JEE Problems in Linear Algebra: 2D
Question 36

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Problem Statement

Let ellipse

$$x^T V x = 16$$

where

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}$$

be inscribed in a rectangle whose sides are parallel to the coordinate axes. If the rectangle is inscribed in another ellipse that passes through the point

$$\begin{pmatrix} 16 \\ 0 \end{pmatrix}$$

find the equation of outer ellipse.

Finding vertices of the rectangle

Let the length of semi-major axis be 'a' and semi-minor axis be 'b'.
Let A,B,C and D be the vertices of the rectangle.
From the given equation of ellipse we get,

$$a^2 = \frac{16}{1} = 16, \qquad b^2 = \frac{16}{16} = 1$$

$$\therefore a = 4, b = 1$$

Hence, the vertices of the given rectangle will be

$$A = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad C = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \quad D = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Finding the equation of outer Ellipse

Let the equation of the outer ellipse be

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1$$

Given, the outer ellipse passes through $\begin{pmatrix} 16 \\ 0 \end{pmatrix}$, substituting the point in above equation we get,

$$\begin{pmatrix} 16 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 16 \\ 0 \end{pmatrix} = 1$$

$$\implies \begin{pmatrix} 16p & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 0 \end{pmatrix} = 1$$

$$\implies 16^2 p = 1 \implies p = \frac{1}{16^2} = 0.00390625$$

Finding the equation of outer Ellipse

Given the outer ellipse also passes through the vertices of the rectangle, Substituing the point A in the equation obtained above we get,

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 1/16^2 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1$$

$$\implies \begin{pmatrix} 4/16^2 & q \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1$$

$$\implies \frac{1}{16} + q = 1$$

$$\therefore q = \frac{15}{16} = 0.9375$$

$\therefore$ the equation of the ellipse is $\mathbf{x}^T \begin{pmatrix} 1/16^2 & 0 \\ 0 & 15/16 \end{pmatrix} \mathbf{x} = 1$

C code for computation

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

float a1,b1;
float a,b;
int main()
{
int c,len;
float p,q;
double **V,**P,**theta;


c = 16;
len = 100;
```

```
V = loadtxt("./data/V.dat",2,2);
P = loadtxt("./data/P.dat",2,1);

a = sqrt(c/V[0][0]);
b = sqrt(c/V[1][1]);

printf("The value of a is %lf\n",a);
printf("The value of b is %lf\n",b);

p = 1/pow(P[0][0],2);
q = 1 - (16*p);

a1=sqrt(1/p);
b1=sqrt(1/q);

printf("The value of p = %lf\n",p);
printf("The value of q = %lf\n",q);
```

```
theta = linspace(0,2*M_PI,len);
savetxt(theta,"theta.dat",len,1);

save_a();
save_b();
save_a1();
save_b1();
save_pa();
save_pb();
save_pc();
save_pd();


return 0;
}
```

C code for saving data in files

```
void save_a(){
FILE * fptr;
fptr = fopen("./data/a.dat","w");
fprintf(fptr,"%lf",a);
fclose(fptr);
}

void save_b(){
FILE * fptr;
fptr = fopen("./data/b.dat","w");
fprintf(fptr,"fclose(fptr);
}
```

```
void save_a1(){
FILE * fptr;
fptr = fopen("./data/a1.dat","w");
fprintf(fptr,"%lf",a1);
fclose(fptr);
}

void save_b1(){
FILE * fptr;
fptr = fopen("./data/b1.dat","w");
fprintf(fptr,"%lf",b1);
fclose(fptr);
}
```

```
void save_pa(){
FILE * fptr;
fptr = fopen("./data/pa.dat","w");
fprintf(fptr,"%lf",a);
fprintf(fptr,"\n");
fprintf(fptr,"%lf",b);
fclose(fptr);
}

void save_pb(){
FILE * fptr;
fptr = fopen("./data/pb.dat","w");
fprintf(fptr,"%lf",-a);
fprintf(fptr,"\n");
fprintf(fptr,"%lf",b);
fclose(fptr);
}
```

```
void save_pc(){
FILE * fptr;
fptr = fopen("./data/pc.dat","w");
fprintf(fptr,"%lf",-a);
fprintf(fptr,"\n");
fprintf(fptr,"%lf",-b);
fclose(fptr);
}

void save_pd(){
FILE * fptr;
fptr = fopen("./data/pd.dat","w");
fprintf(fptr,"%lf",a);
fprintf(fptr,"\n");
fprintf(fptr,"%lf",-b);
fclose(fptr);
}
```

Python Code for plotting

```
import numpy as np
import matplotlib.pyplot as plt

P = np.array([16,0])
O=np.array([0,0])


len=100
theta=np.loadtxt('theta.dat',dtype='double')

a=np.loadtxt('./data/a.dat',dtype='float')
b=np.loadtxt('./data/b.dat',dtype='float')


a1=np.loadtxt('./data/a1.dat',dtype='float')
b1=np.loadtxt('./data/b1.dat',dtype='float')
```

```
A=np.loadtxt('./data/pa.dat',dtype='float')
B=np.loadtxt('./data/pb.dat',dtype='float')
C=np.loadtxt('./data/pc.dat',dtype='float')
D=np.loadtxt('./data/pd.dat',dtype='float')


x1=a*np.cos(theta)
y1=b*np.sin(theta)
x2=a1*np.cos(theta)
y2=b1*np.sin(theta)

plt.plot(x1,y1,'b')
plt.plot(x2,y2,'y')

plt.plot([A[0],B[0]],[A[1],B[1]],'b')
plt.plot([B[0],C[0]],[B[1],C[1]],'b')
plt.plot([C[0],D[0]],[C[1],D[1]],'b')
plt.plot([D[0],A[0]],[D[1],A[1]],'b')
```

```
plt.plot([A[0],B[0]],[A[1],B[1]],'o')
plt.plot([B[0],C[0]],[B[1],C[1]],'o')
plt.plot([C[0],D[0]],[C[1],D[1]],'o')
plt.plot([D[0],A[0]],[D[1],A[1]],'o')

plt.plot(O[0],O[1],"o")
plt.text(O[0]*(1+0.1),O[1]*(1-0.1),"O")

plt.plot(P[0],P[1],'o')
plt.text(P[0]*(1),P[1]*(0.8),'P')

plt.title('jee_linalg_2d\nQuestion 36')
plt.grid()
plt.axis('equal')
plt.show()
```

Plotted Figure