OPTIMALITY OF MOMENTUM AND REVERSAL

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Abstract. We develop a continuous-time asset price model to capture short-run momentum and long-run reversal. By studying a dynamic asset allocation problem, we derive the optimal investment strategy in closed form and show that the combined momentum and reversal strategies are optimal. We then estimate the model to the S&P 500 and demonstrate that, by taking the timing opportunity with respect to trend in return and market volatility, the optimal strategies outperform not only pure momentum and pure mean reversion strategies, but also the market index and time series momentum strategy. Furthermore we show that the optimality also holds for the out-of-sample tests and with short-sale constraints and the outperformance is immune to market states, investor sentiment and market volatility.

Key words: Momentum, reversal, portfolio choice, optimality, profitability.

JEL Classification: G12, G14, E32

Date: July 15, 2014.

Acknowledgement: Financial support from the Australian Research Council (ARC) under Discovery Grant (DP130103210) is gratefully acknowledged. We would like to thank Li Chen, Blake LeBaron, Bernt Øksendal, Lei Shi and Zhen Wu for helpful comments and suggestions. The usual caveats apply.

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Electronic copy available at: http://ssrn.com/abstract=2467392

1. Introduction

This paper studies the optimality of trading strategies to characterize the short-run momentum and long-run reversal in financial markets. We first extend the standard geometric Brownian asset pricing model to incorporate a mean reversion process and a moving average momentum component into the drift. We study a dynamic asset allocation problem with log utility preferences, and derive the optimal investment strategy in closed form, which includes pure mean-reverting, pure momentum, and Merton's optimal portfolio as special cases. We show that a combined momentum and reversal strategy is optimal. To demonstrate the optimality of this strategy, we estimate the model to monthly returns of the S&P 500 index and show that neither a pure momentum nor a pure mean reversion strategy can outperform the market; however, the optimal strategy combining momentum and mean reversion can outperform the market. Also, compared with time series momentum strategy documented in recent literature, the optimal strategy performs better. Essentially, in contrast to a momentum strategy based on trend only, the optimal strategy takes not only the trading signal based on momentum and reversal effects but also the market volatility into account. The outperformance also holds for out-of-sample tests and short sale constraints. Regression analyses further show that the optimal strategy is immune to market states, investor sentiment and market volatility.

The asset pricing model developed in this paper takes the momentum and the reversal effects into account directly. Therefore historical prices underlying the momentum component affect asset prices, resulting in a non-Markov process characterized by stochastic delay differential equations (SDDEs). This is very different from the Markov asset price process documented in the literature (Merton 1969, 1971), where, however, it is difficult to model the momentum strategy explicitly. In the case of a Markov process, the stochastic control problem is most frequently solved using the dynamic programming method and HJB equation. However, solving the optimal control problem for SDDEs using the dynamic programming method becomes more challenging because it involves infinite-dimensional partial differential equations. To overcome this challenge, we explore the latest development in the theory of the Maximum Principle for control problems of SDDEs. By assuming a log utility preference, we derive the optimal strategies in closed form. This helps us to theoretically study the impact of historical information on the profitability of different strategies based on different time horizons, in particular of momentum trading strategies based on moving averages over different time horizons. In fact, the impact of the time horizon on the profitability has been extensively investigated

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in the empirical literature, see, for example, De Bondt and Thaler (1985) and Jegadeesh and Titman (1993). However, due to the technical challenge, there are few theoretical results on it.

This paper is closely related to the literature on reversal and momentum, two of the most prominent financial market anomalies; in particular, time series momentum. Reversal is the empirical observation that assets performing well (poorly) over a long period tend to subsequently underperform (outperform). Momentum is the tendency of assets with good (bad) recent performance to continue outperforming (underperforming) in the short-run. Reversal and momentum have been documented extensively for a wide variety of assets. On the one hand, Fama and French (1988) and Poterba and Summers (1988), among many others, document reversal for holding periods more than one year, which induces negative autocorrelation in returns.[1] The value effect documented in Fama and French (1992) is closely related to reversal, whereby the ratio of an asset's price relative to book value is negatively related to subsequent performance. Mean reversion in equity returns has been shown to induce significant market timing opportunities (Campbell and Viceira 1999, Wachter 2002 and Koijen, Rodríguez and Sbuelz 2009). On the other hand, Jegadeesh and Titman (1993) document momentum for individual U.S. stocks, predicting returns over horizons of 3-12 months using returns over the past 3-12 months. The evidence has been extended to stocks in other countries (Fama and French 1998), stocks within industries (Cohen and Lou 2012), across industries (Cohen and Frazzini 2008), and the global market with different asset classes (Asness, Moskowitz and Pedersen 2013). More recently, Moskowitz, Ooi and Pedersen (2012) investigate time series momentum (TSM) that characterizes strong positive predictability of a security's own past returns. For a large set of futures and forward contracts, Moskowitz et al. (2012) find TSM based on the past 12 month excess returns persists for 1 to 12 months then partially reverses over longer horizons. They provide strong evidence for TSM based on the moving average of "look-back" returns. This effect based purely on a security's own past returns is related to, but different from, the cross-sectional momentum phenomenon studied extensively in the literature. Through return decomposition, Moskowitz et al. (2012) argue that positive auto-covariance is the main driving force for TSM and cross-sectional momentum effects, while the contribution of serial cross-correlations and variation in mean returns is small. Intuitively, a strategy taking into both the short-run momentum and long-run mean reversion in time series should be profitable and outperform

[1] For instance, Jegadeesh (1991) finds that the next 1-month returns can be negatively predicted by their lagged multiyear returns. Lewellen (2002) shows that the past one year returns negatively predict future monthly returns for up to 18 months.

pure momentum or mean reversion strategies. This paper establishes a model to verify this intuition theoretically and empirically.

The size and apparent persistence of momentum and reversal profits have attracted considerable attention, and many studies have tried to explain the phenomena. Among which, the three-factor model of Fama and French (1996) can explain long-run reversal but not short-run momentum. Barberis, Shleifer and Vishny (1998) argue that these phenomena are the result of systematic errors that investors make when they use public information to form expectations of future cash flows. Models of Daniel, Hirshleifer and Subrahmanyam (1998) with single representative agent and Hong and Stein (1999) with different trader types attribute the under-reaction to overconfidence and overreaction to biased self-attribution. Barberis and Shleifer (2003) show that style investing can explain momentum and value effects. Sagi and Seasholes (2007) present an option model to identify observable firm-specific attributes that drive momentum. Vayanos and Woolley (2013) show that slow-moving capital can also generate momentum. Extending the literature, this paper develops an asset price model by taking both mean reversion and momentum into account directly and demonstrates the explanatory power of the model through the outperformance of the optimal strategy.

This paper is largely motivated by the empirical literature testing trading signals with combinations of momentum and reversal. Balvers and Wu (2006) and Serban (2010) empirically show that a combination of momentum and mean reversion strategies can outperform pure momentum and pure mean reversion strategies for equity markets and foreign exchange markets respectively. Asness, Moskowitz and Pedersen (2013) highlight that studying value and momentum jointly is more powerful than examining each in isolation.[2] Huang, Jiang, Tu and Zhou (2013) find that both mean reversion and momentum can coexist in the S&P 500 index over time. This paper is closely related to Koijen, Rodríguez and Sbuelz (2009) who propose a theoretical model in which stock returns exhibit momentum and mean reversion effects. They study the dynamic asset allocation problem with CRRA utility. However, the modelling of momentum in this paper is very different from Koijen et al. (2009) where the momentum is calculated by all the historical returns with geometrically decaying weights. This effectively reduces the pricing dynamics to a Markovian system. In this paper, momentum is measured by the standard moving average over a moving window with a fixed "look-back period", which is consistent with the momentum literature. Also, different from Koijen et al. (2009)

[2] They find that separate factors for value and momentum best explain the data for eight different markets and asset classes. Furthermore, they show that momentum loads positively and value loads negatively on liquidity risk; however, an equal-weighted combination of value and momentum is immune to liquidity risk and generates substantial abnormal returns.

that focuses on the performance of the hedging demand implied by the model, this paper focuses on the performance of the optimal strategy comparing to the market.

We use both the utility of portfolio wealth and Sharpe ratio to measure the performance of various trading strategies. By estimating the model to the S&P 500, we demonstrate that strategies based on the pure momentum and pure mean reversion models cannot outperform the market but the optimal strategy of combining them can outperform the market. Essentially, the optimal strategy not only reflects the trading signal based on momentum and reversal effects, but also takes volatility into account. The robustness of the optimality of this strategy is also tested for different data periods, with different estimation methods, out-of-sample predictions, market states, investor sentiment and market volatility. Finally, we compare the performance of the optimal strategies with the TSM strategies used in Moskowitz et al. (2012). Measured by the Sharpe ratio and average return, we show that the optimal strategy outperforms the TSM and passive holding strategies in the empirical literature.

The paper is organized as follows. We first present the model and derive the optimal asset allocation in Section 2. We then estimate the model to the S&P 500 in Section 3 and conduct a performance analysis of the optimal portfolio in Section 4. In Section 5, we conduct out-of-sample tests and robustness analysis on short-sale constraints, market states, sentiment, and volatility. In Section 6, we compare the performance of the optimal portfolio to the TSM strategies used in Moskowitz et al. (2012). Section 7 concludes. All the proofs and the analysis are included in the appendices.

2. The Model and Optimal Asset Allocation

In this section, we introduce an asset price model and study the optimal investment decision problem. We consider a financial market with two tradable securities, a riskless asset B satisfying

$$\frac{dB_t}{B_t} = r dt \tag{2.1}$$

with a constant riskless rate r, and a risky asset. Let S_t be the price of the risky asset or the level of a market index at time t where dividends are assumed to be reinvested. Empirical studies on return predictability, see for example Fama (1991), have shown that the most powerful predictive variables of future stock returns in the United States are past returns, dividend yield, earnings/price ratio, and term structure variables. Following this literature and Koijen et al. (2009), we model the expected return by a combination of a momentum term m_t based on the past returns and a long-run mean reversion term μ_t based on market fundamentals such as dividend yield. Consequently, we assume that the stock price S_t follows

$$\frac{dS_t}{S_t} = \big[\phi m_t + (1-\phi)\mu_t\big]dt + \sigma_S' dZ_t, \tag{2.2}$$

where ϕ is a constant, measuring the weight of the momentum component m_t, σ_S is a two-dimensional volatility vector (and σ_S' stands for the transpose of σ_S), and Z_t is a two-dimensional vector of independent Brownian motions. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ on which the two-dimensional Brownian motion Z_t is defined. As usual, the mean reversion process μ_t is defined by an Ornstein-Uhlenbeck process,

$$d\mu_t = \alpha(\bar{\mu} - \mu_t)dt + \sigma_\mu' dZ_t, \qquad \alpha > 0, \qquad \bar{\mu} > 0, \tag{2.3}$$

where $\bar{\mu}$ is the constant long-run expected return, α measures the rate at which μ_t converges to $\bar{\mu}$, and σ_μ' is a two-dimensional volatility vector. The momentum term m_t is defined by a stardard moving average (MA) of past returns over $[t-\tau, t]$,

$$m_t = \frac{1}{\tau}\int_{t-\tau}^{t} \frac{dS_u}{S_u}, \tag{2.4}$$

where delay τ represents the time horizon. The way we model the momentum in this paper is motivated by the time series momentum (TSM) strategy documented recently in Moskowitz et al. (2012) who demonstrate that the average return over a past period (say, 12 months) is a positive predictor of the future returns, especially the return for the next month. This is different from Koijen et al. (2009), in which the momentum M_t at time t is defined by

$$M_t = \int_0^t e^{-w(t-u)}\frac{dS_u}{S_u},$$

which is a geometrically decaying weighted sum of the past returns over $[0, t]$. The advantage of M_t is that the resulting price process is a Markovian process. However, it suffers some drawbacks. First, M_t is not a weighted average since the weights over $[0, t]$ do not sum to one. Secondly, M_t is not a moving average of past returns with a fixed time horizon as commonly used in empirical literature on momentum. Thirdly, the return process and the momentum variable in the discretization of Koijen et al. (2009) model have the same expression. As a result, there is an identification problem. To solve the problem, Koijen et al. (2009) estimate the model by using a restricted maximum likelihood method to ensure that the model implied autocorrelation fits the empirical autocorrelation structure of stock returns. On the other hand, the momentum m_t introduced in (2.4) overcomes these drawbacks. It is a standard moving average over a moving window $[t-\tau, t]$ with a fixed "look-back period" of $\tau(>0)$ and the weights sum to one. It is also consistent with momentum strategies used in the empirical literature that explore the price trends based on the average returns over a fixed "look-back period". The resulting asset price model

(2.2)-(2.4) is characterized by a stochastic delay integro-differential system, which is non-Markovian and lacking analytical tractability. We show in Appendix A that the price process of (2.2)-(2.4) has a continuously adapted pathwise unique solution almost surely and the asset price stays positive for given positive initial values over $[-\tau, 0]$.

We now consider a typical long-term investor who maximizes the expected utility of terminal wealth at time $T(> t)$. Let W_t be the wealth of the investor at time t and π_t be the fraction of the wealth invested in the stock. Then it follows from (2.2) that the change in wealth satisfies

$$\frac{dW_t}{W_t} = \big\{\pi_t[\phi m_t + (1-\phi)\mu_t - r] + r\big\}dt + \pi_t \sigma_S' dZ_t. \tag{2.5}$$

Assume that the preferences of the investor can be represented by a log utility function.[3] The problem of the investor is then given by

$$J(W, m, \mu, t, T) = \sup_{(\pi_u)_{u\in[t,T]}} \mathbb{E}_t[\ln W_T], \tag{2.6}$$

where $J(W, m, \mu, t, T)$ is the value function corresponding to the optimal investment strategy. We apply the maximum principle for optimal control of stochastic delay differential equations and derive the optimal investment strategy in closed-form. The result is presented in the following proposition and the proof can be found in Appendix B.

Proposition 2.1. *For an investor with log utility, the optimal wealth fraction invested in the risk asset is given by*

$$\pi_t^* = \frac{\phi m_t + (1-\phi)\mu_t - r}{\sigma_S' \sigma_S}. \tag{2.7}$$

The optimal strategy (2.7) characterizes the myopic behavior of the investor with log utility. This result has a number of implications. First, when the asset price follows a geometric Brownian motion process with mean reversion drift μ_t, that is when $\phi = 0$, the optimal investment strategy (2.7) becomes

$$\pi_t^* = \frac{\mu_t - r}{\sigma_S' \sigma_S}. \tag{2.8}$$

This is the optimal investment strategy with mean-reverting returns obtained in the literature, say for example Campbell and Viceira (1999) and Wachter (2002). In particular, when $\mu_t = \bar{\mu}$ is a constant, the optimal portfolio (2.8) collapses to the optimal portfolio of Merton (1971).

[3]Since the model is non-Markovian, it is difficult to solve the investment problem in closed-form for CRRA utility functions in general. For analytical tractability, we consider log utility in this paper.

Secondly, when the asset return depends only on the momentum, that is when $\phi = 1$, the optimal portfolio (2.7) reduces to

$$\pi_t^* = \frac{m_t - r}{\sigma_S' \sigma_S}. \tag{2.9}$$

If we consider a trading strategy based on the trading signal indicated by the excess return $m_t - r$ only, with $\tau = 12$ months, the strategy of long/short when the trading signal is positive/neative is consistent with the TSM strategy used in Moskowitz et al. (2012). By constructing portfolios based on the monthly excess returns over the past 12 months and holding for 1 month, Moskowitz et al. (2012) show that this strategy performs the best among all the momentum strategies with look-back and holding periods varying from 1 month to 48 months. Therefore, if we only take fixed long/short positions and construct simple buy-and-hold momentum strategies over a large range of look-back and holding periods, (2.9) shows that the TSM strategy of Moskowitz et al. (2012) can be optimal when mean-reversion is not significant in financial markets. On the one hand, this provides a theoretical justification of the optimality of TSM strategy when market volatilities are constant and returns are not mean-reverting. On the other hand, note that the optimal portfolio (2.9) also depends on the volatility. This explains the dependence of the momentum profitability on market conditions and volatility found in empirical studies. In addition, the optimal portfolio (2.9) defines the optimal wealth fraction invested in the risky asset. Hence the TSM strategy of taking fixed positions based on the trading signal may not be optimal in general.

Thirdly, the optimal strategy (2.7) implies that a weighted average of momentum and mean-reverting strategies is optimal. Intuitively, it takes both the short-run momentum and long-run reversal into account, the two well supported market phenomena. It also takes the timing opportunity with respect to market trend and volatility into account.

In summary, for the first time, we have provided a theoretical support for the optimality of momentum and reversal in a simple asset price model. It is the simple model and the closed-form optimal strategy (2.7) that facilitates model estimation and empirical analysis. In the rest of the paper, we first estimate the model to the S&P 500 and then evaluate the performance of the optimal strategy comparing to the market and other trading strategies used in the literature, demonstrating the optimality of the strategy empirically.

3. Model Estimation

To demonstrate the optimality of the strategy (2.7), we estimate the model to the S&P 500 in this section. In line with Campbell and Viceira (1999) and Koijen et al.

(2009), the mean-reversion variable is affine in the (log) dividend yield,

$$\mu_t = \bar{\mu} + \nu(D_t - \mu_D) = \bar{\mu} + \nu X_t,$$

where ν is a constant, D_t is the (log) dividend yield with $\mathbb{E}(D_t) = \mu_D$, and $X_t = D_t - \mu_D$ denotes the de-meaned dividend yield. Thus the asset price model (2.2)-(2.4) becomes

$$\begin{cases} \dfrac{dS_t}{S_t} = \big[\phi m_t + (1-\phi)(\bar{\mu} + \nu X_t)\big]dt + \sigma_S' dZ_t, \\ dX_t = -\alpha X_t dt + \sigma_X' dZ_t, \end{cases} \tag{3.1}$$

where $\sigma_X = \sigma_\mu/\nu$. The uncertainty in system (3.1) is driven by two independent Brownian motions. Without loss of generality, we follow Sangvinatsos and Wachter (2005) and Koijen et al. (2009) and assume the Cholesky decomposition on the volatility matrix Σ of the dividend yield and return,

$$\Sigma = \begin{pmatrix} \sigma_S' \\ \sigma_X' \end{pmatrix} = \begin{pmatrix} \sigma_{S(1)} & 0 \\ \sigma_{X(1)} & \sigma_{X(2)} \end{pmatrix}.$$

Thus, the first element of Z_t is the shock to the return and the second element of Z_t is the dividend yield shock that is orthogonal to the return shock.

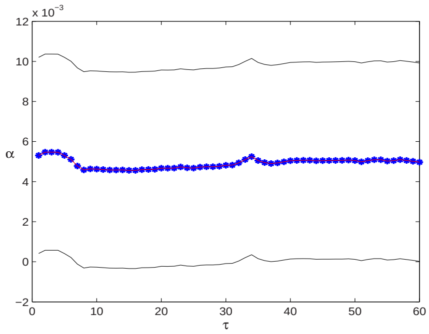
To be consistent with the momentum and reversal literature, we discretize the continuous-time model (3.1) at a monthly frequency. This results in a bivariate Gaussian vector autoregressive (VAR) model on the simple return[4] and dividend yield X_t,

$$\begin{cases} R_{t+1} = \dfrac{\phi}{\tau}(R_t + R_{t-1} + \cdots + R_{t-\tau+1}) + (1-\phi)(\bar{\mu} + \nu X_t) + \sigma_S' \Delta Z_{t+1}, \\ X_{t+1} = (1-\alpha)X_t + \sigma_X' \Delta Z_{t+1}. \end{cases} \tag{3.2}$$

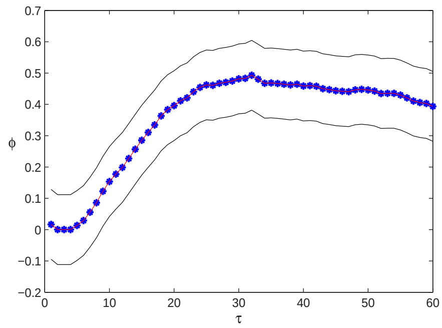
Note that both R_t and X_t are observable. We use monthly S&P 500 data over the period January 1871—December 2012 obtained from the home page of Robert Shiller (http://aida.wss.yale.edu/ shiller/) and estimate model (3.2) using the maximum likelihood method. We set the instantaneous short rate $r = 4\%$ annually. As in Campbell and Shiller (1988*a*, 1988*b*), the dividend yield is defined as the log of the ratio between the last period dividend and the current index. The total return index is constructed by using the price index series and the dividend series.

The estimations are conducted separately for given time horizon τ varying from one month up to 60 months. Fig. 3.1 reports the estimated parameters in monthly terms for τ ranging from 1 month to 5 years, together with the 95% confidence bounds. As one of the key parameters of the model, Fig. 3.1 (b) shows that the momentum effect parameter ϕ is statistically and economically different from 0 when time horizon τ is greater than half a year, indicating a significant momentum effect

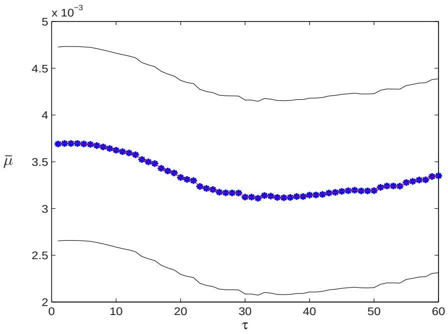
[4]We use simple return to construct m_t and also discretize the stock price process into simple return rather than log return to be consistent with the momentum and reversal literature.



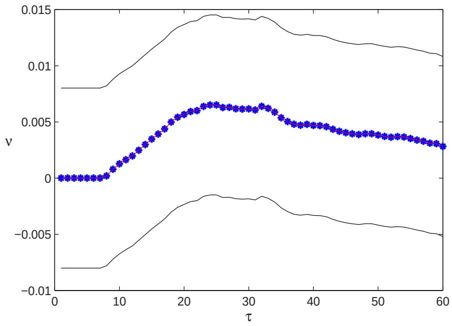
(a) Estimates of α



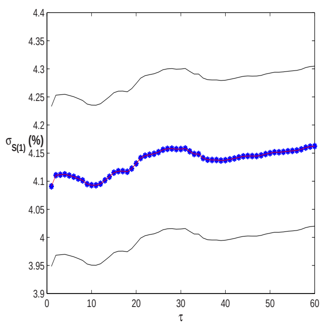
(b) Estimates of ϕ



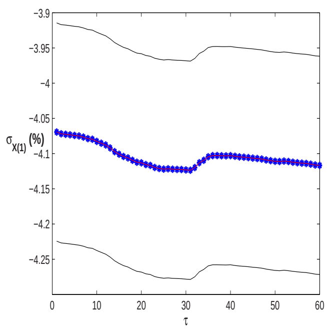
(c) Estimates of $\bar{\mu}$



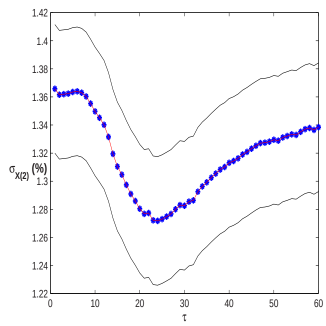
(d) Estimates of ν



(e) Estimates of $\sigma_{S(1)}$



(f) Estimates of $\sigma_{X(1)}$



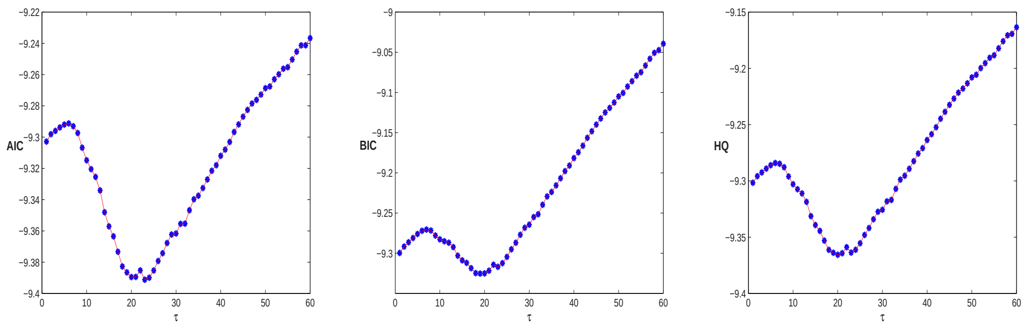
(g) Estimates of $\sigma_{X(2)}$

FIGURE 3.1. The estimates of (a) α; (b) ϕ; (c) $\bar{\mu}$; (d) ν; (e) $\sigma_{S(1)}$; (f) $\sigma_{X(1)}$ and (g) $\sigma_{X(2)}$ as functions of τ.

for τ beyond half a year.[5] Note that ϕ increases to about 50% for τ between 2 and 3 years and then decreases gradually when τ increases further. This implies that market returns can be explained by both the momentum and mean-reverting

[5] For τ from one to five months, ϕ is indifferent from zero statistically and economically. Correspondingly, for small look-back period up to half a year, the model is equivalent to pure mean reversion model. This observation will be helpful when explaining the results of the model for small look-back period in the following discussion.

components. Other results in terms of the level and significance reported in Fig. 3.1 are consistent with Koijen et al. (2009).



(a) AIC (b) BIC (c) HQ

FIGURE 3.2. (a) Akaike information criterion, (b) Bayesian information criterion and (c) Hannan–Quinn information criterion for $\tau \in [1, 60]$.

Obviously, the estimations depend on the specification of time horizon τ. To explore the optimal value for τ, we compare different information criteria, including Akaike information criteria (AIC), Bayesian information criteria (BIC) and Hannan–Quinn (HQ) information criteria for τ from 1 month to 60 months in Fig. 3.2. There are two important observations. First, the AIC, BIC and HQ reach their minima at $\tau = 23, 19$ and 20 respectively, implying that the average returns over the past one and half to two years can predict future return best. Secondly, the increasing pattern of the criteria level for longer τ indicates that the return momentums based on longer time horizons have less explanatory power for market returns. Both observations are consistent with studies that show short-term (one to two years), instead of long-term, return momentum better explains market returns. Combining the results in Figs 3.1 and 3.2, we can conclude that the market returns are better captured by short-term momentum and long-term reversion.

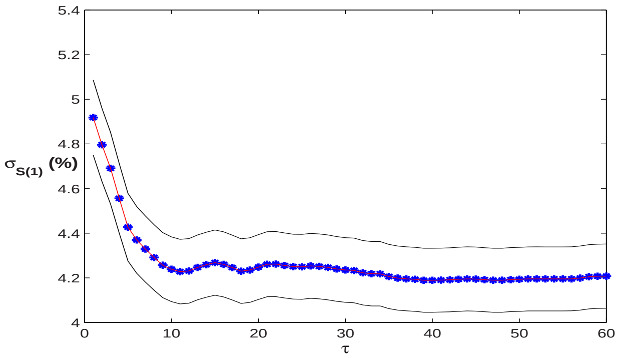


FIGURE 3.3. The estimates of $\sigma_{S(1)}$ for the pure momentum model ($\phi = 1$) for $\tau \in [1, 60]$.

To compare the performance of the optimal strategies with pure momentum or mean-reverting strategies, we also estimate the index for the model with $\phi = 1$ and $\phi = 0$ respectively. For the pure momentum model ($\phi = 1$), the system (2.2)-(2.4) has only one parameter $\sigma_{S(1)}$ to be estimated. Fig. 3.3 reports the estimates of $\sigma_{S(1)}$ and the 95% confidence bounds for $\tau \in [1, 60]$. It shows that as τ increases, the volatility of the index decreases dramatically for small time horizons and then stabilizes for large time horizons. It implies high volatility associated with return momentum over short time horizons and low volatility over long time horizons. We also compare the information criteria for different τ (not reported here) and find that all the AIC, BIC and HQ reach their minima at $\tau = 11$. This implies that the average returns over the past 11 months can predict future returns best for the pure momentum model. This is consistent with the finding of Moskowitz et al. (2012) who find momentum returns over the past 12 months better predict the next month's return than other time horizons.

Parameters	α	$\bar{\mu}$	ν
Estimates (%)	0.55	0.37	$2.67 * 10^{-5}$
Bounds (%)	$(0.07, 1.03)$	(0.31, 0.43)	(-0.46, 0.46)
Parameters	$\sigma_{S(1)}$	$\sigma_{X(1)}$	$\sigma_{X(2)}$
Estimates (%)	4.11	-4.07	1.36
Bounds (%)	(3.97, 4.25)	(-4.22, -3.92)	(1.32, 1.40)

TABLE 1. The estimates of the parameters for the pure mean reversion model.

We then estimate the pure mean reversion model ($\phi = 0$) and report the estimated parameters and the corresponding 95% confidence bounds in Table 1. The results are comparable to those for the full model illustrated in Fig. 3.1.

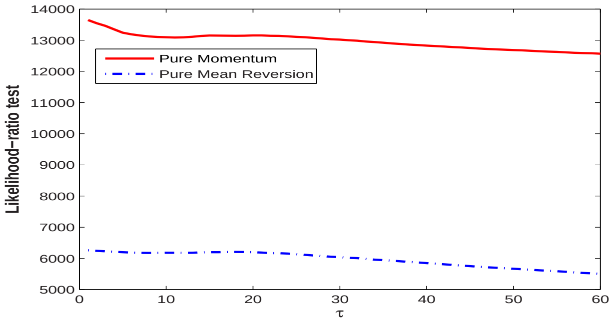


FIGURE 3.4. The log-likelihood ratio test for $\tau \in [1, 60]$.

Finally, we conduct the log-likelihood ratio test to compare the full model (2.2)-(2.4) with the estimated parameters reported in Fig. 3.1 to the estimated pure momentum model ($\phi = 1$) and pure mean-reversion model with respect to different τ. We report the log-likelihood ratio test results in Fig. 3.4. The red solid line shows the test statistic when compared to the pure momentum model. The statistic is much greater than 12.59, the critical value with 6 degrees of freedom at the 5% significance level. This indicates that the full model is significantly better than the pure momentum model for all τ. The blue dash-dotted line illustrates the test statistic when comparing to the pure mean reversion model. The test statistic is much greater than 3.841, the critical value with 1 degree of freedom at 5% significance level, meaning that the full model is also significantly better than the pure mean reversion model.

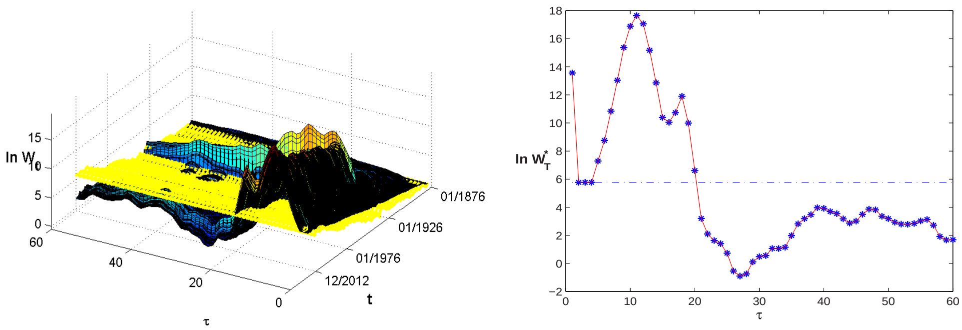
In summary, by estimating the model for the index, we show that the model captures short-term momentum and long-term reversion in market returns. Also, the model fits the data better than the pure momentum and pure mean-reverting models. This provides a foundation for the outperformance of the optimal strategy (2.7) which is analyzed in the following section.

4. Performance Analysis

Based on the estimations in the previous section, we examine in this section the performance of the optimal portfolio based on the optimal strategy (2.7), comparing with the performance of the market index and of the pure momentum and pure mean reversion models. We use two proxies to measure the performance of a portfolio; one is the utility of the portfolio wealth and the other is the Sharpe ratio of the portfolio return.

We first compare the realized utility of the optimal portfolio wealth invested in the S&P 500 index based on the optimal strategy (2.7) with different look-back period τ and 1-month holding period to the utility of a passive holding investment in the

S&P 500 index with an initial wealth of \$1. We consider the look-back period τ from 1 month to 60 months and invest monthly. For comparison, all the portfolios start at the end of January 1876 (60 months after January 1871). As a benchmark, the log utility of an investment of \$1 to the index from January 1876 to December 2012 is equal to 5.765. For a fixed look-back period, say, $\tau = 12$, we calculate the moving average m_t of past 12 months at any point of time (in month) from January 1876 to December 2012 using the index level over the time period. With the initial wealth of \$1 at January 1876 and the estimated parameters for $\tau = 12$ in Fig. 3.1, we calculate the monthly investment of the optimal portfolio wealth W_t based on (2.7) and record the realized utilities of the optimal portfolio wealth from January 1876 to December 2012.



(a) Series of utility (b) Terminal utility

FIGURE 4.1. The utility of wealth from January 1876 until December 2012 for the optimal portfolio and the passive holding portfolio.

For $\tau = 1, 2, \cdots, 60$, Fig. 4.1 (a) illustrates the evolution of the utility of the optimal portfolio wealth (the dark and more volatile surface) and of the passive holding index portfolio (the yellow and smooth surface) from January 1876 until December 2012. For better visibility, we also plot the utility of terminal wealth at December 2012 comparing to the utility of the index portfolio in Fig. 4.1 (b) for τ from 1 month to 60 months. Both plots indicate that the optimal strategies outperform the market index measured by the utility of wealth for τ from 5 months to 20 months consistently.[6]

[6]When τ is less than half a year, Fig. 4.1 (b) shows that the optimal strategies do not perform significantly better than the market index. As we indicate in footnote 5, the model with small look-back period up to half a year performs similarly to the pure mean reversion strategy illustrated in Fig. 4.10. Note the significant outperformance of the optimal strategy with 1-month horizon in Fig. 4.1 (b). This is due to the fact that the first order autocorrelation of the return of the S&P 500 is significantly positive ($AC(1) = 0.2839$) while the autocorrelations with higher orders are insignificantly different from 0. This implies that the last period return can well predict the next period return.

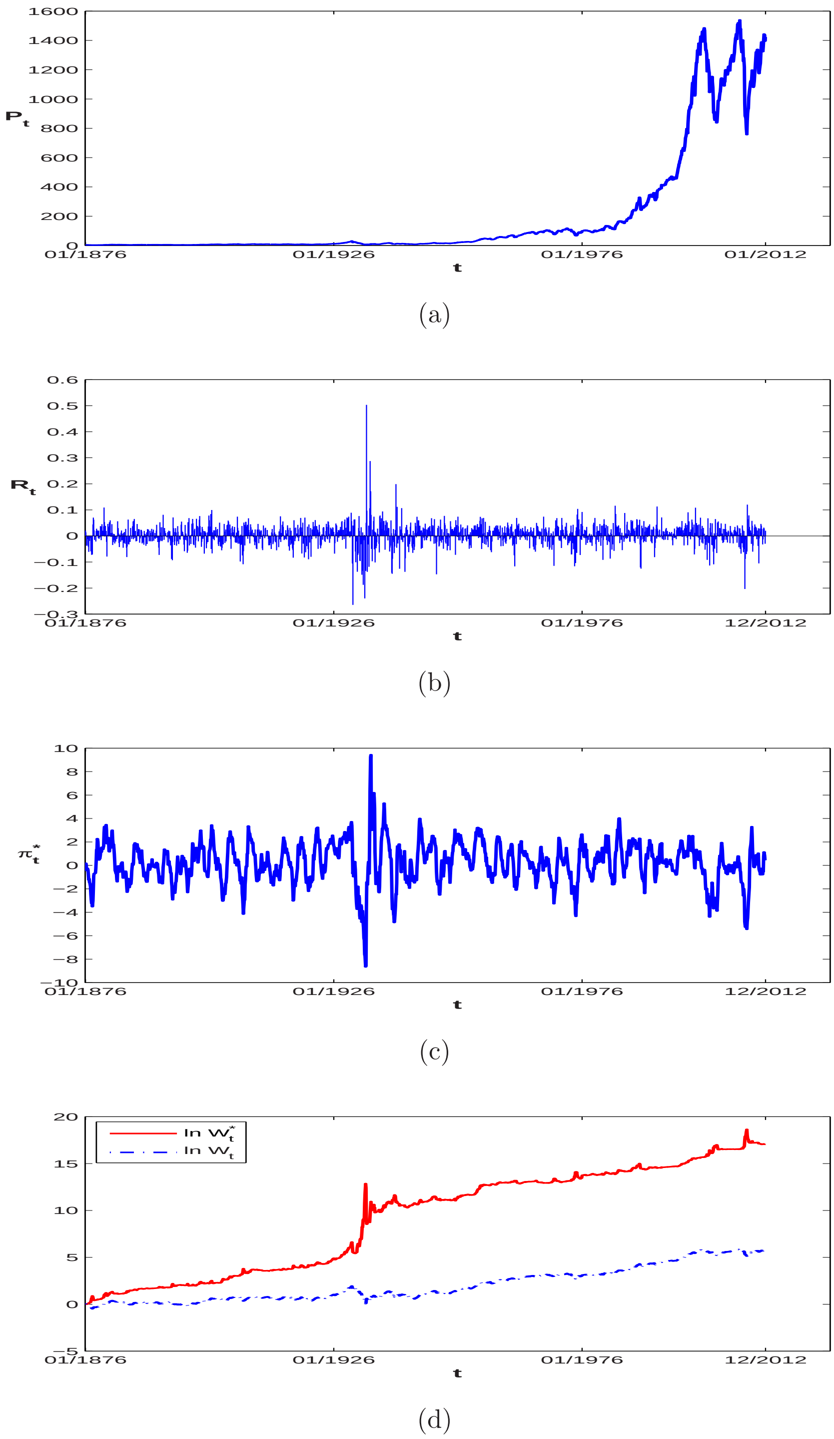


FIGURE 4.2. The time series of (a) the total return index level and (b) the simple return of the total return index of S&P 500; and the time series of (c) the optimal portfolio and (d) the utility of wealth of the optimal portfolio from January 1876 until December 2012 for $\tau = 12$. In (d), the utilities of the optimal portfolio wealth and the market index are plotted in red solid and blue dash-dotted lines respectively.

We now provide more details on the performance of the optimal portfolio with $\tau = 12$. Empirically, Moskowitz et al. (2012) show that the TSM strategy based on 12-month horizon performs the best. We first plot the index level and simple return of the S&P 500 index from January 1876 until December 2012 in Fig. 4.2 (a) and (b). For $\tau = 12$, Fig. 4.2 (c) reports the optimal wealth fractions (2.7) and Fig. 4.2 (d) reports the evolution of the utilities of the optimal portfolio wealth over the same time period. We have two observations from Fig. 4.2. First, the optimal strategies and index returns are positively correlated in general. In fact, the correlation is about 0.335 for $\tau = 12$. It becomes 0.350 and 0.135 for $\tau = 11$ and $\tau = 27$, at which the terminal utility reaches its maximum and minimum respectively as illustrated in Fig. 4.1. Secondly, Fig. 4.2 (d) seems to indicate that the improved utilities of the portfolio wealth are mainly driven by the Great Depression in 1930s. This observation is consistent with Moskowitz et al. (2012) who find that the TSM strategy delivers its highest profits during the most extreme market episodes. To clarify this observation, we also examine the performance using the data from January 1940 to December 2012 to avoid the Great Depression periods. We re-estimate the model, conduct the same analysis, and report the terminal utilities of the optimal portfolios in Fig. 4.3 over this time period. It shows that the optimal strategies still outperform the market index and the performance of the optimal strategies over the recent time period becomes even better for all time horizons. This indicates that the outperformance of the optimal strategy is not necessarily due to extreme market episodes, such as the Great Depression periods. In the next section, we show that the outperformance is in fact independent of market conditions.

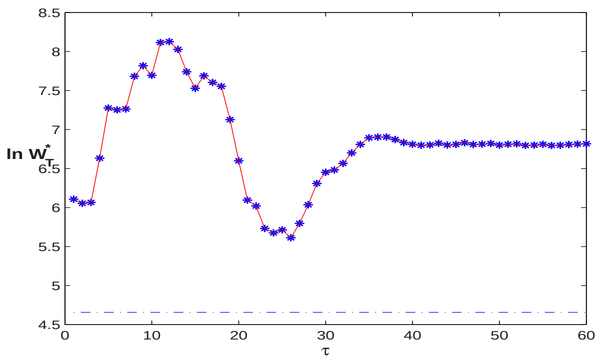
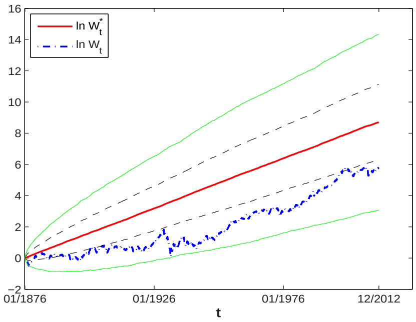
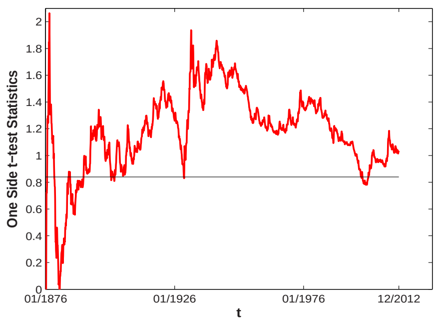


FIGURE 4.3. The terminal utility of the optimal portfolio wealth from January 1945 until December 2012 for various τ and that of the market index portfolio.

To provide further evidence, we conduct a Monte Carlo analysis. For $\tau = 12$, with the corresponding estimated parameters in Fig. 3.1, we simulate model in



(a) Average utility



(b) One sided t statistics

FIGURE 4.4. (a) average utility and (b) one sided t-test statistics based on 1000 simulations for $\tau = 12$.

(3.1). Fig. 4.4 (a) illustrates the average portfolio utilities (the red solid line in the middle) based on 1,000 simulations, together with 95% confidence levels (the two green solid lines outside), comparing to the utility of the market index (the blue dotted line). It shows that first, the average utilities of the optimal portfolios are better than that of the S&P 500. Secondly, the utility for the S&P 500 falls into the 95% confidence bounds and hence the average performance of the optimal strategy is not statistically different from the market index. We also plot two black dashed bounds for the 60% confidence level. It shows that, at the 60% confidence level, the optimal portfolio significantly outperforms the market index. Fig. 4.4 (b) reports the one sided t-test statistics to test $\ln W_t^* > \ln W_t^{SP500}$. The t-statistics are above 0.84 most of the time, which indicates a critical value at 80% confidence level. Therefore, with 80% confidence, the optimal portfolio significantly outperforms the market index.

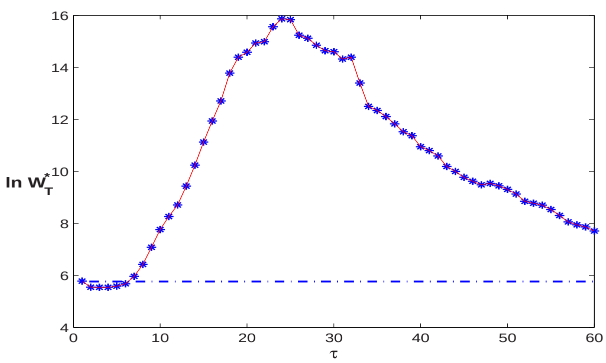


FIGURE 4.5. Average terminal utility of the optimal portfolios based on 1000 simulations for $\tau \in [1, 60]$.

For τ from one month to 60 months, we conduct further Monte Carlo analysis on the performance of the optimal portfolios based on the estimated parameters in Fig. 3.1 and 1,000 simulations and report the average terminal utilities in Fig. 4.5. The result displays a different terminal performance from that in Fig. 4.1. In fact, the terminal utility in Fig. 4.1 is based on only one specific trajectory (the real market index), while Fig. 4.5 provides the average performance based on 1,000 trajectories. It is found that the optimal portfolios significantly perform better than the market index (the dash-dotted constant level) for all time horizons beyond half a year. In particular, the average terminal utility reaches its peak at $\tau = 24$, which is consistent with the result based on the information criteria in Fig. 3.2, especially the AIC. Therefore, according to the utility of portfolio wealth, the optimal strategies outperform the market index for most of the time horizons τ.

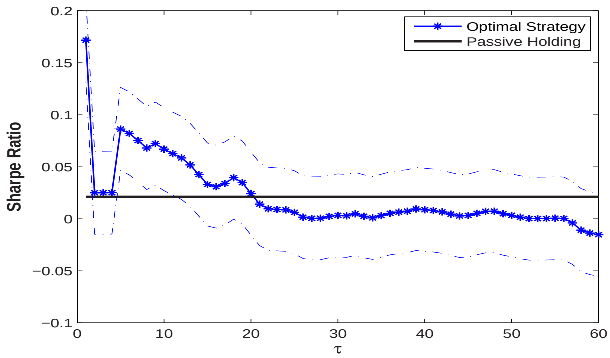


FIGURE 4.6. The Sharpe ratio for the optimal portfolio with $\tau \in [1, 60]$ (the blue solid line), the 90% confidence interval lines, and the passive holding portfolio of market index (the dotted black line) from January 1881 to December 2012.

Based on the utility of portfolio wealth, we have shown that the optimal portfolios outperform the market index. Next, we use the Sharpe ratio as the performance measure to test the performance. The Sharpe ratio is defined as the ratio of the mean excess return on the (managed) portfolio and the standard deviation of the portfolio return. When the Sharpe ratio of an active strategy exceeds the market Sharpe ratio, we say that the active portfolio outperforms or dominates the market portfolio (in an unconditional mean-variance sense). For empirical applications, the (ex post) Sharpe ratio is usually estimated as the ratio of the sample mean of the excess return on the portfolio and the sample standard deviation of the portfolio return (Marquering and Verbeek 2004). The average monthly return on the total return index of the S&P 500 over the period January 1871–December 2012 is 0.42% with an estimated (unconditional) standard deviation of 4.11%. The Sharpe ratio of the market index is 0.021. For the optimal strategy (2.7), the return of the optimal portfolio wealth at time t is given by

$$R_t^* = (W_t^* - W_{t-1}^*)/W_{t-1}^* = \pi_{t-1}^* R_t + (1 - \pi_{t-1}^*) r. \tag{4.1}$$

Fig. 4.6 reports the Sharpe ratios of the optimal portfolios for τ from one month to 60 months, comparing to the Sharpe ratio of the passive holding market index portfolio from January 1881 to December 2012, together with the 90% confidence bounds derived from the asymptotic distributions of the Sharpe ratio with certain conditions (see, Jobson and Korkie 1981). If we consider the optimal portfolio as a combination of the market portfolio and a risk free asset, then the optimal portfolio should be located on the capital market line and hence it should have the same Sharpe ratio as the market. However Fig. 4.6 demonstrates that, by taking the timing opportunity (with respect to the return trend and market volatility), the optimal portfolios (the blue line) outperform the market (the black line) on average for time horizons from 6 to 20 months. The results are surprisingly consistent with that in Fig. 4.1 under the utility measure discussed previously.

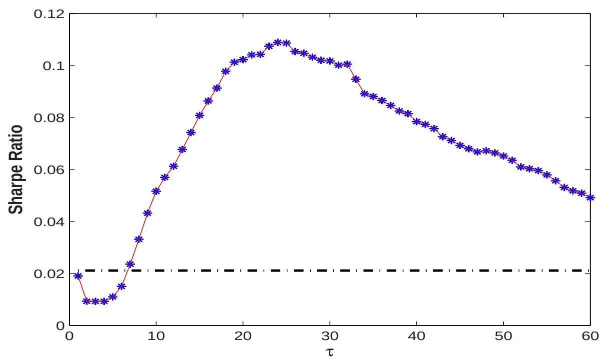


FIGURE 4.7. Average Sharpe ratio based on 1000 simulations for $\tau \in [1, 60]$.

We also conduct a Monte Carlo analysis based on the estimated parameters in Fig. 3.1 and 1,000 simulations for τ from 1 month to 60 months and report the average Sharpe ratios in Fig. 4.7 for the optimal portfolios. It shows the outperformance of the optimal portfolios over the market index based on the Sharpe ratio for the look-back periods of more than 6 months. The results are consistent with that in Fig. 4.5 under the portfolio utility measure. Therefore, we have demonstrated the consistent outperformance of the optimal portfolios under two performance measures.

Finally, we compare the performance of the pure momentum and pure mean-reverting strategies to the market index. For the pure momentum model, based on estimated parameters in Fig. 3.3, we report the terminal utilities of the portfolios of the pure momentum model at December 2012 in Fig. 4.8. It shows that the pure momentum strategies underperform the market index in all time horizons from 1 month to 60 months.

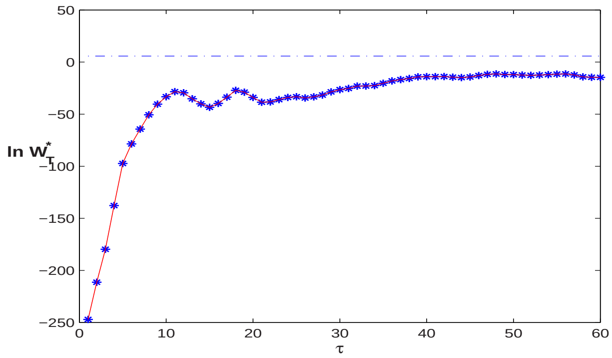
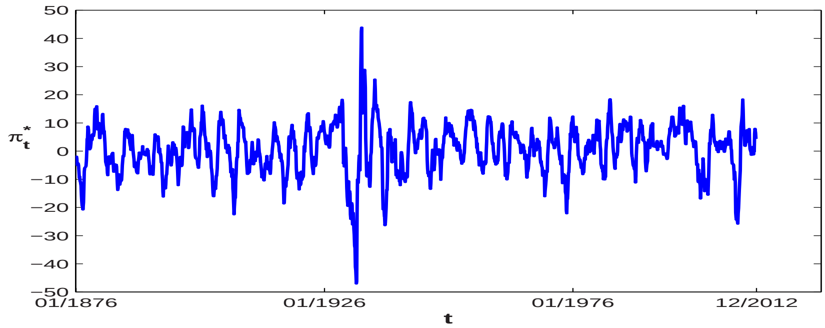
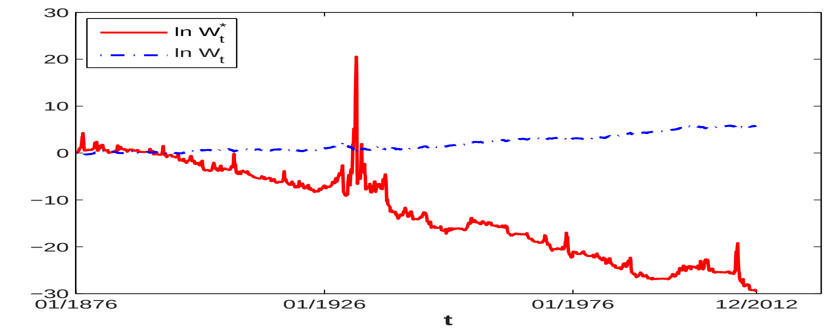


FIGURE 4.8. The utility of terminal wealth of the pure momentum model for $\tau \in [1, 60]$.



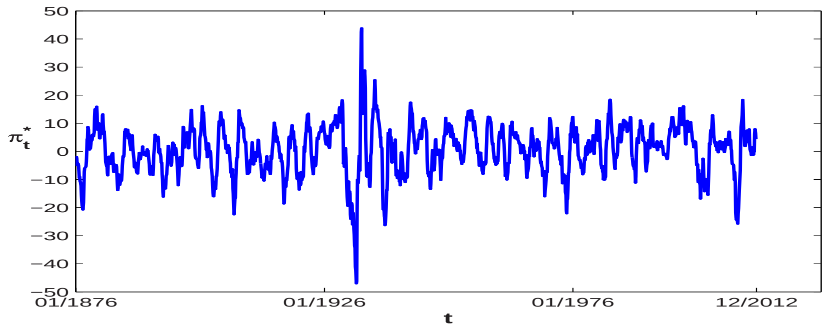
(a)



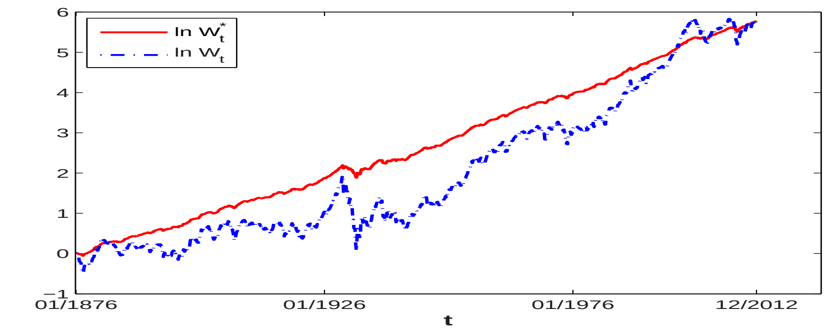
(b)

FIGURE 4.9. The time series of (a) the optimal portfolio and (b) the utility of wealth from January 1876 until December 2012 for $\tau = 12$ for the pure momentum model.

For the pure momentum model with $\tau = 12$, Fig. 4.9 illustrates the time series of the optimal portfolios and the utilities of the portfolio wealth from January 1876 to December 2012. Compared to the full model illustrated in Fig. 4.2, the leverage of the pure momentum strategies is much higher indicated by the higher level of π_t^*. The optimal strategies for the pure momentum model suffer from high risk and perform worse than the market and hence the optimal strategies of the full model. Therefore, the pure momentum strategies underperform the market and the optimal strategies.



(a)



(b)

FIGURE 4.10. The time series of (a) the optimal portfolio and (b) the utility of wealth from January 1876 until December 2012 for the pure mean reversion model.

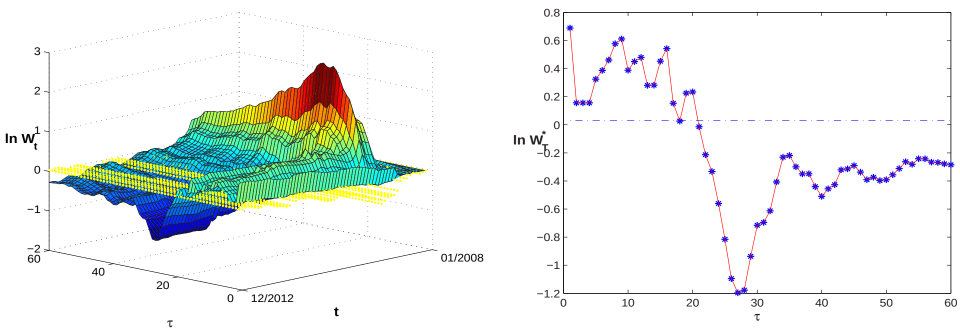
Similarly, we examine the performance of the pure mean reversion model and compare it with the market index. Based on the estimates in Table 1, Fig. 4.10 illustrates the time series of the optimal portfolio and the utility of wealth from January 1876 to December 2012 for the pure mean reversion model. It shows that the performance of the strategy is about the same as the stock index but worse than the optimal strategies (2.7) illustrated in Fig. 4.1.

In summary, we have used two performance measures and provided empirical evidence on the outperformance of the optimal strategy (2.7) compared to the market index, pure momentum strategies and the pure mean reversion strategies. The results provide empirical support for the analytical result on the optimality of momentum and reversal derived in Section 2.

5. Out-of-Sample Test and Robustness Analysis

In this section, we conduct further empirical tests on the optimality results obtained in the previous section. We first test the predication power of the model by doing out-of-sample tests. We then examine the performance of the optimal strategy with short-sale constraint and the dependence of the results on market states, sentiment and volatility.

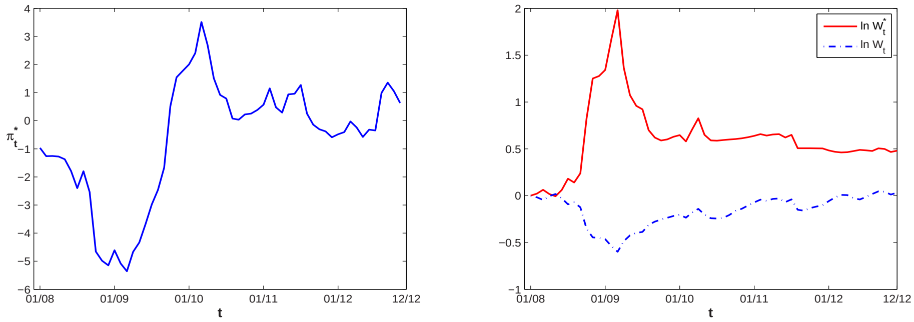
5.1. **Out-of-Sample Tests.** We implement a number of out-of-sample tests for the optimal strategies by splitting the whole data set into two sub-sample periods. We use the first sample period to estimate the model and then apply the estimated parameters to the second part of the data to examine the out-of-sample performance of the optimal strategies. We also implement the rolling window estimation procedure to avoid look ahead bias.



(a) Series of utility (b) Terminal utility

Figure 5.1. The utility of the optimal portfolio wealth, compared to the passive holding market index portfolio, with out-of-sample data from January 2008 to December 2012 for $\tau \in [1, 60]$.

Many studies (see, for example, Jegadeesh and Titman 2011) show that momentum strategies perform poorly after the subprime crisis. We first use the subprime crisis to split the whole sample period into two periods and focus on the performance of the optimal strategies after the subprime crisis. We re-estimate the model using the data up to December 2007 and test the out-of-sample performance of the optimal strategies using the monthly returns of the index from January 2008 to December 2012. For τ from one month to 60 months, Fig. 5.1 reports the out-of-sample utility of the optimal portfolio wealth from January 2008 to December 2012.



(a) (b)

FIGURE 5.2. The time series of out-of-sample (a) optimal portfolio and (b) utility of the optimal portfolio wealth (the solid line) from January 2008 to December 2012 for $\tau = 12$ compared to the utility of the market index (the dotted line).

It clearly shows that the optimal strategies still outperform the market index for time horizons up to 2 years.

To better understand the performance of the out-of-sample test, we fix the time horizon $\tau = 12$ months and examine the out-of-sample time series of the optimal portfolio and the utility of the optimal portfolio wealth from January 2008 to December 2012 in Fig. 5.2. It is clear that the optimal strategy outperforms the market over the sub-sample period, in particular, during the financial crisis period around 2009 by taking large short positions in the optimal portfolios.

In the second out-of-sample test, we split the whole data set into two equal periods: January 1871—December 1941 and January 1942—December 2012. We estimate the model to the first sub-sample period and do the out-of-sample test over the second sub-sample period. The results are reported in Appendix C, showing the outperformance of the optimal portfolios over the market index for τ up to one and a half years. For fixed $\tau = 12$, the utility of the optimal portfolio grows gradually and outperforms the market index. We also use the last 10 years and 20 years data as the out-of-sample tests and find the results are robust.

To avoid look ahead bias, we also estimate the models using rolling window data and examine the performance of the optimal portfolio based on the rolling window estimation. We report the results in Appendix C. The results are consistent with the main findings above. Overall, the out-of-sample tests demonstrate the optimality of the optimal trading strategies based on a large range of time horizons, in particular, we show the outperformance of the optimal portfolio based on 12 month time horizon.

5.2. **Short-sale Constraint.** Investors often face short-sale constraints. To evaluate the optimality under such constraints, we consider the optimal strategies when short selling and borrowing (at the risk-free rate) are not allowed. The portfolio weight π in this case must lie between 0 and 1. Since the value function is concave with respect to π, the optimal strategy becomes

$$\Pi_t^* = \begin{cases} 0, & \text{if} \quad \pi_t^* < 0, \\ \pi_t^*, & \text{if} \quad 0 \le \pi_t^* \le 1, \\ 1, & \text{if} \quad \pi_t^* > 1. \end{cases} \tag{5.1}$$

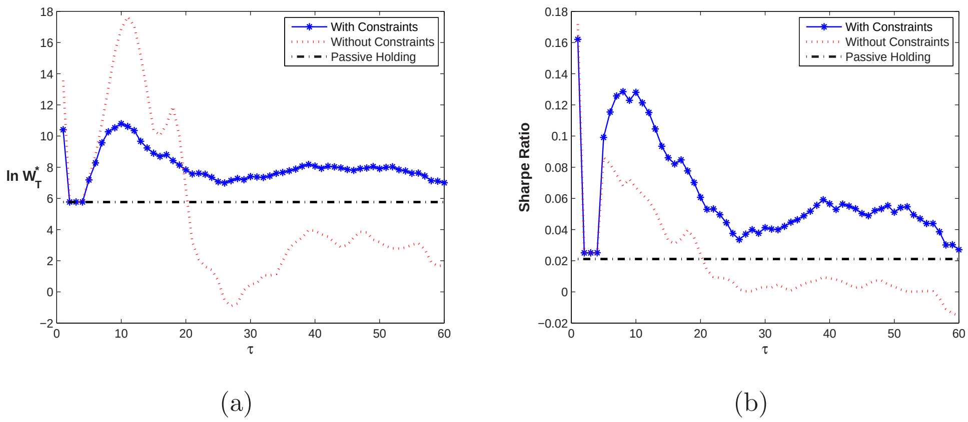
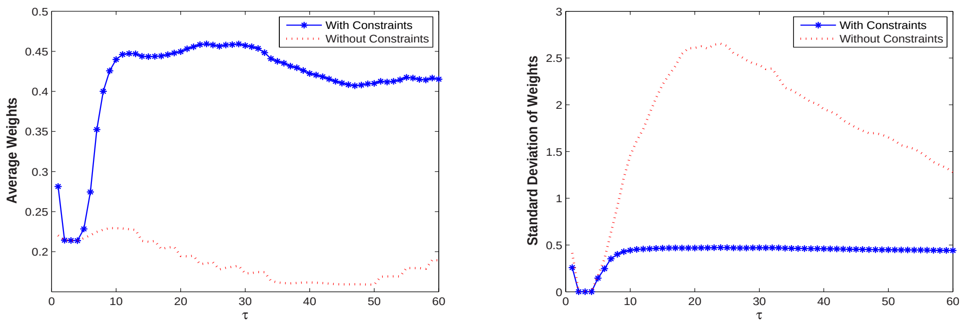


FIGURE 5.3. (a) The terminal utility of wealth and (b) the Sharpe ratio for the optimal portfolio with and without short-sale constraints, compared with the market index portfolio.

Fig. 5.3 reports the terminal utilities of the optimal portfolio wealth and the Sharpe ratio for the optimal portfolio with and without short-sales constraints, comparing with the passive holding market index portfolio. The results lead to three observations. First, based on both performance measures, the optimal portfolio with short-sale constraints outperform the market for all time horizons τ from 1 month to 60 months. Secondly, the optimal portfolios with constraints underperform the ones without short-sale constraints for small time horizons according to the utility measure (the left panel) but outperform the optimal portfolio without constraints for all time horizons (beyond 4 months) according to the Sharpe ratio measure (the right panel). The inconsistency between the two measures might be due to the fact that the Sharpe ratio does not appropriately take into account time-varying volatility.[7] Thirdly, Fig. 5.3 shows that the constraint can improve the performance,

[7]In fact, Sharpe ratio can be sensitive to large return increase. For example, Schuster and Auer (2012) find that increasing a value of return in a time series may, however, decrease the Sharpe ratio. So Sharpe ratio is also sensitive to the size of position which can lead to big return changes

especially for long time horizons. This observation is consistent with Marquering and Verbeek (2004) (p. 419) who argue that "*While it may seem counterintuitive that strategies perform better after restrictions are imposed, it should be stressed that the unrestricted strategies are substantially more affected by estimation error.*" We also examine the mean and standard deviation of the optimal portfolio weights and report the results in Fig. 5.4 with and without short-sale constraints. It is found that, with the constraints, the optimal portfolio weights increase in the mean while the volatility is low and stable. This result is consistent with Marquering and Verbeek (2004) who find that constraints can increase the average weights. On the other hand, without constraint, the volatility of the optimal portfolio weights varies dramatically, which also seems in line with the above argument of Marquering and Verbeek (2004).



(a) (b)

FIGURE 5.4. (a) The mean values and (b) the standard deviations of the optimal portfolio weights with and without short-sale constraints.

5.3. **Market States, Sentiment and Volatility.** The cross-sectional momentum literature has shown that the momentum profitability can be affected by market states, investor sentiment and market volatility. For example, Cooper, Gutierrez and Hameed (2004) find that short-run (6 months) momentum strategies make profits in the up-market and lose in the down-market, but the up-market momentum profits reverse in the long-run (13-60 months). Hou, Peng and Xiong (2009) find momentum strategies with a short time horizon (1 year) are not profitable in a "down" market, but are profitable in an "up" market. Similar results of profitability are also reported in Chordia and Shivakumar (2002), specifically that commonly using macroeconomic instruments related to the business cycle can generate positive returns to momentum strategies during expansionary periods and negative returns during recessions. Baker

when compared with and without short-sale constraint. Fig. 5.3 provides an example that a rational trader may not always choose high Sharpe ratio portfolio.

and Wurgler (2006, 2007) find that investor sentiment affects the cross-sectional stock returns and the aggregate stock market. Wang and Xu (2012) find that market volatility has significant power to forecast momentum profitability. For the TSM, however, Moskowitz et al. (2012) find that there is no significant relationship of the TSM profitability with either market volatility or investor sentiment.

To investigate the performance of optimal strategies under different market states, we follow Cooper et al. (2004) and Hou et al. (2009) and define market state using the cumulative return of the stock index (including dividends) over the most recent 36 months. We label a month as an up (down) market month if the three-year return of the market is non-negative (negative). We compute the average return of the optimal strategy and compare the average returns between up and down market months. We see from Table 4 in Appendix D that the unconditional average excess return is 87 basis points per month. In up months, the average excess return is 81 basis points and it is statistically significant. In down months, the average excess return is 101 basis points; this value is economically significant although it is not statistically significant. The difference between down and up months is 20 basis points, which is not significantly different from zero based on a two sample t-test (p-value of 0.87). We then implement the time series analysis by regressing the excess portfolio returns on a constant, a dummy of up month and the excess market return. We report the results in Table 5 in Appendix D for the optimal strategy, the pure momentum strategy, pure mean reversion strategy and the TSM strategy in Moskowitz et al. (2012) for $\tau = 12$ respectively. Except the TSM, which earns significant positive returns in down market, both constant term α and the coefficient of up month dummy κ are insignificant, indicating that the average return in down months and the incremental average return in up months are both insignificant for all other strategies, the results are consistent with those in Table 4. When we replace the up month dummy with the lagged market return over the previous 36 months (not reported here), it is found that the results are robust. We also control for market risk in up and down months, the results are similar. Simple predictive regression of excess portfolio return on the up month dummy indicates that the market state has no predictive power on portfolio returns. The results are also robust for the CAPM adjusted return. Comparing with finding in Hou et al. (2009) where cross-sectional momentum returns are higher in up months, we do not find significant difference between up and down months for the strategies from our model and the TSM. Among the strategies from our model and the TSM, only the TSM shows significant return in down months.

In terms of the effects of investor sentiment on the portfolio performance, we regress the excess portfolio return on previous month's sentiment index constructed by Baker and Wurgler (2006). It is found (see Table 9 in Appendix D) that investor sentiment has no predictive power on portfolio returns.

We also examine the predictability of market volatility to the portfolio return. The time series analysis, of regressing the excess portfolio return on past month volatility or on past month volatility conditional on up and down market state, suggests that market volatility has no predictive power on the portfolio returns (see Tables 10 and 11 in Appendix D).

Overall, we find that returns of the optimal strategies are not significantly different in up and down market states. We also find that both investor sentiment and market volatility have no predictive power for the returns of the optimal strategies. In fact, the optimal strategies have optimally taken these factors into account and hence the returns of the optimal strategies have no significant relationship with these factors. Therefore, the optimal strategies are immune to the market states, investor sentiment and market volatility.

6. Comparison with Moskowitz, Ooi and Pedersen (2012)

In this final section, we compare the performance of the TSM strategy of Moskowitz et al. (2012) with that of the optimal strategy of our model. The momentum strategies in the empirical studies are based on trading signals only. We first verify the profitability of the TSM strategies and then examine the excess return of buy-and-hold strategies when the position is determined by the sign of the optimal portfolio strategies (2.7) with different combinations of time horizons τ and holding periods h.

Based on the index, for a given look-back period τ, we take long/short positions based on the sign of the optimal portfolio (2.7). Then for a given holding period h, we calculate the monthly excess return of the strategy (τ, h). Table 2 reports the average monthly excess return (%) of the optimal strategies by skipping one month between the portfolio formation period and holding period to avoid the 1-month reversal in stock returns for different look-back periods (in the first column) and different holding periods (in the first row). The average return is calculated using the same method as in Moskowitz et al. (2012). To calculate the momentum component and to evaluate the profitability of holding periods, we calculate the excess return of the optimal strategies over the period from January 1881 (10 years after January 1871 with 5 years for calculating the trading signals and 5 years for holding periods) to December 2012.

For comparison, Table 3 reports the average return (%) for the optimal strategies for the pure momentum model.[8] Notice that Tables 2 and 3 indicate that the

[8]Notice the position is completely determined by the sign of the optimal strategies. Therefore, the position used in Table 3 is the same as that of the TSM strategies in Moskowitz et al. (2012).

$(\tau \setminus h)$	1	3	6	9	12	24	36	48	60
1	0.1337	0.1387	0.1874*	0.1573*	0.0998	0.0222	0.0328	0.0479	0.0362
	(1.28)	(1.84)	(3.29)	(2.83)	(1.84)	(0.42)	(0.63)	(0.90)	(0.66)
3	0.0972	0.0972	0.0972	0.0972	0.0972	0.0972	0.0972	0.0972	0.0972
	(0.93)	(0.93)	(0.93)	(0.93)	(0.93)	(0.93)	(0.93)	(0.93)	(0.93)
6	0.2022	0.2173*	0.2315*	0.1462	0.0700	-0.0414	0.0199	0.0304	0.0014
	(1.93)	(2.28)	(2.60)	(1.75)	(0.88)	(-0.58)	(0.32)	(0.53)	(0.02)
9	0.3413*	0.3067*	0.2106*	0.1242	0.0333	-0.0777	-0.0095	0.0000	-0.0450
	(3.27)	(3.12)	(2.28)	(1.45)	(0.41)	(-1.16)	(-0.17)	(0.00)	(-1.11)
12	0.1941	0.1369	0.0756	-0.0041	-0.0647	-0.0931	-0.0234	-0.0137	-0.0587
	(1.85)	(1.40)	(0.80)	(-0.04)	(-0.76)	(-1.30)	(-0.41)	(-0.30)	(-1.46)
24	-0.0029	-0.0513	-0.0776	-0.0591	-0.0557	-0.0271	0.0261	-0.0020	-0.0082
	(-0.03)	(-0.51)	(-0.79)	(-0.62)	(-0.61)	(-0.34)	(0.40)	(-0.03)	(-0.15)
36	0.0369	0.0602	0.0517	0.0419	0.0416	0.0657	0.0351	0.0273	0.0406
	(0.35)	(0.59)	(0.52)	(0.43)	(0.44)	(0.81)	(0.49)	(0.42)	(0.64)
48	0.1819	0.1307	0.1035	0.0895	0.0407	-0.0172	0.0179	0.0500	0.0595
	(1.74)	(1.30)	(1.06)	(0.93)	(0.43)	(-0.21)	(0.24)	(0.70)	(0.86)
60	-0.0049	-0.0263	-0.0800	-0.1160	-0.1289	-0.0396	0.0424	0.0518	0.0680
	(-0.05)	(-0.26)	(-0.81)	(-1.20)	(-1.41)	(-0.49)	(0.55)	(0.69)	(0.92)

TABLE 2. The average excess return (%) of the optimal strategies for different look-back period τ (different row) and different holding period h (different column).

$(9, 1)$ strategy performs the best. This is consistent with the finding in Moskowitz et al. (2012), which documents $(9, 1)$ is the best strategy for equity markets although 12-month horizon is the best for most asset classes. Note that the results are inconsistent with the results in the previous section on the performance based on the utility and the Sharpe ratio of the optimal portfolio. This is due to the fact that the optimal strategies not only explore the return signal but also take the volatility into account.[9] Therefore, the profitability pattern characterized by the average returns (or excess returns) used by most of the empirical momentum literature may not capture portfolio wealth effects. This is also observed by comparing with the Sharpe ratio in Fig. 4.6 which characterizes both return and risk.

[9]In fact, Wang and Xu (2012) find that market volatility has significant power to forecast momentum profitability.

$(\tau \setminus h)$	1	3	6	9	12	24	36	48	60
1	-0.0144	0.0652	0.0714	0.0689	0.0568	-0.0040	0.0006	0.0010	-0.0133
	(-0.14)	(0.89)	(1.34)	(1.52)	(1.37)	(-0.12)	(0.02)	(0.04)	(-0.57)
3	0.1683	0.1915*	0.1460	0.1536*	0.0764	-0.0360	-0.0290	-0.0143	-0.0395
	(1.61)	(2.16)	(1.91)	(2.20)	(1.17)	(-0.69)	(-0.72)	(-0.45)	(-1.38)
6	0.2906*	0.2633*	0.2635*	0.1884*	0.1031	-0.0484	-0.0130	0.0157	-0.0281
	(2.78)	(2.79)	(3.01)	(2.29)	(1.34)	(-0.75)	(-0.26)	(0.40)	(-0.77)
9	0.4075*	0.3779*	0.2422*	0.1538	0.0545	-0.0735	-0.0217	-0.0047	-0.0460
	(3.91)	(3.78)	(2.62)	(1.76)	(0.66)	(-1.05)	(-0.38)	(-0.10)	(-1.12)
12	0.2453*	0.1660	0.0904	0.0122	-0.0748	-0.1195	-0.0602	-0.0454	-0.0798
	(2.35)	(1.67)	(0.94)	(0.13)	(-0.86)	(-1.63)	(-1.02)	(-0.95)	(-1.88)
24	0.0092	-0.0242	-0.0800	-0.0962	-0.0955	-0.0682	-0.0081	-0.0140	-0.0211
	(0.09)	(-0.24)	(-0.81)	(-1.03)	(-1.06)	(-0.88)	(-0.13)	(-0.24)	(-0.39)
36	-0.0005	0.0194	0.0219	0.0212	0.0113	0.0030	0.0127	0.0241	0.0206
	(-0.01)	(0.19)	(0.23)	(0.22)	(0.12)	(0.04)	(0.18)	(0.37)	(0.33)
48	0.0779	0.0733	0.0231	0.0019	-0.0392	-0.0676	-0.0004	0.0435	0.0382
	(0.74)	(0.73)	(0.24)	(0.02)	(-0.42)	(-0.83)	(-0.01)	(0.61)	(0.55)
60	-0.0568	-0.0852	-0.1403	-0.1706	-0.1986	-0.1091	-0.0043	0.0157	0.0239
	(-0.54)	(-0.84)	(-1.41)	(-1.77)	(-2.15)	(-1.36)	(-0.06)	(0.22)	(0.34)

TABLE 3. The average excess return (%) of the optimal strategies for different look-back period τ (different row) and different holding period h (different column) for the pure momentum model.

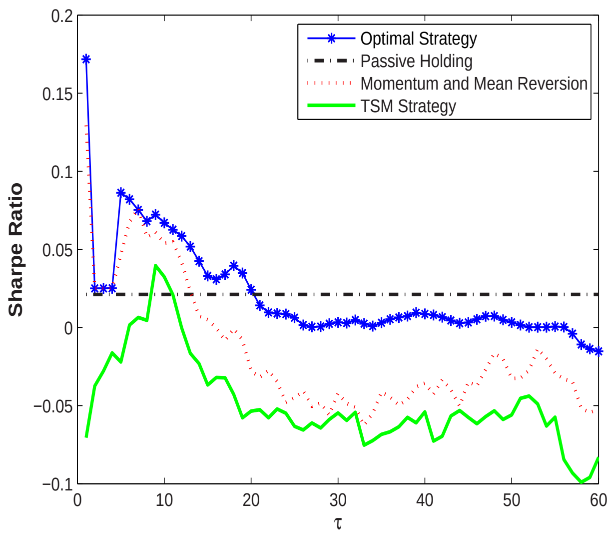


FIGURE 6.1. The average Sharpe ratio for the optimal portfolio, the momentum and mean reversion portfolio and the TSM portfolio with $\tau \in [1, 60]$ and the passive holding portfolio from January 1881 until December 2012.

Next we use the Sharpe ratio to examine the performance of the optimal strategy π_t^* in (2.7) and compare with the passive index strategy and two TSM strategies: one follows from Moskowitz et al. (2012) and the other is the TSM strategy based on the sign of the optimal strategies $sign(\pi_t^*)$ as the trading signal (instead of the average excess return over a past period), which is called *momentum and mean reversion strategy* for convenience. For time horizons from 1 month to 60 months and one month holding period, we report the Sharpe ratios of the portfolios for the four strategies in Fig. 6.1.[10] For comparison, we collect the Sharpe ratio for the optimal portfolio and the passive holding portfolio reported in Fig. 4.6 and report the Sharpe ratios of the TSM strategy in green solid line and of momentum and mean reversion strategy in red dotted line together in Fig. 6.1 from January 1881 to December 2012. We have three observations. First, the TSM strategy outperforms the market only for $\tau = 9, 10$ and the momentum and mean reversion strategy outperform the market for short time horizons $\tau \leq 13$. Secondly, by taking the mean reversion effect into account, the momentum and mean reversion strategy performs better than the TSM strategy for all time horizons. Finally, the optimal strategy significantly outperforms both the momentum and mean reversion strategy (for all time horizons beyond 4 months) and the TSM strategy (for all time horizons). Note that the only difference between the optimal strategy and the momentum and mean reversion strategy is that the former considers the size of the portfolio position, which is inversely proportional to variance, while the latter always takes one unit of long/short position. This implies that, in addition to price trend, the position size (or volatility) is another very important factor for exploring the timing opportunity.

Following Eq. (5) in Moskowitz et al. (2012), we study the performance of the cumulative excess return. That is, the return at time t is defined by

$$\hat{R}_{t+1} = sign(\pi_t^*)\frac{0.1424}{\hat{\sigma}_{S,t}}R_{t+1}, \tag{6.1}$$

where 0.1424 is the sample standard deviation of the total return index. Also, following Moskowitz et al. (2012), the ex ante annualized variance $\hat{\sigma}_{S,t}^2$ for the total return index is calculated as the exponentially weighted lagged squared month returns,

$$\hat{\sigma}_{S,t}^2 = 12\sum_{i=0}^{\infty}(1-\delta)\delta^i(R_{t-1-i} - \bar{R}_t)^2, \tag{6.2}$$

where the constant 12 scales the variance to be annual, and $\bar{R}_t$ is the exponentially weighted average return based on the weights $(1-\delta)\delta^i$. The parameter δ is chosen so that the center of mass of the weights is $\sum_{i=1}^{\infty}(1-\delta)\delta^i = \delta/(1-\delta) = 2$ months.

[10]The monthly Sharpe ratio for the pure mean reversion strategy is 0.0250, slightly higher than that for the passive holding portfolio (0.0211).

To avoid look ahead bias contaminating the results, we use the volatility estimates at time $t-1$ applied to time t returns throughout the analysis.

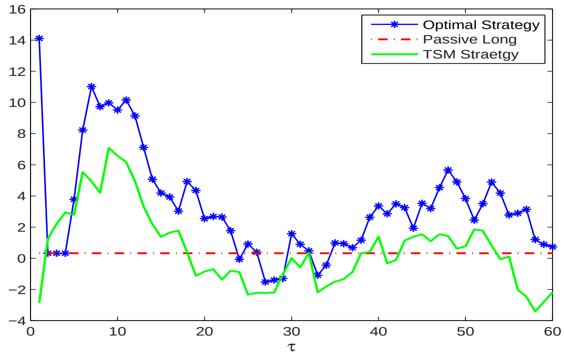


FIGURE 6.2. Terminal log cumulative excess return of the optimal strategies (2.7) and TSM strategies with $\tau \in [1, 60]$ and passive long strategy from January 1876 to December 2012.

Fig. 6.2 shows the terminal values of the log cumulative excess returns of the optimal strategy (2.7) and the TSM strategy with $\tau \in [1, 60]$, together with the passive long strategy, from January 1876 to December 2012.[11] It shows that the optimal strategy outperforms the TSM strategy for all time horizons (beyond 4 months), while the TSM strategy outperforms the market for small time horizons (from about 2 to 18 months). The terminal values of the log cumulative excess return have similar patterns to the average Sharpe ratio reported in Fig. 6.1, especially for small time horizons. The inconsistency for long time horizons is caused by the estimation errors of volatility in (6.2).

[11]Notice the passive long strategy introduced in Moskowitz et al. (2012) is different from the passive holding strategy studied in the previous sections. Passive long means holding one share of the index each period, however, passive holding in our paper means investing $1 in the index in the first period and holding it until the last period.

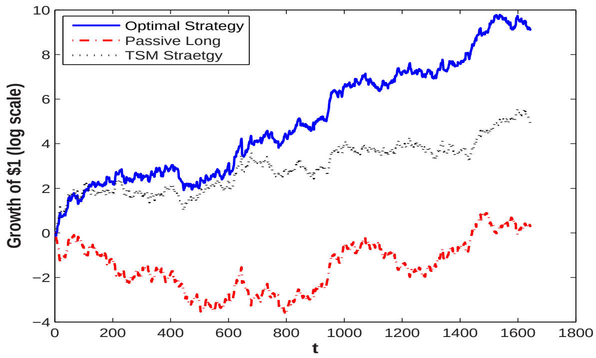


FIGURE 6.3. Log cumulative excess return of the optimal strategy (2.7) and momentum strategy with $\tau = 12$ and passive long strategy from January 1876 to December 2012.

Finally, with a 12-month time horizon Fig. 6.3 illustrates the log cumulative excess return of the optimal strategy (2.7), the momentum strategy and the passive long strategy from January 1876 to December 2012. It shows that, first, the optimal strategy has the highest growth rate and the passive long strategy has the lowest growth rate. Secondly, we can replicate the pattern of Fig. 3 in Moskowitz et al. (2012) and show that the TSM strategy outperforms the passive long strategy.[12]

In summary, we have shown the optimal strategy outperforms the TSM strategy of Moskowitz et al. (2012). We conclude this section with some remarks. By comparing the performance of two TSM strategies, we find that the TSM strategy based on momentum and reversal trading signal is more profitable than the pure TSM strategy of Moskowitz et al. (2012). Also, this paper studies the S&P 500 index over 140 years of data, while Moskowitz et al. (2012) focus on the futures and forward contracts that include country equity indexes, currencies, commodities, and sovereign bonds. Despite a large difference between the data investigated, we find similar patterns for the TSM in the stock index and replicate their results with respect to the stock index.

7. CONCLUSION

To characterize the short-run momentum and long-run reversal in financial markets, we propose a continuous-time model of asset price dynamics with the drift as a weighted average of mean reversion and moving average components. By applying

[12]In fact, the profits of the diversified TSMOM portfolio in Moskowitz et al. (2012) are to some extent driven by the bonds when scaling for the volatility in equation (5) of their paper, and hence applying the TSM strategies to stock index may have less significant profits than the diversified TSMOM portfolio.

the maximum principle for control problems of stochastic delay differential equations, we derive the optimal strategies in closed form. By estimating the model to the S&P 500, we show that the optimal strategy outperforms the pure momentum, pure mean reversion strategies, and the market index. The outperformance holds for out-of-sample tests and with short-sale constraints. The optimality is immune to the market states, investor sentiment and market volatility. The results show that the profitability pattern reflected by the average return of commonly used strategies in much of the empirical literature may not reflect the effect of portfolio wealth.

The model proposed in this paper is simple and stylized. The weights of the momentum and mean reversion components are constant. When market conditions change, the weights can be time-varying. Hence it would be interesting to model their dependence on market conditions. This can be modelled, for example, as a Markov switching process or based on some rational learning process (Xia 2001). Also the optimization problem is solved under log utility in this paper. It would be interesting to study the intertemporal effect under general power utility functions considered in Koijen et al. (2009). We could also consider stochastic volatilities of the index process. Finally, an extension to a multi-asset model to study the cross-sectional optimal strategies would be helpful to understand cross-sectional momentum and reversal.

Appendix A. Properties of the Solutions to the System (2.2)-(2.4)

Let $C([-\tau,0],R)$ be the space of all continuous functions $\varphi:[-\tau,0]\to R$. For a given initial condition $S_t=\varphi_t$, $t\in[-\tau,0]$ and $\mu_0=\hat{\mu}$, the following proposition shows that the system (2.2)-(2.4) admits pathwise unique solutions such that $S_t>0$ almost surely for all $t\geq 0$ whenever $\varphi_t>0$ for $t\in[-\tau,0]$ almost surely.

Proposition A.1. *The system* (2.2)-(2.4) *has an almost surely continuously adapted pathwise unique solution* (S,μ) *for a given* $\mathcal{F}_0$*-measurable initial process* $\varphi:\Omega\to C([-\tau,0],R)$. *Furthermore, if* $\varphi_t>0$ *for* $t\in[-\tau,0]$ *almost surely, then* $S_t>0$ *for all* $t\geq 0$ *almost surely.*

Proof. Basically, the solution can be found by using forward induction steps of length τ as in Arriojas, Hu, Mohammed and Pap (2007). Let $t\in[0,\tau]$. Then the system (2.2)-(2.4) becomes

$$\begin{cases} dS_t = S_t dN_t, \qquad t\in[0,\tau],\\ d\mu_t=\alpha(\bar{\mu}-\mu_t)dt+\sigma_\mu' dZ_t, \qquad t\in[0,\tau],\\ \;\; S_t=\varphi_t \quad \text{for} \quad t\in[-\tau,0] \quad \text{almost surely} \quad \text{and} \quad \mu_0=\hat{\mu}. \end{cases} \tag{A.1}$$

where $N_t=\int_0^t\big[\frac{\phi}{\tau}\int_{s-\tau}^s\frac{d\varphi_u}{\varphi_u}+(1-\phi)\mu_s\big]ds+\int_0^t\sigma_S' dZ_s$ is a semimartingale. Denote by $\langle N_t,N_t\rangle=\int_0^t\sigma_S'\sigma_S ds$, $t\in[0,\tau]$, the quadratic variation. Then system (A.1) has a unique solution

$$\begin{cases} S_t=\varphi_0\exp\big\{N_t-\dfrac{1}{2}\langle N_t,N_t\rangle\big\},\\ \mu_t=\bar{\mu}+(\hat{\mu}-\bar{\mu})\exp\{-\alpha t\}+\sigma_\mu'\exp\{-\alpha t\}\displaystyle\int_0^t\exp\{\alpha u\}dZ_u \end{cases}$$

for $t\in[0,\tau]$. This clearly implies that $S_t>0$ for all $t\in[0,\tau]$ almost surely, when $\varphi_t>0$ for $t\in[-\tau,0]$ almost surely. By a similar argument, it follows that $S_t>0$ for all $t\in[\tau,2\tau]$ almost surely. Therefore $S_t>0$ for all $t\geq 0$ almost surely, by induction. Note that the above argument also gives existence and pathwise-uniqueness of the solution to the system (2.2)-(2.4).

□

Appendix B. Proof of Proposition 2.1

To solve the stochastic control problems, there are two approaches: the dynamic programming method (HJB equation) and the maximum principle. Since the SDDE is not Markovian, we cannot use the dynamic programming method. Recently, Chen and Wu (2010) introduced a maximum principle for the optimal control problem of SDDE. This method is further extended by Øksendal et al. (2011) to consider a one dimensional system allowing both delays of moving average type and jumps. Because the optimal control problem of SDDE is relative new to the field of economics and finance, we first introduce the maximum principle of Chen and Wu (2010) briefly and refer readers to their paper for details.

B.1. **Introduction to the Maximum Principle for an Optimal Control Problem of SDDE.** Consider a past-dependent state X_t of a control system

$$\begin{cases} dX_t = b(t, X_t, X_{t-\tau}, v_t, v_{t-\tau})dt + \sigma(t, X_t, X_{t-\tau}, v_t, v_{t-\tau})dZ_t, & t \in [0, T], \\ X_t = \xi_t, \qquad v_t = \eta_t, \qquad t \in [-\tau, 0], \end{cases} \tag{B.1}$$

where Z_t is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$, and $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^{n\times d}$ are given functions. In addition, v_t is a $\mathcal{F}_t\,(t \geq 0)$-measurable stochastic control with values in U, where $U \subset \mathbb{R}^k$ is a nonempty convex set, $\tau > 0$ is a given finite time delay, $\xi \in C[-\tau, 0]$ is the initial path of X, and η, the initial path of $v(\cdot)$, is a given deterministic continuous function from $[-\tau, 0]$ into U such that $\int_{-\tau}^{0} \eta_s^2 ds < +\infty$. The problem is to find the optimal control $u(\cdot) \in \mathcal{A}$, such that

$$J(u(\cdot)) = \sup\{J(v(\cdot)); v(\cdot) \in \mathcal{A}\}, \tag{B.2}$$

where $\mathcal{A}$ denotes the set of all admissible controls. The associated performance function J is given by

$$J(v(\cdot)) = \mathbb{E}\Big[\int_0^T L(t, X_t, v_t, v_{t-\tau})dt + \Phi(X_T)\Big],$$

where $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ are given functions. Assume **(H1)**: the functions b, σ, L and Φ are continuously differentiable with respect to $(X_t, X_{t-\tau}, v_t, v_{t-\tau})$ and their derivatives are bounded.

In order to derive the maximum principle, we introduce the following adjoint equation,

$$\begin{cases} -dp_t = \big\{(b_X^u)^\top p_t + (\sigma_X^u)^\top z_t + \mathbb{E}_t[(b_{X_\tau}^u|_{t+\tau})^\top p_{t+\tau} + (\sigma_{X_\tau}^u|_{t+\tau})^\top z_{t+\tau}] \\ \qquad + L_X(t, X_t, u_t, u_{t-\tau})\big\}dt - z_t dZ_t, \quad t \in [0, T], \\ p_T = \Phi_X(X_T), \qquad p_t = 0, \qquad t \in (T, T + \tau], \\ z_t = 0, \qquad t \in [T, T + \tau]. \end{cases} \tag{B.3}$$

We refer readers to Theorems 2.1 and 2.2 in Chen and Wu (2010) for the existence and uniqueness of the solutions of the systems (B.3) and (B.1) respectively.

Next, define a Hamiltonian function H from $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^k \times L^2_{\mathcal{F}}(0, T + \tau; \mathbb{R}^n) \times L^2_{\mathcal{F}}(0, T + \tau; \mathbb{R}^{n\times d})$ to $\mathbb{R}$ as follows,

$$\begin{aligned} &H(t, X_t, X_{t-\tau}, v_t, v_{t-\tau}, p_t, z_t) \\ =&\langle b(t, X_t, X_{t-\tau}, v_t, v_{t-\tau}), p_t\rangle + \langle \sigma(t, X_t, X_{t-\tau}, v_t, v_{t-\tau}), z_t\rangle + L(t, X_t, v_t, v_{t-\tau}). \end{aligned}$$

Assume **(H2)**: the functions $H(t,\cdot,\cdot,\cdot,\cdot,p_t,z_t)$ and $\Phi(\cdot)$ are concave with respect to the corresponding variables respectively for $t\in[0,T]$ and given p_t and z_t. Then we have the following proposition on the maximum principle of the stochastic control system with delay by summarizing Theorem 3.1, Remark 3.4 and Theorem 3.2 in Chen and Wu (2010).

Proposition B.1. (i) *Let $u(\cdot)$ be an optimal control of the optimal stochastic control problem with delay subject to* (B.1) *and* (B.2), *and $X(\cdot)$ be the corresponding optimal trajectory. Then we have*

$$\max_{v\in U}\langle H^u_v+\mathbb{E}_t[H^u_{v_\tau}|_{t+\tau}],v\rangle=\langle H^u_v+\mathbb{E}_t[H^u_{v_\tau}|_{t+\tau}],u_t\rangle,\qquad a.e.,\ a.s.;\tag{B.4}$$

(ii) *Suppose $u(\cdot)\in\mathcal{A}$ and let $X(\cdot)$ be the corresponding trajectory, p_t and z_t be the solution of the adjoint equation* (B.2). *If **(H1)**, **(H2)** and* (B.4) *hold for $u(\cdot)$, then $u(\cdot)$ is an optimal control for the stochastic delayed optimal problem* (B.1) *and* (B.2).

B.2. **Proof of Proposition 2.1.** We now apply Proposition B.1 to our stochastic control problem. Let $P_u:=\ln S_u$ and $V_u:=\ln W_u$. Then the stochastic delayed optimal problem in Section 2 becomes to maximize $\mathbb{E}_u[\Phi(X_T)]:=\mathbb{E}_u[\ln W_T]=\mathbb{E}_u[V_T]$, subject to

$$\begin{cases} dX_u=b(u,X_u,X_{u-\tau},\pi_u)du+\sigma(u,X_u,\pi_u)dZ_u, & u\in[t,T],\\ X_u=\xi_u,\qquad v_u=\eta_u, \qquad u\in[t-\tau,t], \end{cases}\tag{B.5}$$

where

$$X_u=\begin{pmatrix}P_u\\ \mu_u\\ V_u\end{pmatrix},\qquad \sigma=\begin{pmatrix}\sigma'_S\\ \sigma'_\mu\\ \pi_u\sigma'_S\end{pmatrix},$$

$$b=\begin{pmatrix}\frac{\phi}{\tau}(P_u-P_{u-\tau})+(1-\phi)\mu_u-(1-\phi)\frac{\sigma'_S\sigma_S}{2}\\ \alpha(\bar\mu-\mu_u)\\ -\frac{\pi_u^2\sigma'_S\sigma_S}{2}+\pi_u\Big[\frac{\phi}{\tau}(P_u-P_{u-\tau})+\frac{\sigma'_S\sigma_S}{2}\phi+(1-\phi)\mu_u-r\Big]+r\end{pmatrix}.$$

Then we have the following adjoint equation

$$\begin{cases} -dp_u=\big\{(b_X^{\pi^*})^\top p_u+(\sigma_X^{\pi^*})^\top z_u+\mathbb{E}_u[(b_{X_\tau}^{\pi^*}|_{u+\tau})^\top p_{u+\tau}+(\sigma_{X_\tau}^{\pi^*}|_{u+\tau})^\top z_{u+\tau}]\\ \qquad +L_X\big\}du-z_udZ_u, \quad u\in[t,T],\\ p_T=\Phi_X(X_T), \qquad p_u=0, \qquad u\in(T,T+\tau],\\ z_u=0, \qquad u\in[T,T+\tau], \end{cases}$$

where

$$p_u = (p_u^i)_{3\times 1}, \qquad z_u = (z_u^{ij})_{3\times 2}, \qquad (b_X^{\pi^*})^\top = \begin{pmatrix} \frac{\phi}{\tau} & 0 & \frac{\phi}{\tau}\pi_u^* \\ 1-\phi & -\alpha & (1-\phi)\pi_u^* \\ 0 & 0 & 0 \end{pmatrix},$$

$$(b_{X_\tau}^{\pi^*}|_{u+\tau})^\top = \begin{pmatrix} -\frac{\phi}{\tau} & 0 & -\frac{\phi}{\tau}\pi_{u+\tau}^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Phi_X(X_T) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad L_X = 0,$$

$$(\sigma_X^{\pi^*})^\top = (\sigma_{X_\tau}^{\pi^*}|_{u+\tau})^\top = \mathbf{0}_{2\times 3\times 3}.$$

Since the parameters and terminal values for dp_u^3 are deterministic, we can assert that $z_u^{31} = z_u^{32} = 0$ for $u \in [t,T]$, which leads to $p_u^3 = 1$ for $u \in [t,T]$. Then the Hamiltonian function H is given by

$$\begin{aligned} H = &\Big[\frac{\phi}{\tau}(P_u - P_{u-\tau}) + (1-\phi)\mu_u - (1-\phi)\frac{\sigma_S'\sigma_S}{2}\Big]p_u^1 + \alpha(\bar{\mu} - \mu_u)p_u^2 \\ &+ \Big\{ -\frac{\pi_u^2\sigma_S'\sigma_S}{2} + \pi_u\big[\frac{\phi}{\tau}(P_u - P_{u-\tau}) + \frac{\sigma_S'\sigma_S}{2}\phi + (1-\phi)\mu_u - r\big] + r\Big\}p_u^3 \\ &+ \sigma_S'\begin{pmatrix} z_u^{11} \\ z_u^{12} \end{pmatrix} + \sigma_\mu'\begin{pmatrix} z_u^{21} \\ z_u^{22} \end{pmatrix}, \end{aligned}$$

so that

$$H_\pi^{\pi^*} = -\pi_u^*\sigma_S'\sigma_S + \frac{\phi}{\tau}(P_u - P_{u-\tau}) + \frac{\sigma_S'\sigma_S}{2}\phi + (1-\phi)\mu_u - r.$$

It can be verified that $\mathbb{E}_u[H_{\pi_\tau}^{\pi^*}|_{u+\tau}] = 0$. Therefore,

$$\langle H_\pi^{\pi^*} + \mathbb{E}_u[H_{\pi_\tau}^{\pi^*}|_{u+\tau}], \pi\rangle = \pi_u\big[-\pi_u^*\sigma_S'\sigma_S + \frac{\phi}{\tau}(P_u - P_{u-\tau}) + \frac{\sigma_S'\sigma_S}{2}\phi + (1-\phi)\mu_u - r\big].$$

Taking the derivative with respect to π_u and letting it equal zero yields

$$\begin{aligned} \pi_u^* &= \frac{\frac{\phi}{\tau}(P_u - P_{u-\tau}) + \frac{\sigma_S'\sigma_S}{2}\phi + (1-\phi)\mu_u - r}{\sigma_S'\sigma_S} \\ &= \frac{\phi m_u + (1-\phi)\mu_u - r}{\sigma_S'\sigma_S}. \end{aligned}$$

This gives the optimal investment strategy.

Appendix C. Additional out-of-sample tests and Rolling Window Estimations

In this appendix, we provide some robustness analysis on out-of-sample tests and rolling window estimations.

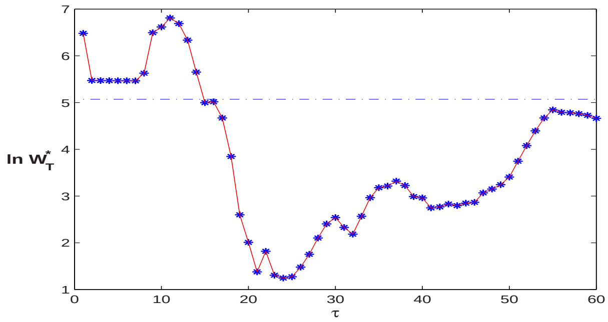


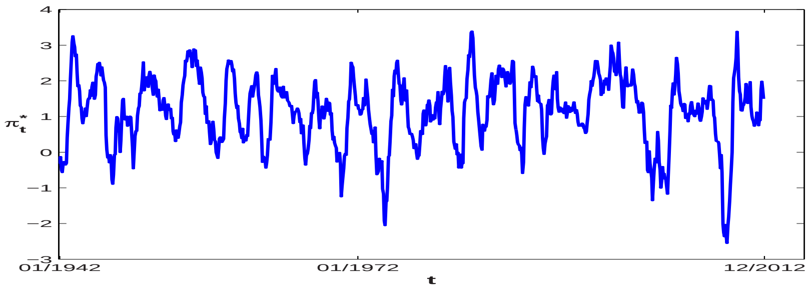
FIGURE C.1. The utility of terminal wealth for $\tau \in [1, 60]$ based on the out-of-sample period of the last 71 years.

C.1. **Additional out-of-sample tests.** As an additional out-of-sample test, we split the whole data set into two equal periods: January 1871—December 1941 and January 1942—December 2012. We estimate the model for the first sub-sample period and do the out-of-sample test over the second sub-sample period. Notice the data in the two periods are quite different. The market index increases gradually in the first period but fluctuates widely in the second period as illustrated in Fig. 4.2 (a). For the out of sample test, Fig. C.1 illustrates the utility of terminal wealth for $\tau \in [1, 60]$ using sample data of the last 71 years. It becomes clear that the optimal strategies still outperform the market for $\tau \in [1, 14]$.

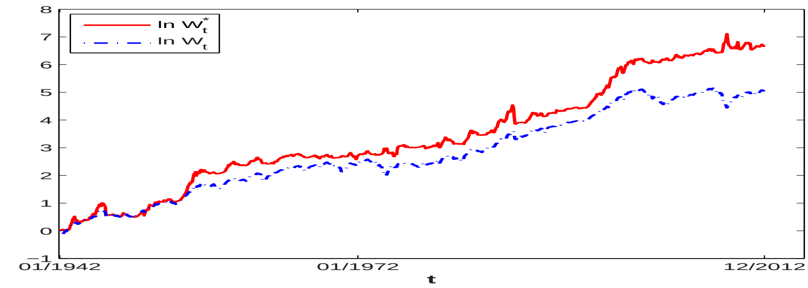
With fixed $\tau = 12$, Fig. C.2 illustrates the corresponding time series of the optimal portfolio and the utility of the portfolio wealth by conducting the out-of-sample test from January 1942 to December 2012. It shows that the utility of the optimal strategy grows gradually and outperforms the market index. We also use the last 10 years and 20 years data as the out-of-sample data and find the results are robust.

C.2. **Rolling Window Estimations.** We also implement rolling window estimations. We first fix $\tau = 12$ and estimate parameters of (3.2) at each month by using the past 20 years' data to avoid look ahead bias. Fig. C.3 illustrates the estimated parameters. The big jump in estimated $\sigma_{S(1)}$ during 1930—1950 is consistent with the high volatility of market return illustrated in Fig. C.4 (b).

Fig. C.4 illustrates the time series of (a) the index level and (b) the simple return of the total return index of S&P 500; (c) the optimal portfolio and (d) the utility of the optimal portfolio wealth from December 1890 to December 2012 for $\tau = 12$ with 20 year rolling window estimated parameters. The index return and π_t^* are positively correlated with correlation 0.0620. In addition, we find that the profits are higher after 1930s.



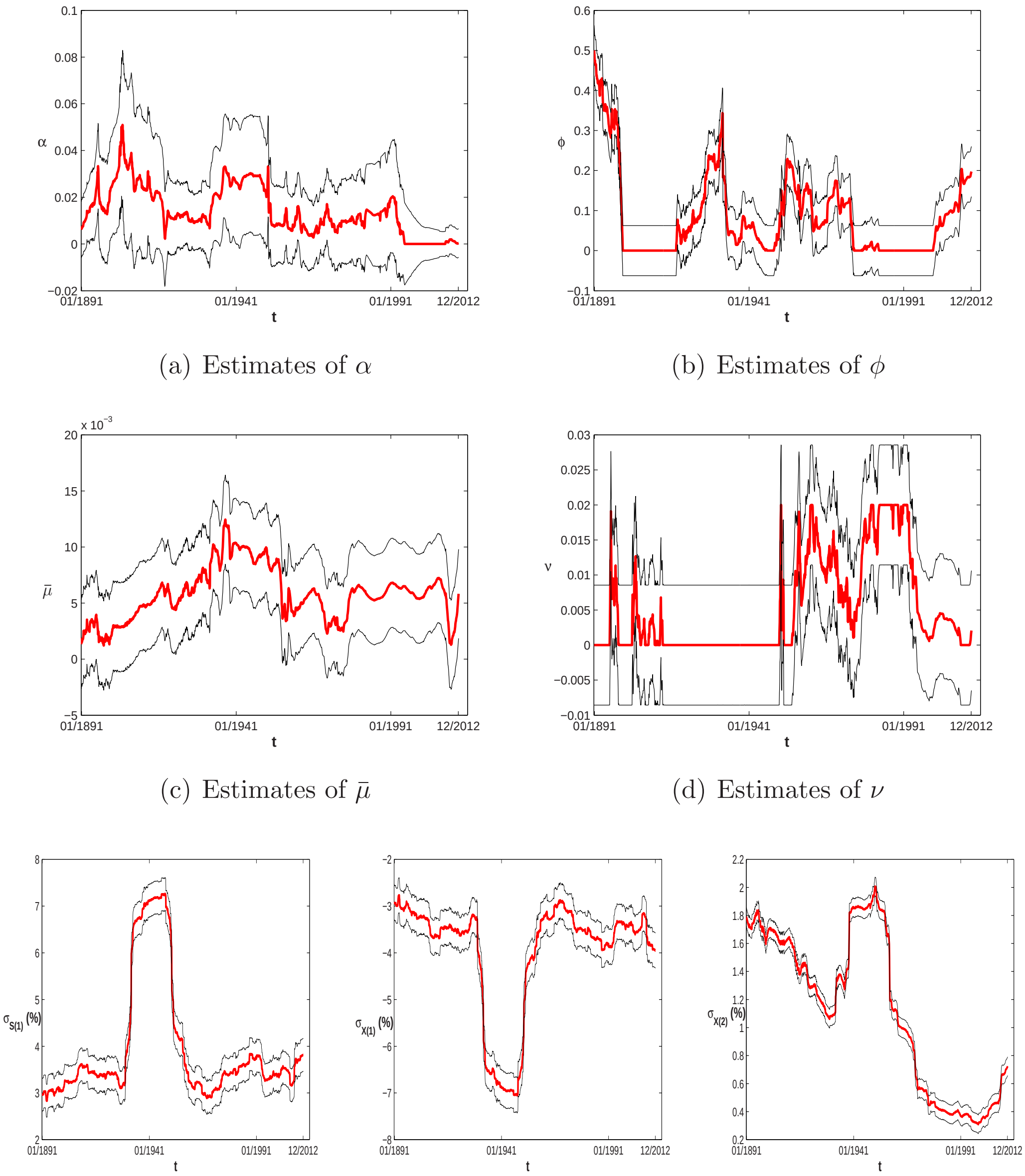
(a)



(b)

FIGURE C.2. The time series of (a) the optimal portfolio and (b) the utility of wealth from January 1942 until December 2012 for $\tau = 12$ for the out-of-sample tests with out-of-sample data for the last 71 years.

Fig. C.3 also illustrates the interesting phenomenon: that the estimated ϕ is very close to zero for three periods of time, implying insignificant momentum but significant mean reversion effect. By comparing Fig. C.3 (b) and (e), we observe that the insignificant ϕ is accompanied by high volatility $\sigma_{S(1)}$. Fig. C.5 illustrates the correlations of the estimated $\sigma_{S(1)}$ with (a) the estimated ϕ and the return of the optimal strategies for (b) the full model, (c) the pure momentum model and (d) the TSM return for $\tau \in [1, 60]$. Interestingly, higher volatility is accompanied by a less significant momentum effect with small time horizons ($\tau \leq 13$). But ϕ and $\sigma_{S(1)}$ are positive correlated when the time horizon becomes large. One possible reason is that a long time horizon makes the trading signal less sensitive to the changes in price and hence the trading signal is significant only when the market price changes dramatically in a high volatility period. Fig. C.5 (c) and (d) show that the profitability of the optimal strategies for the pure momentum model and the TSM strategies are sensitive to the estimated market volatility. The return is positively (negatively) related to market volatility for short (long) time horizons.



(a) Estimates of α

(b) Estimates of ϕ

(c) Estimates of $\bar{\mu}$

(d) Estimates of ν

(e) Estimates of $\sigma_{S(1)}$

(f) Estimates of $\sigma_{X(1)}$

(g) Estimates of $\sigma_{X(2)}$

FIGURE C.3. The estimates of (a) α; (b) ϕ; (c) $\bar{\mu}$; (d) ν; (e) $\sigma_{S(1)}$; (f) $\sigma_{X(1)}$ and (g) $\sigma_{X(2)}$ for $\tau = 12$ based on data from the past 20 years.

But Fig. C.5 (b) shows that the optimal strategies for the full model perform well even in a high volatility market.

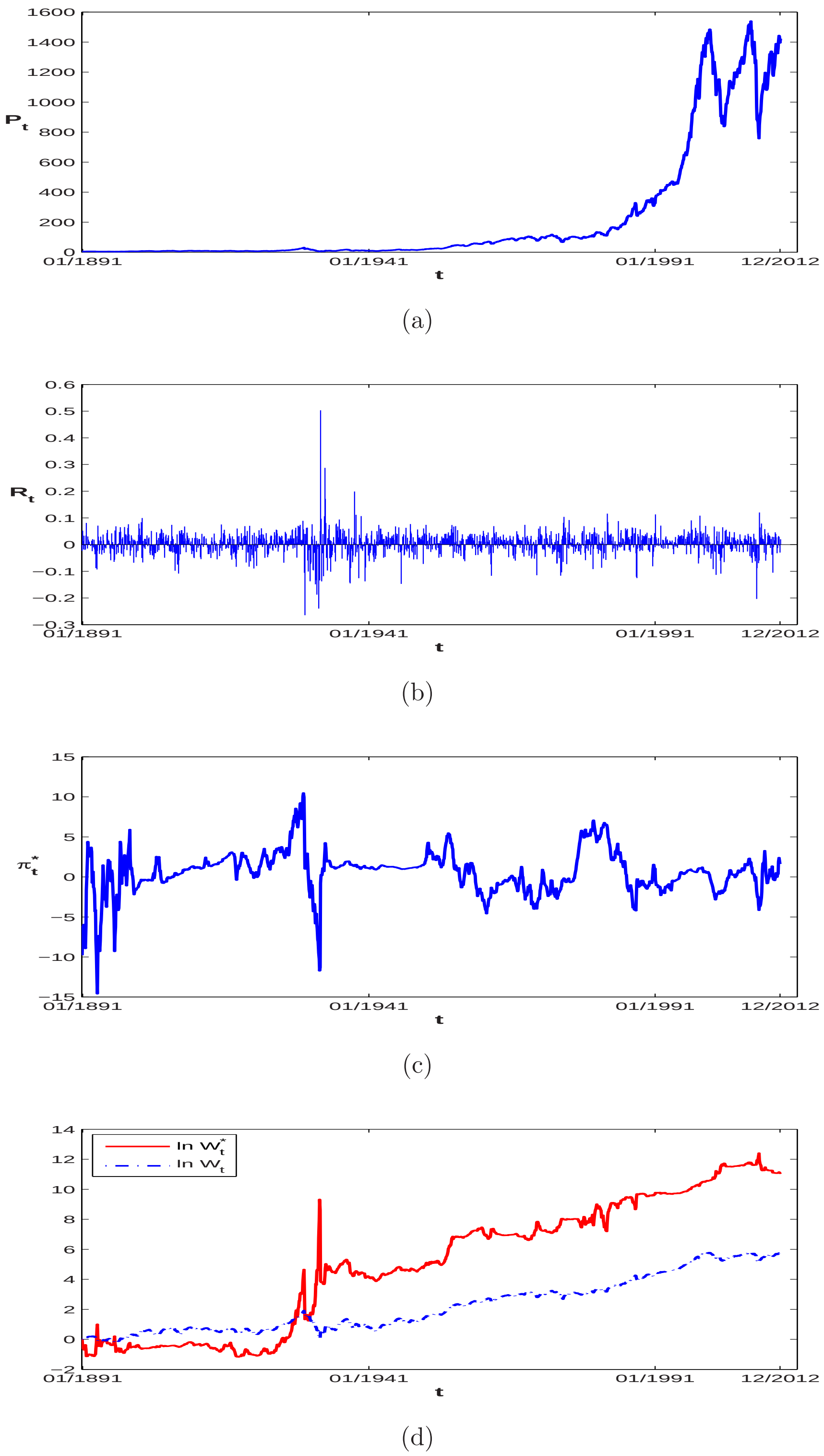
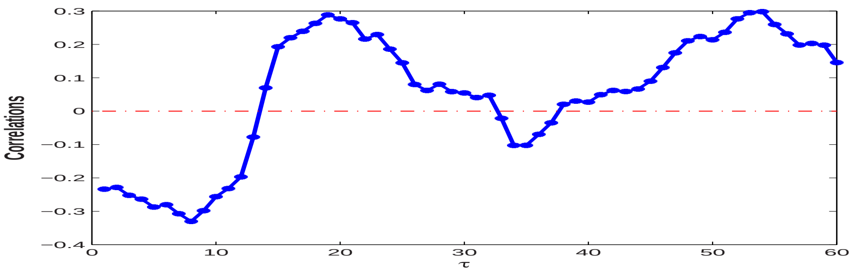
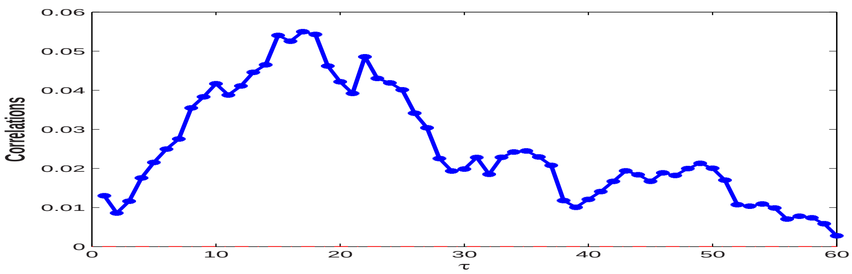


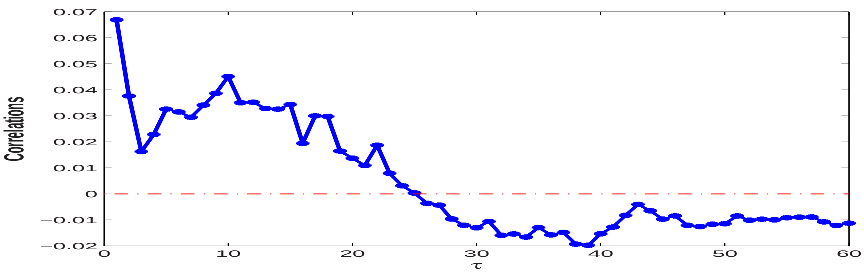
FIGURE C.4. The time series of (a) the index level and (b) the simple return of the total return index of S&P 500; (c) the optimal portfolio and (d) the utility of wealth from December 1890 until December 2012 for $\tau = 12$ with 20 year rolling window estimated parameters.



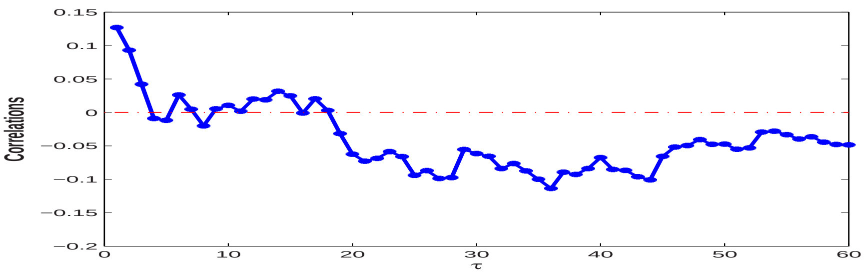
(a)



(b)



(c)



(d)

FIGURE C.5. The correlations of the estimated $\sigma_{S(1)}$ with (a) the estimated ϕ and the return of the optimal strategies for (b) the full model, (c) the pure momentum model and (d) the TSM return.

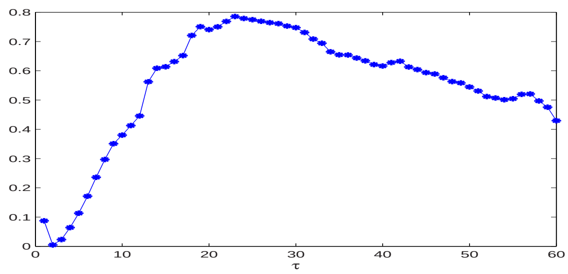


FIGURE C.6. The fraction of ϕ significantly different from zero for $\tau \in [1, 60]$.

We also study other time horizons. We find that the estimates of $\sigma_{S(1)}$, $\sigma_{X(1)}$ and $\sigma_{X(2)}$ are insensitive to τ but the estimates of ϕ are sensitive to τ. Fig. C.6 illustrates the corresponding fraction of ϕ which is significantly different from zero for $\tau \in [1, 60]$. It shows that the momentums with 20-30 month horizons occur most frequently during the period of December 1890 until December 2012.

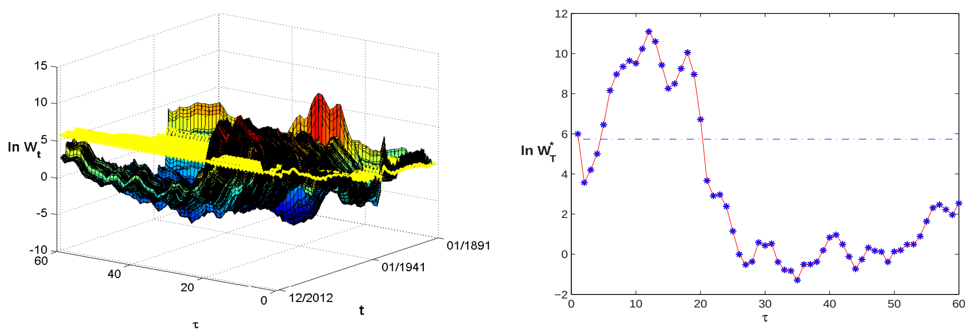


FIGURE C.7. The utility of wealth from December 1890 to December 2012 for the optimal portfolio with $\tau \in [1, 60]$ and the passive holding portfolio with 20 year rolling window estimated parameters.

Fig. C.7 (a) illustrates the utility of wealth from December 1890 until December 2012 for the optimal portfolio with $\tau \in [1, 60]$ and the passive holding portfolio. Especially, the utility of terminal wealth illustrated in Fig. C.7 (b) shows that the optimal strategies work well for short horizons $\tau \leq 20$ and the terminal utility reaches its peak at $\tau = 12$.

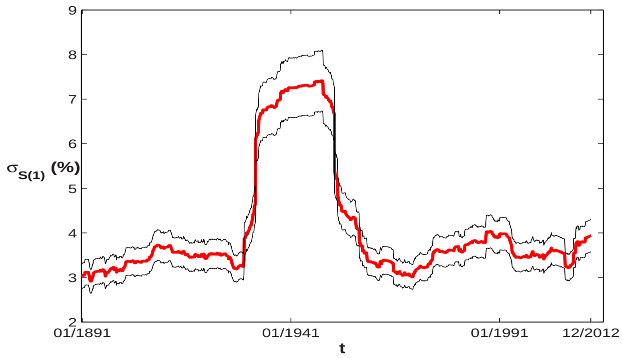


FIGURE C.8. Estimates of $\sigma_{S(1)}$ for the pure momentum model ($\phi = 1$) based on data from the past 20 years.

Fig. C.8 illustrates the estimates of $\sigma_{S(1)}$ for the pure momentum model ($\phi = 1$) based on data from the past 20 years; the big jump in volatility is due to the Great Depression in 1930s.

Fig. C.9 illustrates the time series of (a) the optimal portfolio and (b) the utility of wealth from December 1890 until December 2012 for $\tau = 12$ for the pure momentum model with the 20 year rolling window estimated $\sigma_{S(1)}$. By comparing Fig. C.8 and Fig. C.9 (b), the optimal strategy implied by the pure momentum model suffers huge losses during the high market volatility period. However Fig C.4 illustrates that the optimal strategy implied by the full model makes big profits during the big market volatility period.

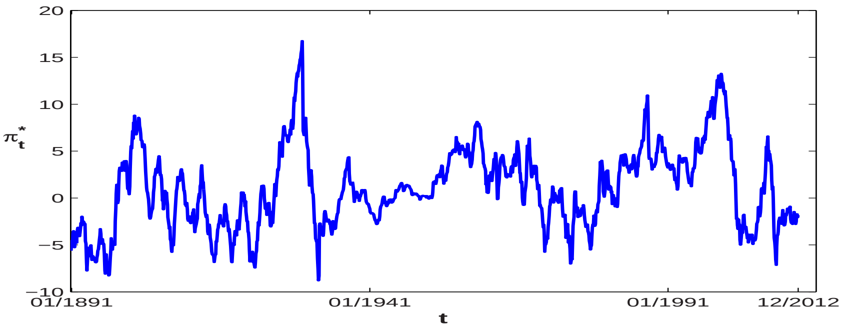
Fig. C.10 illustrates the estimated parameters for the pure mean-reversion model based on data from the past 20 years.

Fig. C.11 illustrates the time series of the optimal portfolio and the utility of wealth from December 1890 until December 2012 for the pure mean-reversion model with 20 year rolling window estimated parameters. After eliminating the look-ahead bias, the pure mean-reversion strategy cannot outperform the stock index anymore.

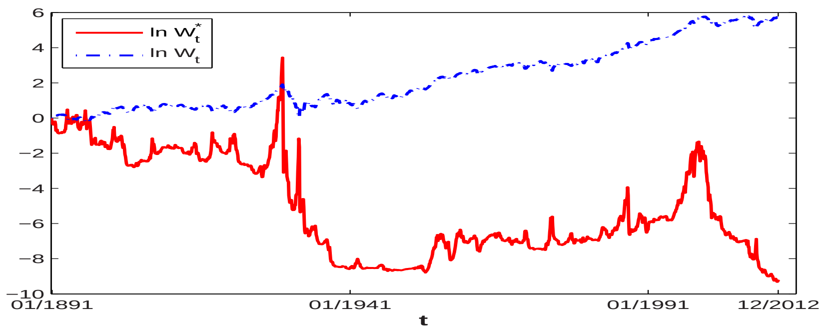
We also implement the estimations for different window sizes of 25, 30 and 50 years and we find that the estimated parameters are insensitive to the size of rolling window and the performance of strategies is similar to the case of 20 year rolling window estimation (not reported here).

APPENDIX D. REGRESSIONS ON THE MARKET STATES, SENTIMENT AND VOLATILITY

D.1. **Market States.** First, we follow Cooper et al. (2004) and Hou et al. (2009) and define market state using the cumulative return of the stock index (including



(a)



(b)

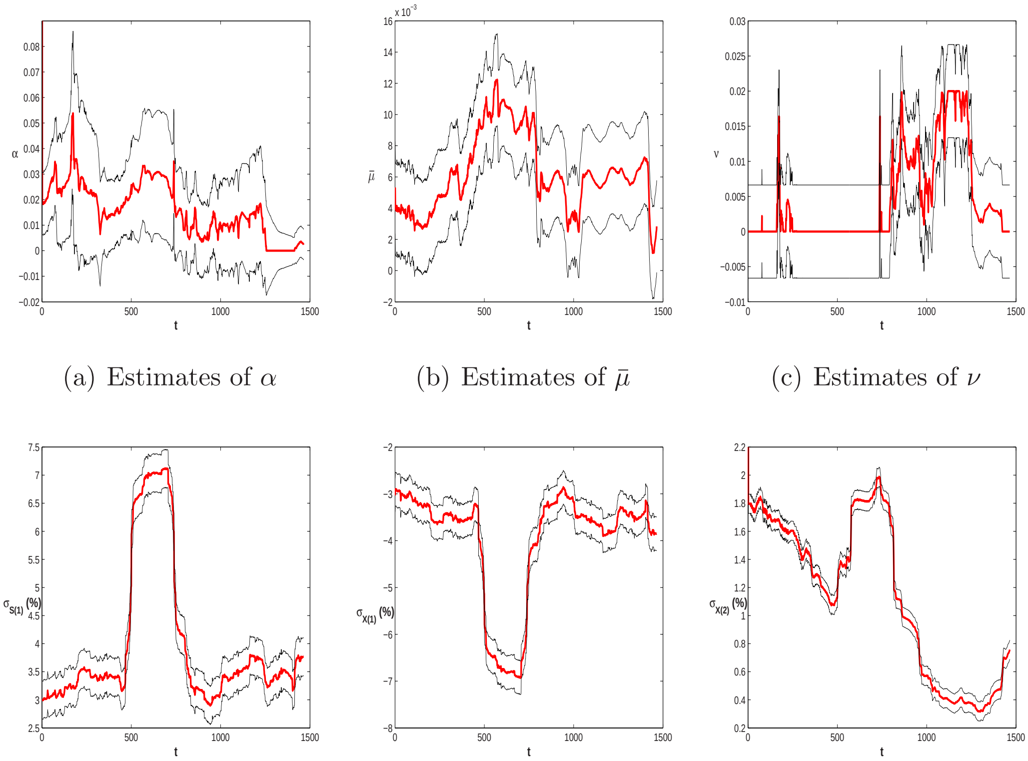
FIGURE C.9. The time series of (a) the optimal portfolio and (b) the utility of wealth from December 1890 until December 2012 for $\tau = 12$ for the pure momentum model with 20 year rolling window estimated parameters.

dividends) over the most recent 36 months.[13] We label a month as an up (down) market month if the market's three-year return is non-negative (negative). There are 1165 up months and 478 down months from February 1876[14] to December 2012.

We compute the average return of the optimal strategy and compare the average returns between up and down market months. Table 4 presents the average unconditional excess returns and the average excess returns for up and down market months. The unconditional average excess return is 87 basis points per month. In up market months, the average excess return is 81 basis points and it is statistically significant. In down market months, the average excess return is 101 basis points; this value is economically significant although it is not statistically significant. The

[13]The results are similar if we use the alternative 6, 12 or 24 month market state definition, even though they are more sensitive to sudden changes in market sentiment.

[14]We escape January 1876 in which there is no return to the optimal strategies.



(a) Estimates of α (b) Estimates of $\bar{\mu}$ (c) Estimates of ν

(d) Estimates of $\sigma_{S(1)}$ (e) Estimates of $\sigma_{X(1)}$ (f) Estimates of $\sigma_{X(2)}$

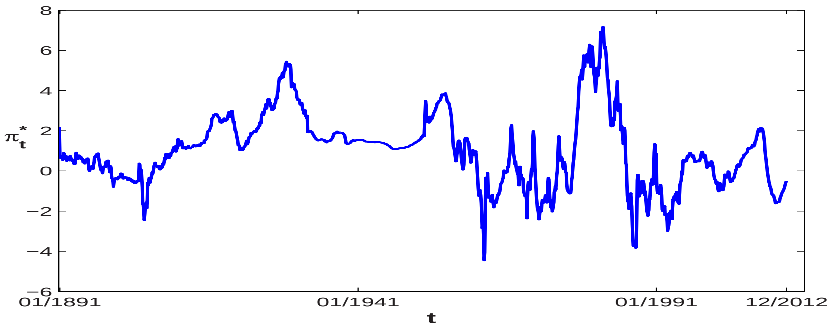
Figure C.10. The estimates of (a) α; (b) ϕ; (c) $\bar{\mu}$; (d) ν; (e) $\sigma_{S(1)}$; (f) $\sigma_{X(1)}$ and (g) $\sigma_{X(2)}$ for the pure mean-reversion model based on data from the past 20 years.

	Observations (N)	Average Excess Return
Unconditional Return	1643	0.0087
		(2.37)
Up Market	1165	0.0081
		(4.09)
Down Market	478	0.0101
		(0.87)

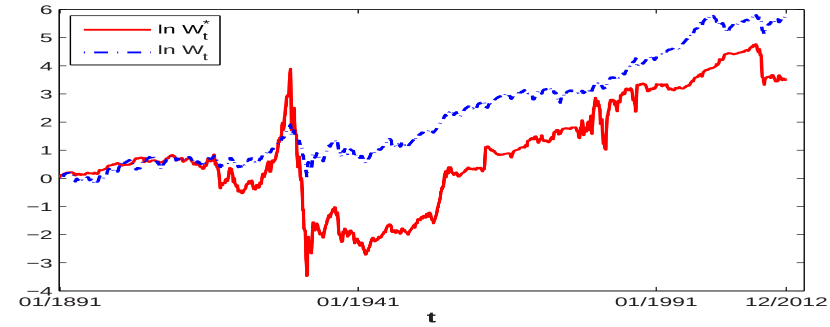
Table 4. The average excess return of the optimal strategy for $\tau = 12$.

difference between down and up months is 20 basis points, which is not significantly different from zero based on a two sample t-test (p-value of 0.87).[15]

[15]The p-values for the pure momentum strategy, pure mean reversion strategy and TSM are 0.87, 0.87 and 0.67 respectively.



(a)



(b)

FIGURE C.11. The time series of (a) the optimal portfolio and (b) the utility of wealth from December 1890 until December 2012 for the pure mean-reversion model with 20 year rolling window estimated parameters.

We use the following regression model to test for the difference in returns:

$$R_t^* - r = \alpha + \kappa I_t(UP) + \beta(R_t - r) + \epsilon_t, \tag{D.1}$$

where $R_t^* = (W_t^* - W_{t-1}^*)/W_{t-1}^*$ in (4.1) is the month t return of the optimal strategy, $R_t - r$ is the excess return of the stock index, and $I_t(UP)$ is a dummy variable that takes the value of one if month t is in an up month, and zero otherwise. The regression intercept α measures the average return of the optimal strategy in down market months, and the coefficient κ captures the incremental average return in up market months relative to down months. We also replace the market state dummy in (D.1) with the lagged market return over the previous 36 months (not reported here), and the results are robust.

Table 5 reports the regression coefficients for the full model, the pure momentum model, pure mean reversion model and the TSM strategy in Moskowitz et al. (2012) for $\tau = 12$ respectively. Except the TSM, which earns significant positive return in

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
α	0.0094	0.0476	-0.0000	0.0060
	(1.46)	(1.34)	(-0.01)	(3.23)
κ	0.0005	0.0041	-0.0005	-0.0014
	(0.06)	(0.10)	(-0.32)	(-0.63)
β	-1.0523	-6.7491	0.3587	-0.1548
	(-12.48)	(-14.60)	(22.97)	(-6.39)

TABLE 5. The coefficients for the regression (D.1).

down market, both α and κ are insignificant for all other strategies; the results are consistent with those in Table 4.

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
α	0.0086	0.0423	0.0002	0.0058
	(1.44)	(1.32)	(0.18)	(3.29)
κ	-0.0008	-0.0034	-0.0002	-0.0017
	(-0.11)	(-0.09)	(-0.14)	(-0.81)
β_1	0.1994	0.7189	0.0708	0.1341
	(1.90)	(1.27)	(3.84)	(4.30)
β_2	-2.5326	-15.5802	0.6991	-0.4964
	(-22.16)	(-25.31)	(34.88)	(-14.63)

TABLE 6. The coefficients for the regression (D.2).

To further control for market risk in up and down market months, we now run the following regression:

$$R_t^* - r = \alpha + \kappa I_t(UP) + \beta_1(R_t - r)I_t(UP) + \beta_2(R_t - r)I_t(DOWN) + \epsilon_t, \quad \text{(D.2)}$$

the regression coefficients are reported in Table 6. These results are similar to those in Table 5.

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
α	0.0083	0.0409	0.0002	0.0057
	(1.22)	(1.08)	(0.14)	(3.03)
κ	0.0006	0.0037	-0.0002	-0.0012
	(0.07)	(0.08)	(-0.14)	(-0.53)

TABLE 7. The coefficients for the regression (D.3).

We also run the following regression:

$$R_t^* - r = \alpha + \kappa I_{t-1}(UP) + \epsilon_t, \tag{D.3}$$

and Table 7 reports the coefficients. We find that the dummy variable of up market month has no predictive power for returns of all strategies.

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
α	0.0094	0.0476	0.0000	0.0060
	(1.46)	(1.34)	(-0.01)	(3.23)
κ	0.0005	0.0041	-0.0005	-0.0014
	(0.06)	(0.10)	(-0.32)	(-0.63)

Table 8. The coefficients for the regression (D.4).

We also study the beta adjusted returns:

$$\begin{aligned} R_t^* - r &= \alpha^{CAPM} + \beta^{CAPM}(R_t - r) + \varepsilon_t, \\ R_t^* - r - \beta^{CAPM}(R_t - r) &= \alpha + \kappa I_t(UP) + \epsilon_t, \end{aligned} \tag{D.4}$$

and Table 8 reports the coefficients. Again, the difference of return between up and down months are not significant for all the strategies.

Compared with findings in Hou et al. (2009) where cross-sectional momentum returns are higher in up months, we do not find significant differences between up and down months for the strategies from our model and the TSM. Among the strategies from our model and the TSM, only the TSM has significant return in down market months.

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
a	0.0059	0.0267	0.0005	0.0040
	(1.77)	(1.74)	(1.49)	(2.57)
b	0.0040	0.0134	-0.0003	0.0023
	(1.20)	(0.87)	(-1.01)	(1.48)

Table 9. The coefficients for the regression (D.5).

D.2. **Investor Sentiment.** In this subsection, we examine the relationship between the excess return of the optimal strategies and investor sentiment by running the following regression:

$$R_t^* - r = a + bT_{t-1} + \epsilon, \tag{D.5}$$

where T_t is the sentiment index measure used by Baker and Wurgler (2006). The data on the Baker-Wurger sentiment index from 07/1965 to 12/2010 is obtained from the Jeffrey Wurglers web site (http://people.stern.nyu.edu/jwurgler/). Table 9 reports the coefficients. The results suggest that sentiment index has no predictive power for returns of optimal strategies and of the TSM. We also examine monthly changes of the level of sentiment by replacing T_t with its monthly changes and their orthogonalized indexes. The results are similar.

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
α	0.0037	0.0421	-0.0014	0.0053
	(1.06)	(1.11)	(-1.00)	(2.78)
κ	0.0138	0.0141	0.0137	-0.0232
	(0.25)	(0.05)	(1.21)	(-1.48)

Table 10. The coefficients for the regression (D.6).

D.3. **Market Volatility.** Finally, we examine the predictability of market volatility to the profitability. First, we run the following regression:

$$R_t^* - r = \alpha + \kappa \hat{\sigma}_{S,t-1} + \epsilon_t, \tag{D.6}$$

where the ex ante annualized volatility $\hat{\sigma}_{S,t}$ is given by (6.2). Table 10 reports the results. We see that volatility has no predictive power for returns of optimal strategies and of the TSM.

	Full Model	Pure Momentum	Pure Mean Reversion	TSM
α	-0.0020	-0.0151	0.0012	0.0025
	(-0.27)	(-0.36)	(0.80)	(1.19)
κ_1	0.1043	0.5763	-0.0127	0.0016
	(1.34)	(1.33)	(-0.80)	(-0.07)
κ_2	0.1026	0.5564	-0.0098	0.0084
	(1.80)	(1.75)	(-0.84)	(0.53)

Table 11. The coefficients for the regression (D.7).

Second, we run the regression by following Wang and Xu (2012):

$$R_t^* - r = \alpha + \kappa_1 \hat{\sigma}^+_{S,t-1} + \kappa_2 \hat{\sigma}^-_{S,t-1} + \epsilon_t, \tag{D.7}$$

where $\hat{\sigma}^+_{S,t}$ ($\hat{\sigma}^-_{S,t}$) is equal to $\hat{\sigma}_{S,t}$ if the market state is up (down) and otherwise equal to 0. Table 11 reports the coefficients. The results are similar to those in Table 10 even if we distinguish volatility in up and down market months.

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