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CELLULAR AUTOMATA ON CAYLEY TREE

HASAN AKIN

ABSTRACT. In this paper, we study cellular automata on Cayley tree of order 2 over the field \mathbb{Z}_p (the set of prime numbers modulo p). We construct the rule matrix corresponding to finite cellular automata on Cayley tree. Further, we analyze the reversibility problem of this cellular automata for some given values of $a, b, c, d \in \mathbb{Z}_p \setminus \{0\}$ and the levels n of Cayley tree. We compute the measure-theoretical entropy of the cellular automata which we define on Cayley tree.

1. INTRODUCTION

A cellular automaton (plural cellular automata, shortly CA) has been studied and applied as a discrete model in many areas of science. Cellular automata (CAs) have very rich computational properties and provide different models in computation. CAs were first used for modeling various physical and biological processes and especially in computer science. Recently, CAs have been widely investigated in many disciplines with different purposes such as simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, coding theory, cryptography, ergodic theory ([1, 2, 3, 4, 5, 6]).

Most of the studies and applications for CA is extensively done for one-dimensional (1-D) CA. "The Game of Life" developed by John H. Conway in the 1960s is an example of a two-dimensional (2-D) CA. John von Neumann in the late 40's and early 50's studied CA as a self-reproducing simple organisms [7]. 2-D CA with von Neumann neighborhood has found many applications and been explored in the literature [8]. Nowadays, 2-D CAs have attracted much of the interest. Some basic and precise mathematical models using matrix algebra built on field \mathbb{Z}_2 were reported for characterizing the behavior of two-dimensional nearest neighborhood linear CAs with null or periodic boundary conditions [2, 3, 4, 6, 8]. The reversibility problem of some special classes of 1-D CAs reflective and periodic boundary conditions has been studied with the help of matrix algebra approach by several researchers [9, 10].

In Ref. [11], Fici and Fiorenzi have a first attempt to study topological properties of CA on the full tree shift A^{Σ^*} , where Σ^* is the free monoid of finite rank $|\Sigma|$. In this case the

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Cayley graph of Σ^* is a regular $|\Sigma|$ -ary rooted tree. Fici and Fiorenzi [11] have studied cellular automata defined on the full k -ary tree shift (for $k \geq 2$). In this paper, we study cellular automata on regular Cayley tree of order 2.

Several notions of entropy of measure-preserving transformation on probability space in ergodic theory have been investigated. The notion of entropy, both topological and measure-theoretical is one of the fundamental invariants in ergodic theory. In the last years, a lot of works have been devoted to this subject [1, 12, 13, 14]. Recall that by the Variational Principle the topological entropy is the supremum of the entropies of invariant measures. In [1], the author has shown that the uniform Bernoulli measure is a measure of maximal entropy for some 1-D LCAs. Morris and Ward [15] proved that an ergodic additive CA in two dimensions has infinite topological entropy (see [16] for details). Recently, Blanchard and Tisseur [17] have introduced the entropy rate of multidimensional CAs and proved several results that show that entropy rate of 2-D CA preserve similar properties of the entropy of 1-D CA.

In this short work, firstly we define cellular automata on Cayley tree (or Bethe lattice) of order 2. This generalizes the case of one-sided CA (where order of the Cayley tree is one). We construct the rule matrix corresponding to finite cellular automata on Cayley tree by using matrix algebra built on the field \mathbb{Z}_p (the set of prime numbers modulo p). Further, we discuss the reversibility problem of this cellular automata. Lastly, we study the measure theoretical entropy of the CAs on Cayley tree. We show that for CAs on Cayley tree the measure entropy with respect to uniform Bernoulli measure is infinity.

2. FINITE CA OVER CAYLEY TREE

Let $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ ($p \geq 2$) be the field of the prime numbers modulo p (\mathbb{Z}_p is called a *state space*). The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k , L is the set of edges of Γ^k and i is the incidence function associating each edge $\ell \in L$ with its end points $x, y \in V$. A configuration σ on V is defined as a function $x \in V \rightarrow \sigma(x) \in \mathbb{Z}_p$; in a similar manner one defines configurations σ_n and ω on V_n and W_n , respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \mathbb{Z}_p^V$ (resp. $\Omega_{V_n} = \mathbb{Z}_p^{V_n}$, $\Omega_{W_n} = \mathbb{Z}_p^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Denote by $\mathbb{Z}_p^{\Gamma^2}$, i.e., the set of all configurations on Γ^2 .

In the sequel we will consider Cayley tree $\Gamma^2 = (V, L, i)$ with the root x_0 . If $i(\ell) = \{x, y\}$, then x and y are called the nearest neighboring vertices and we write $\ell = \langle x, y \rangle$. For $x, y \in V$, the distance $d(x, y)$ on Cayley tree is defined by the formula

$$d(x, y) = \min\{d | x = x_0, x_1, x_2, \dots, x_{d-1}, x_d = y \in V \text{ such that the pairs } \\ \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle \text{ are neighboring vertices}\}.$$

For the fixed root vertex $x^0 \in V$ we have

$$W_n = \{x \in V : d(x^0, x) = n\}$$

$$V_n = \{x \in V : d(x^0, x) \leq n\},$$

$$L_n = \{\ell = \langle x, y \rangle \in L : x, y \in V_n\}.$$

In this section, we will order the elements of V_n in the lexicographical meaning (see [18]) as the Fig. 1. Given two vertices x_u, x_v , the lexicographical order of x_u, x_v is defined as $x_u \preceq x_v$ if and only if $u \preceq v$.

Let us rewrite the elements of W_n in the following order,

$$\overrightarrow{W_n} := (x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}).$$

One can easily compute equations $|W_n| = 3 \cdot 2^{(n-1)}$ and $|V_n| = 1 + 3 \cdot (2^n - 1)$. For the sake of shortness, throughout the paper we are going to represent vertices $x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}$ of W_n by means of the coordinate system as follows:

$$x_{W_n}^{(1)} = x_{11\dots 11}, x_{W_n}^{(2)} = x_{11\dots 12}, x_{W_n}^{(3)} = x_{11\dots 21}, x_{W_n}^{(4)} = x_{11\dots 22},$$

...

$$x_{W_n}^{(|W_n|-3)} = x_{32\dots 211}, x_{W_n}^{(|W_n|-2)} = x_{32\dots 212}, x_{W_n}^{(|W_n|-1)} = x_{32\dots 221}, x_{W_n}^{(|W_n|)} = x_{32\dots 222}.$$

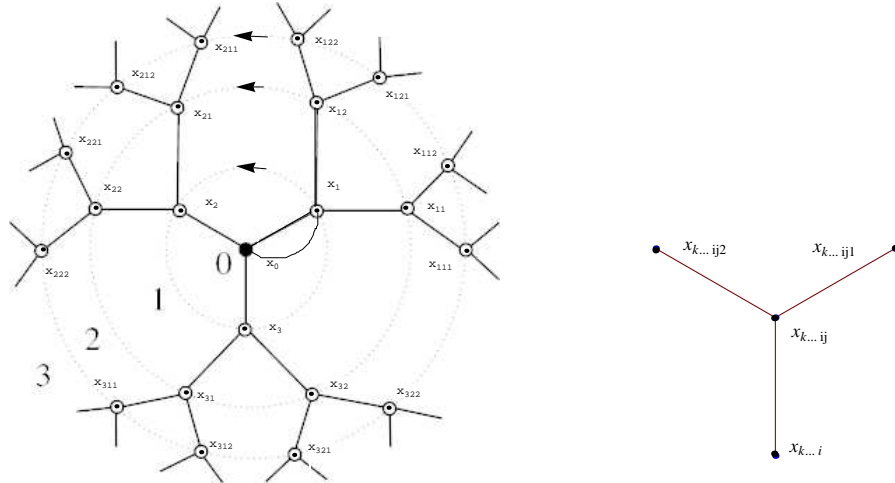


FIGURE 1. a) Cayley tree of order two with levels 3, b) Elements of the nearest neighborhoods surround the center $x_{k...ij}$, $k = 1, 2, 3$ and $i, j = 1, 2$.

In the Fig. 1, we show Cayley tree of order two with levels 3 and the nearest neighborhood which comprises three cells which surround the center cell $x_{k...ij}$. The state $x_{k...ij}^{(t+1)}$ of the cell

$(i, j)^{\text{th}}$ at time $(t + 1)$ is defined by the local rule function $f : \mathbb{Z}_p^4 \rightarrow \mathbb{Z}_p$ as follows:

$$(1) \quad \begin{aligned} x_{k \dots ij}^{(t+1)} &= f(x_{k \dots i}^{(t)}, x_{k \dots ij}^{(t)}, x_{k \dots ij1}^{(t)}, x_{k \dots ij2}^{(t)}) \\ &= ax_{k \dots ij1}^{(t)} + bx_{k \dots ij2}^{(t)} + cx_{k \dots i}^{(t)} + dx_{k \dots ij}^{(t)} \pmod{p}, \end{aligned}$$

where $a, b, c, d \in \mathbb{Z}_p \setminus \{0\}$, $x_{k \dots i}^{(t)} \in W_{n-2}$, $x_{k \dots ij}^{(t)} \in W_{n-1}$ and $x_{k \dots ij1}^{(t)}, x_{k \dots ij2}^{(t)} \in W_n$, $k = 1, 2, 3$ and $i, j = 1, 2$ (see the Fig. 1 (b)).

Specifically, for state $x_0^{(t+1)}$ in the root vertex we can show

$$(2) \quad x_0^{(t+1)} = f(x_0^{(t)}, x_1^{(t)}, x_2^{(t)}, x_3^{(t)}) = ax_1^{(t)} + bx_2^{(t)} + cx_3^{(t)} + dx_0^{(t)} \pmod{p}.$$

Function

$$(3) \quad T_f : \mathbb{Z}_p^{\Gamma^2} \rightarrow \mathbb{Z}_p^{\Gamma^2}$$

is called a cellular automaton (CA) generated by the rules (1) and (2).

If the boundary cells are connected to 0-state, then CA are called Null Boundary CA, i.e., $V \setminus W_n = \{0\}$ for a fixed n . If the same rule is applied to all of the cells in ever evaluation, then those CA are called uniform or regular.

In Sections 3 and 4, we consider linear transformations of finite dimensional vectors spaces corresponding to these finite linear cellular automata by imposing the null boundary condition, which means that the states of cells outside a given ball around the origin are fixed to be zero.

3. CONSTRUCTION OF THE RULE MATRIX IN THE FINITE CASE

In this section, we can characterize finite cellular automata with Null boundary condition over Cayley tree of order two over the field \mathbb{Z}_p . In order to characterize the corresponding rule, first we represent finite Cayley tree n level as a column vector of size $(1 + 3(2^n - 1)) \times 1$.

Let us denote all configurations of Cayley tree with levels n by Ω_n . In order to accomplish this goal we define the following map

$$\Phi : \Omega_n \rightarrow \mathbb{M}_{(1+3(2^n-1)) \times 1}(\mathbb{Z}_p),$$

which takes the t^{th} state $X^{(t)}$ given by

$$\Omega_n \rightarrow X^{(t)} := \left(x_0^{(t)}, x_1^{(t)}, \dots, x_{21 \dots 11}^{(t)}, x_{21 \dots 12}^{(t)}, \dots, x_{32 \dots 211}^{(t)}, x_{32 \dots 212}^{(t)}, x_{32 \dots 221}^{(t)}, x_{32 \dots 222}^{(t)} \right)^T,$$

where the superscript T denotes the transpose and $\mathbb{M}_{(1+3(2^n-1)) \times 1}(\mathbb{Z}_p)$ is the set of matrices with entries \mathbb{Z}_p .

The configuration $\sigma_n^{(t)} \in \Omega_n$ is called the configuration matrix (or information matrix) of the finite CA on Cayley tree with levels n at time t and $\sigma_n^{(0)}$ is initial information matrix of the finite CA. The whole evolution of a particular cellular automata can be comprised in its global transition function [6] (see [6, 19, 20] for the square lattice \mathbb{Z}^2 and see [21] for the hexagonal lattice).

Therefore, one can conclude that $\Phi(\sigma_n^{(t)}) = X_{(1+3(2^n-1)) \times 1}^{(t)}$. Using the identification (3), due to linearity of the finite CA we can define as follows:

$$(M_R^{(n)})_{(1+3(2^n-1)) \times (1+3(2^n-1))} X_{(1+3(2^n-1)) \times 1}^{(t)} = X_{(1+3(2^n-1)) \times 1}^{(t+1)},$$

where n is the number of levels of the Cayley tree.

Theorem 3.1. *Let $a, b, c, d \in \mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$, $n \geq 2$. Then, the rule matrix $(M_R^{(n)})_{1+3(2^n-1) \times 1+3(2^n-1)}$ corresponding to the finite cellular automata on Cayley tree of order two with n -level finite over NB is given by*

$$(4) \quad \begin{pmatrix} d & P & 0_{1 \times 6} & \cdots & 0 & 0 & 0 \\ Q & D_{3 \times 3} & B_{3 \times 6} & 0 & \cdots & 0 & 0 \\ 0 & C_{6 \times 3} & D_{6 \times 6} & B_{6 \times 12} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & C_{3, 2^{n-3} \times 3, 2^{n-4}} & D_{3, 2^{n-3} \times 3, 2^{n-3}} & B_{3, 2^{n-3} \times 3, 2^{n-2}} & 0 \\ 0 & 0 & \cdots & 0 & C_{3, 2^{n-2} \times 3, 2^{n-3}} & D_{3, 2^{n-2} \times 3, 2^{n-2}} & B_{3, 2^{n-2} \times 3, 2^{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & C_{3, 2^{n-1} \times 3, 2^{n-2}} & D_{3, 2^{n-1} \times 3, 2^{n-1}} \end{pmatrix}$$

where each submatrices are as follows: $P = \begin{pmatrix} a & b & c \end{pmatrix}$, $Q = \begin{pmatrix} c \\ c \\ c \end{pmatrix}$,

$$C_{3, 2^{n-i} \times 3, 2^{n-(i+1)}} = \begin{pmatrix} c & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & c & 0 \\ 0 & 0 & \cdots & \cdots & 0 & c & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & c \\ 0 & 0 & \cdots & \cdots & 0 & 0 & c \end{pmatrix},$$

$$B_{3, 2^{n-(i+1)} \times 3, 2^{n-i}} = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a & b & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & a & b & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & a & b \end{pmatrix},$$

and

$$D_{3, 2^{n-i} \times 3, 2^{n-i}} = d \cdot I_{3, 2^{n-i} \times 3, 2^{n-i}}.$$

For $i = 1, 2, \dots, n-1$.

In the theorem 3.1, we have obtained a general form of the matrix representation (or rule matrix) for these linear transformations with respect to a basis given by the lexicographical order on the vertices. We do not include a detailed proof of the theorem which gives the rule matrix of CA. The proof is obtained by determining the image of the basis elements of the space $\mathbb{Z}_p^{(1+3(2^n-1))}$ under the CA. These images contribute to the columns of the rule matrix.

Let us illustrate this in Examples 3.2 and 3.3.

Example 3.2. *If we take the number of level as $n = 2$, then we get the rule matrix M_R of order 10. We consider a configuration $\sigma_2^{(t)}$ of number of levels 2 with null boundary:*

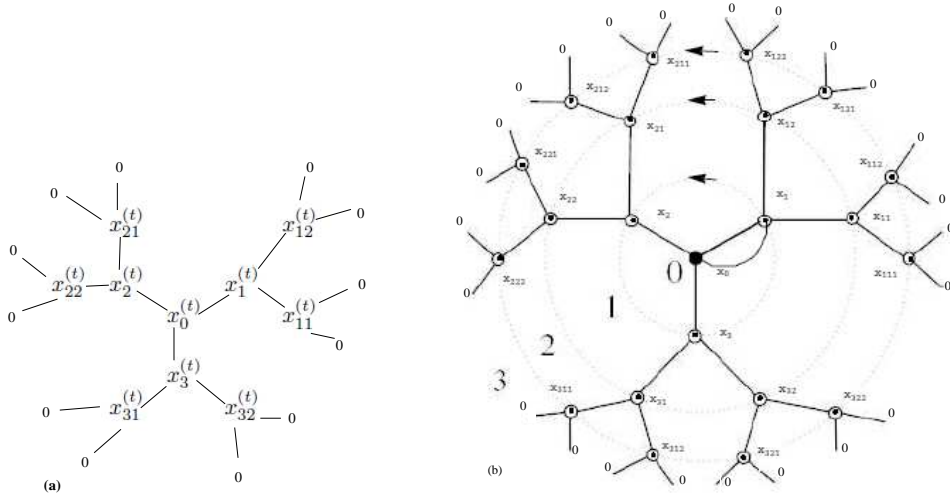


FIGURE 2. A configuration $\sigma_2^{(t)}$ of levels 2 and 3 with null boundary on Cayley tree of order two.

Let us apply the local rules (1) and (2) on configuration $\sigma_2^{(t)}$ in the Fig 2. Then, we get a new configuration under this transformation which is

$$\begin{aligned}
x_0^{(t+1)} &= ax_1^{(t)} + bx_2^{(t)} + cx_3^{(t)} + dx_0^{(t)} \\
x_1^{(t+1)} &= ax_{11}^{(t)} + bx_{12}^{(t)} + cx_0^{(t)} + dx_1^{(t)} \\
x_2^{(t+1)} &= ax_{21}^{(t)} + bx_{22}^{(t)} + cx_0^{(t)} + dx_2^{(t)} \\
x_3^{(t+1)} &= ax_{31}^{(t)} + bx_{32}^{(t)} + cx_0^{(t)} + dx_3^{(t)} \\
x_{11}^{(t+1)} &= cx_1^{(t)} + dx_{11}^{(t)} \\
x_{12}^{(t+1)} &= cx_1^{(t)} + dx_{12}^{(t)} \\
x_{21}^{(t+1)} &= cx_2^{(t)} + dx_{21}^{(t)} \\
x_{22}^{(t+1)} &= cx_2^{(t)} + dx_{22}^{(t)} \\
x_{31}^{(t+1)} &= cx_3^{(t)} + dx_{31}^{(t)} \\
x_{32}^{(t+1)} &= cx_3^{(t)} + dx_{32}^{(t)}.
\end{aligned}$$

Hence, we obtain the rule matrix $M_R^{(2)}$ of order 10 as follows:

$$M_R^{(2)} = \left(\begin{array}{c|cccc|cccccc} d & a & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline c & d & 0 & 0 & a & b & 0 & 0 & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & a & b & 0 & 0 \\ c & 0 & 0 & d & 0 & 0 & 0 & 0 & a & b \\ \hline 0 & c & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & d \end{array} \right).$$

[illegible]

4. REVERSIBILITY OF CA ON CAYLEY TREE WITH NULL BOUNDARY

$$X^{(t+1)} = M_B^{(n)} X^{(t)} \pmod{p}.$$

TABLE 1. The reversibility of finite CAs for some given $a, b, c, d \in \mathbb{Z}_p^*$ and the levels $n = 2, 3$ of Cayley tree of order two.

a	b	c	d	n	p	reversibility of finite CA
1	1	1	1	2	2	irreversible
1	1	1	1	2	3,5,...,101	reversible
2	1	5	2	2	17	irreversible
2	1	3	2	2	17	reversible
2	3	4	3	2	11	irreversible
1	1	1	1	3	3	irreversible
2	2	3	3	3	5	irreversible
2	1	1	3	3	5	reversible
2	2	3	3	3	7,11,13,19,23,29	reversible

If the rule matrix M_R is non-singular, then we have

$$X^{(t)} = (M_R^{(n)})^{-1} X^{(t+1)} \pmod{p}.$$

Thus, in this paper one of our main aims is to study whether the rule matrix $M_R^{(n)}$ in (4) is invertible or not. It is well known that the finite CA is reversible if and only if its rule matrix $M_R^{(n)}$ is non-singular (see [3, 4, 5, 20, 21] for details). If the determinant of a matrix is not equal to zero, then it is invertible, so the CA on Cayley tree is reversible, otherwise it is irreversible. If the CA is not invertible, then one can study "Garden of Eden" for the finite CA (see [11, 19]).

By means of Mathematica, we compute the determinant of the rule matrix for some random $a, b, c, d \in \mathbb{Z}_p^*$ and the levels n of Cayley tree.

$$\det(M_R^{(2)}) = -d^4(c(2(a+b)+c) - d^2)(-(a+b)c + d^2)^2,$$

$$\det(M_R^{(3)}) = -d^8((a+b)c - d^2)^3(-2(a+b)c + d^2)^2((a+b)c^2(a+b+c) - c(3(a+b)+c)d^2 + d^4).$$

We have seen that the CAs are reversible for some given values $a, b, c, d \in \mathbb{Z}_p^*$ and n , for some values the CAs are irreversible.

In the Table 1, we examine under what conditions these linear transformations are invertible, and check invertibility for a list of parameters using computations by means of "Mathematica". For example, if we take as $a = b = c = d = 1$ and $n = 3$, then we can see that the CAs are irreversible for prime numbers $p < 47$. The reversibility of finite CAs on Cayley tree of order two is determined for some given values of $a, b, c, d \in \mathbb{Z}_p^*$ and the levels n of Cayley tree. One can fully characterize reversibility of finite cellular automata with NBC determined by nearest neighbor rule on Cayley tree by computing the determinant of

the matrix in the Eq. (4). Also, one can study the reversibility of finite CAs via rank of the matrix in the Eq. (4) (see [21]).

5. THE MEASURE ENTROPY OF THE CA ON CAYLEY TREE

In this section we study the measure entropy of cellular automata defined by local rules in (1) and (2) on Cayley tree of order two. In order to state our result, we first recall necessary definitions. Let (X, \mathcal{B}, μ, T) be a measure-theoretical dynamical system. If $\alpha = \{A_1, \dots, A_n\}$ and $\beta = \{B_1, \dots, B_m\}$ are two measurable partitions of X , then $\alpha \vee \beta = \{A_i \cap B_j : i = 1, \dots, n; j = 1, \dots, m\}$ is the partition of X . Also, $T^{-1}\alpha$ is the partition of X and $T^{-1}\alpha = \{T^{-1}A_1, \dots, T^{-1}A_n\}$ (see [22, 23] for details).

Definition 5.1. *Let α be a measurable partition of X . The quantity*

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$$

is called the entropy of the partition α . The logarithm is usually taken to the base 2. Let α be a partition with finite entropy, then the quantity

$$h_\mu(T, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

is called the entropy of α with respect to T . The quantity

$$(6) \quad h_\mu(T) = \sup_{\alpha} \{h_\mu(T, \alpha) : \alpha \text{ is a partition with } H_\mu(\alpha) < \infty\}$$

is called the measure-theoretical entropy of (X, \mathcal{B}, μ, T) , the entropy of T (with respect to μ).

Let $\pi = \{\pi_0, \pi_1, \dots, \pi_{p-1}\}$ be a probability vector. Recall that the Bernoulli measure is defined as follows:

$$\mu_\pi([i_0, \dots, i_k]_k) = \pi_{i_1} \pi_{i_0} \dots \pi_{i_k},$$

where $[i_0, \dots, i_k]_k$ is a cylinder set (see [22, 23] for details). If we take the Bernoulli measure as

$$\mu_\pi([i_0, \dots, i_k]_k) = \frac{1}{p} \frac{1}{p} \dots \frac{1}{p} = \frac{1}{p^{k+1}},$$

then the measure is called uniform Bernoulli measure, i.e., for all $i \in \mathbb{Z}_p$, $\mu_\pi([i]) = \frac{1}{p}$, then μ_π is the uniform Bernoulli measure on the space $\mathbb{Z}_p^{\Gamma^2}$. In this paper, we consider uniform Bernoulli measure.

It is clear that due to $(a, p) = 1, (b, p) = 1, (c, p) = 1$ and $(d, p) = 1$, the rules given in the Eqs. (1) and (2) are bipermutative. The following Theorems have been proved:

Theorem 5.2. [24,] *Any left-permutative (right-permutative) cellular is surjective (see [25] for details).*

Theorem 5.3. [26] *If a cellular automaton is surjective then it preserves a uniform Bernoulli measure.*

D'amico *et al.* [14] have proved that for D -dimensional linear CA with $D \geq 2$ the topological entropy must be 0 or infinity (see [27]). In the one-dimensional case, the measure theoretical entropy of the cellular automata is finite [1, 26]. In the following theorem, we prove that the linear CA on Cayley tree of order two has infinite entropy. Let us choose $a, b, c, d \in \mathbb{Z}_p^*$ such that the cellular automata T_f defined in the Eq. (3) is measure-preserving function with respect to (w.r.t.) the uniform Bernoulli measure on the space $\mathbb{Z}_p^{\Gamma^2}$. Then we have the following theorem.

Theorem 5.4. *Let T_f be cellular automata defined by local rules in (1) and (2) on Cayley tree of order two over the field \mathbb{Z}_p . Then the measure theoretical entropy of T_f w.r.t. the uniform Bernoulli measure on the space $\mathbb{Z}_p^{\Gamma^2}$ is infinity.*

Proof. From theorems 5.2 and 5.3, we note that μ_π is a T_f -invariant measure. Let the zero-time partition be given as $\xi(0, 1) = \{0[0]_0, 0[1], \dots, 0[p-1]\}$, we put $\xi(-i, i) = \bigvee_{u=-i}^i \sigma^{-u} \xi$, where σ is the shift map. Since T_f is permutative, one has

$$\bigvee_{k=0}^{n-1} T_f^{-k} \xi(0, 1) = \xi(0, 1 + 3(2^n - 1)).$$

From the definition of measure theoretical entropy w.r.t the measure, we get

$$\begin{aligned} h_{\mu_\pi}(T_f, \xi(0, 1)) &= \lim_{n \rightarrow \infty} \frac{1}{n} H_{\mu_\pi} \left(\bigvee_{k=0}^{n-1} T_f^{-k} \xi(0, 1) \right) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \xi(0, 1+3(2^n-1))} \mu_\pi(A) \log \mu_\pi(A) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} p^{1+3(2^n-1)} \frac{1}{p^{1+3(2^n-1)}} \log \frac{1}{p^{1+3(2^n-1)}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} (1 + 3(2^n - 1)) \log p = \infty. \end{aligned}$$

Therefore, from the Eq. (6), one can conclude that $h_{\mu_\pi}(T_f) = \infty$. □

Remark 5.5. *If we choose the probability vector as $\pi = (1, 0, \dots, 0)$, then $h_{\mu_\pi}(T_f, \xi(0, 1)) = 0$.*

6. CONCLUSIONS

In this short paper, firstly we have defined cellular automata on Cayley tree of order 2. We have constructed the rule matrix corresponding to finite cellular automata on Cayley tree by using matrix algebra built on the field \mathbb{Z}_p (the set of prime numbers modulo p).

Further, we have discussed the reversibility problem of this cellular automata. Lastly, we have studied the measure theoretical entropy of the cellular automata on Cayley tree.

To the best knowledge of the author, it is believed that this is the first instance in the literature where such a connection is established. Thus, this connection between cellular automata and Cayley tree leads to many questions and applications that waits to be explored. Also, investigation of CA on Cayley tree with more higher orders will be studied in the future works.

REFERENCES

- [1] H. Akin, *Entropy* **5** (2003) 233.
- [2] P. Chattopadhyay, P. P. Choudhury and K. Dihidar, *Comput. Math. Appl.* **38** (1999) 207.
- [3] A. R. Khan, P. P. Choudhury, K. Dihidar, S. Mitra, P. Sarkar, *Comput. Math. Appl.* **33** (1997) 79.
- [4] A. R. Khan, P. P. Choudhury, K. Dihidar, R. Verma, *Comput. Math. Appl.* **37** (1999) 115.
- [5] Z. Ying, Y. Zhong and D. Pei-min, *Inform. Sci.* **179** (2009) 613.
- [6] K. Dihidar, P.P. Choudhury, *Inform. Sci.* **165** (2004) 91.
- [7] J. von Neumann, Collected Works, Design of Computers Theory of Automata and Numerical Analysis, Vol. 5, (Pergamon Press, 1951).
- [8] N.Y. Soma, J.P. Melo, *Disc. Appl. Math.* **154** (2006) 861.
- [9] H. Akin, F. Sah, I. Siap, *Internat. J. Modern Phys. C*, **23** (2012) 1.
- [10] Z. Cinkir, H. Akin, I. Siap, *J. Stat. Phys.* **143** (2011) 807..
- [11] G. Fici and F. Fiorenzi, *AUTOMATA 2012, Elec. Proc. in Theoret. Comput. Sci.* **90** (2012) 255.
- [12] H. Akin, *Appl. Math. Comput.* **174** (2006) 1427.
- [13] H. Akin, *Internat. J. Modern Phys. C* **22** (2011) 711.
- [14] M. D'amico, G. Manzini and L. Margara, *Theor. Comput. Sci.* **290** (2003) 1629.
- [15] G. Morris, T. Ward, *Israel J. Math.* **106** (1998) 1.
- [16] T. Meyerovitch, *Ergodic Theory and Dynam. Systems* **28** (2008) 1243.
- [17] F. Blanchard and P. Tisseur, arXiv:1206.6765.
- [18] L. Accardi, F. Mukhamedov and M. Saburov, *Infin. Dimens. Anal. Quantum Probab. Relat.* **14** (2011) 443.
- [19] I. Siap, H. Akin and F. Sah, *Inform. Sci.* **180** (2010) 3562.
- [20] I. Siap, H. Akin and F. Sah, *J. Franklin Inst.* **348** (2011) 1258.
- [21] I. Siap, H. Akin, and S. Uguz, *Comput. Math. Appl.* **62** (2011) 4161.
- [22] M. Denker, C. Grillenberger and K. Sigmund, Ergodic theory on compact spaces. Springer Lectures Notes in Math. **527** Springer Verlag, 1976.
- [23] P. Walters, *An Introduction to Ergodic Theory*, Springer Graduate Texts in Math. **79** New York, 1982.
- [24] G. A. Hedlund, *Math. Syst. Theory* **3** (1969) 320.
- [25] M. A. Shereshevsky, *Mh. Math.* **114** (1992) 305.
- [26] F. Blanchard, P. Kurka and A. Maass, *Physica D* **103** (1997) 86.
- [27] G. Manzini, L. Margara, *Theoret. Comput. Sci.* **221** (1999) 157.

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