Mathematics - 1

Computational Geometry

Chapter 3 – Projections

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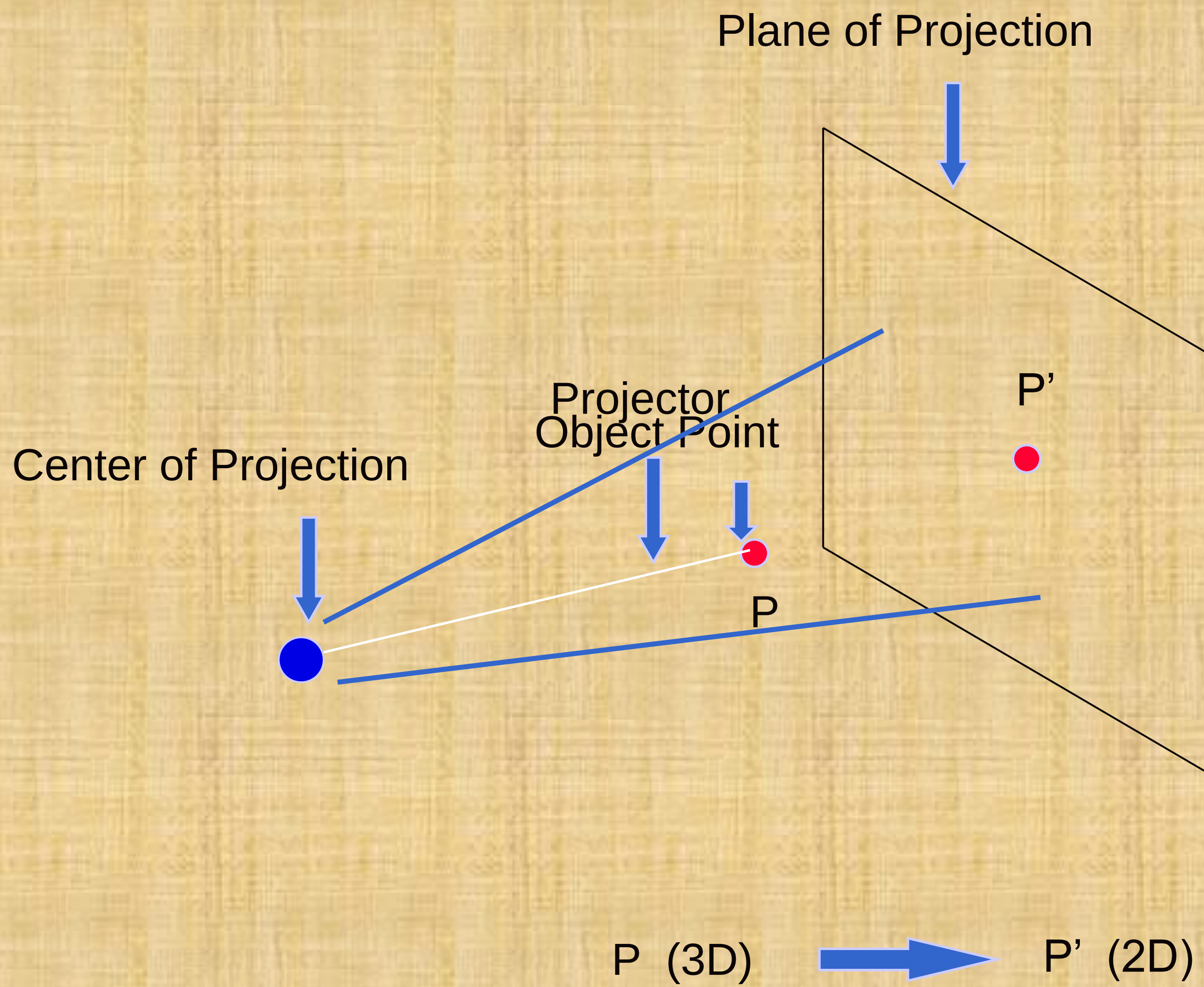
Projection Geometry

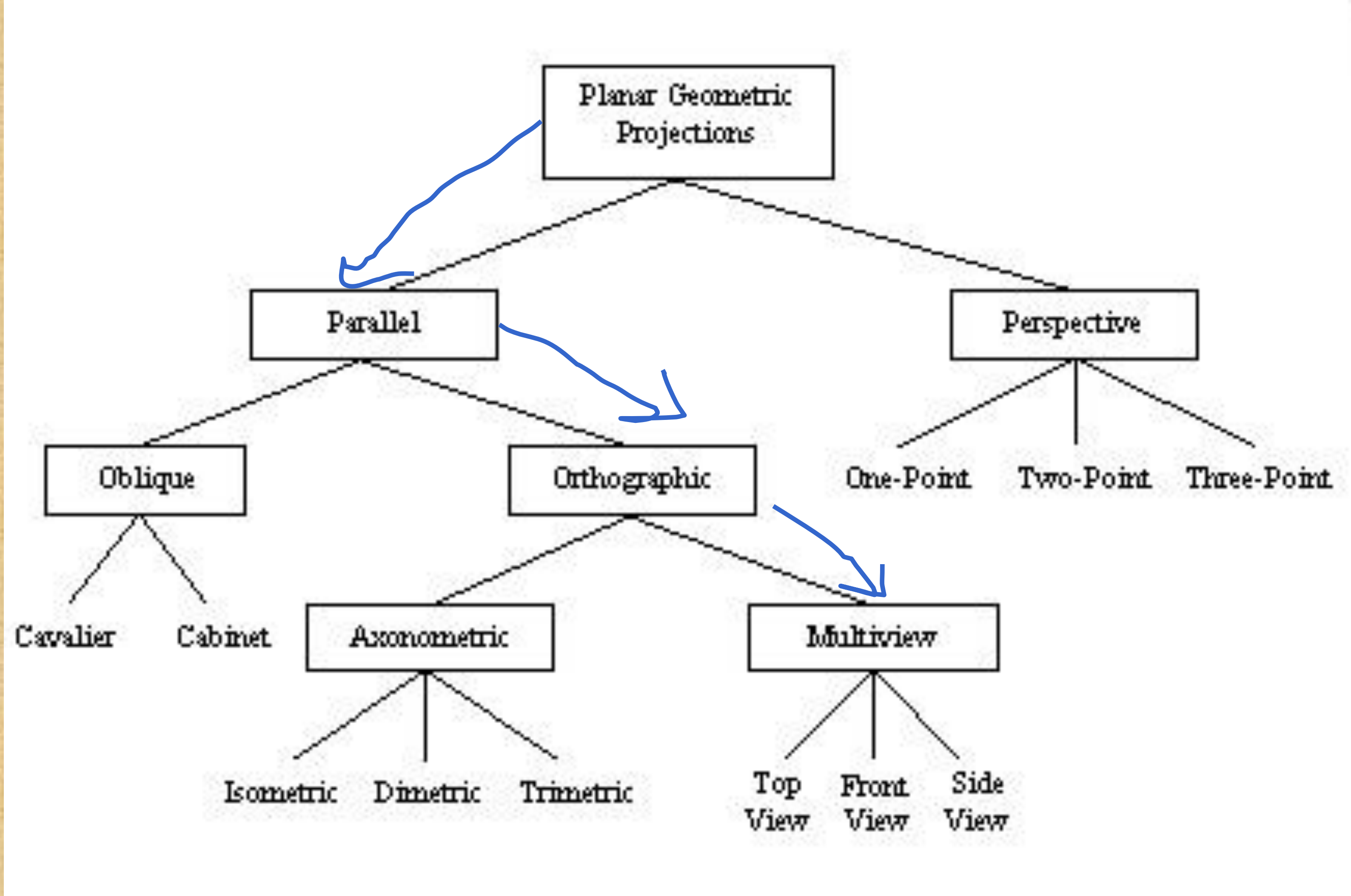


Projections: Properties

- Projections map points from one space to another coordinate space of lower dimension, and hence involves loss of information. T: R^n → R^(n-1)
- If T: $R^3 \rightarrow R^2$, then T plane geometric projection
- Projections are not invertible. All projection matrices are singular.
- All points on a projector map to the same point on the plane of projection.

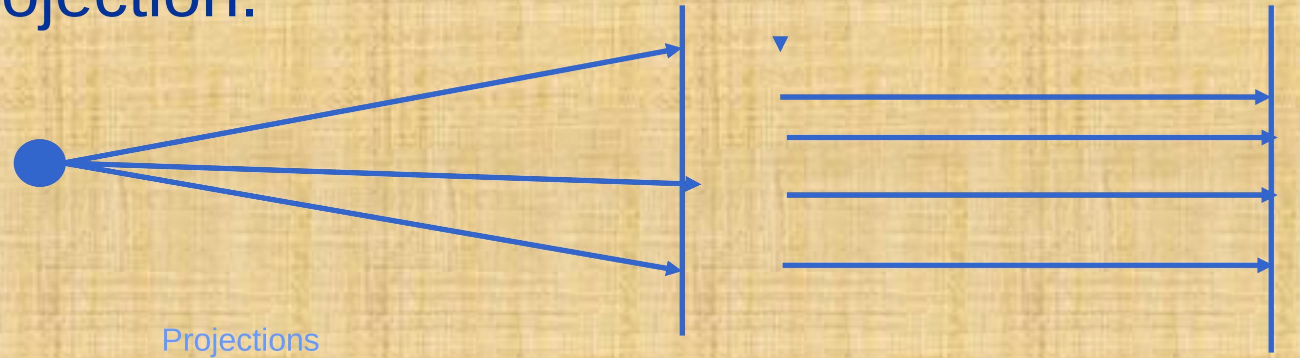
- $XT \rightarrow X'$
- $X = T^{-1}(X')$
- T is a projection, then det(T)=0. X can not be found.

Planar Projections



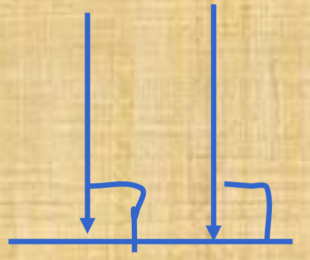
Parallel and Perspective

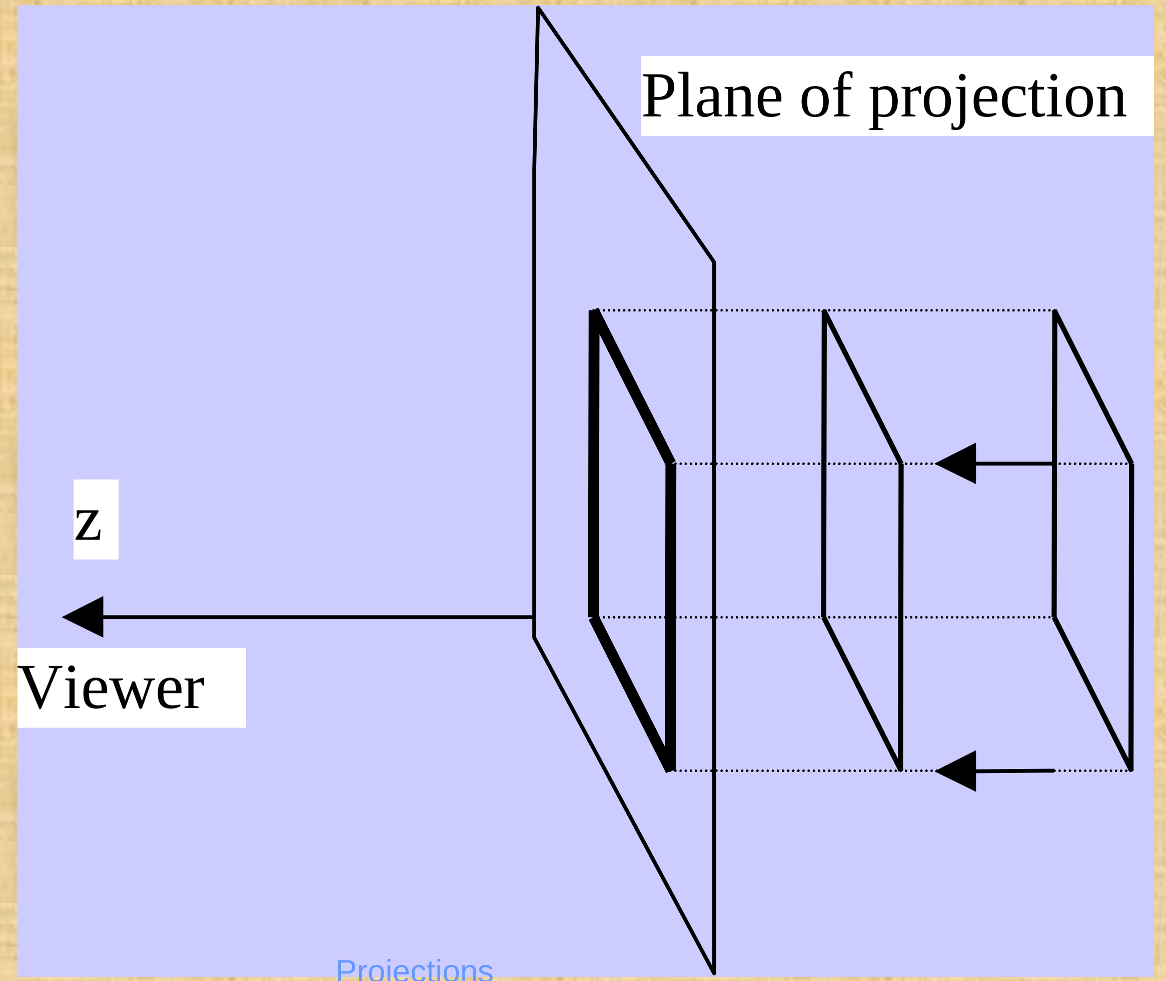
- Parallel Projections:
  - The center of projection is at infinity.
  - The projectors are parallel to each other.
- Perspective Projections:
  - The center of projection is a finite point.
  - The projectors intersect at the center of projection.



Orthographic Projection

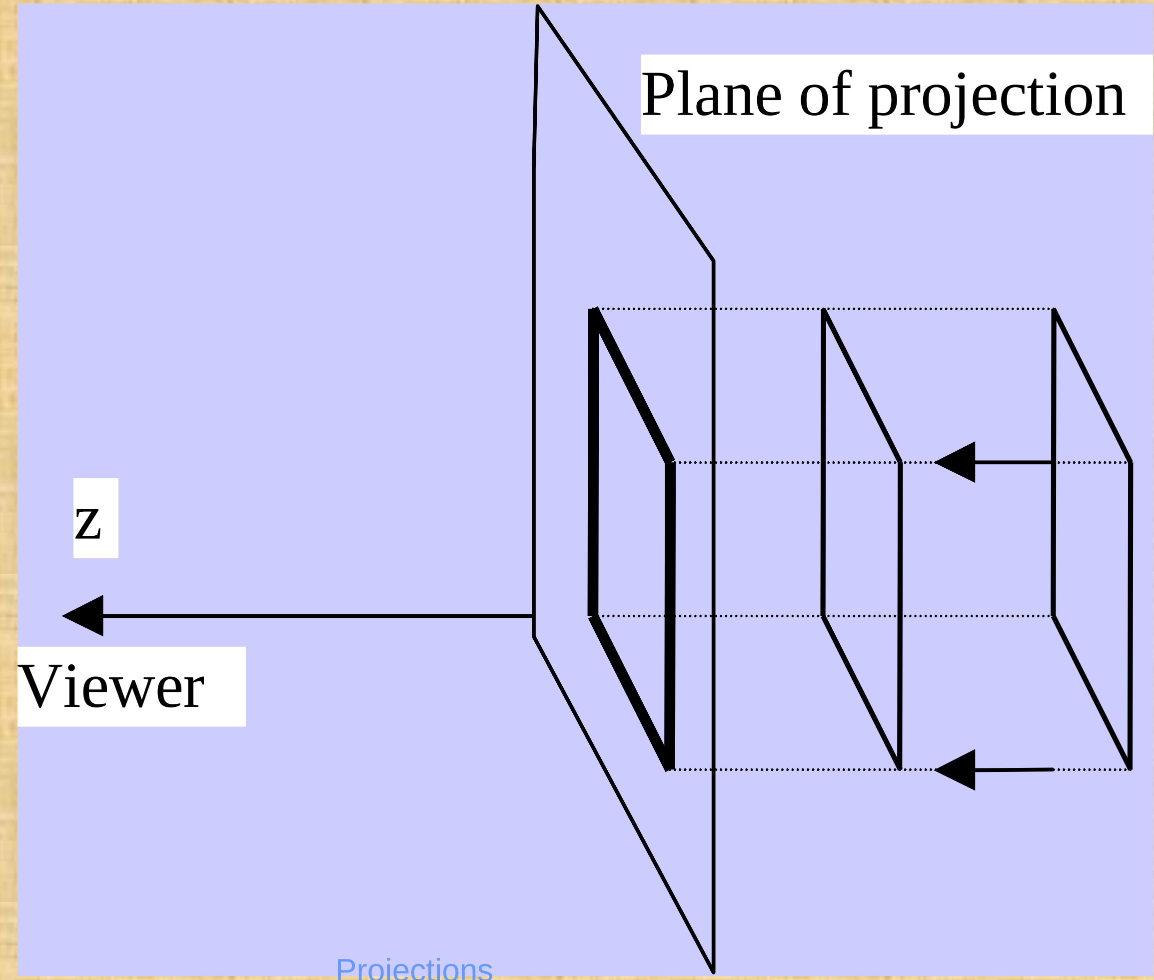
The lines of projection are parallel, and at the same time orthogonal to the plane of projection.





Orthographic Projection

The lines of projection are parallel, and at the same time orthogonal to the plane of projection.



Orthographic Projection

Projection along z axis:
(OR onto the plane z=0)

Transformation: $x' = x$
$y' = y$

z-coordinate information is lost!

Orthographic Projection Matrix: $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Orthographic Projection

Projection along x axis:
(OR onto the plane x =0)

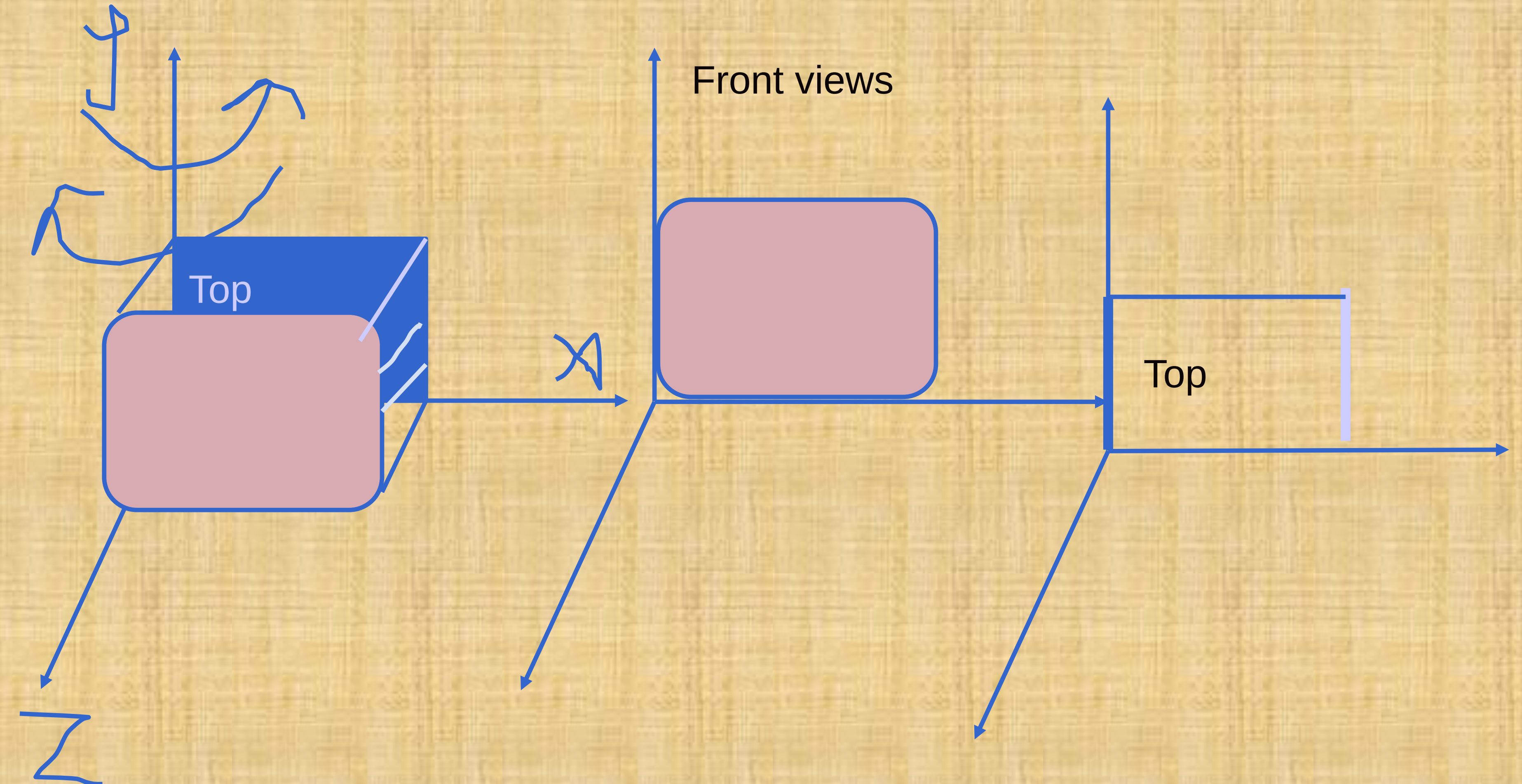
Transformation: $z' = z$
$y' = y$

x-coordinate information is lost!

Orthographic Projection Matrix: $T=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Multiple orthographic projections

Sr No	View	Rotation	Combined transformation matrix
1	Front	--	T=Pz
2	Rear	Rotation about Y-axis 180 deg	T= Ry(180) Pz
3	Top	Rotation about X-axis 90 deg	T=Rx(90) Pz
4	Bottom	Rotation about X-axis -90 deg	T=Rx(-90) Pz
5	Right	Rotation about Y-axis -90 deg	T=Ry(-90) Pz
6	Left	Rotation about Y-axis 90 deg	T=Ry(90) Pz


Y
X
Z
Top
Front views
Top

Develop --- view for the object X

1. Front view – X= $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Ans Front view, X'= X Pz

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$X' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

2. Bottom view X = $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Ans T= Rx(-90) Pz

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 & 0 \\ 0 & sin 90 & \cos 90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X'= XT = $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \end{bmatrix}$

X'= $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

3. Rear view $X = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Ans T= Ry(180) Pz

$$= \begin{bmatrix} \cos 180 & 0 & -\sin(180) & 0 \\ 0 & 1 & 0 & 0 \\ \sin 180 & 0 & \cos 180 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

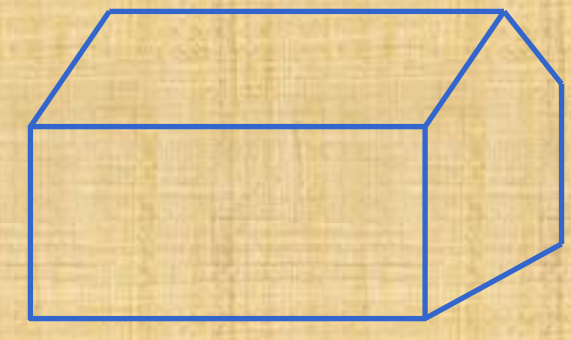
$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

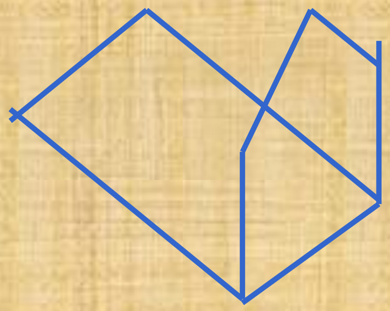
$$X' = XT = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \qquad X' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$$

Axonometric projection

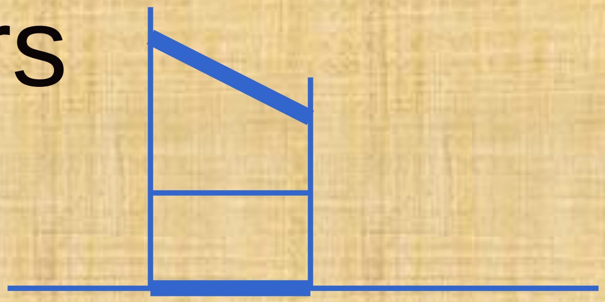
Orthogrpahic projection in which at least three faces are visible.

- Translation, rotation used if required.
- The lines which are not parallel to the plane of projection are fore-shortened.





Foreshortening factors



f= $\frac{Projected\ length}{Actual\ length} = \frac{A'B'}{AB}$

- Foreshortening factors along the coordinate axes are called principal foreshortening factors *fx, fy, fz.*
- If T = $\begin{bmatrix} t11 & t12 & 0 & 0 \\ t21 & t22 & 0 & 0 \\ t31 & t32 & 0 & 0 \\ l & m & 0 & 1 \end{bmatrix}$ is a transformation matrix for axonometric projection then the principal foreshortening factors are $fx = \sqrt{t11^2 + t12^2}$, $fy = \sqrt{t21^2 + t22^2}$, $fz = \sqrt{t31^2 + t32^2}$

Find the principal foreshortening factors

for T = $\begin{bmatrix} 0.2 & 0.4 & 0 & 0 \\ 0.5 & 0.8 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0.8 & 0.5 & 0 & 1 \end{bmatrix}$.

$fx = \sqrt{t11^2 + t12^2} = \sqrt{0.04 + 0.16} = 0.4472$

$fy = \sqrt{t21^2 + t22^2} = \sqrt{0.25 + 0.64} = 0.9433$

$fz = \sqrt{t31^2 + t32^2} = \sqrt{0.09 + 0.36} = 0.6708$

Transformation matrix – Axonometric projection

Std axonometric projection is obtained by Rotation about Y-axis through φ and then rotation about X-axis through Ɵ, then projection onto the plane z=0.

T= Ry(φ) Rx(Ɵ) Pz

$$= \begin{bmatrix} \cos\varphi & sin\varphi sin\theta & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ sin\varphi & -\cos\varphi \; sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Trimetric projection

Axonometric projection in which fx, fy, fz are not necessarily equal is a trimetric projection.

$$fx \neq fy \neq fz$$

Transformation matrix for axonometric projection

$$T = \begin{bmatrix} \cos\varphi & sin\varphi sin\theta & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ sin\varphi & -\cos\varphi \; sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$fx = \sqrt{cos^2\emptyset + (sin\emptyset \; sin\theta)^2}$$

$$fy = cos\theta$$

$$fz = \sqrt{sin^2\emptyset + (cos\emptyset sin\theta)^2}$$

Find trimetric projection of X when rotation about Y-axis through 35 and about X-axis through -70.

Φ=35 deg, ϴ= -70 deg

$$T=\begin{bmatrix} \cos\varphi & sin\varphi sin\theta & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ sin\varphi & -\cos\varphi\ sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0.819 & -0.538 & 0 & 0 \\ 0 & 0.342 & 0 & 0 \\ 0.573 & 0.769 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is cos φ =
sin φ =
cos ϴ =
sin ϴ =

Find the principal foreshortening factors of the problem above.

$$T=\begin{bmatrix} 0.819 & -0.538 & 0 & 0 \\ 0 & 0.342 & 0 & 0 \\ 0.573 & 0.769 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

fx =(0.819^2+0.538^2)^(1/2) = 0.9804

fy = 0.342

fz = (0.573^2+0.769^2)^(1/2) = 0.6046

Dimetric projection

An axonometric projection in which (at least) two foreshortening factors are equal.
A std dimetric projection : fx=fy, fz is free parameter.

$$T = \begin{bmatrix} \cos\varphi & sin\varphi sin\theta & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ sin\varphi & -\cos\varphi\ sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} fx &= \sqrt{cos^2\emptyset + (sin\emptyset\ sin\theta)^2} \\ fy &= cos\theta \end{aligned}\right\} \text{fx=fy}$$

$$fz = \sqrt{sin^2\emptyset + (cos\emptyset sin\theta)^2}$$

Dimetric projection

$$fx^2 = fy^2$$
$$cos^2\emptyset + (sin\emptyset\ sin\theta)^2 = \cos^2\theta = 1 - \sin^2\theta$$
$$1 - \sin^2\emptyset + \sin^2\emptyset\sin^2\theta = 1 - \sin^2\theta$$
$$\sin^2\emptyset(-1 + \sin^2\theta) = -\sin^2\theta$$
$$\mathbf{\sin^2\emptyset = \frac{\sin^2\theta}{1 - \sin^2\theta}}$$
$$fz^2 = \sin^2\emptyset + \cos^2\emptyset\sin^2\theta$$
$$fz^2 = \sin^2\emptyset + (1 - sin^2\emptyset)\sin^2\theta = \sin^2\emptyset(1 - \sin^2\theta) + \sin^2\theta$$

Fz^2 = sin^2 (th) + sin^2 (th) = 2 sin^2 (th)

sin(th)= fz /√2

$$\sin^2\emptyset = \frac{\sin^2\theta}{1-(\sin^2\theta)} = \frac{fz^2/2}{1-fz^2/2} = \frac{fz^2}{2-fz^2} \qquad \mathbf{\sin\emptyset = \frac{fz}{\sqrt{2-fz^2}}}$$

Develop dimetric projection if fz=5/8.

sin(th) = fz/√ 2 = 5/(8 √ 2)

th = 26.227

$$\sin \emptyset = \frac{fz}{\sqrt{2-fz^2}} = \frac{5/8}{\sqrt{2-25/64}} =$$

phi=

$$T = \begin{bmatrix} \cos\varphi & sin\varphi sin\theta & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ sin\varphi & -\cos\varphi\ sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Dimetric projections fz=0.2. Find th and phi.

Sin(th) = fz/√2 = 0.2/ √2 =

th = sin^(-1) (   ) =

$$\mathbf{\sin}\,\emptyset = \frac{\boldsymbol{fz}}{\sqrt{\mathbf{2} - \boldsymbol{fz}^{\mathbf{2}}}} = \frac{0.2}{\sqrt{2 - 0.04}} = \frac{0.2}{sqrt(1.96)}$$

$$=$$

phi =

Isometric projection

Isometric projection is an axonometric projection where fx=fy=fz.

In an isometric projection, values of φ and θ are constant. θ=35.26 and φ=45 degrees.

$$T = \begin{bmatrix} \cos\varphi & \sin\varphi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\varphi & -\cos\varphi\,\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Isometric projection

- T = $\begin{bmatrix} \cos\varphi & \sin\varphi \sin\Theta & 0 & 0 \\ 0 & \cos\Theta & 0 & 0 \\ \sin\varphi & -\cos\varphi \ \sin\Theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $\emptyset = \mp 45$,

$\theta = \mp 35.26$

- T= $\begin{bmatrix} 0.7071 & 0.4082 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0.7071 & -0.4082 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Develop isometric projection of X, where th, phi>0.

$$X=\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 4 & -1 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

- Transformation matrix for isometric projection T=

$$\begin{bmatrix} 0.7071 & 0.4082 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0.7071 & -0.4082 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

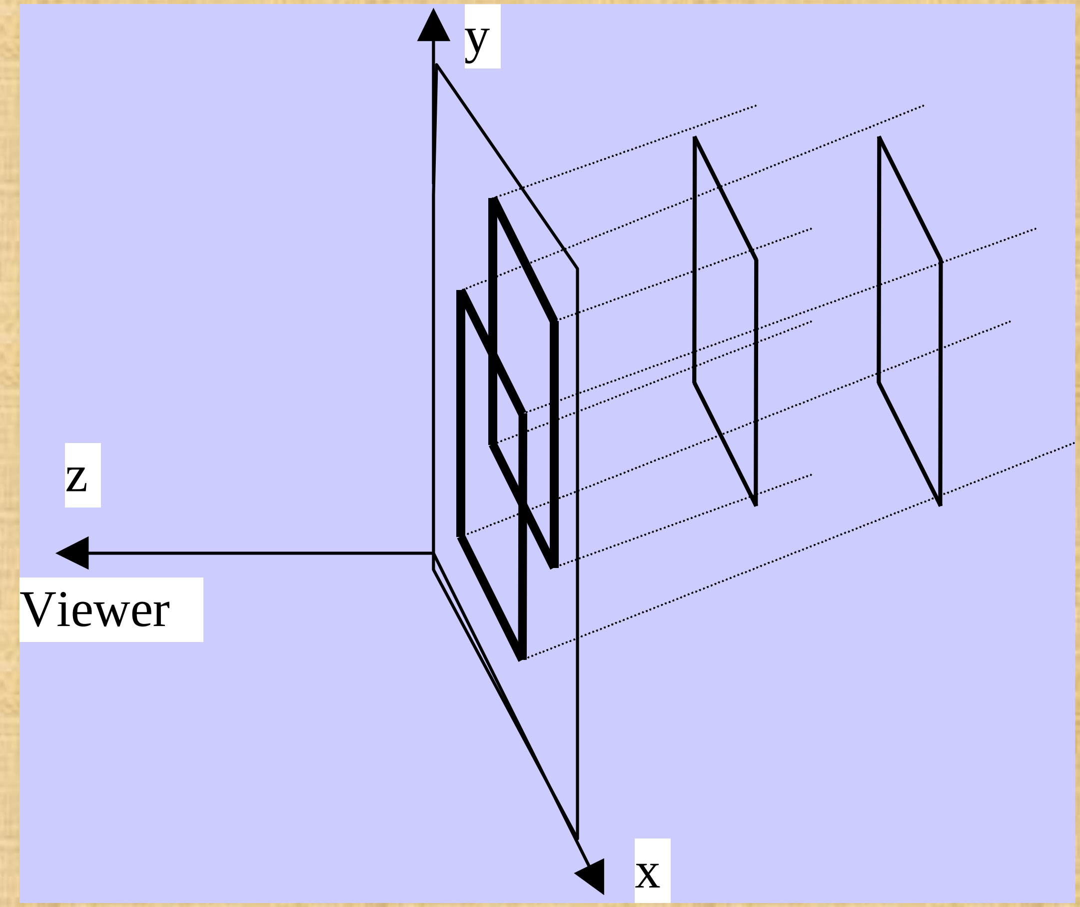
- Isometric projection of X is X'=XT=

$$\begin{bmatrix} 1.414 & 0.8162 & 0 & 1 \\ 1.414 & 0 & 0 & 1 \\ 2.121 & 1.2245 & 0 & 1 \\ -0.707 & 2.8577 & 0 & 1 \end{bmatrix}$$

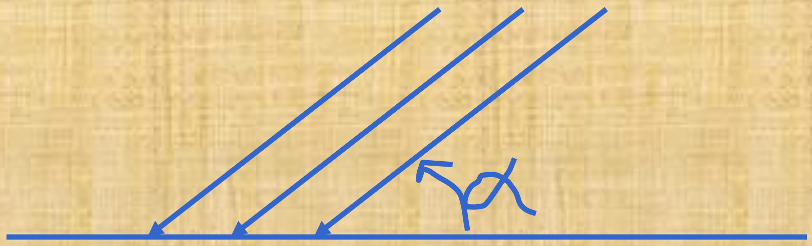
- X’= $\begin{bmatrix} 1.414 & 0.8164 \\ 1.414 & 0 \\ 2.121 & 1.2245 \\ -0.707 & 2.8577 \end{bmatrix}$

Oblique Projection

The lines of projection are parallel, but <u>not</u> orthogonal to the plane of projection.



Oblique Projection



Transformation: $x' = x+k_1z$

$y' = y+k_2z$

The z-coordinate value of the object point, leads to a shift of x, y coordinates of the projected point, proportional to z.

Oblique Projection Matrix: $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k1 & k2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$k1 = -f \cos \alpha$ and $k2 = -f \sin \alpha$

Oblique projection

- Matrix for oblique projection

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\ cos\alpha & -f\ sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If f=1 then oblique projection is called cavalier projection.
- If f=1/2 then oblique projection is called cabinet projection.

Find cavalier and cabinet projections of seg AB, A=[2,-3,2], B=[-1,-1,4] with horizontal inclination angle 70˚

- α=70˚; X=$\begin{bmatrix} 2 & -3 & 2 & 1 \\ -1 & -1 & 4 & 1 \end{bmatrix}$
- Cavalier projection f=1.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\ cos\alpha & -f\ sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.342 & -0.9396 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X' = XT = $\begin{bmatrix} 1.316 & -4.879 & 0 & 1 \\ -2.368 & -4.758 & 0 & 1 \end{bmatrix}$

- α=70°; X=$\begin{bmatrix} 2 & -3 & 2 & 1 \\ -1 & -1 & 4 & 1 \end{bmatrix}$
- Cabinet projection f=1/2.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\ cos\alpha & -f\ sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.171 & -0.4698 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X' = XT = \begin{bmatrix} 1.658 & -3.9396 & 0 & 1 \\ -1.684 & -2.8792 & 0 & 1 \end{bmatrix}$$

Find transformation matrices for cavalier and cabinet projections with α=120˚.

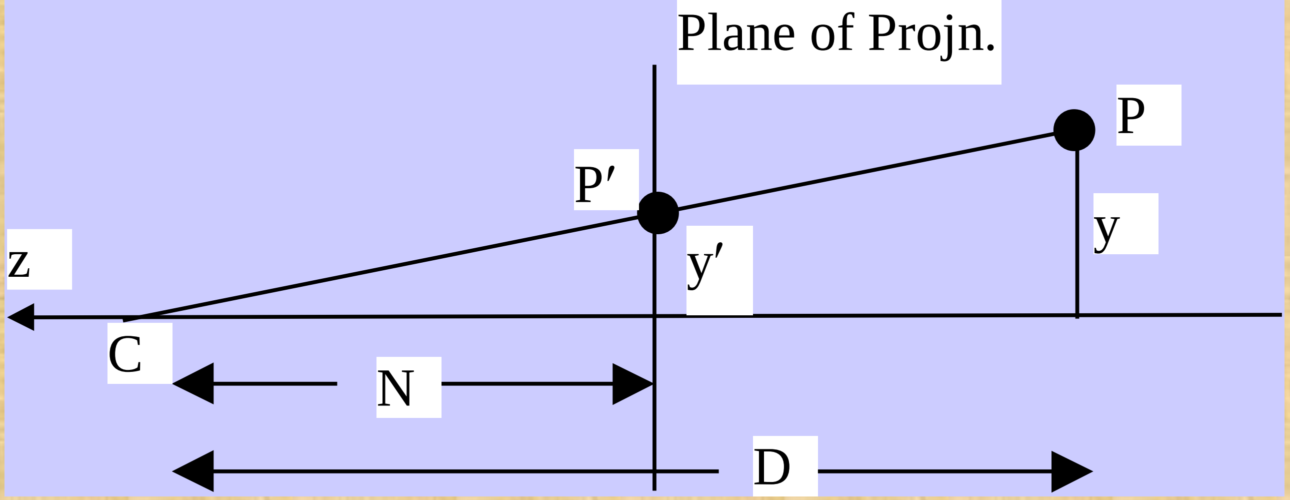
- Cavalier projection: f=1, α=120˚
- $T=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\ cos\alpha & -f\ sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & -0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Cabinet projection: f= ½
- $T=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\ cos\alpha & -f\ sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.25 & -0.433 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

If T is a matrix for cabinet projection, then find α.

- T= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.25 & -0.433 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- -(1/2) cos α = -0.25 → cos α= 0.5
- -(1/2) sin α = -0.433 → sin (al)=0.866
- Tan(al) = 0.866/0.5 → al =tan^(-1) (al) =

Perspective Projection

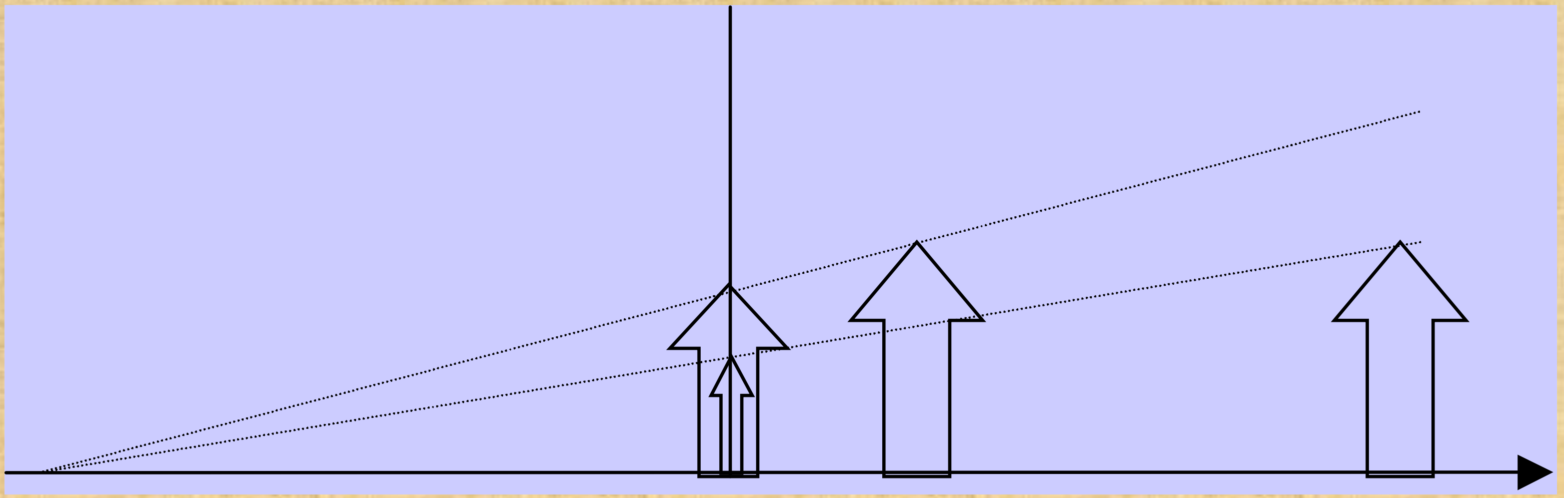
The projectors intersect at a Center of Projection C.



$$\frac{y}{D} = \frac{y'}{N}$$

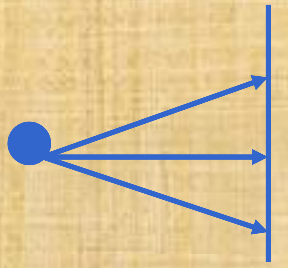
$$\frac{x}{D} = \frac{x'}{N}$$

Perspective Projection

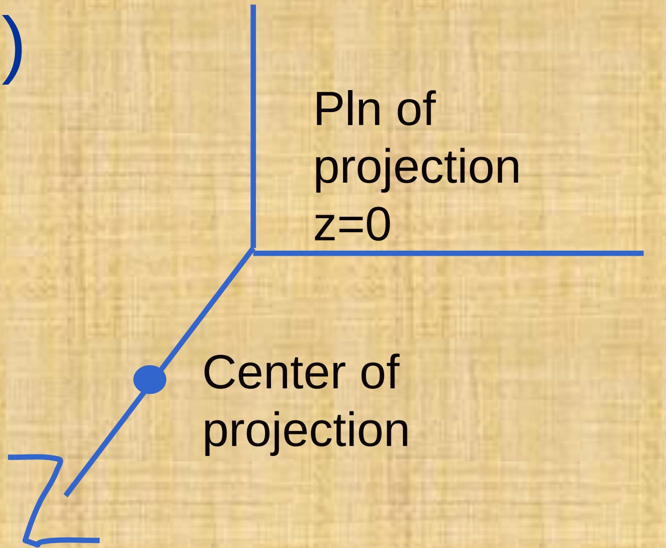


The z-coordinate value of the object point leads to proportional scaling along x, y directions. Projections of objects located closer to the center of projection O, appear to be larger in size compared to objects that are farther away from O.

Perspective projection



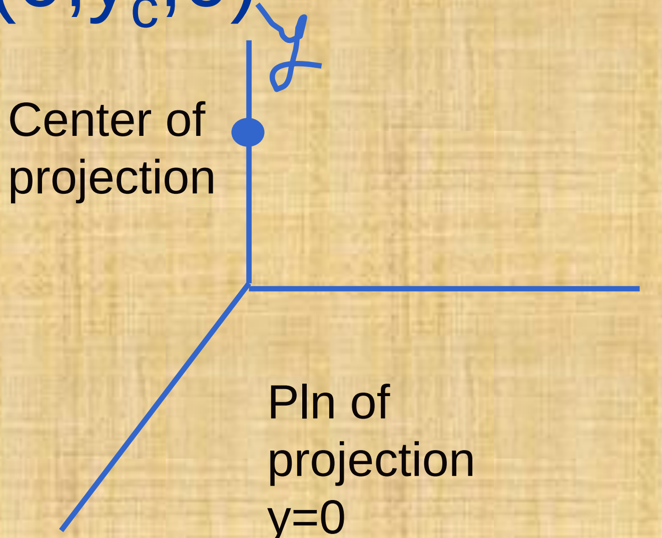
- Center of projection is at finite distance from the plane of projection.
- Projectors are not parallel to each other.
- Parallel lines are not foreshortened equally
- Perspective projection matrix when center of projection is on Z axis $(0,0,z_c)$



- T= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/zc \\ 0 & 0 & 0 & 1 \end{bmatrix}$

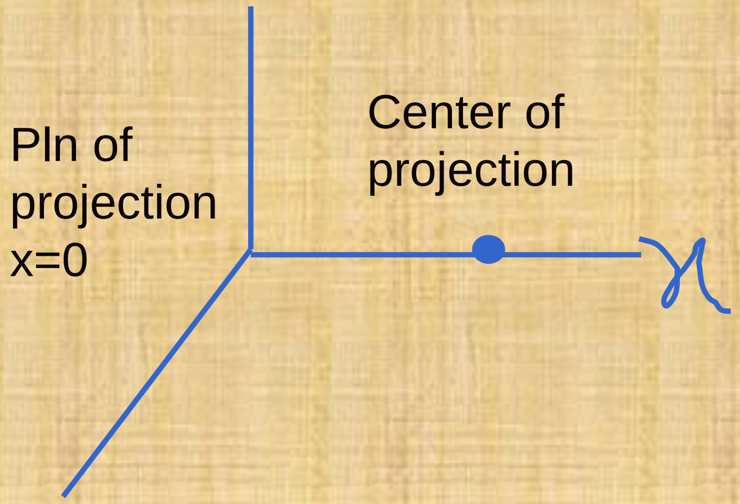
Perspective Projection

- Perspective projection matrix when center of projection is on Y axis $(0,y_c,0)$
- T= $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/yc \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Perspective Projection

- Perspective projection matrix when center of projection is on X axis $(x_c,0,0)$
- T= $\begin{bmatrix} 0 & 0 & 0 & -1/xc \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



- Notation: p=-1/xc, q=-1/yc, r=-1/zc

Write the transformation matrix for perspective projection when the center of projection is (0,0,2).

- C=(0,0,2) on Z-axis
- R=-1/zc = -1/2 = -0.5
- T= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Write the transformation matrix for perspective projection when the center of projection is (3,0,0).

- C=(3,0,0) on X-axis
- p =-1/xc = -1/3 = -0.3333
- T= $\begin{bmatrix} 0 & 0 & 0 & -0.33 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Develop the perspective projection of X when the center of projection is (3,0,0). X=segAB, A=[0,1,1], B=[-1,2,4]

- C=(3,0,0) on X-axis
- p =-1/xc = -1/3 = -0.3333
- T= $\begin{bmatrix} 0 & 0 & 0 & -0.33 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- X'=XT