Finding Roots of Bergman Kernels of Rational Hartog's Triangles

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December 2019

1 Introduction

We are interested in studying a collection of domains in $\mathbb{C}^2$

$$\mathbb{H}_\gamma = \left\{ (z_1, z_2) \in \mathbb{C}^2 \ : \ 0 \leq \left| \frac{z_1^\gamma}{z_2} \right| < 1 \right\}$$

where γ is a positive real number. Let us consider the Bergman kernel of each of these domains to be the function $\mathbb{B}_\gamma : \ \mathbb{H}_\gamma \times \mathbb{H}_\gamma \to \mathbb{C}$ which has a regenerating property for l_2 holomorphic functions on $\mathbb{H}_\gamma$.

In order to study these functions, we can make use of a particular structure of $\mathbb{H}_\gamma$ to consider its Bergman kernel as a function of 2 complex variables as opposed to 4.

$$\mathbb{H}_\gamma \times \mathbb{H}_\gamma = \left\{ (z_1, z_2, w_1, w_2) \ : \ 0 \leq \left| \frac{z_1^\gamma}{z_2} \right| < 1, \ 0 \leq \left| \frac{w_1^\gamma}{z_2} \right| < 1 \right\}$$

So if we fix some $(z_1, z_2, w_1, w_2) \in \mathbb{H}_\gamma$, we note that $0 \leq \left| \frac{(z_1 \overline{w_1})^\gamma}{z_2 \overline{w_2}} \right| < 1$. Define $s := z_1 \overline{w_1}$ and $t := z_2 \overline{w_2}$. We can study $\mathbb{B}_\gamma(z_1, z_2, w_1, w_2)$ by studying an lower dimensional version of the bergman kernel which we will denote $\mathbb{B}_\gamma(s, t)$ to be this function.

In particular, we shall examine these functions for $\gamma \in \mathbb{Q}$ with $\gamma > 1$. So fix $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$ and $m > n$. Now, from Debraj's paper, we have that

$$\mathbb{B}_{m/n}(s, t) = \frac{1}{mn\pi^2} \frac{\sum_{\beta_1=0}^{2m-2} \sum_{\beta_2=0}^{2n} \hat{D}_1(\beta_1) \hat{D}_2(\beta_1, \beta_2) s^{\beta_1} t^{\beta_2}}{(t^m - s^n)^2 (1 - t^m)^2}$$

We shall postpose discussing $\hat{D}_1(\beta_1)$ and $\hat{D}_2(\beta_2)$ for the moment as we will reformulate them later for notational convenience. Note that since $m/n > 1$ and $|s| < 1$ we have that $(s, s) \in \mathbb{H}_{m,n}$. With this in mind define

$$\tilde{N}(s) = \sum_{\beta_1=0}^{2m-2} \sum_{\beta_2=0}^{2n} \hat{D}_1(\beta_1) \hat{D}_2(\beta_1, \beta_2) s^{\beta_1 + \beta_2}$$

on $\mathbb{H}_{m/n}$. Note that if we find a root of $\tilde{N}$ in $\mathbb{D}$, it will induce a root of the kernel in $\mathbb{H}_{m/n} \times \mathbb{H}_{m/n}$.

2 Simplifying $\hat{D}_1(\beta_1)$ and $\hat{D}_2(\beta_1, \beta_2)$

Well to find roots of $\tilde{N}(s)$, we now actually need to analyze the coefficient functions from Debraj's Formulas. This section will establish some properties of these coefficients which will help us argue for the existence of a root within the domain. First consider these basic computations.

$$\hat{D}_1(\beta_1) = D_{mn}(2mn - n(\beta_1+1) - 1) = \begin{cases} 0 & 2mn - n(\beta_1+1) - 1 \le -1 \\ 2mn - n(\beta_1+1) - 1 + 1 & 0 \le 2mn - n(\beta_1+1) - 1 \le mn - 1 \\ 2mn - 1 - (2mn - n(\beta_1+1) - 1) & mn \le 2mn - n(\beta_1+1) - 1 \le 2mn - 2 \\ 0 & 2mn - n(\beta_1+1) - 1 \ge 2mn - 1 \end{cases}$$

$$= \begin{cases} 0 & 2mn \le n(\beta_1+1) \\ 2mn - n\beta_1 - n & m - 1 \le \beta_1 \le 2m - 1 - \frac{1}{n} \\ n\beta_1 + n & -1 + \frac{1}{n} \le \beta_1 \le m - 1 - \frac{1}{n} \\ 0 & 0 \ge n(\beta_1+1) \end{cases}$$

$$= \begin{cases} 0 & \text{never} \\ 2mn - n\beta_1 - n & m - 1 \le \beta_1 \le 2m - 2 \\ n\beta_1 + n & 0 \le \beta_1 \le m - 2 \\ 0 & \text{never} \end{cases}$$

$$= \begin{cases} 2mn - n\beta_1 - n & m - 1 \le \beta_1 \le 2m - 2 \\ n\beta_1 + n & 0 \le \beta_1 \le m - 2 \end{cases}$$

$$= n \begin{cases} \beta_1 + 1 & 0 \le \beta_1 \le m - 2 \\ 2m - 1 - \beta_1 & m - 1 \le \beta_1 \le 2m - 2 \end{cases}$$

We shall now compute $\hat{D}_2(\beta_1, \beta_2)$ in a similar fashion.

$$\begin{aligned} &\hat{D}_2(\beta_1, \beta_2) \\ =&D_m(m(\beta_2+1) + n(\beta_1+1) - 2mn - 1) \\ =&\begin{cases} 0 & m(\beta_2+1) + n(\beta_1+1) - 2mn - 1 \le -1 \\ m(\beta_2+1) + n(\beta_1+1) - 2mn & 0 \le m(\beta_2+1) + n(\beta_1+1) - 2mn - 1 \le m - 1 \\ 2m - 1 - m(\beta_2+1) - n(\beta_1+1) + 2mn + 1 & m \le m(\beta_2+1) + n(\beta_1+1) - 2mn - 1 \le 2m - 2 \\ 0 & m(\beta_2+1) + n(\beta_1+1) - 2mn - 1 \ge 2m - 1 \end{cases} \\ =&\begin{cases} 0 & m\beta_2 + n\beta_1 \le 2mn - n - m \\ -2mn + n + m + m\beta_2 + n\beta_1 & 2mn - m - n + 1 \le m\beta_2 + n\beta_1 \le 2mn - n \\ 2mn - n + m - m\beta_2 - n\beta_1 & 2mn - n + 1 \le m\beta_2 + n\beta_2 \le 2mn - n + m - 1 \\ 0 & m\beta_2 + n\beta_1 \ge 2mn + m \end{cases} \end{aligned}$$

We define the following notation $\hat{\gamma}(\beta_1, \beta_2) = m\beta_2 + n\beta_1$ and $\alpha = 2mn - n$. Now, the computation above simplifies to

$$\hat{D}_2(\beta_1, \beta_2) = \begin{cases} 0 & \hat{\gamma}(\beta_1, \beta_2) \le \alpha - m \\ -\alpha + m + \hat{\gamma}(\beta_1, \beta_2) & \alpha - m + 1 \le \hat{\gamma}(\beta_1, \beta_2) \le \alpha \\ \alpha + m - \hat{\gamma}(\beta_1, \beta_2) & \alpha + 1 \le \hat{\gamma}(\beta_1, \beta_2) \le \alpha + m - 1 \\ 0 & \alpha + m \le \hat{\gamma}(\beta_1, \beta_2) \end{cases}$$

Note that the monomial terms in $\tilde{N}(s)$ are indexed by $\beta_1 + \beta_2$ which motivates the following change of variables

$$(\beta_1, \beta_2) \to (\beta_1 + \beta_2, \beta_2) := (d, \beta_2)$$

With this change of variables in mind, we define $D_1(d, \beta_2) = \hat{D}_1(d - \beta_2)$ and $D_2(d, \beta_2) = \hat{D}_2(d - \beta_2, \beta_2)$. Now

$$\tilde{N}(s) = \sum_{\beta_1, \beta_2} \hat{D}_1(\beta_1)\hat{D}_2(\beta_1, \beta_2)s^{\beta_1+\beta_2} = \sum_{d, \beta_2} \hat{D}_1(d - \beta_2)\hat{D}_2(d - \beta_2, \beta_2)s^d = \sum_{d, \beta_2} D_1(d, \beta_2)D_2(d, \beta_2)$$

We can be a bit more specific in fact. Some powers of d vanish as $D_2(d,\beta_2)$ will vanish entirely for every relevant β_2. To find out which ones vanish, we need to examine $\hat{\gamma}$.

$$\hat{\gamma}(\beta_1,\beta_2) = m\beta_2 + n\beta_1 = (m-n+n)\beta_2 + n\beta_1 = (m-n)\beta_2 + n(\beta_1+\beta_2) = (m-n)\beta_2 + nd := \gamma(d,\beta_2)$$

$D_2(d,\beta_2)$ will vanish when either $\gamma(d,\beta_2) \leq \alpha - m$ or $\gamma(d,\beta_2) \geq \alpha + m$. Note that $\gamma(d,\beta_2)$ is an increasing function with respect to β_2. So let us find values of d where $\gamma(d,2n) \leq \alpha - m$ or $\gamma(d,0) \geq \alpha + m$.

- Case 1:

$$\begin{aligned} \gamma(d,2n) &\leq \alpha - m \\ (m-n)(2n) + nd &\leq 2mn - n - m \\ nd &\leq 2n^2 - n - m \\ d &\leq 2n - 1 - \frac{m}{n} \\ d &\leq 2n - 2 \end{aligned}$$

  So $\forall\ d < 2n-1$ we have that $D_2(d,\beta_2)$ vanishes for every β_2.

- Case 2:

$$\begin{aligned} \gamma(d,0) \geq \alpha + m& \\ nd &\geq 2mn - n + m \\ d &\geq 2m - 1 + \frac{m}{n} \\ d &\geq 2m \end{aligned}$$

  Similarly, $\forall\ d > 2m-1$ we have that $D_2(d,\beta_2)$ vanishes for every β_2.

With this in mind, define the following coefficients

$$\begin{aligned} a_0 &= \sum_{\beta_2=0}^{2n-1} D_1(2n-1,\beta_2) D_2(2n-1,\beta_2) \\ a_k &= \sum_{\beta_2=0}^{2n} D_1(k+2n-1,\beta_2) D_2(k+2n-1,\beta_2) \qquad\qquad k = 1, \dots 2m-2n-1 \\ a_{2m-2n} &= \sum_{\beta_2=1}^{2n} D_1(2m-1,\beta_2) D_2(2m-1,\beta_2) \end{aligned}$$

Now we can write

$$\tilde{N}(s) = \sum_{d=2n-1}^{2m-1} \sum_{\beta_2} D_1(d,\beta_2) D_2(d,\beta_2) s^d = \sum_{k=0}^{2m-2n} a_k s^{k+2n-1} = s^{2n-1} \sum_{k=0}^{2m-2n} a_k s^k := s^{2n-1} N(s)$$

with $N(0) \neq 0$. So if we can find zeros of $N(s)$ we will be able to find a zero of the berman kernel of $\mathbb{H}_{m/n}$ which lies in $\mathbb{H}_{m,n} \times \mathbb{H}_{m,n}$.

3 Useful Properties of a_k's'

First, we'd like to note that from Debraj's paper, that these are all non negative integers. Additionally, we'd like to now establish that $N(s)$ is a self reciprocal polynomial. This means that $\forall\ k$ we have $a_{2m-2n-k} = a_k$. First we show $\hat{D}_1(\beta_1) = \hat{D}_1(2m-2-\beta_1)$

$$\begin{aligned}
\hat{D}_1(2m-2-\beta_1) &= n\begin{cases} (2m-2-\beta_1) & 0 \le 2m-2-\beta_1 \le m \le 2 \\ 2m-1-(2m-2-\beta_1) & m-1 \le \beta_1 \le 2m-2 \end{cases} \\
&= n\begin{cases} 2m-1-\beta_1 & -2m+2 \le -\beta_1 \le -m \\ \beta_1+1 & -m+1 \le -\beta_1 \le 0 \end{cases} \\
&= n\begin{cases} 2m-1-\beta_1 & m \le \beta_1 \le 2m-2 \\ \beta_1+1 & 0 \le \beta_1 \le m-1 \end{cases} \\
&= n\begin{cases} 2m-1-\beta_1 & m-1 \le \beta_1 \le 2m-2 \\ \beta_1+1 & 0 \le \beta_1 \le m-2 \end{cases} \\
&= \hat{D}_1(\beta_1)
\end{aligned}$$

Now we have that

$$\begin{aligned}
D_1(k+2n-1-\beta_2) &= \hat{D}_1(2m-2-(k+2n-1-\beta_2)) = \hat{D}_1(2m-k-1-(2n-\beta_2)) \\
&= D_1(2m-2n-k+2n-1-(2n-\beta_2))
\end{aligned}$$

We'd like to establish a similar equality: that is, we'd like to show $D_2(k+2n-1,\beta_2) = D_2(2m-2n-k+2n-1, 2n-\beta_2)$. To show this, let us first identify $\gamma(2m-2n-k+2n-1, 2n-\beta_2)$. Let $\Gamma = \gamma(k+2n-1,\beta_2)$. Now we have

$$\begin{aligned}
\gamma(2m-2n-k+2n-1, 2n-\beta_2) &= (m-n)(\beta_2-2n) + n(2m-2n-k+2n-1) \\
&= (m-n)(2n-\beta_2) + n(2m-k-1) \\
&= 2n(m-n) - (m-n)\beta_2 + 2mn - nk - n \\
&= 4mn - (m-n)\beta_2 - nk - 2n^2 - n \\
&= 4mn - 2n - (m-n)\beta_2 - nk - 2n^2 + n \\
&= 2\alpha - (m-n)\beta_2 - n(k+2n-1) \\
&= 2\alpha - \Gamma
\end{aligned}$$

Now we have

$$D_2(2m-2n-k+2n-1,2n-\beta_2) = \begin{cases} 0 & 2\alpha-\Gamma \leq \alpha - m \\ -\alpha+m+2\alpha-\Gamma & \alpha-m+1 \leq 2\alpha-\Gamma \leq \alpha \\ \alpha+m-(2\alpha-\Gamma) & \alpha+1 \leq 2\alpha-\Gamma \leq \alpha+m-1 \\ \alpha+m \leq 2\alpha+\Gamma & \end{cases}$$

$$= \begin{cases} 0 & -\Gamma \leq -\alpha-m \\ \alpha+m-\Gamma & -\alpha-m+1 \leq -\Gamma \leq -\alpha \\ -\alpha+m+\Gamma & -\alpha+1 \leq -\Gamma \leq -\alpha+m-1 \\ 0 & -\alpha+m \leq -\Gamma \end{cases}$$

$$= \begin{cases} 0 & \alpha+m \leq \Gamma \\ \alpha+m-\Gamma & \alpha \leq \Gamma \leq \alpha+m-1 \\ -\alpha+m+\Gamma & \alpha-m+1 \leq \Gamma \leq \alpha-1 \\ 0 & \Gamma \leq \alpha-m \end{cases}$$

$$= \begin{cases} 0 & \alpha+m \leq \Gamma \\ \alpha+m-\Gamma & \alpha+1 \leq \Gamma \leq \alpha+m-1 \\ -\alpha+m+\Gamma & \alpha-m+1 \leq \Gamma \leq \alpha \\ 0 & \Gamma \leq \alpha-m \end{cases}$$

$$= D_2(k+2n-1,\beta_2)$$

The previous two computations establish that $a_k = a_{2m-2n-k}$ but it also shows something stronger. It shows that

$$D_1(k+2n-1,\beta_2)D_2(k+2n-1,\beta_2) = D_1(2m-2n-k+2n+1,2n-\beta_2)D_2(2m-2n-k+2n-1,2n-\beta_2)$$

Using this, we can also establish the following nice fact. We have that

$$a_{m-n} \mod 2 \equiv D_1(m-n+2n-1,n)D_2(2m-2n-(m-n)+2n-1,n) \mod 2 \equiv m^2 n \mod 2$$

Lastly, from the computations we have that each a_k is divisible by n.

4 Finding Zeros of $N(s)$ with $|s| < 1$

In the previous section, we established that we can find a root of the berman kernel within the general hartog's triangle by finding certain zeros of a polynomial $N(s)$. In particular, we are looking for s_0 such that $N(s_0) = 0$ and $0 < |s_0| < 1$. In the previous section, we established that the coefficients are real and symmetric. From the fact that the coefficients are real, we note that roots will come in conjugate pairs. Additionally since the coefficients are symmetric, we kow that if z_0 is a root of $N(s)$ then $\frac{1}{z_0}$ is also a root of $N(s)$. From these facts, we conclude that since $N(0) \neq 0$. it will be enough to show that $N(s)$ does not have all $2m-2n$ of its roots on the unit circle.

5 Results

We'd like to put all of the following facts together in a clever way. Since every coefficient of $N(s)$ is divisible by n, let $f(s) = \frac{N(s)}{n}$. Now note that if -1 is a root of an integer polynomial, then $-1 \equiv 1 \mod 2$ is a root of the quotient polynomial in $\mathbb{F}_2[x]$. So let us look $\overline{f}(s)$ in the quotient. This polynomial is given by just taking all of the coefficients of f mod 2. Now note that

$$\overline{f}(-1) = \overline{f}(1) = \left(\frac{1}{n}\sum_{k=0}^{2m-2n} a_k\right) \mod 2 = \left(\frac{2}{n}\sum_{k=0}^{m-n-1} a_k\right) \mod 2 + \left(\frac{a_{m-n}}{n}\right) \mod 2$$
$$= \left(\frac{a_{m-n}}{n}\right) \mod 2 = \left(\frac{m^2 n}{n}\right) \mod 2 = m^2 \mod 2$$

So note that if m is odd, we have $f(-1) \neq 0$. Now consider the following lemma:

Lemma: If f is a real self reciprical polynomial of even degree, then $f(-1)$ and $f'(-1)$ have the opposite sign.
pf: We can verify this by direct computation. We have

$$f(-1) = 2\sum_{j=0}^{M-1}(-1)^j a_j + (-1)^M a_M \qquad f'(-1) = -2M\sum_{j=0}^{M-1}(-1)^j a_j + (-1)^{M-1} a_M$$

5.1 Computational lemmas about the integer cases

From *Edholm*, for each $j \in \mathbb{N}$ we'd like to compute $p_j(-1)$ and $q_j(-1)$.

Lemma 1: $\sum_{\ell=1}^{j-1} \ell(-1)^{\ell-1} = \frac{1}{4} + (-1)^j\left(\frac{2j-1}{4}\right)$
pf:

$$\sum_{\ell=1}^{j-1} \ell(-1)^{j-1} = \begin{cases} -\frac{j-2}{2} + j - 1 & j \text{ even} \\ -\frac{j-1}{2} & j \text{ odd} \end{cases} = \begin{cases} \frac{j}{2} & j \text{ even} \\ \frac{-j+1}{2} & j \text{ odd} \end{cases} = \frac{1}{4} + (-1)^j\left(\frac{2j-1}{4}\right)$$

Lemma 2: $\sum_{\ell=1}^{j}(j-\ell)^2(-1)^{\ell-1} = \frac{j(j-1)}{2}$
pf: We shall prove this by induction. For $n = 1$, this follows by inspection. Now suppose the result holds for $n = j$. Now, for the following sum, the key is to notice that the negation all terms but the leading one appear in the sum for the $n = j$ case. Now

$$\sum_{\ell=1}^{j+1}(j+1-\ell)^2(-1)^{\ell-1} = j^2 - \sum_{\ell=1}^{j}(j-\ell)^2(-1)^{\ell-1} = j^2 - \frac{j(j-1)}{2} = \frac{(j+1)j}{2}$$

Lemma 3: $p_j(-1) = \frac{j}{4} + (-1)^j\frac{j}{4}$
pf: Again we shall do this by induction. $p_1(-1) = 0 = \frac{1}{4} + (-1)^1\frac{1}{4}$. Now suppose we know the formula holds for $p_j(-1)$. Now

$$\begin{aligned} p_{j+1}(-1) &= \sum_{\ell=1}^{j}\ell(j+1-\ell)(-1)^\ell = \sum_{\ell=1}^{j-1}\ell(j+1-\ell)(-1)^{\ell-1} + (-1)^{j-1}j \\ &= \sum_{\ell=1}^{j-1}\ell(j-\ell)(-1)^{\ell-1} + \sum_{\ell=1}^{j-1}\ell(-1)^{\ell-1} + (-1)^{j-1}j \\ &= \frac{j}{4} + (-1)^j\frac{j}{4} + \left(\frac{1}{4} + (-1)^j\left(\frac{2j-1}{4}\right)\right) + (-1)^{j-1}j \\ &= \frac{j+1}{4} + (-1)^j\frac{3j}{4} + (-1)^{j+1}\frac{1}{4} + (-1)^{j-1}j \\ &= \frac{j+1}{4} + (-1)^{j+1}\frac{j+1}{4} \end{aligned}$$

Lemma 4: $q_j(-1) = (-1)^{k+1}k$
pf: Again we shall proceed by induction. $q_1(-1) = 1^2 = 1 = (-1)^2 1$ so the formula holds for the base case. Now assume the formula holds for $n = j$. Now

$$\begin{aligned}
q_{j+1}(-1) &= \sum_{\ell=1}^{j+1} \left(\ell^2 + (j+1-\ell)^2(-1)^{j+1}\right)(-1)^{\ell-1} \\
&= \sum_{\ell=1}^{j+1} \left(\ell^2 - (j+1-\ell)^2(-1)^{j}\right)(-1)^{\ell-1} \\
&= \sum_{\ell=1}^{j+1} \left(\ell^2 + (j+1-\ell)^2(-1)^{j}\right)(-1)^{\ell-1} - 2(-1)^j \sum_{\ell=1}^{j+1}(k+1-\ell)^2(-1)^{\ell-1} \\
&= \sum_{\ell=1}^{j} \left(\ell^2 + (j+1-\ell)^2(-1)^{j}\right)(-1)^{\ell-1} + (-1)^j(j+1)^2 - 2(-1)^j \sum_{\ell=1}^{j+1}(k+1-\ell)^2(-1)^{\ell-1} \\
&= \sum_{\ell=1}^{j} \left(\ell^2 + (j-\ell)^2(-1)^{j}\right)(-1)^{\ell-1} + (-1)^j \sum_{\ell=1}^{j}(2j-2\ell+1)(-1)^{\ell-1} \\
&+ (-1)^j(j+1)^2 - 2(-1)^j \sum_{\ell=1}^{j+1}(j+1-\ell)^2(-1)^{\ell-1} \\
&= q_j(-1) + (-1)^j \sum_{\ell=1}^{j}(2j-2\ell+1)(-1)^{\ell-1} + (-1)^j(j+1)^2 - 2(-1)^j \frac{(j+1)j}{2} \\
&= (-1)^{j+1}j + (-1)^j \sum_{\ell=1}^{j}(2j-2\ell+1)(-1)^{\ell-1} + (-1)^j(j+1) \\
&= (-1)^{j+1}j + (-1)^j \left((2j+1)\sum_{\ell=1}^{j}(-1)^{\ell-1} - 2\sum_{\ell=1}^{j}\ell(-1)^{\ell-1}\right) + (-1)^j(j+1) \\
&= (-1)^{j+1}j + (-1)^j \left((2j+1)\left(\frac{1}{2} + (-1)^{j+1}\frac{1}{2}\right) - 2\left(\frac{1}{4} + (-1)^j\frac{2j-1}{4}\right)\right) + (-1)^j(j+1) \\
&= (-1)^{j+1}j + (-1)^j j + (-1)^j(j+1) \\
&= (-1)^{j+2}(j+1)
\end{aligned}$$

With this in mind, we can now compute $N_j(-1)$. From above, we know $N_j(s) = \frac{p_j(s)s^2 + q_j(s)s + p_j(s)s^j}{s}$. So

$$N_j(-1) = \frac{p_j(-1) - q_j(-1) + (-1)^j p_j(-1)}{-1} = -\left(\left(\frac{j}{4} + (-1)^j\frac{j}{4}\right) - (-1)^{j+1}j + (-1)^j\left(\frac{j}{4} + (-1)^j\frac{j}{4}\right)\right)$$

So when k is odd $N_{\frac{k+1}{1}}(-1) = -2(k+1) < 0$.

5.2 Relation when k is odd