Market Timing with Moving Averages: Anatomy and Performance of Trading Rules*

Valeriy Zakamulin†

First draft: June 2014. This revision: May 29, 2016

Abstract

The underlying concept behind the technical trading indicators based on moving averages of prices has remained unaltered for more than half of a century. The development in this field has consisted in proposing new ad-hoc rules and using more elaborate types of moving averages in the existing rules, without any deeper analysis of commonalities and differences between miscellaneous choices for trading rules and moving averages. The first contribution of this paper is to uncover the anatomy of market timing rules with moving averages. Our analysis offers a new and very insightful reinterpretation of the existing rules and demonstrates that the computation of every trading indicator can equivalently be interpreted as the computation of a weighted moving average of price changes. Therefore the performance of any moving average trading rule depends exclusively on the shape of the weighting function for price changes. The second contribution of this paper is a straightforward application of the useful knowledge revealed by our analysis. Specifically, we evaluate the out-of-sample performance of 300 various shapes of the weighting function for price changes using historical data on four financial market indices. The goal of this exercise is to suggest answers to long-standing questions about optimal types of moving averages and whether the best performing trading rule can beat the passive counterpart in out-of-sample tests.

Key words: technical analysis, trading rules, market timing, moving averages, out-of-sample testing

JEL classification: G11, G17.

*The author is grateful to Steen Koekebakker, Pete Nikolai, and the participants of the 3rd Economics & Finance Conference (April 2015, Rome, Italy) for their insightful comments on earlier drafts of this paper. The usual disclaimer applies.

†a.k.a. Valeri Zakamouline, School of Business and Law, University of Agder, Service Box 422, 4604 Kristiansand, Norway, Tel.: (+47) 38 14 10 39, E-mail: `Valeri.Zakamouline@uia.no`

1 Introduction

Technical analysis represents a methodology of forecasting the future price movements through the study of past price data and uncovering some recurrent regularities, or patterns, in price dynamics. One of the fundamental principles of technical analysis is that prices move in trends. Analysts firmly believe that these trends can be identified in a timely manner and used to generate profits and limit losses. Market timing is an active trading strategy that implements this idea in practice. Specifically, this strategy is based on switching between the market and cash depending on whether the prices trend upward or downward. A moving average of prices is one of the oldest and most popular tools used in technical analysis for detecting a trend. Over the past two decades, market timing with moving averages has been the subject of substantial interest on the part of academics[1] and investors alike.

However, despite a series of publications in academic journals, the market timing rules based on moving averages have remained virtually unaltered for more than half of a century. Modern technical analysis still remains art rather than science. The situation with market timing is as follows. There have been proposed many technical trading rules based on moving averages of prices calculated on a fixed size data window. The main examples are: the momentum rule, the price-minus-moving-average rule, the change-of-direction rule, and the double-crossover method. In addition, there are several popular types of moving averages: simple (or equally-weighted) moving average, linearly-weighted moving average, exponentially-weighed moving average, etc. As a result, there exists a large number of potential combinations of trading rules with moving average weighting schemes. One of the controversies about market timing is over which trading rule in combination with which moving average weighting scheme produces the best performance. The situation is further complicated because in order to compute a moving average one must define the size of the averaging window. Again, there is a big controversy over the optimal size of this window. The development in this field has consisted in proposing new ad-hoc rules and using more elaborate types of moving averages (for example, moving averages of moving averages) in the existing rules without any deeper analysis of commonalities

[1] See, among others, Brock, Lakonishok, and LeBaron (1992), Neely, Weller, and Dittmar (1997), Brown, Goetzmann, and Kumar (1998), Sullivan, Timmermann, and White (1999), Lo, Mamaysky, and Wang (2000), Ready (2002), Okunev and White (2003), Ellis and Parbery (2005), Faber (2007), Marshall, Cahan, and Cahan (2008), Fifield, Power, and Knipe (2008), Zhu and Zhou (2009), Gwilym, Clare, Seaton, and Thomas (2010), Neuhierl and Schlusche (2011), Moskowitz, Ooi, and Pedersen (2012), Metghalchi, Marcucci, and Chang (2012), Kilgallen (2012), Clare, Seaton, Smith, and Thomas (2013), Pätäri and Vilska (2014), and Zakamulin (2014).

and differences between miscellaneous choices for trading rules and moving average weighting schemes.

In this paper, we contribute to the literature in two important ways. The first contribution of this paper is to uncover the anatomy of market timing rules with moving averages of prices. Specifically, we present a methodology for examining how the value of a trading indicator is computed. Then using this methodology we study the computation of trading indicators in many market timing rules and analyze the commonalities and differences between the rules. We reveal that despite being computed seemingly different at the first sight, all technical trading indicators considered in this paper are computed in the same general manner. In particular, the computation of every technical trading indicator can equivalently be interpreted as the computation of a weighted moving average of price changes. Consequently, the only real difference, between diverse market timing rules coupled with various types of moving averages, lies in the shape of weighting function used to compute the moving average of price changes.

Our methodology of analyzing the computation of trading indicators for the timing rules based on moving averages offers a broad and clear perspective on the relationship between different rules. We show, for example, that every trading rule can also be presented as a weighted average of the momentum rules computed using different averaging periods. Thus, the momentum rule might be considered as an elementary trading rule on the basis of which one can construct more elaborate rules. In addition, we establish a one-to-one equivalence between a price-minus-moving-average rule and a corresponding moving-average-change-of-direction rule. Overall, our analysis offers a new and very insightful re-interpretation of the existing market timing rules.

The second contribution of this paper is a straightforward application of the useful knowledge revealed by our analysis of anatomy of timing rules and is motivated as follows. In all previous academic studies on the profitability of market timing rules (see the references in footnote 1 above), the researchers usually selected an arbitrary and limited set of so-called "most popular combinations" of trading rules with moving average weighting schemes. Therefore the conclusions on the profitability of market timing rules reached in previous studies are exclusively related to the chosen set of combinations. Put differently, these conclusions cannot be generalized to the entire universe of all potential combinations of trading rules with moving average weighting schemes.

Earlier, in order to select the best combination of a trading rule with a moving average weighting scheme, using relevant historical data a researcher had to perform the tests of all possible combinations in order to find the one with the best performance. This is a daunting and next to impossible task. Our analysis allows a researcher to simplify dramatically this procedure because the performance of any moving average trading rule depends exclusively on the shape of the weighting function for price changes. Therefore, to find the best trading rule one needs only to test various shapes of the weighting function. In this paper we, for the first time, evaluate the out-of-sample[2] performance of 300 various shapes of the weighting function for price changes using historical data on four financial market indices. These shapes are chosen to represent different variations of a few most typical shapes of the weighting functions used in market timing with moving averages. Our findings suggest answers to long-standing questions about optimal types of moving averages and whether the best performing trading rule can beat the passive counterpart in out-of-sample tests.

The rest of the paper is organized as follows. In the subsequent Section 2 we present the moving averages and trading rules considered in the paper. Then in Section 3 we demonstrate the anatomy of trading rules with different moving averages and briefly review an alternative approach to the construction of trading indicators based on moving averages. Section 4 describes our empirical data, the set of weighting functions, the methodology for out-of-sample testing, and the results of the tests. Finally, Section 5 concludes the paper.

2 Moving Averages and Technical Trading Rules

2.1 Moving Averages

A moving average of prices is calculated using a fixed size data "window" that is rolled through time. The length of this window of data, also called the lookback period or averaging period, is the time interval over which the moving average is computed. We follow the standard practice and use prices, not adjusted for dividends, in the computation of moving averages and all technical trading indicators. More formally, let $(P_1, P_2, \ldots, P_T)$ be the observations of the

[2] It is worth mentioning that, to the best knowledge of the author, there are only two papers to date, Sullivan et al. (1999) and Zakamulin (2014), where the researchers implement out-of-sample tests of profitability of some trading rules in the stock market.

monthly[3] closing prices of a stock price index. A moving average at time t is computed using the last closing price P_t and k lagged prices P_{t-j}, $j \in [1, k]$. It is worth noting that the time interval over which the moving average is computed amounts to k months and includes $k+1$ monthly observations. Generally, each price observation in the rolling window of data has its own weight in the computation of a moving average. More formally, a weighted Moving Average at month-end t with k lagged prices (denoted by $MA_t(k)$) is computed as

$$MA_t(k) = \frac{w_t P_t + w_{t-1} P_{t-1} + w_{t-2} P_{t-2} + \ldots + w_{t-k} P_{t-k}}{w_t + w_{t-1} + w_{t-2} + \ldots + w_{t-k}} = \frac{\sum_{j=0}^{k} w_{t-j} P_{t-j}}{\sum_{j=0}^{k} w_{t-j}}, \qquad (1)$$

where w_{t-j} is the weight of price P_{t-j} in the computation of the weighted moving average. It is worth observing that in order to compute a moving average one has to use at least one lagged price, this means that one should have $k \geq 1$. Note that when the number of lagged prices is zero, a moving average becomes the last closing price, that is, $MA_t(0) = P_t$.

The most commonly used types of moving averages are: the Simple Moving Average (SMA), the Linear (or linearly weighted) Moving Average (LMA), and the Exponential Moving Average (EMA). A less commonly used type of moving average is the Reverse Exponential Moving Average (REMA). These moving averages at month-end t are computed as

$$\begin{aligned} SMA_t(k) &= \frac{1}{k+1} \sum_{j=0}^{k} P_{t-j}, \quad LMA_t(k) = \frac{\sum_{j=0}^{k} (k-j+1) P_{t-j}}{\sum_{j=0}^{k} (k-j+1)}, \\ EMA_t(k) &= \frac{\sum_{j=0}^{k} \lambda^j P_{t-j}}{\sum_{j=0}^{k} \lambda^j}, \quad REMA_t(k) = \frac{\sum_{j=0}^{k} \lambda^{k-j} P_{t-j}}{\sum_{j=0}^{k} \lambda^{k-j}}, \end{aligned} \qquad (2)$$

where $0 < \lambda \leq 1$ is a decay factor.

As compared with the simple moving average, either the linearly weighted moving average or the exponentially weighted moving average puts more weight on the more recent price observations. The usual justification for the use of these types of moving averages is a widespread belief that the most recent stock prices contain more relevant information on the future direction of the stock price than earlier stock prices. In the linearly weighted moving average the weights decrease in arithmetic progression. In particular, in $LMA(k)$ the latest observation has weight $k+1$, the second latest k, etc. down to one. A disadvantage of the linearly weighted

[3]Throughout the paper, we assume that the price data comes at the monthly frequency. Yet the results presented in the first part of the paper are valid for any data frequency.

moving average is that the weighting scheme is too rigid. In contrast, by varying the value of λ in the exponentially weighted moving average, one is able to adjust the weighting to give greater or lesser weight to the most recent price. The properties of the exponential moving average:

$$\lim_{\lambda \to 1} EMA_t(k) = SMA_t(k), \quad \lim_{\lambda \to 0} EMA_t(k) = P_t. \tag{3}$$

Contrary to the normal exponential moving average that gives greater weights to the most recent prices, the reverse exponential moving average assigns greater weights to the most oldest prices and decreases the importance of the most recent prices. The properties of the reverse exponential moving average:

$$\lim_{\lambda \to 1} REMA_t(k) = SMA_t(k), \quad \lim_{\lambda \to 0} REMA_t(k) = P_{t-k}. \tag{4}$$

Instead of the regular moving averages of prices considered above, traders sometimes use more elaborate moving averages that can be considered as "moving averages of moving averages". Specifically, instead of using a regular moving average to smooth the price series, some traders perform either double- or triple-smoothing of the price series. The main examples of this type of moving averages are: Triangular Moving Average, Double Exponential Moving Average, and Triple Exponential Moving Average (see, for example, Kirkpatrick and Dahlquist (2010)). To shorten and streamline the presentation, we will not consider these moving averages in our paper. Yet our methodology can be applied to the analysis of the trading indicators based on this type of moving averages in a straightforward manner.

2.2 Technical Trading Rules

Every market timing rule prescribes investing in the stocks (that is, the market) when a Buy signal is generated and moving to cash or shorting the market when a Sell signal is generated. In the absence of transaction costs, the time t return to a market timing strategy is given by

$$r_t = \delta_{t|t-1} r_{Mt} + \left(1 - \delta_{t|t-1}\right) r_{ft}, \tag{5}$$

where r_{Mt} and r_{ft} are the month t returns on the stock market (including dividends) and the risk-free asset respectively, and $\delta_{t|t-1} \in \{0, 1\}$ is a trading signal for month t (0 means Sell and

1 means Buy) generated at the end of month $t-1$.

In each market timing rule the generation of a trading signal is a two-step process. At the first step, one computes the value of a technical trading indicator using the last closing price and k lagged prices

$$\text{Indicator}_t^{TR(k)} = Eq(P_t, P_{t-1}, \ldots, P_{t-k}), \tag{6}$$

where TR denotes the timing rule and $Eq(\cdot)$ is the equation that specifies how the technical trading indicator is computed. At the second step, using a specific function one translates the value of the technical indicator into the trading signal. In all market timing rules considered in this paper, the Buy signal is generated when the value of a technical trading indicator is positive. Otherwise, the Sell signal is generated. Thus, the generation of a trading signal can be interpreted as an application of the following (mathematical) *indicator function* to the value of the technical indicator

$$\delta_{t+1|t} = \mathbf{1}_+ \left(\text{Indicator}_t^{TR(k)}\right), \tag{7}$$

where the indicator function $\mathbf{1}_+(\cdot)$ is defined by

$$\mathbf{1}_+(x) = \begin{cases} 1 \text{ (or Buy signal)} & \text{if } x > 0, \\ 0 \text{ (or Sell signal)} & \text{if } x \leq 0. \end{cases} \tag{8}$$

We start the presentation of trading rules considered in the paper with the Momentum rule (MOM) which is the simplest and most basic market timing rule. In the Momentum rule one compares the last closing price, P_t, with the closing price k months ago, P_{t-k}. In this rule a Buy signal is generated when the last closing price is greater than the closing price k months ago. Formally, the technical trading indicator for the Momentum rule is computed as

$$\text{Indicator}_t^{\text{MOM}(k)} = MOM_t(k) = P_t - P_{t-k}. \tag{9}$$

Then the trading signal is generated by

$$\delta_{t+1|t}^{\text{MOM}(k)} = \mathbf{1}_+ \left(MOM_t(k)\right). \tag{10}$$

Most often, in order to generate a trading signal, a trader compares the last closing price with the value of a k-month moving average. In this case a Buy signal is generated when the last closing price is above a k-month moving average. Otherwise, if the last closing price is below a k-month moving average, a Sell signal is generated. Formally, the technical trading indicator for the Price-Minus-Moving-Average rule (P-MA) is computed as

$$\text{Indicator}_t^{\text{P-MA}(k)} = P_t - MA_t(k). \tag{11}$$

Some traders argue that the price is noisy and the Price-Minus-Moving-Average rule produces many false signals (whipsaws). They suggest to address this problem by employing two moving averages in the generation of a trading signal: one shorter average with averaging period s and one longer average with averaging period $k > s$. This technique is called the Double Crossover Method[4] (DCM). In this case the technical trading indicator is computed as

$$\text{Indicator}_t^{\text{DCM}(s,k)} = MA_t(s) - MA_t(k). \tag{12}$$

It is worth noting the obvious relationship

$$\text{Indicator}_t^{\text{DCM}(0,k)} = \text{Indicator}_t^{\text{P-MA}(k)}. \tag{13}$$

Less often, in order to generate a trading signal, the traders compare the most recent value of a k-month moving average with the value of a k-month moving average in the preceding month. Intuitively, when the stock prices are trending upward (downward) the moving average is increasing (decreasing). Consequently, in this case a Buy signal is generated when the value of a k-month moving average has increased over a month. Otherwise, a Sell signal is generated. Formally, the technical trading indicator for the Moving-Average-Change-of-Direction rule (ΔMA) is computed as

$$\text{Indicator}_t^{\Delta\text{MA}(k)} = MA_t(k) - MA_{t-1}(k). \tag{14}$$

[4]Also known as the Moving Average Crossover (MAC).

3 Anatomy of Trading Rules

3.1 Preliminaries

It has been known for years that there is a relationship between the Momentum rule and the Simple-Moving-Average-Change-of-Direction rule.[5] In particular, note that

$$SMA_t(k-1) - SMA_{t-1}(k-1) = \frac{P_t - P_{t-k}}{k} = \frac{MOM_t(k)}{k}. \tag{15}$$

Therefore

$$\text{Indicator}_t^{\Delta\text{SMA}(k-1)} \equiv \text{Indicator}_t^{\text{MOM}(k)}, \tag{16}$$

where the symbol "≡" means *equivalence*. The equivalence of two technical indicators stems from the following property: *the multiplication of a technical indicator by any positive real number produces an equivalent technical indicator*. This is because the trading signal is generated depending on the sign of the technical indicator. The formal presentation of this property:

$$\mathbf{1}_+ \left(a \times \text{Indicator}_t(k)\right) = \mathbf{1}_+ \left(\text{Indicator}_t(k)\right), \tag{17}$$

where a is any positive real number. Using relation (16) as an illustrating example, observe that if $SMA_t(k-1) - SMA_{t-1}(k-1) > 0$ then $MOM_t(k) > 0$ and vice versa. In other words, the Simple-Moving-Average-Change-of-Direction rule, $\Delta\text{SMA}(k-1)$, generates the Buy (Sell) trading signal when the Momentum rule, $MOM_t(k)$, generates the Buy (Sell) trading signal.

What else can we say about the relationship between different market timing rules? The ultimate goal of this section is to answer this question and demonstrate that all market timing rules considered in this paper are closely interconnected. In particular, we are going to show that the computation of a technical trading indicator for every market timing rule can be interpreted as the computation of the weighted moving average of monthly price changes over the averaging period. We will do it sequentially for each trading rule.

[5]See, for example, `http://en.wikipedia.org/wiki/Momentum_(technical_analysis)`.

3.2 Momentum Rule

The computation of the technical trading indicator for the Momentum rule can equivalently be represented by

$$\begin{aligned}\text{Indicator}_t^{\text{MOM}(k)} &= MOM_t(k) = P_t - P_{t-k} \\ &= (P_t - P_{t-1}) + (P_{t-1} - P_{t-2}) + ... + (P_{t-k+1} - P_{t-k}) = \sum_{i=1}^{k} \Delta P_{t-i},\end{aligned} \tag{18}$$

where $\Delta P_{t-i} = P_{t-i+1} - P_{t-i}$ denotes the monthly price change. Consequently, using property (17), the computation of the technical indicator for the Momentum rule is equivalent to the computation of the equally weighted moving average of the monthly price changes:

$$\text{Indicator}_t^{\text{MOM}(k)} \equiv \frac{1}{k} \sum_{i=1}^{k} \Delta P_{t-i}. \tag{19}$$

3.3 Price-Minus-Moving-Average Rule

First, we derive the relationship between the Price-Minus-Moving-Average rule and the Momentum rule:

$$\begin{aligned}\text{Indicator}_t^{\text{P-MA}(k)} &= P_t - MA_t(k) = P_t - \frac{\sum_{j=0}^{k} w_{t-j} P_{t-j}}{\sum_{j=0}^{k} w_{t-j}} = \frac{\sum_{j=0}^{k} w_{t-j} P_t - \sum_{j=0}^{k} w_{t-j} P_{t-j}}{\sum_{j=0}^{k} w_{t-j}} \\ &= \frac{\sum_{j=1}^{k} w_{t-j} (P_t - P_{t-j})}{\sum_{j=0}^{k} w_{t-j}} = \frac{\sum_{j=1}^{k} w_{t-j} MOM_t(j)}{\sum_{j=0}^{k} w_{t-j}}.\end{aligned} \tag{20}$$

Using property (17), the relation above can be conveniently re-written as

$$\text{Indicator}_t^{\text{P-MA}(k)} \equiv \frac{\sum_{j=1}^{k} w_{t-j} MOM_t(j)}{\sum_{j=1}^{k} w_{t-j}}. \tag{21}$$

Consequently, the computation of the technical indicator for the Price-Minus-Moving-Average rule, $P_t - MA_t(k)$, is equivalent to the computation of the weighted moving average of technical indicators for the Momentum rules, $MOM_t(j)$, for $j \in [1, k]$. It is worth noting that the weighting scheme for computing the moving average of the momentum technical indicators, $MOM_t(j)$, is the same as the weighting scheme for computing the weighted moving average

$MA_t(k)$.

Second, we use identity (18) and rewrite the numerator in (21) as

$$\sum_{j=1}^{k} w_{t-j} MOM_t(j) = \sum_{j=1}^{k} w_{t-j} \sum_{i=1}^{j} \Delta P_{t-i} = w_{t-1}\Delta P_{t-1} + w_{t-2}(\Delta P_{t-1} + \Delta P_{t-2}) + \ldots$$
$$+ w_{t-k}(\Delta P_{t-1} + \Delta P_{t-2} + \ldots + \Delta P_{t-k}) = (w_{t-1} + \ldots + w_{t-k})\Delta P_{t-1} \tag{22}$$
$$+ (w_{t-2} + \ldots + w_{t-k})\Delta P_{t-2} + \ldots + w_{t-k}\Delta P_{t-k} = \sum_{i=1}^{k} \left(\sum_{j=i}^{k} w_{t-j} \right) \Delta P_{t-i}.$$

The last expression tells us that the numerator in (21) is a weighted sum of the monthly price changes over the averaging window, where the weight of ΔP_{t-i} equals $\sum_{j=i}^{k} w_{t-i}$. Thus, another alternative expression for the computation of the technical indicator for the Price-Minus-Moving-Average rule is given by

$$\text{Indicator}_t^{\text{P-MA}(k)} \equiv \frac{\sum_{i=1}^{k} \left(\sum_{j=i}^{k} w_{t-j} \right) \Delta P_{t-i}}{\sum_{i=1}^{k} \left(\sum_{j=i}^{k} w_{t-j} \right)} = \frac{\sum_{i=1}^{k} x_{t-i} \Delta P_{t-i}}{\sum_{i=1}^{k} x_{t-i}}. \tag{23}$$

where

$$x_{t-i} = \sum_{j=i}^{k} w_{t-j} \tag{24}$$

is the weight of the price change ΔP_{t-i}. In words, the computation of the technical indicator for the Price-Minus-Moving-Average rule is equivalent to the computation of the weighted moving average of the monthly price changes in the averaging window.

It is important to note from equation (24) that the application of the Price-Minus-Moving-Average rule usually leads to overweighting the most recent price changes as compared to the original weighting scheme used to compute the moving average of prices. If the weighting scheme in a trading rule is already designed to overweight the most recent prices, then as a rule the trading signal is computed with a much stronger overweighting the most recent price changes. This will be demonstrated below.

Let us now, on the basis of (23), present the alternative expressions for the computation of Price-Minus-Moving-Average technical indicators that use the specific weighting schemes described in the preceding section. We start with the Simple Moving Average which uses the equally weighted moving average of prices. In this case the weight of ΔP_{t-i} is given by

$$x_{t-i} = \sum_{j=i}^{k} w_{t-j} = \sum_{j=i}^{k} 1 = k - i + 1. \tag{25}$$

Consequently, the equivalent representation for the computation of the technical indicator for the Price-Minus-Simple-Moving-Average rule:

$$\text{Indicator}_t^{\text{P-SMA}(k)} \equiv \frac{\sum_{i=1}^{k}(k-i+1)\Delta P_{t-i}}{\sum_{i=1}^{k}(k-i+1)} = \frac{k\Delta P_{t-1} + (k-1)\Delta P_{t-2} + \ldots + \Delta P_{t-k}}{k + (k-1) + \ldots + 1}. \tag{26}$$

This suggests that alternatively we can interpret the computation of the technical indicator for the Price-Minus-Simple-Moving-Average rule as the computation of the linearly weighted moving average of monthly price changes.

We next consider the Linear Moving Average which uses the linearly weighted moving average or prices. In this case the weight of ΔP_{t-i} is given by

$$x_{t-i} = \sum_{j=i}^{k} w_{t-j} = \sum_{j=i}^{k}(k-j+1) = \frac{(k-i+1)(k-i+2)}{2}, \tag{27}$$

which is the sum of the terms of arithmetic sequence from 1 to $k-i+1$ with the common difference of 1. As the result, the equivalent representation for the computation of the technical indicator for the Price-Minus-Linear-Moving-Average rule

$$\text{Indicator}_t^{\text{P-LMA}(k)} \equiv \frac{\sum_{i=1}^{k}\frac{(k-i+1)(k-i+2)}{2}\Delta P_{t-i}}{\sum_{i=1}^{k}\frac{(k-i+1)(k-i+2)}{2}}. \tag{28}$$

Then we consider the Exponential Moving Average which uses the exponentially weighted moving average or prices. In this case the weight of ΔP_{t-i} is given by

$$x_{t-i} = \sum_{j=i}^{k} w_{t-j} = \sum_{j=i}^{k} \lambda^j = \frac{\lambda}{1-\lambda}\left(\lambda^{i-1} - \lambda^k\right), \tag{29}$$

which is the sum of the terms of geometric sequence from λ^i to λ^k. Consequently, the equivalent presentation for the computation of the technical indicator for the Price-Minus-Exponential-

Moving-Average rule

$$\text{Indicator}_t^{\text{P-EMA}(k)} \equiv \frac{\sum_{i=1}^{k} \left(\lambda^{i-1} - \lambda^{k}\right) \Delta P_{t-i}}{\sum_{i=1}^{k} \left(\lambda^{i-1} - \lambda^{k}\right)}. \tag{30}$$

If k is relatively large such that $\lambda^k \approx 0$, then the expression for the computation of the technical indicator for the Price-Minus-Exponential-Moving-Average rule becomes

$$\text{Indicator}_t^{\text{P-EMA}(k)} \equiv \frac{\sum_{i=1}^{k} \lambda^{i-1} \Delta P_{t-i}}{\sum_{i=1}^{k} \lambda^{i-1}} = \frac{\Delta P_{t-1} + \lambda \Delta P_{t-2} + \ldots + \lambda^{k-1} \Delta P_{t-k}}{1 + \lambda + \ldots + \lambda^{k-1}}, \text{ when } \lambda^k \approx 0. \tag{31}$$

In words, the computation of the trading signal for the Price-Minus-Exponential-Moving-Average rule, when k is rather large, is equivalent to the computation of the exponential moving average of monthly price changes. It is worth noting that this is probably the only trading rule where the weighting scheme for the computation of moving average of prices is identical to the weighting scheme for the computation of moving average of price changes.

The weight of ΔP_{t-i} for the Reverse Exponential Moving Average is given by

$$x_{t-i} = \sum_{j=i}^{k} w_{t-j} = \sum_{j=i}^{k} \lambda^{k-j} = \frac{1 - \lambda^{k-i+1}}{1 - \lambda}, \tag{32}$$

which is the sum of the terms of geometric sequence from 1 to λ^{k-i}. Consequently, the equivalent representation for the computation of the technical indicator for the Price-Minus-Reverse-Exponential-Moving-Average rule

$$\text{Indicator}_t^{\text{P-REMA}(k)} \equiv \frac{\sum_{i=1}^{k} \left(1 - \lambda^{k-i+1}\right) \Delta P_{t-i}}{\sum_{i=1}^{k} \left(1 - \lambda^{k-i+1}\right)}. \tag{33}$$

3.4 Moving-Average-Change-of-Direction Rule

The value of this technical trading indicator is based on the difference of two weighted moving averages computed at times t and $t-1$ respectively. We assume that the size of the averaging window is $k-1$ months, the reason for this assumption will become clear very soon. The straightforward computation yields

$$\begin{aligned}\text{Indicator}_t^{\Delta\text{MA}(k-1)} &= MA_t(k-1) - MA_{t-1}(k-1) = \frac{\sum_{i=0}^{k-1} w_{t-i} P_{t-i}}{\sum_{i=0}^{k-1} w_{t-i}} - \frac{\sum_{i=0}^{k-1} w_{t-i} P_{t-i-1}}{\sum_{i=0}^{k-1} w_{t-i}} \\ &= \frac{\sum_{i=0}^{k-1} w_{t-i}\left(P_{t-i} - P_{t-i-1}\right)}{\sum_{i=0}^{k-1} w_{t-i}} = \frac{\sum_{i=1}^{k} w_{t-i+1} \Delta P_{t-i}}{\sum_{i=1}^{k} w_{t-i+1}}.\end{aligned} \tag{34}$$

Consequently, the computation of the technical indicator for the Moving-Average-Change-of-Direction rule can be directly interpreted as the computation of the weighted moving average of monthly price changes:

$$\text{Indicator}_t^{\Delta\text{MA}(k-1)} = \frac{\sum_{i=1}^{k} w_{t-i+1} \Delta P_{t-i}}{\sum_{i=1}^{k} w_{t-i+1}}. \tag{35}$$

Note that the weighting scheme for the computation of the moving average of monthly price changes is the same as for the computation of moving average of prices. From (35) we easily recover the relationship for the case of the Simple Moving Average where $w_{t-i+1} = 1$ for all i

$$\text{Indicator}_t^{\Delta\text{SMA}(k-1)} \equiv \frac{\sum_{i=1}^{k} \Delta P_{t-i}}{k} \equiv \text{Indicator}_t^{\text{MOM}(k)}, \tag{36}$$

where the last equivalence follows from (19).

In the case of the Linear Moving Average, where $w_{t-i+1} = k - i + 1$, we derive a new relationship:

$$\text{Indicator}_t^{\Delta\text{LMA}(k-1)} \equiv \frac{\sum_{i=1}^{k}(k-i+1)\Delta P_{t-i}}{\sum_{i=1}^{k}(k-i+1)} \equiv \text{Indicator}_t^{\text{P-SMA}}(k), \tag{37}$$

where the last equivalence follows from (26). Putting it into words, the Price-Minus-Simple-Moving-Average rule, $P_t - SMA_t(k)$, prescribes investing in the stocks (moving to cash) when the Linear Moving Average of prices over the averaging window of $k-1$ months increases (decreases).

In the case of the Exponential Moving Average and Reverse Exponential Moving Average, the resulting expressions for the Change-of-Direction rules can be written as

$$\text{Indicator}_t^{\Delta\text{EMA}(k\,-\,1)} = \frac{\sum_{i=1}^{k} \lambda^{i-1} \Delta P_{t-i}}{\sum_{i=1}^{k} \lambda^{i-1}}, \tag{38}$$

$$\text{Indicator}_t^{\Delta\text{REMA}(k\,-\,1)} = \frac{\sum_{i=1}^{k} \lambda^{k-i} \Delta P_{t-i}}{\sum_{i=1}^{k} \lambda^{k-i}}. \tag{39}$$

Observe in particular that if k is rather large, then, using result (31), we obtain yet another new relationship:

$$\text{Indicator}_t^{\text{P-EMA}(k)} \equiv \text{Indicator}_t^{\Delta\text{EMA}(k\,-\,1)}, \text{ when } \lambda^k \approx 0. \tag{40}$$

In words, when k is rather large, the Price-Minus-Exponential-Moving-Average rule is equivalent to the Exponential-Moving-Average-Change-of-Direction rule. As it might be observed, for the majority of weighting schemes considered in the paper, there is a one-to-one equivalence between a Price-Minus-Moving-Average rule and a corresponding Moving-Average-Change-of-Direction rule. Therefore, the majority of the moving-average-change-of-direction rules (and may be all of them) can also be expressed as the moving average of Momentum rules.

Finally it is worth commenting that the traders had long ago taken notice of the fact that, for example, very often a Buy signal is generated first by the Price-Minus-Moving-Average rule, then with some delay a Buy signal is generated by the Moving-Average-Change-of-Direction rule. Therefore the traders sometimes use the trading signal of the Moving-Average-Change-of-Direction rule to "confirm" the signal of the Price-Minus-Moving-Average rule (see Murphy (1999), Chapter 9). Our analysis provides a simple explanation for the existence of a delay between the signals generated by these two rules. Specifically, the delay naturally occurs because the Price-Minus-Moving-Average rule overweights more heavily the most recent price changes than the Moving-Average-Change-of-Direction rule computed using the same weighting scheme. Therefore the Price-Minus-Moving-Average rule reacts more quickly to the recent trend changes than the Moving-Average-Change-of-Direction rule.[6]

[6]Assume, for example, that the trader uses the simple moving average weighting scheme in both the rules. In this case our result says that the Price-Minus-Simple-Moving-Average rule is equivalent to the Linear-Moving-Average-Change-of-Direction rule. As a consequence, it is naturally to expect that the Price-Minus-Simple-Moving-Average rule reacts more quickly to the recent trend changes than the Simple-Moving-Average-Change-of-Direction rule.

3.5 Double Crossover Method

The relationship between the Double Crossover Method and the Momentum rule is as follows (here we use result (20))

$$\begin{aligned}\text{Indicator}_t^{\text{DCM}(s,\,k)} &= MA_t(s) - MA_t(k) = (P_t - MA_t(k)) - (P_t - MA_t(s)) \\ &= \frac{\sum_{j=1}^{k} w_{t-j}^{k} MOM_t(j)}{\sum_{j=0}^{k} w_{t-j}^{k}} - \frac{\sum_{j=1}^{s} w_{t-j}^{s} MOM_t(j)}{\sum_{j=0}^{s} w_{t-j}^{s}}.\end{aligned} \tag{41}$$

Different superscripts in the weights mean that for the same subscript the weights are generally not equal. For example, in case of either linearly weighted moving averages or reverse exponential moving averages $w_{t-j}^{k} \neq w_{t-j}^{s}$, yet for the other weighting schemes considered in this paper $w_{t-j}^{k} = w_{t-j}^{s}$. In order to get a closer insight into the anatomy of the Double Crossover Method, we assume that one uses the exponential weighting scheme in the computation of moving averages (as it most often happens in practice). In this case the expression for the value of the technical indicator in terms of monthly price changes is given by (here we use results (22) and (29))

$$\begin{aligned}\text{Indicator}_t^{\text{DCM}(s,\,k)} &= \frac{\sum_{j=1}^{k} \lambda^j \sum_{i=1}^{j} \Delta P_{t-i}}{\sum_{j=0}^{k} \lambda^j} - \frac{\sum_{j=1}^{s} \lambda^j \sum_{i=1}^{j} \Delta P_{t-i}}{\sum_{j=0}^{s} \lambda^j} = \frac{\sum_{i=1}^{k} \left(\sum_{j=i}^{k} \lambda^j\right) \Delta P_{t-i}}{\sum_{j=1}^{k} \lambda^j} \\ &- \frac{\sum_{i=1}^{s} \left(\sum_{j=i}^{s} \lambda^j\right) \Delta P_{t-i}}{\sum_{j=1}^{s} \lambda^j} = \frac{\sum_{i=1}^{k} \left(\lambda^i - \lambda^{k+1}\right) \Delta P_{t-i}}{1 - \lambda^{k+1}} - \frac{\sum_{i=1}^{s} \left(\lambda^i - \lambda^{s+1}\right) \Delta P_{t-i}}{1 - \lambda^{s+1}}.\end{aligned} \tag{42}$$

If we assume in addition that both s and k are relatively large such that $\lambda^s \approx 0$ and $\lambda^k \approx 0$, then we obtain

$$\text{Indicator}_t^{\text{DCM}(s,\,k)} \approx \sum_{i=1}^{k} \lambda^i \Delta P_{t-i} - \sum_{i=1}^{s} \lambda^i \Delta P_{t-i} = \sum_{i=s+1}^{k} \lambda^i \Delta P_{t-i}. \tag{43}$$

The expression above can be conveniently re-written as

$$\text{Indicator}_t^{\text{DCM}(s,\,k)} \equiv \frac{\sum_{i=s+1}^{k} \lambda^{i-s-1} \Delta P_{t-i}}{\sum_{j=s+1}^{k} \lambda^{i-s-1}} \text{ when } k > s, \lambda^s \approx 0, \lambda^k \approx 0. \tag{44}$$

In words, the computation of the trading signal for the Double Crossover Method based on the exponentially weighted moving averages of lengths s and $k > s$, when both s and k are rather large, is equivalent to the computation of the exponentially weighted moving average of monthly price changes, ΔP_{t-i}, for $i \in [s+1, k]$. Note that the most recent s monthly price changes completely disappear in the computation of the technical trading indicator. In other words, in the computation of the trading indicator one disregards, or skips, the most recent s monthly price changes. When the values of s and k are not rather large, the most recent s monthly price changes do not disappear in the computation of the technical indicator, yet the weights of these price changes are reduced as compared to the weight of the subsequent $(s+1)$-th price change.

3.6 Discussion

Summing up the results presented above, all technical trading indicators considered in this paper are computed in the same general manner. We find, for instance, that the computation of every technical trading indicator can be interpreted as the computation of a weighted average of the momentum rules computed using different averaging periods. Thus, the momentum rule might be considered as an elementary trading rule on the basis of which one can construct more elaborate rules. The most insightful conclusion emerging from our analysis is that the computation of every technical trading indicator, based on moving averages of prices, can also be interpreted as the computation of the weighted moving average of price changes. More formally, our analysis shows that the value of every trading indicator can alternatively be computed using the following general formula

$$\text{Indicator}_t^{\text{TR}(k)} \equiv \frac{\sum_{i=1}^{k} x_{t-i} \Delta P_{t-i}}{\sum_{i=1}^{k} x_{t-i}}, \tag{45}$$

where x_{t-i} is the weight of the price change ΔP_{t-i}.

Our main conclusion is that, despite being computed seemingly different at the first sight, the only real difference between miscellaneous rules lies in the weighting scheme used to compute the moving average of price changes. Figure 1 illustrates a few distinctive weighting schemes for the computation of technical trading indicators based on moving averages. In particular, this figure illustrates the weighting schemes for the Momentum rule, the Price-Minus-

Reverse-Exponential-Moving-Average rule (with $\lambda = 0.8$), the Price-Minus-Simple-Moving-Average rule, the Price-Minus-Linear-Moving-Average rule, and the Double Crossover Method (based on using two exponential moving averages with $\lambda = 0.8$). For all technical indicators we use $k = 10$ which means that to compute the value of a technical indicator we use the most recent price change, ΔP_{t-1}, denoted as Lag0, and 9 preceding lagged price changes up to lag ΔP_{t-10}, denoted as Lag9. In addition, in the computation of the technical indicator for the Double Crossover Method we use $s = 3$.

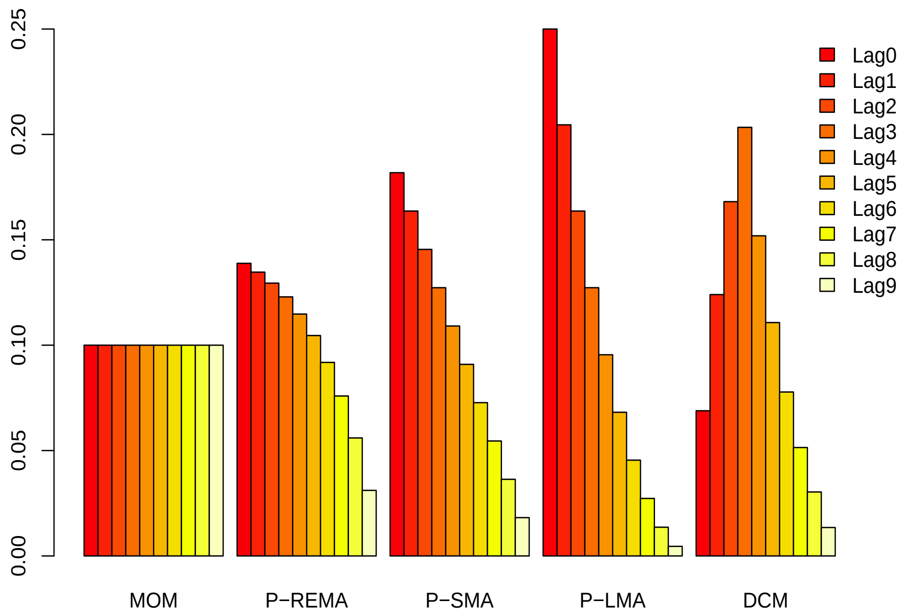


Figure 1: Weights of monthly price changes used for the computations of the technical trading indicators with $k = 10$. **MOM** denotes the Momentum rule. **P-REMA** denotes the Price-Minus-Reverse-Exponential-Moving-Average rule (with $\lambda = 0.8$). **P-SMA** denotes the Price-Minus-Simple-Moving-Average rule. **P-LMA** denotes the Price-Minus-Linear-Moving-Average rule. **DCM** denotes the Double Crossover Method (based on using two exponential moving averages with $\lambda = 0.8$ and $s = 3$). **Lag**$(i-1)$ denotes the weight of the lag ΔP_{t-i}, where Lag0 denotes the most recent price change ΔP_{t-1} and Lag9 denotes the most oldest price change ΔP_{t-10}.

Apparently, the Momentum rule assigns equal weights to all monthly price changes in the averaging window. The next three rules overweight the most recent price changes. They are arranged according to increasing degree of overweighting. Whereas the Price-Minus-Simple-Moving-Average rule employs the linear weighting scheme, the degree of overweighting in the Price-Minus-Reverse-Exponential-Moving-Average rule can be gradually varied from the equal weighting scheme (when $\lambda = 0$) to the linear weighting scheme (when $\lambda = 1$), see property (4). Formally this can be expressed by

$$\lim_{\lambda \to 0} \text{Indicator}_t^{\text{P-REMA}(k)} = \text{Indicator}_t^{\text{MOM}(k)}, \quad \lim_{\lambda \to 1} \text{Indicator}_t^{\text{P-REMA}(k)} = \text{Indicator}_t^{\text{P-SMA}(k)}. \tag{46}$$

Comparing to the Price-Minus-Simple-Moving-Average rule, a higher degree of overweighting can be attained by using the Exponential-Moving-Average-Change-of-Direction rule. The degree of overweighting in this rule can be gradually varied from the linear weighting scheme (when $\lambda = 1$) to the very extreme overweighting where only the most recent price change has a non-zero weight (when $\lambda = 0$), see property (3). Formally this can be expressed by

$$\lim_{\lambda \to 1} \text{Indicator}_t^{\Delta\text{EMA}(k)} = \text{Indicator}_t^{\text{MOM}(k)}, \quad \lim_{\lambda \to 0} \text{Indicator}_t^{\Delta\text{EMA}(k)} = \Delta P_{t-1}. \tag{47}$$

When $\lambda \approx 0.82$, the degree of overweighting the most recent price changes in the Exponential-Moving-Average-Change-of-Direction rule is virtually the same as in the Price-Minus-Linear-Moving-Average rule. Therefore, we demonstrate only the weighting scheme in the Price-Minus-Linear-Moving-Average rule.

In contrast to the previous rules, the weighting scheme in the Double Crossover Method underweights both the most recent and the most old price changes. In this weighting scheme the price change $\Delta P_{t-s-1} = \Delta P_{t-4}$ has the largest weight in the computation of moving average.

Our alternative representation of the computation of technical trading indicators by means of the moving average of price changes, together with the graphical visualization of the weighting schemes for different rules presented in Figure 1, reveals a couple of paradoxes. The first paradox consists in the following. Many traders argue that the most recent stock prices contain more relevant information on the future direction of the stock price than earlier stock prices. Therefore, one should better use the $LMA(k)$ instead of the $SMA(k)$ in the computation of trading signals. Yet in terms of the monthly price changes the application of the Price-Minus-Simple-Moving-Average rule already leads to overweighting the most recent price changes. If it is the most recent stock price changes (but not prices) that contain more relevant information on the future direction of the stock price, then the use of the Price-Minus-Linear-Moving-

Average rule leads to a severe overweighting the most recent price changes, which might be suboptimal.

The other paradox is related to the effect produced by the use of a shorter moving average in the computation of a trading signal for the Double Crossover Method. Specifically, our alternative representation of the computation of technical trading indicators reveals an apparent conflict of goals that some traders want to pursue. In particular, on the one hand, one wants to put more weight on the most recent prices that are supposed to be more relevant. On the other hand, one wants to smooth the noise by using a shorter moving average instead of the last closing price (as in the Price-Minus-Moving-Average rule). It turns out that these two goals cannot be attained simultaneously because the noise smoothing results in a substantial reduction of weights assigned to the most recent price changes (and, therefore, most recent prices). Figure 1 clearly demonstrates that the weighting scheme for the Double Crossover Method has a hump-shaped form such that the largest weight is given to the monthly price change at lag s. Then, as the lag number decreases to 0 or increases to $k-1$, the weight of the lag decreases. Consequently, the use of the Double Crossover Method can be justified only when the price change at lag s contains the most relevant information on the future direction of the stock price.

3.7 Alternative Construction of Trading Indicators

Let $\{p_t\}$ be the series of observations of the log-prices of a stock index. That is, $p_t = \log(P_t)$ where P_t is the month t closing price. The trading indicators based on moving averages can, in principle, be constructed alternatively using the log-prices

$$\text{Indicator}_t^{TR(k)} = Eq(p_t, p_{t-1}, \dots, p_{t-k}). \tag{48}$$

In this case the straightforward application of our methodology (for examining how the value of a trading indicator is computed) leads to the following general formula for the computation of the value of trading indicator

$$\text{Indicator}_t^{\text{TR}(k)} \equiv \frac{\sum_{i=1}^{k} x_{t-i} q_{t-i}}{\sum_{i=1}^{k} x_{t-i}}, \tag{49}$$

where $q_{t-i} = p_{t-i+1} - p_{t-i}$ is the log-return on the index over $(t-i, t-i+1)$ and x_{t-i} is the weight of the log-return q_{t-i} in the computation of moving average. In words, in this case the value of any trading indicator based on moving averages of log-prices can alternatively be computed using a weighting moving average of log-returns.

In fact, Hong and Satchell (2015) presented already in 2013 the result that can be considered as a particular case of equation (49) when the trading rule is DCM(s, k) where in both shorter and longer windows one uses the SMA weighting scheme. Later on Beekhuizen and Hallerbach (2015) considered other types of trading rules and derived[7] several particular cases of general equation (49).

4 Best Performing Weighting Schemes in Out-of-Sample Tests

4.1 Data

The data for our empirical study in this section are similar to the data used in the study by Zakamulin (2014). Specifically, we use data on two stock market indices, two bond market indices, and the risk-free rate of return. The two stock market indices are the Standard and Poor's Composite stock price index and the Dow Jones Industrial Average index. The two bond market indices are the Long-Term and Intermediate-Term US Government Bond indices. Our sample period begins in January 1926 and ends in December 2012 (87 full years), giving a total of 1044 monthly observations.

We use the monthly Standard and Poor's Composite stock price index data and corresponding dividend data provided by Amit Goyal.[8] From 1926 to 1956, the index data come from various reports of the Standard and Poor's. From 1957 this index is identical to the Standard and Poor's 500 index. For more details about the construction of the index and its dividend series see Welch and Goyal (2008). The DJIA index values for the total sample period and dividends for the period 1988 to 2012 are provided by S&P Dow Jones Indices LLC, a subsidiary of the McGraw-Hill Companies.[9] The dividends for the period 1926 to 1987 are obtained from Barron's.[10]

[7] Their paper appeared several months after our paper was made available on the Internet.
[8] See http://www.hec.unil.ch/agoyal/.
[9] See http://www.djaverages.com.
[10] See http://online.barrons.com.

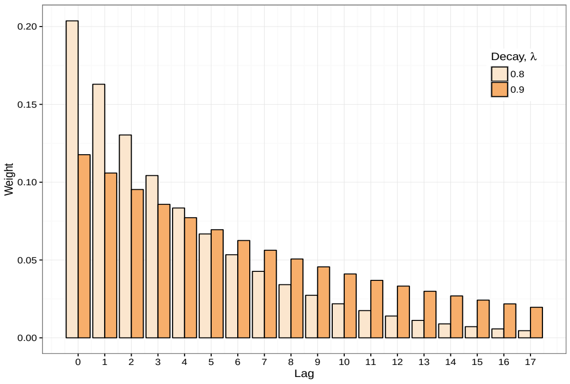
The bond data are from the Ibbotson SBBI 2013 Classic Yearbook. We use both the capital appreciation returns and total returns on the Long-Term and Intermediate-Term Government Bonds. The risk-free rate of return is also provided by Amit Goyal. In particular, the risk-free rate of return for our sample period is the Treasury bill rate.

4.2 Empirical Research Design

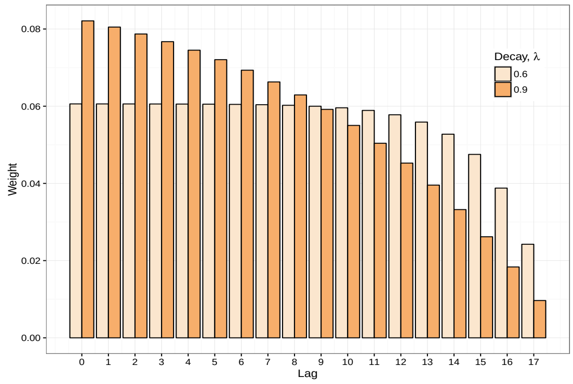
4.2.1 The Set of Weighting Schemes

The generation of different shapes of the moving average weighting function is based on the following idea. Even though there are various combinations of trading rules based on moving averages of prices coupled with various types of moving averages, all these combinations result in basically only three types of the shape of the weighting function: equal weighting of price changes (as in the MOM rule), underweighting the most old price changes (as in the P-MA rule or in the most ΔMA rules), and underweighting both the most recent and the most old price changes (as in the DCM). In order to generate these shapes, we employ three types of weighting schemes based on exponential moving averages: (1) convex EMA weighting scheme (CV-EMA) produced by ΔEMA(k) trading rule, (2) concave EMA weighting scheme (CC-EMA) produced by P-REMA(k) trading rule, and (3) hump-shaped EMA weighting scheme (HS-EMA) produced by DCM(s, k) trading rule where in both short and long windows we use concave EMA weighting schemes. There is an uncertainly about the proper choice of the size of the shorter window s in the DCM rule. Since the most popular combination in practice is to use a 200-day long window and a 50-day short window, we set $s = \frac{1}{4}k$ for all values of k.

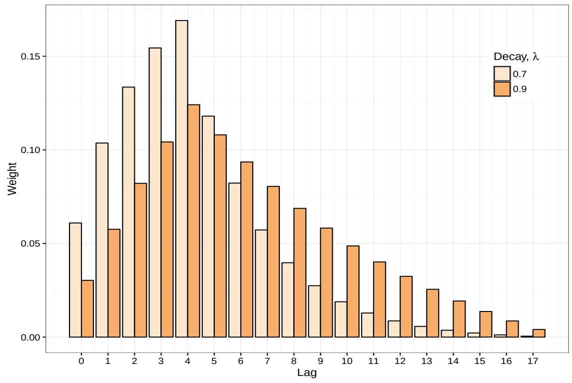
For some fixed number of price change lags k, the shape of each type of a moving average weighting function depends on the value of the decay factor λ. In order to generate many different shapes of the weighting function, in each trading rule we vary the value of $\lambda \in \{0.00, 1.00\}$ with a step of $\Delta\lambda = 0.01$. As a result, for each type of the EMA we get 100 different shapes. Since we have three different types of the EMA, the total number of generated shapes amounts to 300. As a result, we obtain 300 different trading strategies; each strategy is specified by a particular shape of the moving average weighting function. Figure 2 illustrates the shapes of each type of weighting functions for two arbitrary values of λ. Both CV-EMA and CC-EMA weighting schemes underweight the most old price changes. Yet, whereas in



Panel A: Convex EMA (CV-EMA) weighting scheme



Panel B: Concave EMA (CC-EMA) weighting scheme



Panel C: Hump-shaped EMA (HS-EMA) weighting scheme

Figure 2: The types of the moving average weighting schemes used in our empirical study. Panel A illustrates the convex exponential moving average weighting scheme produced by ΔEMA(k) trading rule. Panel B illustrates the concave exponential moving average weighting scheme produced by P-REMA(k) trading rule. Panel C illustrates the hump-shaped exponential moving average weighting scheme produced by DCM(s, k) trading rule. $\boldsymbol{\lambda}$ denotes the decay factor. In all illustrations the number of price changes $k = 18$. **Lag** denotes the weight of the lag ΔP_{t-i}, where Lag0 denotes the most recent price change ΔP_{t-1} and Lag17 denotes the most oldest price change ΔP_{t-18}.

the CV-EMA the weight of the price lag i is a convex exponential function with respect to i (see equation (38)), in the CC-EMA the weight of the price lag i is a concave exponential function with respect to i (see equation (33)). It is worth repeating (recall the discussion in Section 3.6) that by varying the value of λ from 0 to 1, the weighting scheme of the CC-EMA varies from the equal weighting scheme (when $\lambda = 0$) to the linear weighting scheme (when $\lambda = 1$); the weighting scheme of the CV-EMA varies from the very extreme overweighting (when $\lambda = 0$, only the most recent price change has a non-zero weight) to the linear weighting scheme (when $\lambda = 1$). Last but not least, the HS-EMA with $\lambda = 1$ is equivalent to using the linear weighting schemes in both the shorter and longer windows (in this case the trading signal can be generated by SMA(s)-SMA(k)).

4.2.2 Performance Measurement in Out-of-Sample Tests

We closely follow the methodology used in the study by Zakamulin (2014). Each shape of the weighting function in our study is associated with a trading rule denoted by $TR(k)$. Since our goal is to estimate the real-life performance of trading rules, we need also to account for the fact that the rebalancing an active portfolio incurs transaction costs. We suppose that buying and selling stocks and bonds is costly, whereas buying and selling Treasury bills is costless. Denoting by ν the one-way transaction costs, the return to the trading rule over month t is given by

$$r_t = \begin{cases} r_{Pt} & \text{if } (\delta_t = \text{Buy}) \text{ and } (\delta_{t-1} = \text{Buy}), \\ r_{Pt} - \nu & \text{if } (\delta_t = \text{Buy}) \text{ and } (\delta_{t-1} = \text{Sell}), \\ r_{ft} & \text{if } (\delta_t = \text{Sell}) \text{ and } (\delta_{t-1} = \text{Sell}), \\ r_{ft} - \nu & \text{if } (\delta_t = \text{Sell}) \text{ and } (\delta_{t-1} = \text{Buy}), \end{cases}$$

where r_{Pt} denotes the dividend-adjusted return to the passive counterpart of the active trading rule (either stock or bond index return over month t). We assume that the one-way transaction costs in the stock market amount to 0.25% ($\nu = 0.0025$), whereas in the bond market the one-way transaction costs amount to 0.10% ($\nu = 0.001$).

The performance is measured by means of the Sharpe ratio. Specifically, the Sharpe ratio of a trading rule with excess returns $r_t^e = r_t - r_{ft}$ is computed as (according to Sharpe (1994))

$$SR(r_t^e) = \frac{\mu(r_t^e)}{\sigma(r_t^e)},$$

where $\mu(r_t^e)$ and $\sigma(r_t^e)$ denote the mean and standard deviation of r_t^e respectively.

It is crucial to observe that in order to compute the value of the technical indicator we need to specify the size of the averaging window k. The out-of-sample performance measurement method is based on simulating the real-life trading where a trader has to make a choice of what size of the averaging window k to use given the information about the past performances of the trading rule for different values of k. Specifically, the out-of-sample testing procedure begins with splitting the full historical data sample $[1, T]$ into the initial in-sample subset $[1, \tau]$ and out-of-sample subset $[\tau + 1, T]$, where T is the last observation in the full sample and τ denotes the splitting point. The initial in-sample period of $[1, \tau]$ is used to complete the procedure of selecting the value of k which produces the best performance. That is, the choice of the optimal k^*_τ is given by

$$k^*_\tau = \arg \max_{k \in [k^{\min}, k^{\max}]} SR(r^e_1, r^e_2, \ldots, r^e_\tau),$$

where $k^{\min}$ and $k^{\max}$ are the minimum and maximum values for k, and $SR(r^e_1, r^e_2, \ldots, r^e_\tau)$ denotes the trading rule's Sharpe ratio computed using the excess returns from month 1 to month τ. Subsequently, the trading signal for month $\tau + 1$ is determined using the $TR(k^*_\tau)$ rule. We then expand the in-sample period by one month, perform the selection of the value of k which produces the best performance once again using the new in-sample period of $[1, \tau + 1]$, and determines the trading signal for month $\tau + 2$ using the $TR(k^*_{\tau+1})$ rule. We repeat this procedure, pushing the endpoint of the in-sample period ahead by one month with each iteration of this process, until the trading signal for the last month T is determined.

The out-of-sample performance of a trading strategy is measured by computing trading rule's Sharpe ratio using the excess returns over the out-of-sample period, $(r^e_{\tau+1}, r^e_{\tau+2}, \ldots, r^e_T)$. To facilitate the performance comparison, we compute the Sharpe ratio of the passive counterpart of the active trading rule using the excess returns over the same out-of-sample period $(r^e_{P,\tau+1}, r^e_{P,\tau+2}, \ldots, r^e_{P,T})$ and report the difference between the Sharpe ratio of the trading rule and the Sharpe ratio of the passive strategy

$$\Delta SR = SR_{TR} - SR_P,$$

where SR_{TR} and SR_P denote the Sharpe ratios of the trading rule and its passive benchmark respectively. Because the estimate for a Sharpe ratio is subject to estimation errors, we have scientific evidence that a trading rule outperforms its passive counterpart only when we can reject the following null hypothesis

$$H_0 : \Delta SR \leq 0.$$

This hypothesis is tested using Jobson and Korkie (1981) test with the Memmel (2003) correction. Specifically, given SR_{TR}, SR_P, and ρ as two estimated Sharpe ratios and correlation coefficient between the excess returns of the active and passive strategies over a sample of size

T, the test of the null hypothesis is obtained via the test statistic

$$z = \frac{SR_{TR} - SR_P}{\sqrt{\frac{1}{T}\left[2(1-\rho^2) + \frac{1}{2}(SR_{TR}^2 + SR_P^2 - 2\rho^2 SR_{TR} SR_P)\right]}},$$

which is asymptotically distributed as a standard normal.

4.3 Empirical Results

For each stock and bond market index, we perform out-of-sample simulation of the returns to 300 different trading rules (where each one is associated with a specific shape of the weighting function) over the period January 1930 to December 2012. Since the most typical recommendation for the size of the averaging window varies from 10 to 12 months, to be on the safe side we set $k^{\min} = 4$ and $k^{\max} = 18$. For each index, Table 1 reports the top 10 best performing weighting schemes together with their decay factors and the mean sizes of the averaging window[11] $k+1$, the difference between the Sharpe ratio of the trading rule and the Sharpe ratio of its passive counterpart ΔSR, and the p-value of testing the null hypothesis $H_0 : \Delta SR \leq 0$.

For the Standard and Poor's Composite index, 7 out of 10 best performing weighting schemes belong to the HS-EMA type where the decay factor varies in the range from 0.95 to 1.00. It is worth noting that in the best performing weighting scheme the decay factor equals to 1.00 which means that the best performing trading rule can be implemented as the difference between SMA(s) and SMA(k). Interestingly, since the mean value of $k+1$ equals to 9 and, therefore, the mean value of $s+1$ equals to 3, the best performing weighting scheme closely corresponds to the very popular among practitioners DCM rule where one uses 50-day and 200-day simple moving averages. The CC-EMA weighting scheme with $\lambda = 0.82$ and the CV-EMA weighting scheme with $\lambda \in \{0.94, 0.95\}$ are also among the top 10 best performing weighting schemes. The major types among the top 10 best performing weighting schemes for the Standard and Poor's Composite index are illustrated in Figure 3, Panel A. Whereas the Sharpe ratio of the passive strategy amounts to 0.38, the Sharpe ratio of a weighting scheme, that belongs to the top 10 best ones, exceeds the Sharpe ratio of the passive strategy by 0.12-0.15. For 9 out of 10 best performing weighting schemes we can reject the null hypothesis at

[11]Note that in our exposition the value of k denotes the number of the lagged price changes. Therefore the value of $k+1$ equals the number of prices used to compute the value of the trading indicator.

the 10% level in favor of the alternative hypothesis that the Sharpe ratio of the trading rule is greater than the Sharpe ratio of the passive strategy.

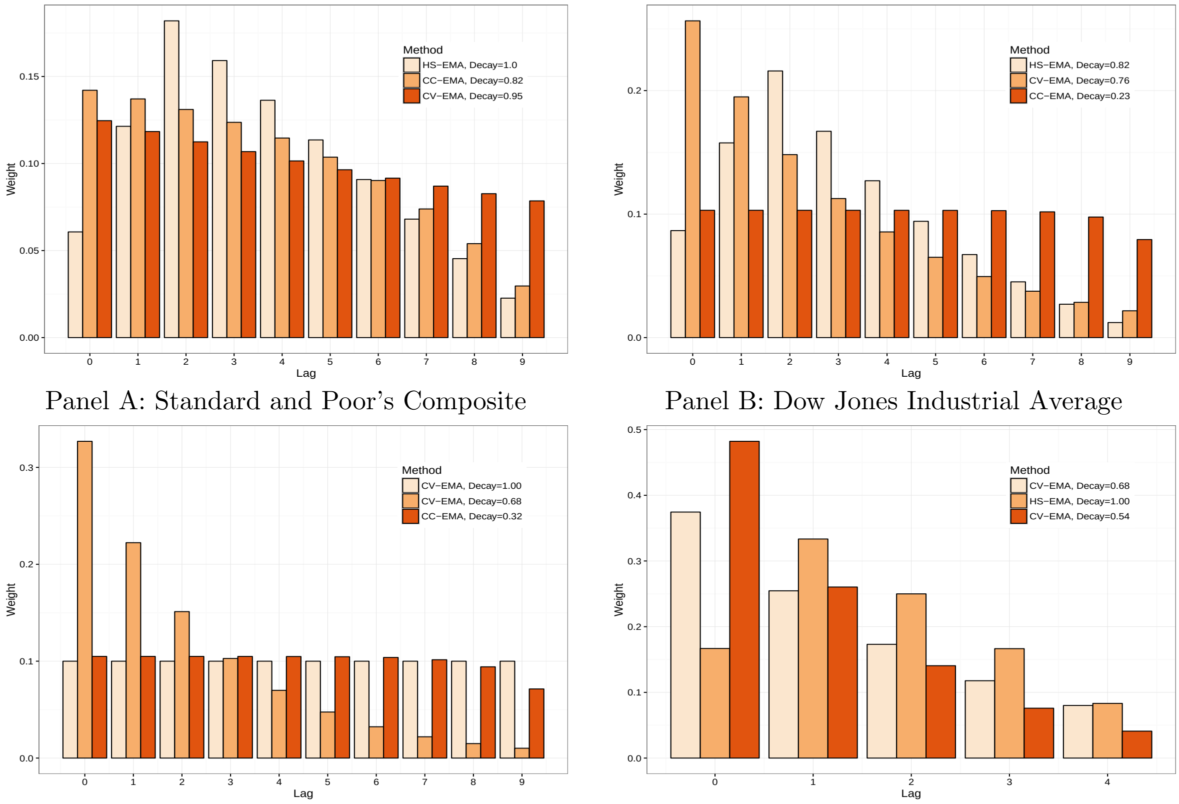
For the Dow Jones Industrial Average index, among the top 10 best performing schemes 3 belong to the HS-EMA type, 4 to the CC-EMA type, and 3 to the CV-EMA type. As for the Standard and Poor's Composite index, the best performing weighting scheme also belongs to the HS-EMA type. In contrast to the parameters of the best performing HS-EMA scheme for the Standard and Poor's Composite index, in this case the HS-EMA scheme uses a substantially longer length of the averaging window (15 versus 9) and a notable lower decay factor (0.82 versus 1.00). The major types among the top 10 best performing weighting schemes for the Dow Jones Industrial Average index are illustrated in Figure 3, Panel B. Interestingly, 3 out of 4 CC-EMA weighting schemes (that are among the top 10 best ones) have a decay factor in the range 0.21-0.23. As a result, the weighting in these schemes is close to the equal weighting of price changes as in the MOM rule. The Sharpe ratio of the passive strategy also amounts to 0.38, while the Sharpe ratio of a weighting scheme, that belongs to the top 10 best ones, exceeds the Sharpe ratio of the passive strategy by 0.05-0.06. However, none of the top 10 best performing weighting schemes produces the performance which is statistically significantly better than that of the passive strategy (at conventional statistical levels).

For the bond market indices, the best performing weighting schemes belong almost exclusively to the CV-EMA type. Notably, the best performing weighting scheme for timing the Long-Term Government Bond index has a decay factor of 1.00 which means that the best performing trading rule in out-of-sample tests can be implemented as P-SMA(k) rule. Another observation that is worth mentioning is that market timing does not work at all on the Long-Term Government Bond index. Even the best performing rule in this case has the same Sharpe ratio as that of the passive strategy (which amounts to 0.29). In contrast, for the Intermediate-Term Government Bond index the Sharpe ratios of the best performing weighting schemes exceed the Sharpe ratio of the passive strategy (that is equal to 0.43) by 0.07-0.09. Yet, none of the weighting schemes produces the performance which is statistically significantly better than that of the passive strategy. Interestingly, for this bond index the mean size of the averaging window is much smaller than that for any other index in our empirical study. Another interesting observation is that the 5th best performing weighting scheme is of the HS-EMA type with a decay factor of 1.00. The major types among the top 10 best perform-

Rank	Weighting Scheme	Average size $k+1$	Decay λ	Difference ΔSR	P-value
Panel A: Standard and Poor's Composite					
1	HS-EMA	9.07	1.00	0.15	0.06
2	HS-EMA	8.88	0.99	0.15	0.07
3	HS-EMA	9.37	0.95	0.14	0.07
4	HS-EMA	9.26	0.96	0.14	0.08
5	CC-EMA	9.86	0.82	0.13	0.08
6	HS-EMA	8.42	0.97	0.13	0.09
7	CV-EMA	9.89	0.95	0.13	0.09
8	HS-EMA	8.31	0.94	0.13	0.10
9	CV-EMA	9.38	0.94	0.12	0.10
10	HS-EMA	8.50	0.98	0.12	0.11
Panel B: Dow Jones Industrial Average					
1	HS-EMA	15.05	0.82	0.06	0.27
2	CV-EMA	10.00	0.76	0.06	0.27
3	CV-EMA	10.41	0.89	0.06	0.25
4	HS-EMA	13.90	0.87	0.06	0.27
5	CC-EMA	11.85	0.23	0.06	0.28
6	CC-EMA	11.85	0.22	0.06	0.28
7	CC-EMA	11.85	0.21	0.06	0.28
8	CV-EMA	10.00	0.77	0.06	0.28
9	CC-EMA	12.15	0.98	0.05	0.28
10	HS-EMA	14.18	0.85	0.05	0.28
Panel C: Long-Term Government Bonds					
1	CV-EMA	11.86	1.00	0.00	0.50
2	CV-EMA	10.06	0.68	-0.00	0.52
3	CV-EMA	9.86	0.63	-0.01	0.54
4	CC-EMA	10.81	0.32	-0.01	0.54
5	CC-EMA	10.81	0.31	-0.01	0.54
6	CV-EMA	8.84	0.62	-0.01	0.55
7	CV-EMA	8.59	0.71	-0.01	0.56
8	CV-EMA	8.55	0.65	-0.01	0.57
9	CC-EMA	9.00	0.96	-0.02	0.58
10	CV-EMA	9.27	0.69	-0.02	0.59
Panel D: Intermediate-Term Government Bonds					
1	CV-EMA	4.38	0.68	0.09	0.14
2	CV-EMA	4.27	0.71	0.09	0.16
3	CV-EMA	4.49	0.73	0.09	0.16
4	CV-EMA	4.71	0.72	0.09	0.16
5	HS-EMA	5.31	1.00	0.08	0.17
6	CV-EMA	5.09	0.84	0.08	0.19
7	CV-EMA	4.00	0.54	0.07	0.21
8	CV-EMA	5.87	0.83	0.07	0.20
9	CV-EMA	5.01	0.66	0.07	0.21
10	CV-EMA	8.59	0.62	0.07	0.21

Table 1: For each index, this table reports the top 10 best performing weighting schemes (out of total 300 tested) in our out-of-sample tests. **Rank** denotes the rank of a weighting scheme; the best performing scheme is assigned the 1st rank. **Average size** $k+1$ denotes the mean value of $k+1$ over the out-of-sample period. **Weighting scheme** denotes the type of the weighting scheme. **Decay** λ reports the value of the decay factor in the weighting scheme. **Difference** ΔSR denotes the difference between the Sharpe ratio of the trading rule (associated with the weighting scheme) and the Sharpe ratio of its passive counterpart. **P-value** denotes the p-value of testing the null hypothesis $H_0 : \Delta SR \leq 0$.

ing weighting schemes for the Long-Term Government Bond index and the Intermediate-Term Government Bond index are illustrated in Figure 3, Panels C and D respectively.



Panel A: Standard and Poor's Composite

Panel B: Dow Jones Industrial Average

Panel C: Long-Term Government Bonds

Panel D: Intermediate-Term Government Bonds

Figure 3: For each index, this figure provides illustrations of 3 major types of weighting schemes that belong to the top 10 best performing schemes in out-of-sample tests. **Lag** denotes the weight of the lag ΔP_{t-i}, where Lag0 denotes the most recent price change.

4.4 Discussion

Because of the marginal differences in the performances of the top 10 best weighting schemes in out-of-sample tests, and because of the fact that virtually for every financial index in our study each major type of the weighting scheme happens to be among the top 10, it is extremely difficult to draw general conclusions about what type of the weighting scheme produces the best performance. For practitioners, it is comforting to know that the popular DCM rule, where one uses 50-day and 200-day simple moving averages, is very close to the best performing rule for timing the Standard and Poor's 500 index. Zakamulin (2015) entertains a method of finding the most robust moving average weighting scheme, where "robustness" of a weighting scheme is defined as its ability to generate sustainable performance under all possible market scenarios regardless of the size of the averaging window. He finds that the CV-EMA weighting scheme with a decay factor of 0.85-0.90 produces the most robust performance. The same type of the weighting scheme with decay factors that are close to the range of 0.85-0.90 can also be found among the top 10 best for all financial indices in our study except the Long-Term Government Bond index.

Excluding the Intermediate-Term Government Bond index, the mean size of the averaging window, $k+1$, is close to the most often used size of 10 months (200 days). Practitioners also find this information comforting to know. Yet, practitioners should be aware of the fact that there is no single size of the averaging window that works best for any financial index at any given time. We have evidence that the optimal size of the averaging window is time-varying.

Last but not least, the results of our empirical study agree with the conclusions reached in the study by Zakamulin (2014). Specifically, only for the Standard and Poor's Composite index we find weak evidence[12] that the best performing weighting schemes are able to outperform the passive strategy. Additionally, for 2 out of 4 financial indices the top 10 best weighting schemes outperform the passive benchmark in terms of the value of their Sharpe ratio. Yet, there is no statistical evidence of outperformance. For the Long-Term Government Bond index we find that the best performing weighting schemes are not able to beat the passive benchmark even in terms of the value of the Sharpe ratio.

5 Conclusions

In this paper we present the methodology to study the computation of trading indicators in many market timing rules based on moving averages of prices and analyze the commonalities and differences between the rules. Our analysis reveals that the computation of every technical trading indicator considered in this paper can equivalently be interpreted as the computation of the weighted average of price changes over the averaging window. Despite a great variety of trading indicators that are computed seemingly differently at the first sight, we find that the only real difference between the diverse trading indicators lies in the shape of the weighting function used to compute the moving average of price changes. The most popular trading indicators employ either equal-weighting of price changes, overweighting the most recent price changes, or a hump-shaped weighting function with underweighting both the most recent and most distant price changes. The trading indicators basically vary only by the degree of over- and under-weighting the most recent price changes.

[12]The evidence is "weak" because we can reject the null hypothesis only at the 10% level. Note also that we perform a one-tailed test which produces lower p-values as compared to a two-tailed test.

As a straightforward practical application of our analysis, in this paper we perform a comprehensive out-of-sample test of 300 different shapes of the moving average weighting function using historical data on four financial market indices. These 300 shapes are chosen to represent different variations of a few most typical shapes of the weighting functions used in market timing with moving averages. The results of our tests suggest answers to long-standing questions about optimal types of moving averages and whether the best performing weighting scheme can beat the passive counterpart in out-of-sample tests.

Unfortunately, we find no clear-cut answer to the first question. Yet, practitioners find it comforting to know that the popular double-crossover method, where one uses 50-day and 200-day simple moving averages, is very close to the best performing rule for timing the Standard and Poor's 500 index. Another well performing weighting scheme in out-of-sample tests is the convex exponential moving average of price changes with a decay factor that lies in the range 0.85-0.95 (for monthly data). Practitioners also find it comforting to know that for the majority of indices in our study the mean size of the averaging window is close to the most often used size of 10 months (200 days).

Regarding the answer to the second question, only for one index we find weak evidence that the best performing weighting schemes outperform the passive strategy in out-of-sample tests. For all other financial indices in our study there is no statistically significant evidence of market timing outperformance even for the best performing weighting schemes. Therefore the results of our empirical study are in sharp contrast with the findings reported in the majority of previous studies where the authors document that "market timing works". Our findings reaffirm the following conclusion reached in the two previous studies where the researchers implement out-of-sample tests of profitability of some trading rules in the stock market (Sullivan et al. (1999) and Zakamulin (2014)): the profitability of market timing is highly overstated, to say the least.

References

Beekhuizen, P. and Hallerbach, W. G. (2015). "Uncovering Trend Rules", White paper, Robeco Asset Management.

Brock, W., Lakonishok, J., and LeBaron, B. (1992). "Simple Technical Trading Rules and the Stochastic Properties of Stock Returns", *Journal of Finance*, *47*(5), 1731–1764.

Brown, S. J., Goetzmann, W. N., and Kumar, A. (1998). "The Dow Theory: William Peter Hamilton's Track Record Reconsidered", *Journal of Finance*, *53*(4), 1311–1333.

Clare, A., Seaton, J., Smith, P. N., and Thomas, S. (2013). "Breaking Into the Blackbox: Trend Following, Stop losses and the Frequency of Trading - The Case of the S&P500", *Journal of Asset Management*, *14*(3), 182–194.

Ellis, C. A. and Parbery, S. A. (2005). "Is Smarter Better? A Comparison Of Adaptive, And Simple Moving Average Trading Strategies", *Research in International Business and Finance*, *19*(3), 399 – 411.

Faber, M. T. (2007). "A Quantitative Approach to Tactical Asset Allocation", *Journal of Wealth Management*, *9*(4), 69–79.

Fifield, S. G. M., Power, D. M., and Knipe, D. G. S. (2008). "The Performance Of Moving Average Rules In Emerging Stock Markets", *Applied Financial Economics*, *18*(19), 1515–1532.

Gwilym, O., Clare, A., Seaton, J., and Thomas, S. (2010). "Price and Momentum as Robust Tactical Approaches to Global Equity Investing", *Journal of Investing*, *19*(3), 80–91.

Hong, K. J. and Satchell, S. (2015). "Time Series Momentum Trading Strategy and Autocorrelation Amplification", *Quantitative Finance*, *15*(9), 1471–1487.

Jobson, J. D. and Korkie, B. M. (1981). "Performance Hypothesis Testing with the Sharpe and Treynor Measures", *Journal of Finance*, *36*(4), 889–908.

Kilgallen, T. (2012). "Testing the Simple Moving Average across Commodities, Global Stock Indices, and Currencies", *Journal of Wealth Management*, *15*(1), 82–100.

Kirkpatrick, C. D. and Dahlquist, J. (2010). *Technical Analysis: The Complete Resource for Financial Market Technicians*. FT Press; 2nd edition.

Lo, A. W., Mamaysky, H., and Wang, J. (2000). "Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation", *The Journal of Finance*, *55*(4), 1705–1770.

Marshall, B. R., Cahan, R. H., and Cahan, J. M. (2008). "Can Commodity Futures Be Profitably Traded With Quantitative Market Timing Strategies?", *Journal of Banking and Finance*, *32*(9), 1810 – 1819.

Memmel, C. (2003). "Performance Hypothesis Testing with the Sharpe Ratio", *Finance Letters*, *1*, 21–23.

Metghalchi, M., Marcucci, J., and Chang, Y.-H. (2012). "Are Moving Average Trading Rules Profitable? Evidence From The European Stock Markets", *Applied Economics*, *44*(12), 1539–1559.

Moskowitz, T. J., Ooi, Y. H., and Pedersen, L. H. (2012). "Time Series Momentum", *Journal of Financial Economics*, *104*(2), 228–250.

Murphy, J. J. (1999). *Technical Analysis of the Financial Markets: A Comprehensive Guide to Trading Methods and Applications*. New York Institute of Finance.

Neely, C., Weller, P., and Dittmar, R. (1997). "Is Technical Analysis in the Foreign Exchange Market Profitable? A Genetic Programming Approach", *Journal of Financial and Quantitative Analysis*, *32*, 405–426.

Neuhierl, A. and Schlusche, B. (2011). "Data Snooping and Market-Timing Rule Performance", *Journal of Financial Econometrics*, *9*(3), 550–587.

Okunev, J. and White, D. (2003). "Do Momentum-Based Strategies Still Work in Foreign Currency Markets?", *Journal of Financial and Quantitative Analysis*, *38*(2), 425–447.

Pätäri, E. and Vilska, M. (2014). "Performance of Moving Average Trading Strategies over Varying Stock Market Conditions: the Finnish Evidence", *Applied Economics*, *46*(24), 2851–2872.

Ready, M. J. (2002). "Profits from Technical Trading Rules", *Financial Management*, *31*(3), 43–61.

Sharpe, W. F. (1994). "The Sharpe Ratio", *Journal of Portfolio Management*, *21*(1), 49–58.

Sullivan, R., Timmermann, A., and White, H. (1999). "Data-Snooping, Technical Trading Rule Performance, and the Bootstrap", *Journal of Finance*, *54*(5), 1647–1691.

Welch, I. and Goyal, A. (2008). "A Comprehensive Look at the Empirical Performance of Equity Premium Prediction", *Review of Financial Studies*, *21*(4), 1455–1508.

Zakamulin, V. (2014). "The Real-Life Performance of Market Timing with Moving Average and Time-Series Momentum Rules", *Journal of Asset Management*, *15*(4), 261–278.

Zakamulin, V. (2015). "Market Timing With a Robust Moving Average", Working paper, University of Agder.

Zhu, Y. and Zhou, G. (2009). "Technical Analysis: An Asset Allocation Perspective On The Use Of Moving Averages", *Journal of Financial Economics*, *92*(3), 519 – 544.