in a BST demo

Floor. Find the largest key in a BST that is $\leq k$?

floor of G

E



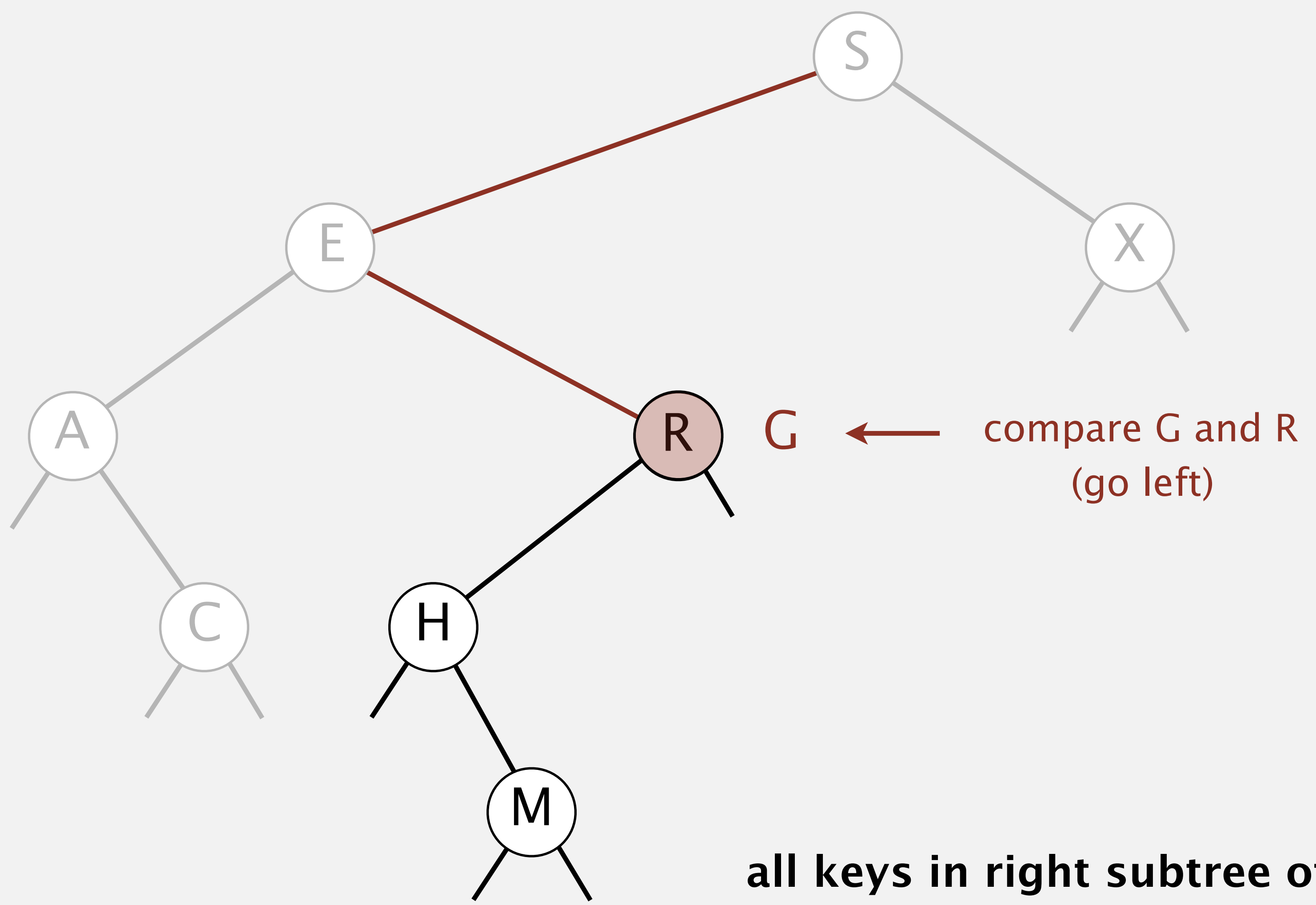
floor of G can't be in left subtree;
floor is either E or floor of G in right subtree

Floor in a BST demo

Floor. Find the largest key in a BST that is $\leq k$?

floor of G

E



all keys in right subtree of R are greater than G

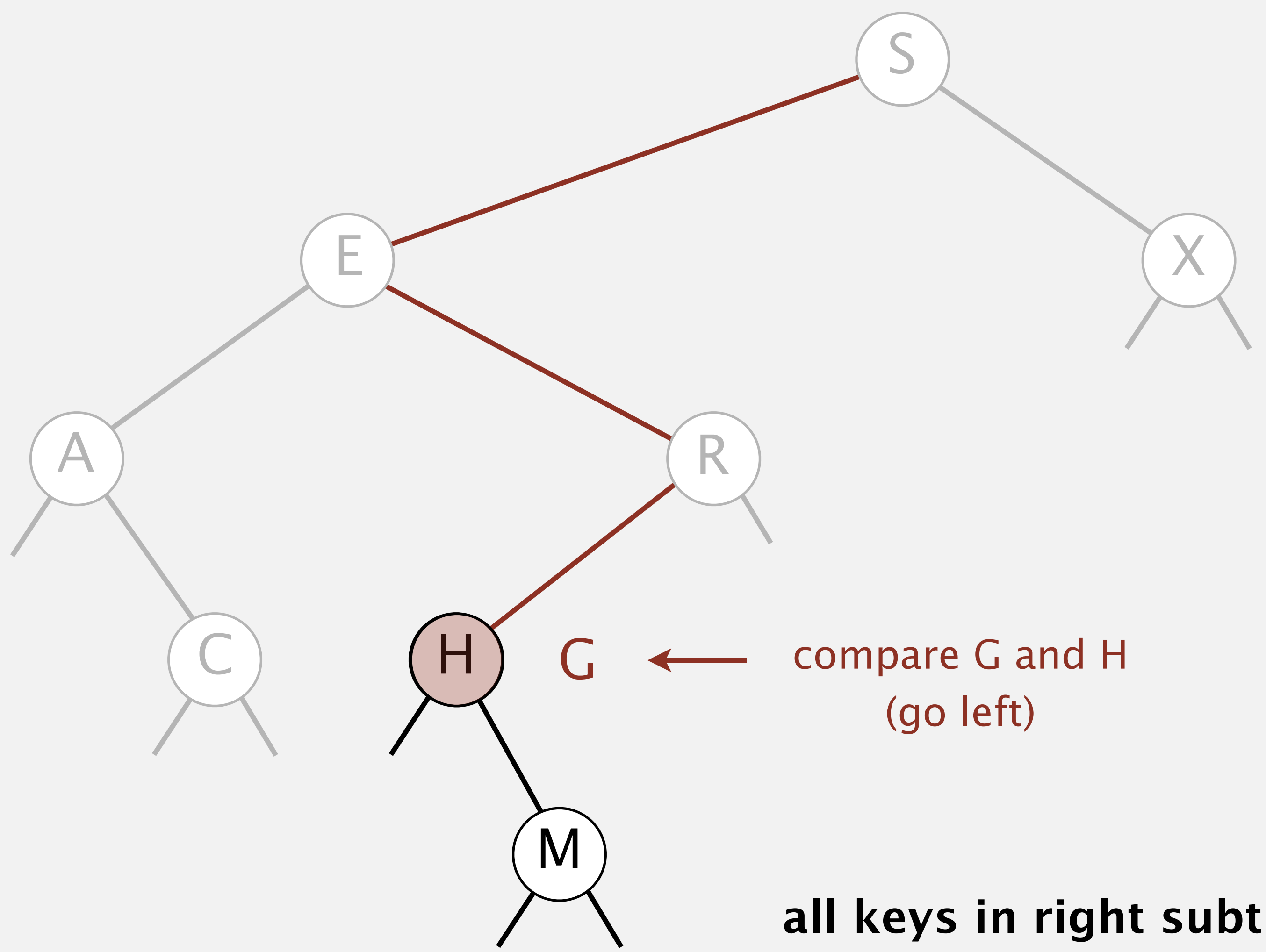
⇒ compute floor of G in left subtree

Floor in a BST demo

Floor. Find the largest key in a BST that is $\leq k$?

floor of G

E



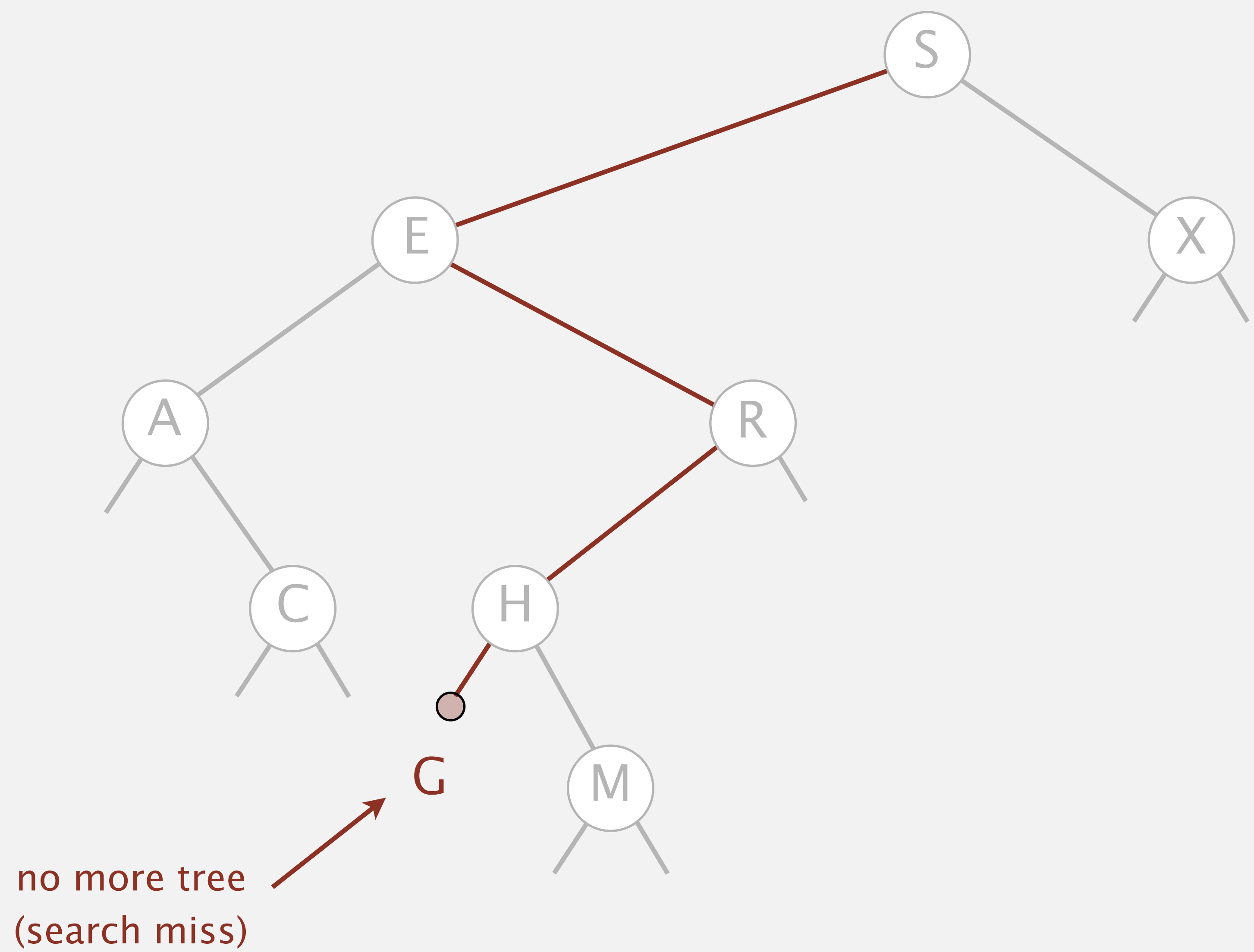
all keys in right subtree of H are greater than G
⇒ compute floor of G in left subtree

Floor in a BST demo

Floor. Find the largest key in a BST that is $\leq k$?

floor of G

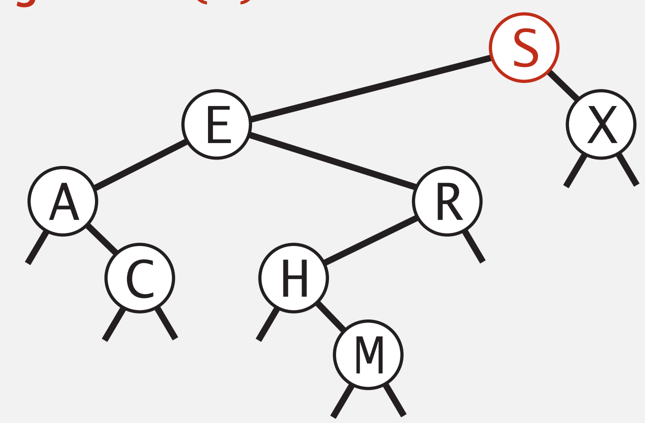
E



Computing the floor

Floor. Largest key in BST $\leq k$?

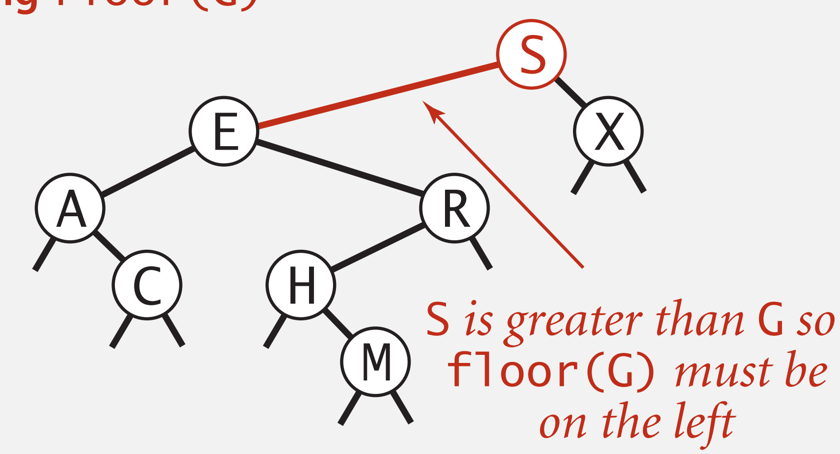
finding floor(S)



Case 1. [key in node $x = k$]

The floor of k is k.

finding floor(G)



Case 2. [key in node $x > k$]

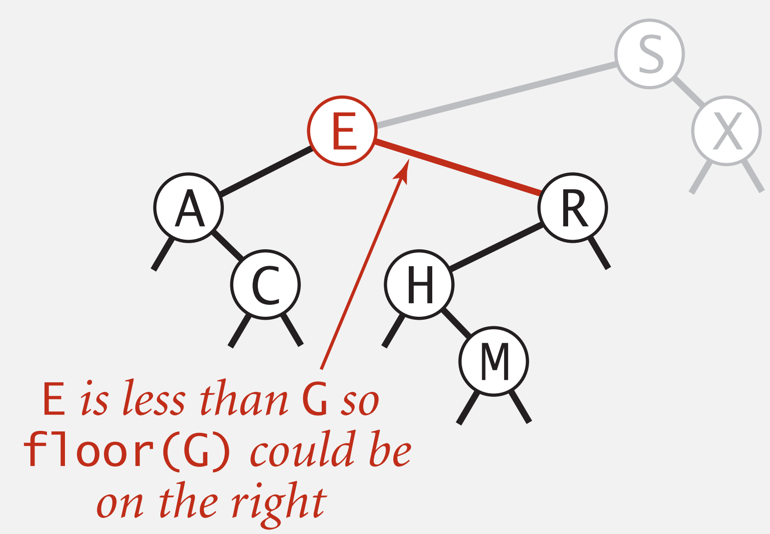
The floor of k is in the left subtree of x.

Case 3. [key in node $x < k$]

The floor of k can't be in left subtree of x:

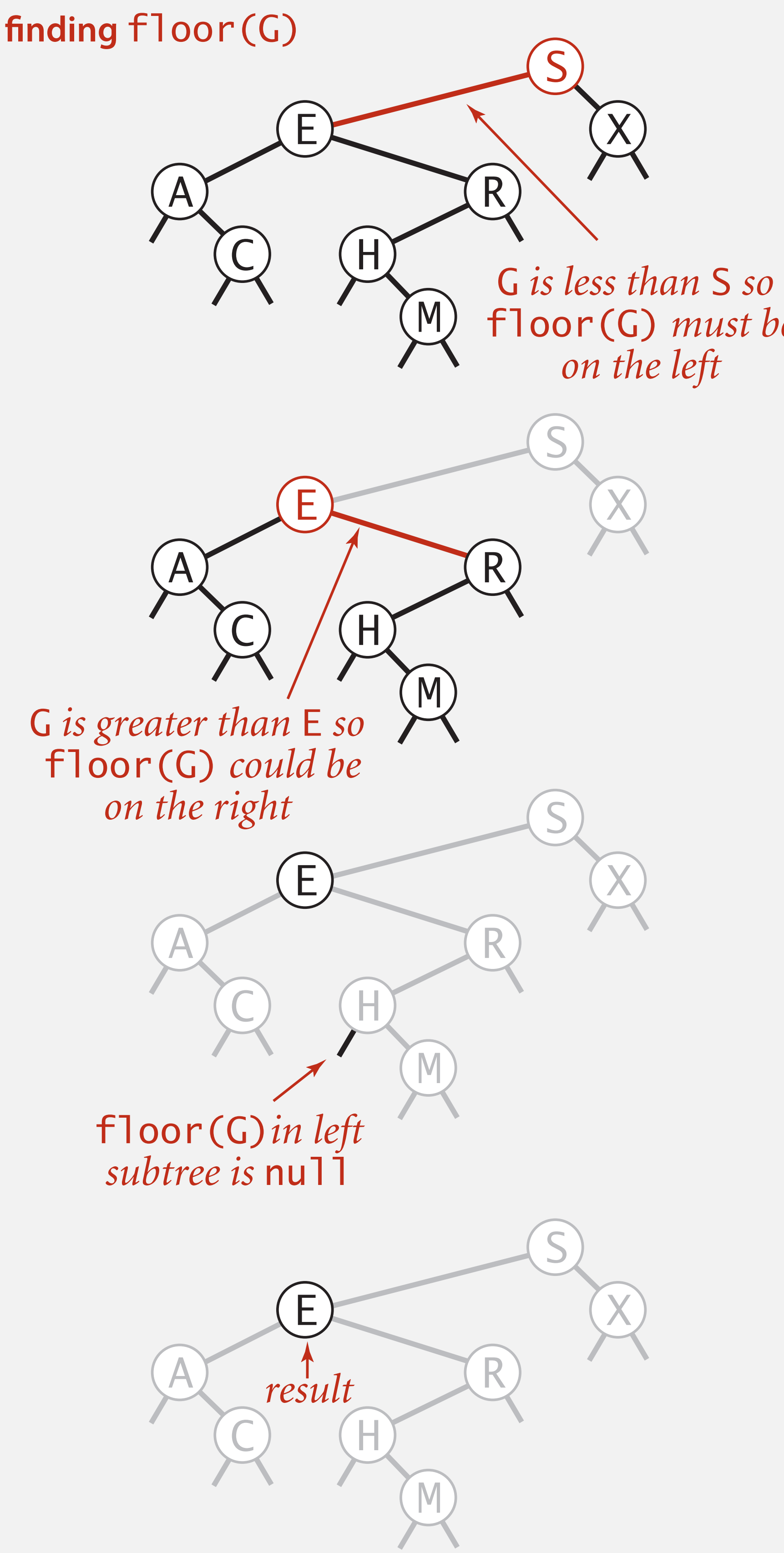
it is either in the right subtree of x or

it is the key in node x.



Computing the floor

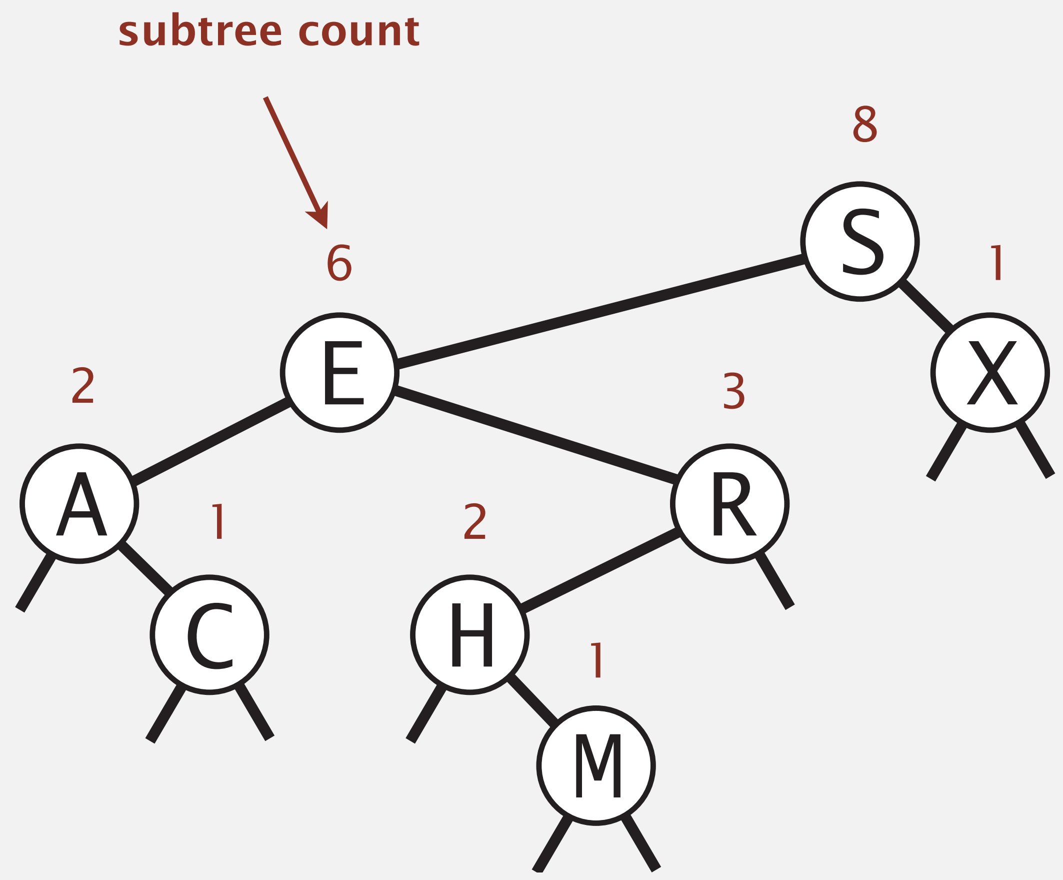
```
public Key floor(Key key)
{  return floor(root, key); }

private Key floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp  < 0) return floor(x.left, key);

   Key t = floor(x.right, key);
   if (t != null) return t;
   else           return x.key;

}
```



Rank and select

Q. How to implement `rank()` and `select()` efficiently for BSTs?

A. In each node, store the number of nodes in its subtree.



BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

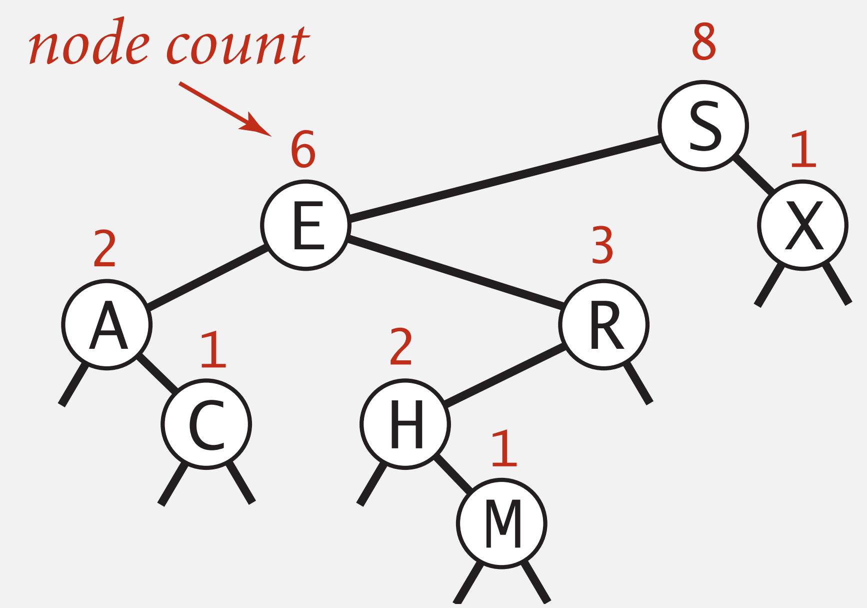
ok to call when x is null

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

initialize subtree count to 1

Computing the rank

Rank. How many keys in BST < k ?



Case 1. [k < key in node]

- Keys in left subtree? *count*
- Key in node? 0
- Keys in right subtree? 0

Case 2. [k > key in node]

- Keys in left subtree? *all*
- Key in node. 1
- Keys in right subtree? *count*

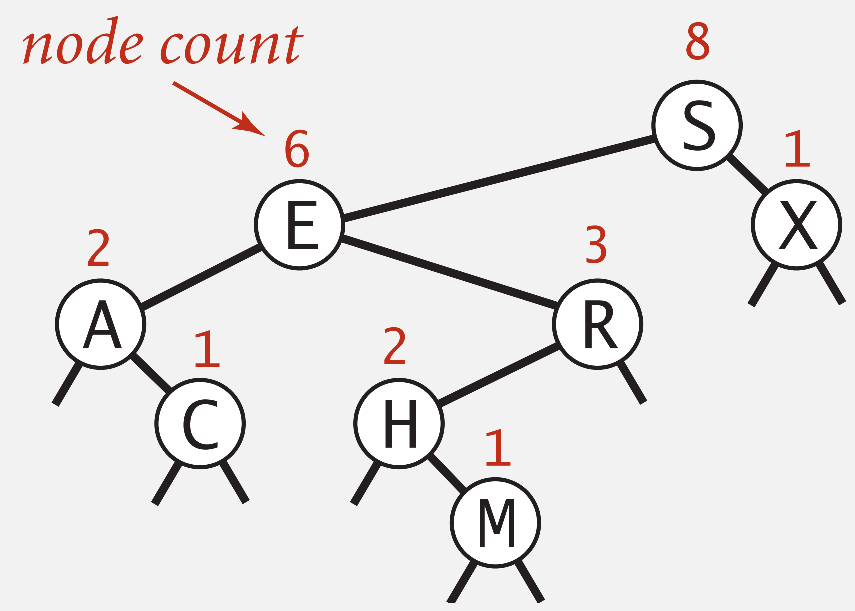
Case 3. [k = key in node]

- Keys in left subtree? *count*
- Key in node. 0
- Keys in right subtree? 0

Rank

Rank. How many keys in BST < k ?

Easy recursive algorithm (3 cases!)

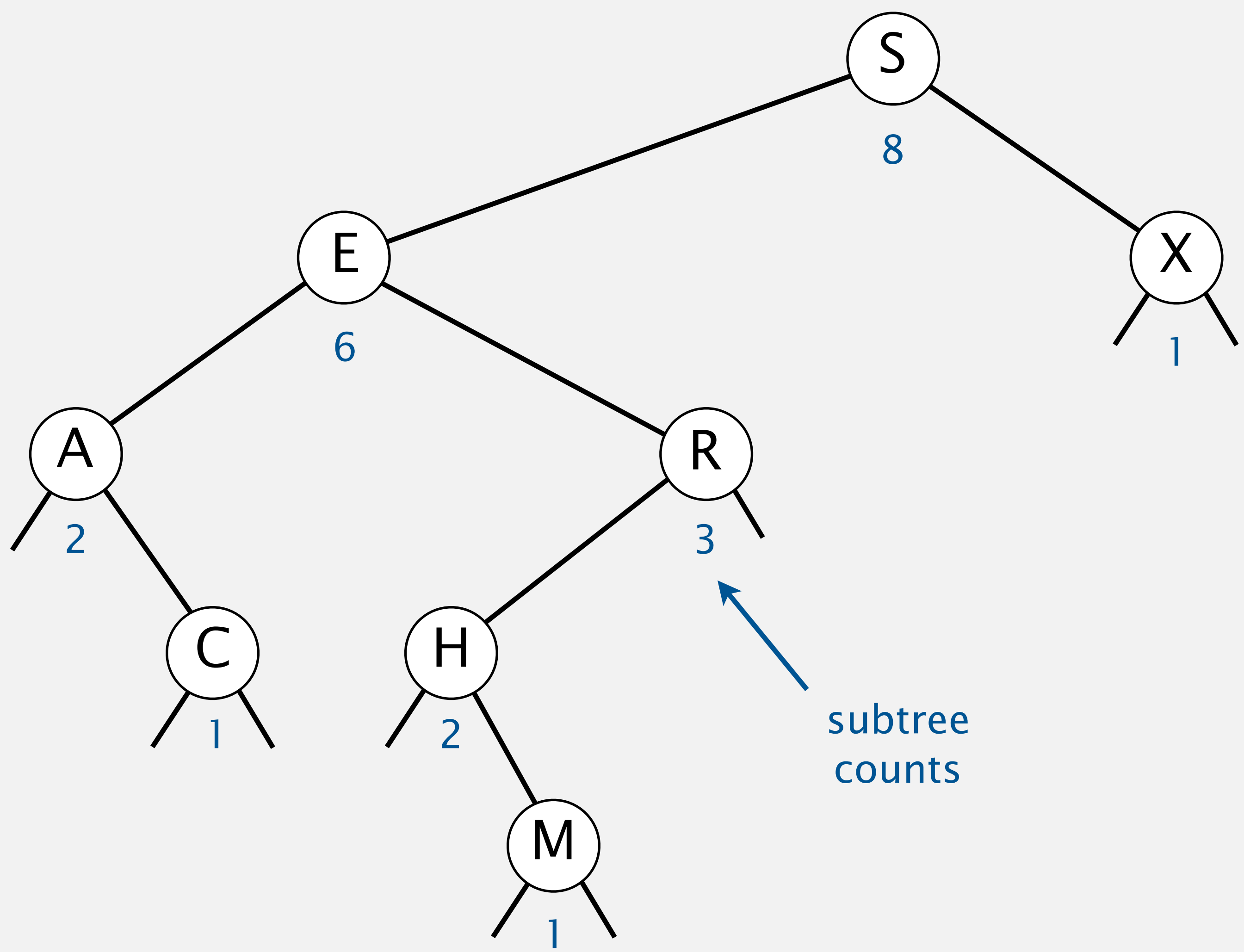



```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

Selection in a BST demo

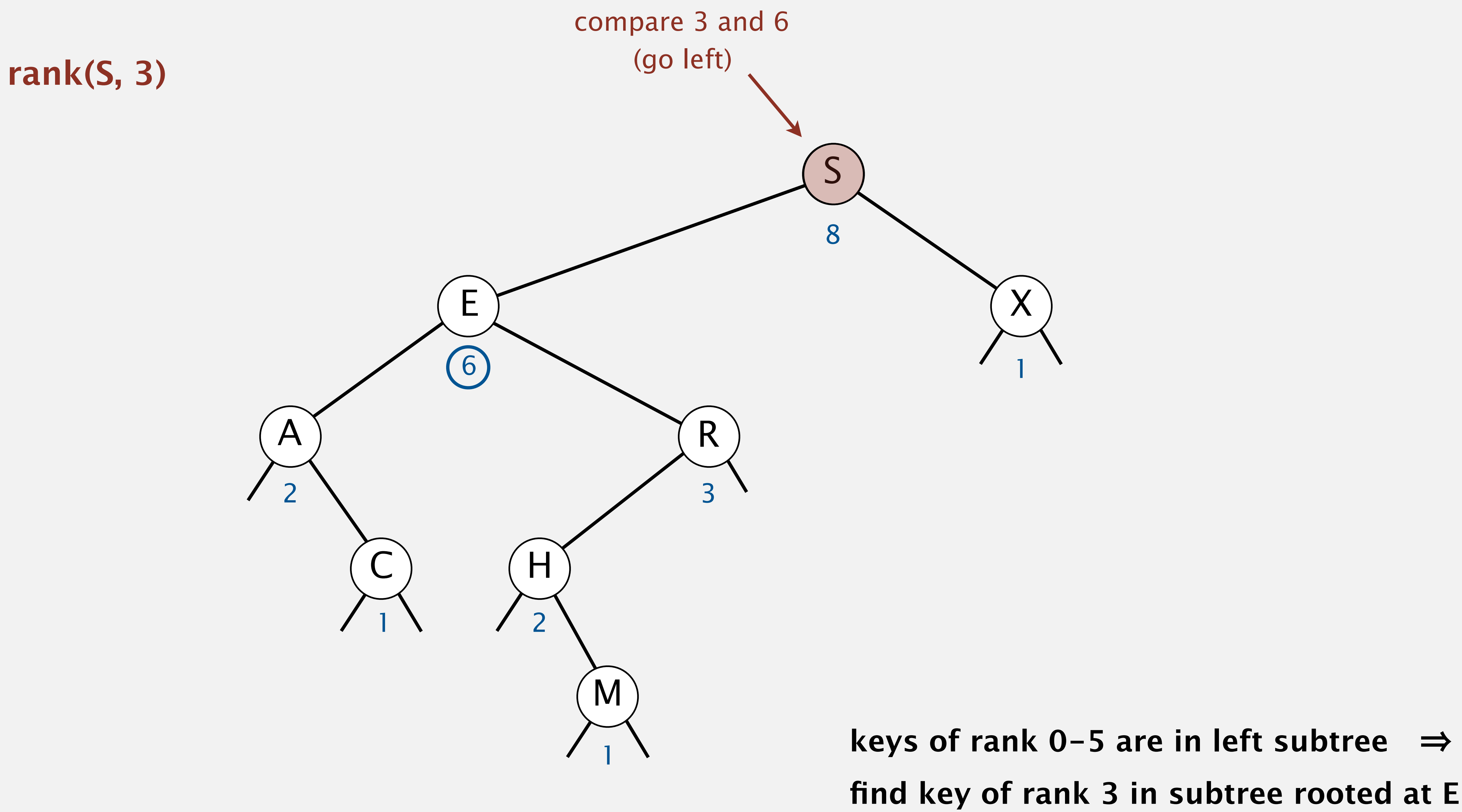
Select. Find the key in a BST of rank k.

rank(S, 3)



Selection in a BST demo

Select. Find the key in a BST of rank k.

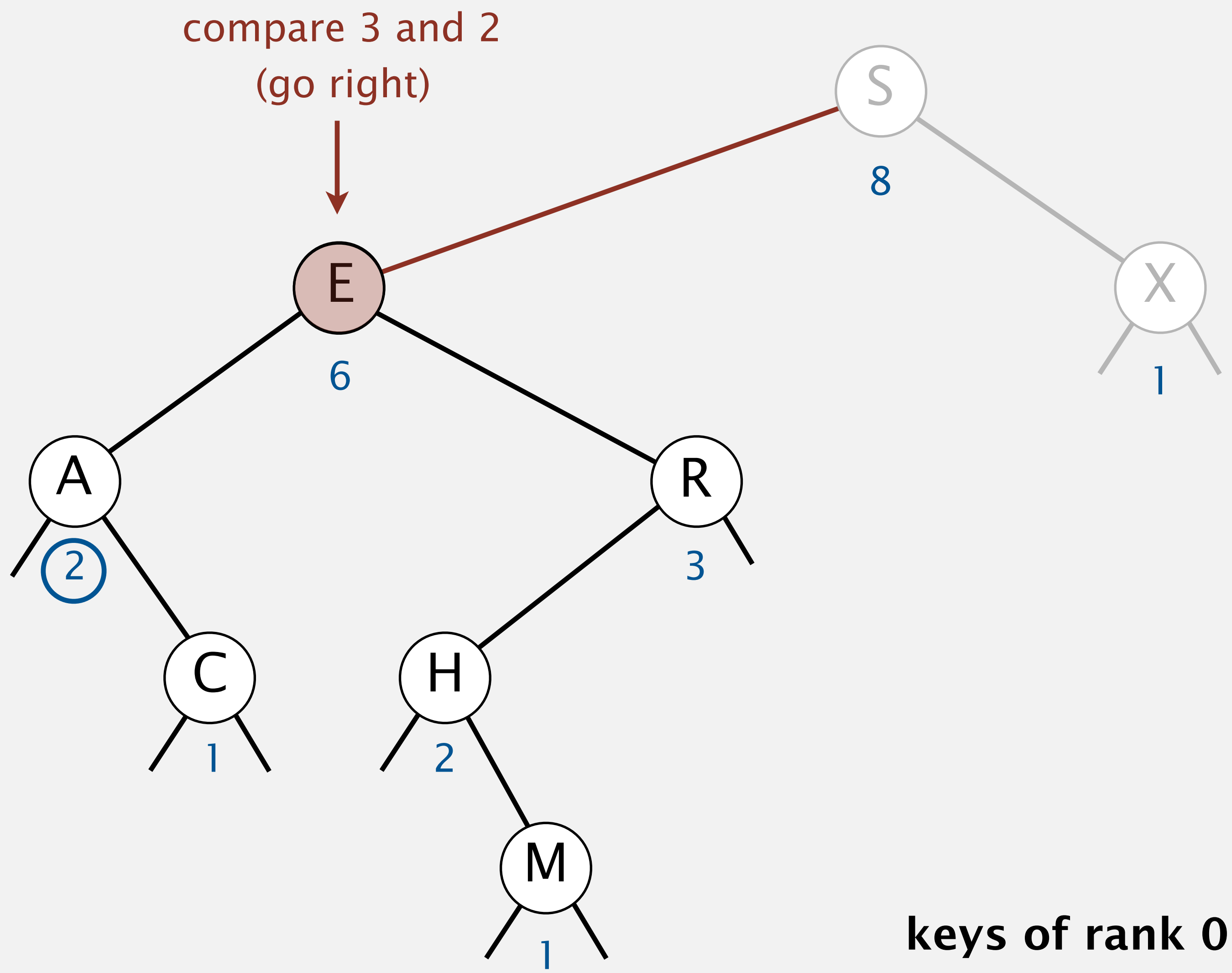


Selection in a BST demo

Select. Find the key in a BST of rank k.

rank(S, 3)

rank(E, 3)



keys of rank 0–1 are in left subtree ⇒

find key of rank 0 in subtree rooted at R

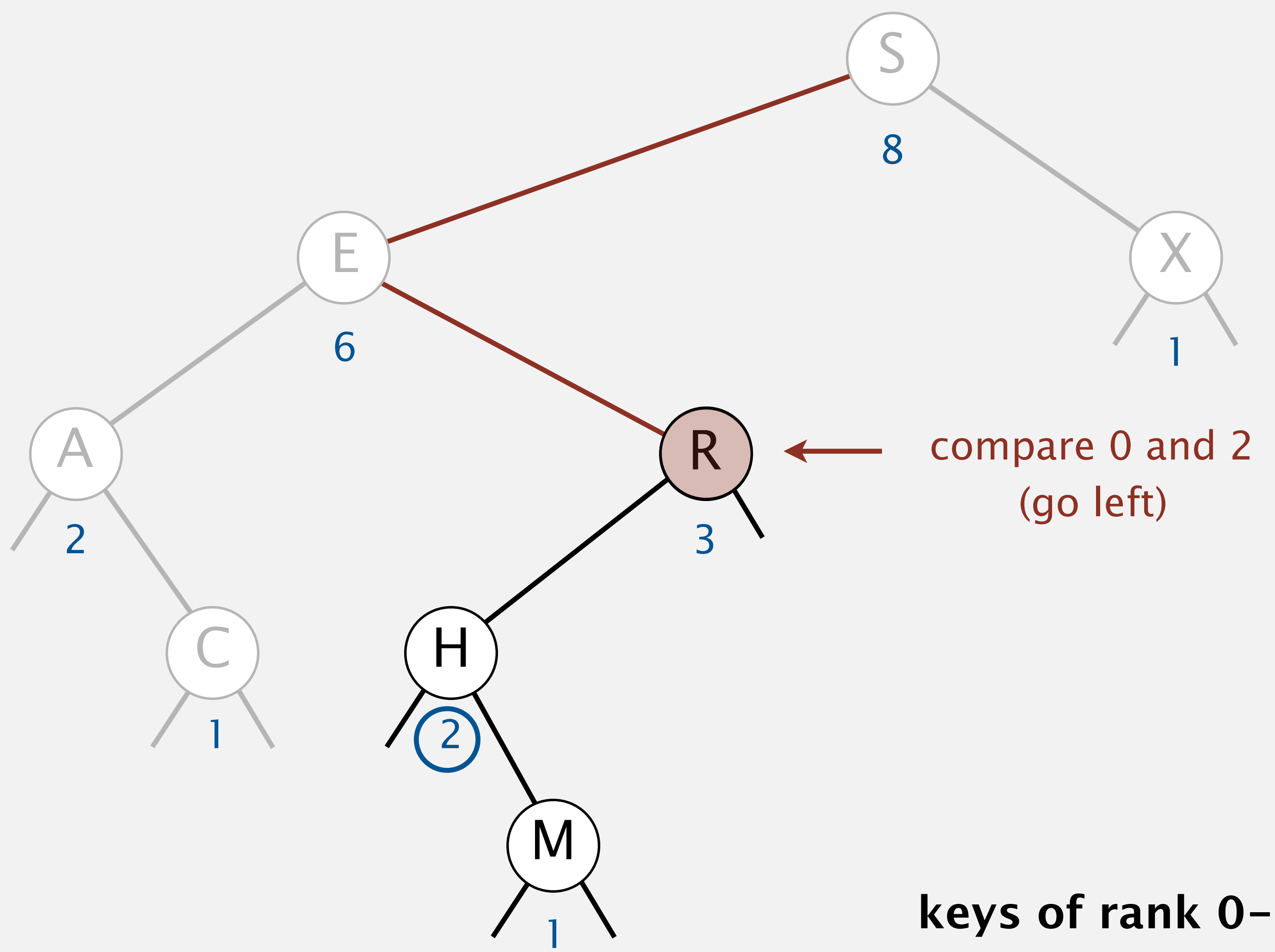
Selection in a BST demo

Select. Find the key in a BST of rank k.

rank(S, 3)

rank(E, 3)

rank(R, 0)



keys of rank 0–1 are in left subtree ⇒

find key of rank 0 in subtree rooted at H

Selection in a BST demo

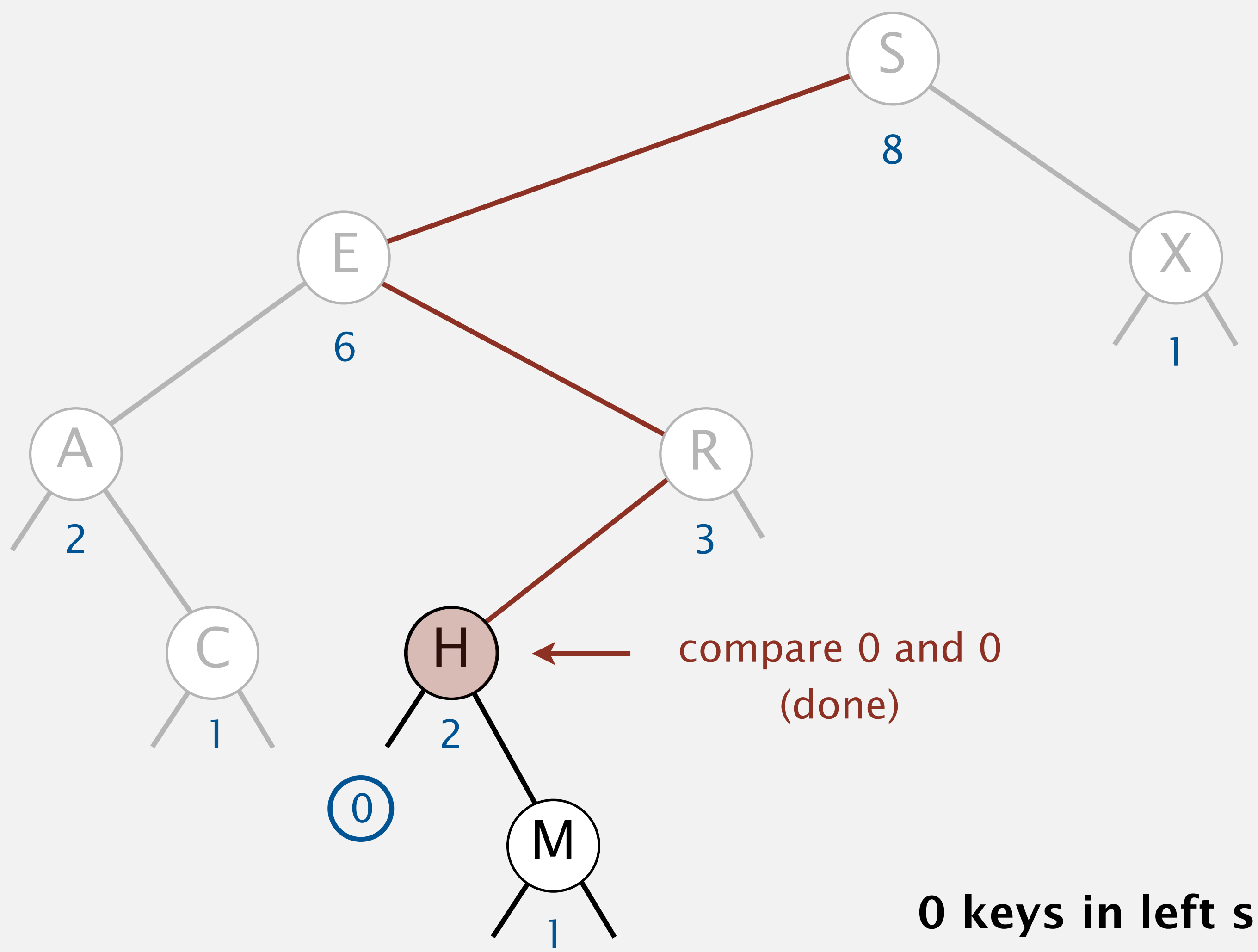
Select. Find the key in a BST of rank k.

rank(S, 3)

rank(E, 3)

rank(R, 0)

rank(H, 0)



0 keys in left subtree ⇒

key of rank 0 in subtree rooted at H is H

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	$\log N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\log N$	h
rank	N	$\log N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

ST implementations: summary

implementation	guarantee		average case		ordered ops?	key interface
	search	insert	search hit	insert		
sequential search (unordered list)	N	N	N	N		`equals()`
binary search (ordered array)	$\log N$	N	$\log N$	N	✔	`compareTo()`
BST	N	N	$\log N$	$\log N$	✔	`compareTo()`
red-black BST	$\boxed{\log N}$	$\boxed{\log N}$	$\log N$	$\log N$	✔	`compareTo()`

Next lecture. Guarantee logarithmic performance for all operations.