# IEOR 4735 Final Project

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## 1 Term Sheet

We would like to price a contract paying on the settlement date $T+\Delta$ the payoff amount in USD, defined on the maturity date $T$ as:

$$N_{notional} \cdot \max\left[0, \left(\frac{S(T)}{S(0)} - k\right) \cdot \left(k' - \frac{L(T,T,T+\Delta)}{L(0,T,T+\Delta)}\right)\right]$$

with

- $N_{notional}$ the notional in USD (e.g 1 million USD)
- $S(t)$ the SX5E spot price at time $t$ (quantoed for Eur into USD)
- $L(t,T,T+\Delta)$ is the $\Delta$ USD forward rate (SOFR) between $T$ and $T+\Delta$, observed at time $t$
- $\Delta$ is the forward period (e.g. 3 months or 0.25 years). $T$ is the maturity date (e.g. 5 years) and therefore $T+\Delta$ is the settlement date.
- $k$, $k'$ given relative strike prices (e.g. both could be 1.00 or

Provide a pricing routine (e.g. Python script) calculating the price of this contract, taking as inputs: the deal terms $(T,\Delta,k,k')$ and the relevant market data (interest rates, volatilities, spot prices, correlations ....).

Explain your precise assumptions and methodology choices clearly in an accompanying write-up.

## 2 Summary

We would like to price hybrid products described by the term sheet above. Generally, because the product has optionality over US domestic rate instruments as well as the performance of an European equity index, we must build a model which considers the US term structure, European stocks, and the exchange rate between USD and EUR.

We model the equity index using a lognormal model under $Q^d$ and model the US term structure under a Hull-White (extend Vasicek) framework. Since the underlying Brownian motions of the two processes are correlated under $Q^d$, we simulate the two processes using Monte Carlo methods and compute the discounted expected future payoff to price the option.

# 3 Rate Modeling

We need to choose a rate model to describe the dynamics of US rates. In particular, we need some way to make predictions about the value of $L(T,T,T+\Delta)$ to accurately price our product. In this project, we will use the **Hull-White (extended Vasicek)** model to describe the US term structure as it is relatively easy to calibrate to market data and is commonly used in predicting future rates.

## 3.1 Hull-White Model

We assume the US short rate follows

$$dr_t = [\theta(t) - ar_t]dt + \sigma_r dW_t$$

We need to find reasonable values for these parameters if we are to actually implement a pricing routine Briefly

- We will estimate $\sigma_r$ by implying the volatility from caplets which can be formulated as bond options.
- We will estimate $a$, sometimes thought of as a "speed of mean reversion" parameter, by performing some sort historical regression.

For the time being, please assume $a, \sigma_r$ are known positive constants. We will now describe the more difficult task of estimating $\theta(t)$ for all relevant values of $t$.

## 3.2 Affine Term Structures and Forward Curves: Estimating $\theta(t)$

**Definition:** If the term structure $\{p(t,T);\ 0 \le t \le T, T > 0\}$ has the form

$$p(t,T) = e^{A(t,T) - B(t,T)r_t}$$

where $A$, $B$ are deterministic functions, then the model is said to possess an **affine term structure**.

**Proposition 1** Suppose you have a short rate model of the form

$$dr_t = \mu(t, r_t)dt + \sigma_r(t, r_t)dW_t$$

where

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)} \end{cases}$$

Then, the model admits an affine term structure where $A$, $B$ satisfy

$$\begin{cases} \frac{\partial B}{\partial t}(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \qquad \begin{cases} \frac{\partial A}{\partial t}(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases}$$

*pf:* See Bjork 4e p.286-287.

So, is Hull-White of this form? Well let $\alpha(t) = -a$, $\beta(t) = \theta(t)$, $\gamma(t) = 0$, and $\delta(t) = \sigma_r^2$ for all relevant $t$. Then

$$\mu(t, r_t) = \alpha(t)r_t + \beta(t) = -ar_t + \theta(t)$$
$$\sigma(t, r_t) = \sqrt{\gamma(t)r_t + \delta(t)} = \sqrt{0r_t + \sigma_r^2} = \sigma_r$$

So, with these choices of $\alpha, \beta, \gamma, \delta$, we see that Hull-White does admit an affine term structure by Proposition 1. Now what do $B(t,T)$ and $A(t,T)$ look like? Let's start by solving the ODE for $B(t,T)$.

$$\begin{aligned}
\frac{\partial B}{\partial t}(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) &= -1 \\
\frac{\partial B}{\partial t}(t,T) - aB(t,T) - \frac{1}{2}(0)B^2(t,T) &= -1 \\
\frac{\partial B}{\partial t}(t,T) - aB(t,T) &= -1 \\
e^{-at}\frac{\partial B}{\partial t}(t,T) - ae^{-at}B(t,T) &= -e^{-at} \\
e^{-at}\frac{\partial B}{\partial t}(t,T) + \frac{d(e^{-at})}{dt}B(t,T) &= -e^{-at} \\
\frac{\partial}{\partial t}\left(e^{-at}B(t,T)\right) &= -e^{-at} \\
e^{-at}B(t,T) &= \int -e^{-at}dt + C \\
e^{-at}B(t,T) &= \frac{e^{-at}}{a} + C \\
B(t,T) &= \frac{1}{a} + Ce^{at}
\end{aligned}$$

Now, using the initial condition, we have

$$0 = B(T,T) = \frac{1}{a} + Ce^{aT} \implies C = -\frac{e^{-aT}}{a}$$

Putting the pieces together, we have

$$B(t,T) = \frac{1}{a} + Ce^{at} = \frac{1}{a} + \left(\frac{-1}{a}e^{-aT}\right)e^{at} = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

Now, onto $A(t,T)$. The differential equation, along with the initial condition simplifies to

$$\begin{aligned}
\frac{\partial A}{\partial s}(s,T) &= \beta(s)B(s,T) - \frac{1}{2}\delta(s)B^2(s,T) \\
\frac{\partial A}{\partial s}(s,T) &= \theta(s)B(s,T) - \frac{1}{2}\sigma_r^2B^2(s,T) \\
\int_t^T \frac{\partial A}{\partial s}(s,T)ds &= \int_t^T \left(\theta(s)B(s,T) - \frac{1}{2}\sigma_r^2B^2(s,T)\right)ds \\
A(T,T) - A(t,T) &= \int_t^T \left(\theta(s)B(s,T) - \frac{1}{2}\sigma_r^2B^2(s,T)\right)ds \\
0 - A(t,T) &= \int_t^T \left(\theta(s)B(s,T) - \frac{1}{2}\sigma_r^2B^2(s,T)\right)ds \\
-A(t,T) &= \int_t^T \left(\theta(s)B(s,T) - \frac{1}{2}\sigma_r^2B^2(s,T)\right)ds \\
A(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_r^2B^2(s,T) - \theta(s)B(s,T)\right)ds
\end{aligned}$$

While we have a closed form expression for $B(s,T)$, it will not help us solve this integral as $\theta(s)$ is undefined currently. So we will leave $A(t,T)$ in this form for now. So, how do we find $\theta(t)$? Well, in all affine term structures, $A, B$ have a convenient relationship to instantaneous forward rates.

**Proposition 2:** For any model which admits an affine term structure, the instantaneous forward rates $f(t,T)$ must satisfy

$$f(0.T) = \frac{\partial B}{\partial T}(0,T)r_0 - \frac{\partial A}{\partial T}(0,T)$$

*pf:* Well, recall instantaneous forward rates are defined in relation to zero-coupon bonds as $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T}$. So, in our affine term structure, we have

$$\begin{aligned} f(0,T) &= -\frac{\partial}{\partial T}\left(\ln p(0,T)\right) \\ &= -\frac{\partial}{\partial T}\left(\ln e^{A(0,T)-B(0,T)r_0}\right) \\ &= -\frac{\partial}{\partial T}\left(A(0,T) - B(0,T)r_0\right) \\ &= \frac{\partial B}{\partial T}(0,T)r_0 - \frac{\partial A}{\partial T}(0,T) \end{aligned}$$

□

So what does this look like in our situation? Well

$$\begin{aligned} \frac{\partial B}{\partial T}(0,T)r_0 &= \left( \frac{\partial}{\partial T}\left(\frac{1}{a}\left(1 - e^{-a(T-t)}\right)\right)\Bigg|_{(t,T)=(0,T)}\right) \cdot r_0 \\ &= \left( e^{-a(T-t)}\Big|_{(t,T)=(0,T)}\right) \cdot r_0 \\ &= e^{-aT} r_0 \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial A}{\partial T}(0,T) &= \frac{\partial}{\partial T}\left(\int_t^T \left(\frac{1}{2}\sigma_r^2 B^2(s,T) - \theta(s)B(s,T)\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\partial}{\partial T}\left(\int_t^T \left(\frac{1}{2}\sigma_r^2 \left(\frac{1}{a}\left(1-e^{-a(T-s)}\right)\right)^2 - \theta(s)\left(\frac{1}{a}\left(1-e^{-a(T-s)}\right)\right)\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\partial}{\partial T}\left(\int_t^T \frac{\sigma_r^2}{2a^2}\left(1-2e^{-aT}e^{as}+e^{-2aT}e^{2as}\right) ds - \int_t^T \frac{\theta(s)}{a}\left(1-e^{-a(T-s)}\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\partial}{\partial T}\left(\frac{\sigma_r^2}{2a^2}\left(\int_t^T ds - 2e^{-aT}\int_t^T e^{as}ds + e^{-2aT}\int_t^T e^{2as}ds\right) - \int_t^T \frac{\theta(s)}{a}\left(1-e^{-a(T-s)}\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\partial}{\partial T}\left(\frac{\sigma_r^2}{2a^2}\left((T-t) - 2e^{-aT}\left(\frac{e^{aT}-e^{at}}{a}\right) + e^{-2aT}\left(\frac{e^{2aT}-e^{2at}}{2a}\right)\right) - \int_t^T \frac{\theta(s)}{a}\left(1-e^{-a(T-s)}\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\partial}{\partial T}\left(\frac{\sigma_r^2}{2a^2}\left((T-t) - \frac{2}{a}\left(1-e^{-a(T-t)}\right) + \left(\frac{1-e^{-2a(T-t)}}{2a}\right)\right) - \int_t^T \frac{\theta(s)}{a}\left(1-e^{-a(T-s)}\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\sigma_r^2}{2a^2}\left(1-2e^{-a(T-t)}+e^{-2a(T-t}\right)\Big|_{(t,T)=(0,T)} - \frac{\partial}{\partial T}\left(\int_t^T \frac{\theta(s)}{a}\left(1-e^{-a(T-s)}\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\sigma_r^2}{2a^2}\left(1-e^{-aT}\right)^2 - \frac{\partial}{\partial T}\left(\int_t^T \frac{\theta(s)}{a}\left(1-e^{-a(T-s)}\right) ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\sigma_r^2}{2a^2}\left(1-e^{-aT}\right)^2 - \frac{\partial}{\partial T}\left(\frac{1}{a}\int_t^T \theta(s)ds - \frac{e^{-aT}}{a}\int_t^T \theta(s)e^{as}ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\sigma_r^2}{2a^2}\left(1-e^{-aT}\right)^2 - \left(\frac{1}{a}\theta(T) - \frac{e^{-aT}}{a}\theta(T)e^{aT} - (-a)\frac{e^{-aT}}{a}\int_t^T \theta(s)e^{as}ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\sigma_r^2}{2a^2}\left(1-e^{-aT}\right)^2 - \left(\int_t^T \theta(s)e^{-a(T-s)}ds\right)\Bigg|_{(t,T)=(0,T)} \\
&= \frac{\sigma_r^2}{2a^2}\left(1-e^{-aT}\right)^2 - \int_0^T \theta(s)e^{-a(T-s)}ds
\end{aligned}
$$

So, we have

$$
\begin{aligned}
f(0,T) &= \frac{\partial B}{\partial T}(0,T)r_0 - \frac{\partial A}{\partial T}(0,T) \\
&= \underbrace{e^{-aT}r_0 + \int_0^T e^{-a(T-s)}\theta(s)ds}_{x(T)} - \underbrace{\frac{\sigma_r^2}{2a^2}\left(1-e^{-aT}\right)^2}_{g(T)}
\end{aligned}
$$

We are getting close! Let's establish one useful condition on $x(T)$. We have

$$\begin{aligned}
x'(T) &= \frac{\partial}{\partial T}\left(e^{-aT}r_0 + \int_0^T e^{-a(T-s)}\theta(s)ds\right) \\
&= \frac{\partial}{\partial T}\left(e^{-aT}r_0 + e^{-aT}\int_0^T e^{as}\theta(s)ds\right) \\
&= -ae^{-aT}r_0 + e^{-aT}e^{aT}\theta(T) + -ae^{-aT}\int_0^T e^{as}\theta(s)ds \\
&= -ae^{-aT}r_0 - ae^{-aT}\int_0^T e^{as}\theta(s)ds + \theta(T) \\
&= -ax(T) + \theta(T)
\end{aligned}$$

We are finally in position to understand $\theta(T)$. We see that

$$\begin{aligned}
\theta(T) = x'(T) + ax(T) &= \frac{\partial}{\partial T}\left(f(0,T) + g(T)\right) + a(f(0,T) + g(T)) \\
&= \frac{\partial}{\partial T}f(0,T) + g'(T) + a(f(0,T) + g(T))
\end{aligned}$$

This formula explains why we need to assume $a, \sigma_r$ are known ahead of time. Additionally, we'd like to calibrate this $\theta(T)$ to the market. In reality, $f(0,T)$ is just a mathematical abstraction. However, we can observe forward rates on the market and use that as the best fit for our $\theta(T)$ parameter. Let $f^*(0,T)$ be the observed instantaneous forward rates for maturity $T$. Then

$$\theta(T) = \frac{\partial}{\partial T}f^*(0,T) + g'(T) + a(f^*(0,T) + g(T))$$

In order to take a derivative of the observed curve, one will need to use some sort of smooth interpolation between various data points. In this project, we choose to use cubic splines with the `scipy.interpolate` library. There are other reasonable choices.

## 3.3 Speed of Mean Reversion: Estimating $a$

This will be arguably the simplest parameter to estimate. It also must be the first one we estimate, as it is used in estimating $\sigma_r$. We will estimate $a$ by building a linear regression model. In particular, over a period of $k$ time units, we can set up a model like

$$r_t = ar_{t-k} + \mathcal{E}$$

where $a$ is the computed best fit slope and $\mathcal{E}$ is random error. To justify this model faithfully, one should examine the assumptions of linear regression and confirm they are met or it is reasonable to ignore them. However, we will take it as given industry practice that this is reasonable.

If we'd like to use market data to set up this linear regression, we are forced to use historical time series data for the overnight rate. Technically, whenever we use historical data, we are at risk that historical estimates of the future value of the parameter are not accurate. A good modeler must be always vigilant if there are signs of fundamental changes in the rates market. However, in practice, historical data is commonly used to estimate $a$.

As the modeler, we are left to make two decisions when setting up this regression.

- What should the size of our period $k$ be? For which period do we see the largest correlation of overnight rates to itself?

- How far back should we collect time series data for to run this regression? It is possible that the historical estimate using data from last year is different from the estimate using data from the past five years.

Both of these decisions are left up to the modeler. We do not provide a systematic way of making decisions w.r.t this parameter. The modeler must exercise sound judgment for the best results.

*Remark:* In our Hull-White calculations, we rely on the assumption that $a > 0$. Without it, the assumption that rates are mean reverting need not hold. However, it is not clear a priori that for all design choices of the regression and all historical data, that the computed coefficient will be positive. Once more, the modeler must use their judgment on how to correct for this problem. Obviously this can lead to problems if the modeler is simply choosing model hyperparameters to ensure the outcome they expected to see ahead of time. Best practice would be to only use a Hull-White model in situations where the model hyperparameters can be justified with fidelity.

### 3.4 Implying Volatility and Rate Caps: Estimating $\sigma_r$

We'd like to calibrate this parameter to observed market data. To do so, we will need to recall what market caps (and floors are).

**Definition**: A **caplet** with strike $K$, maturity $T > 0$ and settlement date $T' > T$ is a European call on the forward rate settled in arrears. It has payoff

$$\pi = [L(T, T, T') - K]^+ \cdot p(T, T')$$

Similarly, a **floorlet** with strike $K$, maturity $T > 0$, and settlement date $T' > T$ is a European put on the forward rate settled in arrears. It has payoff

$$\pi = [K - L(T, T, T')]^+ \cdot p(T, T')$$

Caps and floors are weighted sums of caplets and floorlets which are used to reduce interest rate risk for investors over a series of periods. Typically $T' := T + \Delta$ and a cap would be a sequence of caplets with the increasing maturities and settlements by this period $\Delta$ (say three months). Caplets and floorlets are useful because we can re-interpret them as European options on zero-coupon-bonds. For example, we can

$$\begin{aligned}
\text{Capl}_{T,T',k} &= [L(T, T, T') - K]^+ \cdot p(T, T') \\
&= \left[ \frac{1 - p(T, T')}{(T' - T)p(T, T')} - K \right]^+ \cdot p(T, T') \\
&= \left[ \frac{1 - p(T, T')}{(T' - T)p(T, T')} \cdot p(T, T') - Kp(T, T') \right]^+ \\
&= \left[ \frac{1}{T' - T} - \frac{p(T, T')}{T' - T} - Kp(T, T') \right]^+ \\
&= \left[ \frac{1}{T' - T} - p(T, T') \underbrace{\left( \frac{1}{T' - T} - K \right)}_{N} \right]^+ \\
&= N \cdot \left[ \frac{1}{N(T' - T)} - p(T, T') \right]^+
\end{aligned}$$

which is a scaled put on a zero coupon bond from $T \to T'$. Why is this helpful? Well from Bjork 4e Proposition 21.10, we have

**Proposition 3:** In the Hull-White short rate model, the price at $t = 0$ of a European call with strike price $K$, and time of maturity $T_1$, on a bond maturity at $T_2$ is given by the formula

$$c_0 = p(0, T_2)N[d_1] - Kp(0, T_1)N[d_2]$$

where

$$\begin{aligned} d_2 &= \frac{\ln\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}} \\ d_1 &= d_2 + \sqrt{\Sigma^2} \\ \Sigma^2 &= \frac{\sigma^2}{2a^3}\left\{1 - e^{-2aT_1}\right\}\left\{1 - e^{-a(T_2 - T_1)}\right\}^2 \end{aligned}$$

**Corollary:** Since caps and floors are simply claims, put-call parity will hold. Therefore, working with caplet data is equivalent to working with floorlet data in an arbitrage free market. In theory, both caplet and floorlet data should imply the same volatility. In practice, the caplet and floorlet volatilities can diverge which may suggest it is better to work with floorlet data directly than imply $\sigma_r$ from caplet data.

# 4 Foreign Equities

We also need to build a corresponding model to predict the distribution of $S(T)$. We note that $S(T)$ is a European index quantoed to USD. So, it's value will be dependent on the European short rate and the currency exchange rate model (not necessarily the value of the exchange rate itself). Additionally, one is faced with all of the usual choices to make when modeling equities. Because there is no optionality on the foreign interest rates, nor optionality on the volatility, we will model $S(T)$ lognormally. More details to come.

## 4.1 Assets on the Market

Under objective measure $\mathbb{P}$, we assume the following assets are naturally available in the market and follow dynamics

$$\begin{aligned} dX_t &= X_t \alpha_X dt + X_t \sigma_X dV \\ dS_t^f &= S_t^f \alpha_f dt + S_t^f \sigma_f dZ \\ dB_t^d &= r^d B_t^d dt \\ dB_t^f &= r^f B_t^f dt \end{aligned}$$

where $\mathrm{Corr}(dV, dZ) = \rho_{sx}$. Some remarks before we continue.

- $X_t$ is the spot exchange rate, given in dollars : euros.
- We assume that the exchange rate and the equity follow a one factor model. It is not particularly difficult to expand to multi-factor models - it simply requires more parameters to estimate.
- We assume $r^f$, $\sigma_X$, $\sigma_f$ are all constants. If we were concerned about these parameters having large impact on our pricing model, it may be better to try and model their dynamics using other rate and volatility models.
- Although we write $r^d$ as if it is a constant, in reality, it is our short rate modeled using Hull-White. The fact $r^d$ is a stochastic process does not affect the computations which follow.

Our next step is to try and determine the dynamics of these assets under martingale measure $Q^d$. First, a useful computational tool.

**Lemma:** Given two adapted processes $X_t, Y_t$,

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t + dX_t dY_t$$

*pf:* Ito's Lemma with $f(t, x, y) = xy$. $\square$

We will use this lemma to describe the dynamics of two new assets on the market. Namely, set

$$\begin{aligned} \tilde{B}_t^f &= X_t B_t^f \\ \tilde{S}_t^f &= X_t S_t^f \end{aligned}$$

We view these assets as the USD value of the European bank account and European stock. We can compute their dynamics directly by applying the lemma. Indeed

$$\begin{aligned} d\tilde{B}_t^f &= dX_t B_t^f + dB_t^f X_t + dB_t^f dX_t \\ &= (X_t \alpha_X dt + X_t \sigma_X dV) B_t^f + \left( r^f B_t^f dt \right) X_t + \left( r^f B_t^f dt \right) (X_t \alpha_X dt + X_t \sigma_X dV) \\ &= X_t B_t^f \left( (\alpha_X + r^f) dt + \sigma_X dV \right) + (\cdots)(dt)^2 + (\cdots)(dt)(dV) \\ &= \tilde{B}_t^f \left( (\alpha_X + r^f) dt + \sigma_X dV \right) + 0 + 0 \\ &= \tilde{B}_t^f (\alpha_X + r^f) dt + \tilde{B}_t^f \sigma_X dV \end{aligned}$$

On the other hand, $\tilde{B}_t^f$ is a traded domestic asset. So, under $Q^d$, it must grow at exactly $r^d$. So, by comparing drifts, we have

$$\alpha_X + r^f = r^d \implies \alpha_X = r^d - r^f$$

So, our dynamics of $X_t$ under $Q^d$ can be described as

$$dX_t = X_t(r^d - r^f)dt + X_t\sigma_X dW^d$$

Now, we can use this information and a similar computation to examine the foreign equity converted to USD by the spot.

$$\begin{aligned}
d\tilde{S}_t^f &= dX_t S_t^f + dS_t^f X_t + dS_t^f dX_t \\
&= \left(X_t(r^d - r^f)dt + X_t\sigma_X dV\right) S_t^f + \left(S_t^f \alpha_f dt + S_t^f \sigma_f dZ\right) X_t + \left(S_t^f \alpha_f dt + S_t^f \sigma_f dZ\right)\left(X_t(r^d - r^f)dt + X_t\sigma_X dV\right) \\
&= X_t S_t^f \left((r^d - r^f + \alpha_f)dt + \sigma_X dV + \sigma_f dZ\right) + (\cdots)(dt)^2 + (\cdots)(dt)(dV) + (\cdots)(dt)(dZ) + ()(dV)(dZ) \\
&= X_t S_t^f \left((r^d - r^f + \alpha_f)dt + \sigma_X dV + \sigma_f dZ\right) + 0 + 0 + 0 + X_t S_t^f(\sigma_x\sigma_f)(dV)(dZ) \\
&= X_t S_t^f \left((r^d - r^f + \alpha_f)dt + \sigma_X dV + \sigma_f dZ\right) + X_t S_t^f \sigma_x \sigma_f \rho_{sx} dt \\
&= \tilde{S}_t^f(r^d - r^f + \alpha_f + \rho_{sx}\sigma_x\sigma_f)dt + \tilde{S}_t^f \sigma_X dV + \tilde{S}_t^f \sigma_f dZ
\end{aligned}$$

Once more, $\tilde{S}_t^f$ is a traded domestic asset in USD, so it must grow at $r^d$. So, by comparing drifts we have

$$r^d - r^f + \alpha_f + \rho_{sx}\sigma_x\sigma_f = r^d \implies \alpha_f = r^f - \rho_{sx}\sigma_x\sigma_f$$

So, our foreign equity (and therefore our quantoed asset) must follow

$$dS_t^f = (r^f - \rho_{sx}\sigma_x\sigma_f)dt + \sigma_f dW^d$$

Lastly, above we assumed that $S_t^f$ does not have a dividend yield. This is not true in actuality. However, we know that all of the computation above will still hold true for the gain process $G_t^f$ instead of $S_t^f$. Therefore, we can simply adjust the drift in this final stage. Let $q$ be the annual continuous dividend yield for the equity index. Then

$$dS_t^f = (r^f - q - \rho_{sx}\sigma_x\sigma_f)dt + \sigma_f dW^d$$

*Remark:* The domestically discounted foreign equity is not a martingale under $Q^d$. However, this is OK as this asset is not traded, it is merely a useful construction for building other contracts.

## 4.2 Estimating parameters:

We need to estimate 4 parameters to properly estimate $S(T)$. In particular, we need to determine $r^f, \rho_{sx}, \sigma_x, \sigma_f$.

- We will imply $\sigma_f$ by calibrating the Black-Scholes model with the price of European options. Under the BS framework, this $\sigma_f$ should be constant for all strikes and all maturities. However, in practice, this will not be the case. One could directly imply $\sigma_f$ from a specific option or one can pick a selection of market options and try to imply a $\sigma_f$ which minimizes model error out of this basket. The modeler still must justify and choose which option/basket will imply an appropriate volatility.
- Very similarly, one can use Black-Scholes style formulas and FX calls and puts to imply $\sigma_X$.
- $r^f$ is assumed to be constant over the term 0 to $T + \Delta$. We will collect historical time series data for $r^f$ and take the average value as the best estimate for $r^f$ over the period
- We will also compute $\rho_{sx}$ using historical data. $\rho_{sx}$ represents the correlation between the daily spot exchange "returns" and the daily equity returns. These returns (and the correlation) can be computed using time series data, but the modeler must decide how far back data is needed to make the $\rho_{sx}$ estimate robust.

# 5 A Pricing Routine

This section will walk through some of the data used and the detail the functionality of the code.

## 5.1 Inputs

We use the following data to estimate various observable parameters.

- $q$ = annualized (continuous) dividend yield. This data can be found at `https://www.nasdaq.com/market-activity/etf/fez/dividend-history`. Using their data, a given estimate for the dividend yield is $q = 0.68\%$
- $\{X_t\}_{t\leq 0}$ = historical time series spot exchange data was available at `https://www.investing.com/currencies/eur-usd-historical-data`. Data was collected from 01/01/2014-12/14/2024. Some subset of this data (perhaps all) is used to compute $\rho_{sx}$
- $\{S_t^f\}_{t\leq 0}$ = historical time series spot price data is pulled using the Python library `yfinance` using ticker `^STOXX50E`. Data was collected from 01/01/2014-12/14/2024. Some subset of this data (perhaps all) is used to compute $\rho_{sx}$
- $\{r^f\}_{t\leq 0}$ = historical time series data of the European short rate was available at `https://data.ecb.europa.eu/data/datasets/EST?` by looking at the *Euro short-term rate - Volume-weighted trimmed mean rate, Daily - businessweek*. Some subset of the data is used to establish a point estimate of $r^f$.

  *Remark:* Real discretion must be used here. A cursory look at the time series data suggests there was a rates regime change in 2022. This matches up with anecdotal experience that many central banks raised rates dramatically to combat inflation. However, in Europe, this jump was particular drastic as they were offering negative interest rates previously. Therefore $\hat{r}^f$ will be very dependent on how one chooses a window of the time series to estimate $\hat{r}^f$ from. Best practice would be to stress test this parameter under different choices. In this paper, we will assume that the rate regime in Europe for the foreseeable future will not involve negative rates. Therefore, we will estimate $\hat{r}^f$ using data predominantly from the past 2-3 years.
- $\sigma_f$ can be estimated with a large collection of European options data on SX5E was scraped out of the Columbia Bloomberg terminal manually and exported to Excel. The implied volatility calculations are provided by Bloomberg. Nevertheless, as long-dated and deep in or out of the money options are less liquid, sometimes the IV calculation is less reliable. In this project, we look at implied volatility for ATM strikes and maturity $\min\{T, \text{latest time with liquidly traded options}\}$. For example, if we are trying to imply volatility for a 5 year contract, but there is low trading volume for options past the 3 year mark, we will use implied volatility from 3 year maturities.
- $\sigma_x$ can be estimated with options on the EUR/USD exchange rate, found at `https://www.investing.com/currencies/forex-options`. They provide implied volatility calculations and data can be taken from ATM strikes and the largest maturity provided as they provide short-dated information.
- $a$ = is the historical rate of mean reversion. Data to run an auto-regression can be found at `https://www.newyorkfed.org/markets/reference-rates/sofr`
- $\sigma_r$ can be estimated by implying volatlity from liquid caplets and floorlets. Under the Hull-White framework, $\sigma_r$ is assumed to be constant over time. This is not true in practice. Best practice is to create successive estimates of $\sigma_r$ over different time periods by stripping cap data into component caplets. Each caplet can then be used to imply a volatility. Due to time restrictions of the project, I was not able to implement cap stripping. Instead I used the implied vol for the appropriately dated cap market price (pulled from Bloomberg' `SWPM -CAP` under the `Cashflow` tab).
- $\theta(t)$ composed of the empirical forward term structure. In the text, this is derived from the $p^*(0,t)$. However, we pull data from the empirical zero coupon yield curve $y^*(0,t)$ where

$$y(t,T) := -\frac{\ln(p(t,T))}{T-t}$$

Therefore, we can compute

$$f^*(0,T) = -\frac{\partial \ln(p^*(0,t))}{\partial T} = \frac{\partial}{\partial T} T y^*(0,T)$$

where $y^*(0,t)$ was taken from `https://fred.stlouisfed.org/release/tables?eid=212994&rid=354`.

## 5.2 Monte Carlo Methods

Recall, that under $Q^d$, the arbitrage free price of any simple contingent claim $\Phi$ can be described as

$$\Pi_0 = B_0 \mathbb{E}^{Q^d}\left[\frac{\Pi_T}{B_T}\middle| F_0\right] = \mathbb{E}_0^{Q^d}\left[e^{-\int_0^T r_s ds}\Phi(S_T, L(T,T,T+\Delta))\right]$$

We will estimate this estimate this expectation numerically following these steps

(1) Choose a step size $\delta$ and number of steps $N = (T+\Delta)/\delta$.

(2) Generate two $N$-vectors of numbers pulled from the standard normal distribution such that the two vectors have correlation $\rho_{sx}$. One can achieve this using Cholesky methods. In particular, if $\vec{x}^{(j)}$ and $\vec{y}^{(j)}$ contain uncorrelated standard normal data, then

$$\begin{bmatrix} \tilde{x}^{(j)\intercal} \\ \tilde{y}^{(j)\intercal} \end{bmatrix} := \begin{bmatrix} 1 & 0 \\ \rho_{sx} & \sqrt{1-\rho_{sx}^2} \end{bmatrix} \begin{bmatrix} \vec{x}^{(j)\intercal} \\ \vec{y}^{(j)\intercal} \end{bmatrix}$$

$\tilde{x}^{(j)}$ and $\tilde{y}^{(j)}$ should represent normal data with correlation $\rho_{sx}$ for large enough $N$.

(3) Generate $\{\hat{r}_t\}_{0\le t\le T+\Delta}^{(j)}$ and $\{\hat{S}_t\}_{0\le t\le T}^{(j)}$ following the dynamics described above, using $\tilde{x}^{(j)}$ and $\tilde{y}^{(j)}$

(4) Let $M = T/\delta$ and compute

$$\begin{aligned}
\hat{p}^{(j)}(T,T+\Delta) &= e^{\delta \sum_{M+1}^{N} \hat{r}_t^{(j)}} \\
\hat{L}^{(j)}(T,T,T+\Delta) &= \frac{1-\hat{p}^{(j)}(T,T+\Delta)}{\Delta \cdot \hat{p}^{(j)}(T,T+\Delta)} \\
\hat{S}^{(j)}(T) &= M+1\text{st entry in simulated stock data} \\
\hat{\phi}^{(j)}(U_T) &= \max\left\{0, \left(\frac{\hat{S}^{(j)}(T)}{S(0)} - k\right)\cdot\left(k' - \frac{\hat{L}^{(j)}(T,T,T+\Delta)}{L(0,T,T+\Delta)}\right)\right\} \\
\hat{d}^{(j)}(0,T) &= e^{-\delta \sum_{t=0}^{M} \hat{r}_t^{(j)}} \\
\hat{\pi}_0^{(j)} &= \hat{d}^{(j)}(0,T)\cdot \hat{\Phi}^{(j)}(U_T)
\end{aligned}$$

(5) Repeat steps (2)-(4) $N_{sim}$ times. We estimate

$$\hat{\Pi}_0 = N_{notional}\cdot \frac{1}{N_{sim}} \sum_{j=1}^{N_{sim}} \hat{\pi}_0^{(j)}$$

# 6 Conclusion

## 6.1 Main Result

After running the pricing function `price_hybrid(param_summary)` (in the Jupyter notebook), we typically get prices between $490,000 - $510,000. Please see code for specific results.

## 6.2 Some Limitations and Next Steps

Currently, the pricing script still has a number of short comings which should be addressed if you wanted to bring a script like this to production.

(1) The data collection in place for this script is very non-automated. It pulls from a number of sources and requires human interpretation, intervention, and cleaning. For example, if the customer decided they needed to wait a quarter to purchase, but was still interested in the product, most of the data would have to be recollected and processed manually. Part of this problem would be solved by simply working at a bank where most of these tools might be pre-made and available. Nevertheless, it represents a serious impediment to using the script more seriously.

(2) The Bloomberg command `Help VCUB` has documentation on how to properly strip caplets from cap data. Once that is implemented, it would be better to manually compute the Hull-White implied volatility from $\sigma_r$. In a similar vein, it would be good to find floorlet data to confirm the market is liquid enough to imply volatility from caplets instead of floorlets.

(3) Currently, the estimation for the $a$ parameter is INCREDIBLY sensitive to choice of $k$ days of lag when computing the autocorrelation. I do not know any ways to feasibly justify one $k$ value over the other. Professor Javaheri suggested $a$ should be between 0.03 and 0.06. Choosing $k = 390$ does achieve something in this range, but this is incredibly unmotivated. More statistical methods would need to be employed to strengthen the prediction.

(4) In the script, there are some parameters such as $\sigma_f$ and $\sigma_r$ which are chosen by hand after a human looks at the data. While there is a systematic way to choose the appropriate $\sigma_f$ and $\sigma_r$, it would be a clearer process if these ways were written into the code.

(5) The model should be rigorously stress tested. I am not sure which parameters have the largest impact on final prices. I would be interested in carrying out this stress testing in the future.

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