Proposition:

$$\underline{C(N)} \leq C(N/2) + C(N/2) + N$$

$C(N/2)$: left half; $C(N/2)$: right half; N: merge $\sim N/2$ – $\sim N$

$D(N)$

Total # comparisons less(aux[i], aux[j])

Proposition: If $D(N)$ satisfies

$D(N) = 2 \cdot D(N/2) + N$ for $N > 1$

with $D(1) = 0$, the $D(N) = N \log_2 N$

Proof by picture [assuming N is a power of 2]

$\log_2 N$

$D(N)$

$D(N/2)$ $D(N/2)$

$D(N/4)$ $D(N/4)$ $D(N/4)$ $D(N/4)$

$D(N/8)$ $D(N/8)$ $D(N/8)$ $D(N/8)$ $D(N/8)$ $D(N/8)$ $D(N/8)$ $D(N/8)$

... $D(N/N)$

extra cost for merge

N

$2 \cdot N/2 = N$

$4 \cdot N/4 = N$

$8 \cdot N/8 = N$

$D(N) = N \log_2 N$

Proposition: If $D(N)$ satisfies $D(N) = 2 \cdot D(N/2) + N$ for $N > 1$ with $D(1) = 0$ then $D(N) = N \cdot \log_2 N$

Proof by telescoping [assuming N is a power of 2]

for $N > 1$:
$$\frac{D(N)}{N} = \frac{\cancel{2} \cdot D(N/2)}{N/2} + \frac{\cancel{N}}{\cancel{N}} 1$$

$$= \frac{D(N/4)}{N/4} + 1 + 1$$

$$= \frac{D(N/8)}{N/8} + 1 + 1 + 1$$

$$= \cancel{\frac{D(N/N)}{N/N}} + \underbrace{1 + \cdots + 1}_{\log_2 N}$$

$D(N) = N \cdot \log_2 N$

$\to O($

$\to \Theta(N \cdot \log_2 N)$

$\to \Omega($