Solution
Hidden Markov Models

Task 1.

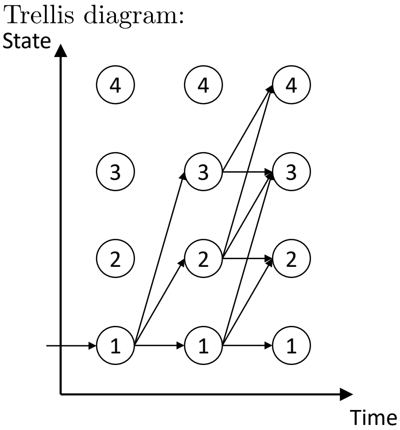
a) Model parameters:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.5 & \underline{0.2} & 0 \\ 0 & \underline{0.4} & 0.2 & 0.4 \\ 0 & 0 & 0.3 & \underline{0.7} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } \sum_{k=1}^{N} a_{ik} = 1 \text{ for } \forall_i$$

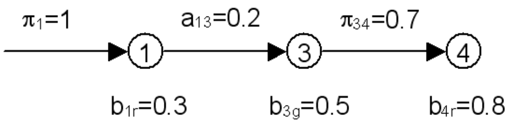
$$\Pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ with } \sum_{k=1}^{N} \pi_k = 1$$

N = Number of states

b) Trellis diagram:



c) Path:



$\Rightarrow$ $P(Q|\lambda) = 1 \cdot 0.2 \cdot 0.7 = 0.14$

$\Rightarrow$ $P(O|Q,\lambda) = 0.3 \cdot 0.5 \cdot 0.8 = 0.12$

$$\Rightarrow P(O,Q|\lambda) = P(Q|\lambda) \cdot P(O|Q,\lambda) = 0.14 \cdot 0.12 = 0.0168$$

Note!
An HMM's total production probability is the sum of all contributions from **all** paths.
$P(O|\lambda) = \sum_{\forall Q} P(O, Q_i|\lambda)$

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Task 2.

a) Class A:

$$A = \begin{pmatrix} 0.1 & 0.4 & 0.5 & 0 & 0 \\ 0 & 0.1 & 0.5 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \Pi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Class B:

$$A = \begin{pmatrix} 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0.2 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Pi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Class C:

$$A = \begin{pmatrix} 0.2 & 0.5 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}, \Pi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b) Class A:

$$Q_1 = \{q_1, q_2, q_3\} = \{1, 3, 5\}$$

Class B:

$$Q_1 = \{q_1, q_2, q_3\} = \{1, 3, 4\}$$
$$Q_2 = \{q_1, q_2, q_3\} = \{1, 2, 4\}$$

Class C:

$$Q_1 = \{q_1, q_2, q_3\} = \{1, 2, 3\}$$
$$Q_2 = \{q_1, q_2, q_3\} = \{1, 1, 3\}$$
$$Q_3 = \{q_1, q_2, q_3\} = \{1, 3, 3\}$$

c) For a single path it holds:

production probability	=	path probability	·	emission probability
$P(O, Q_i \vert \lambda)$	=	$P(Q_i \vert \lambda)$	·	$P(O \vert Q_i, \lambda)$

Class A:

$$P(O, Q_1|\lambda) = \pi_1 \cdot a_{13} \cdot a_{35} \cdot b_1(o_1) \cdot b_3(o_2) \cdot b_5(o_3)$$
$$\mu = 2, \sigma = 2 \Rightarrow b_1(o_1) = \frac{1}{\sqrt{2\pi} \cdot 2} e^{-\frac{1}{2}\left(\frac{1.1-2}{2}\right)^2}$$
$$\mu = 0, \sigma = 1 \Rightarrow b_3(o_2) = \frac{1}{\sqrt{2\pi} \cdot 1} e^{-\frac{1}{2}\left(\frac{-0.9-0}{1}\right)^2}$$
$$\mu = 0, \sigma = 2 \Rightarrow b_5(o_3) = \frac{1}{\sqrt{2\pi} \cdot 2} e^{-\frac{1}{2}\left(\frac{0.3-0}{2}\right)^2}$$
$$P(O, Q_1|\lambda) = 1 \cdot 0.5 \cdot 0.3 \cdot 0.180 \cdot 0.266 \cdot 0.197 = 0.15 \cdot 9.4 \cdot 10^{-3} = 1.41 \cdot 10^{-3}$$

Class B:

$$P(O, Q_1|\lambda) = 1 \cdot 0.1 \cdot 0.7 \cdot 0.397 \cdot 0.133 \cdot 0.197 = 0.07 \cdot 10.4 \cdot 10^{-3} = 0.73 \cdot 10^{-3}$$
$$P(O, Q_2|\lambda) = 1 \cdot 0.6 \cdot 0.2 \cdot 0.397 \cdot 0.199 \cdot 0.197 = 0.12 \cdot 15.6 \cdot 10^{-3} = 1.87 \cdot 10^{-3}$$

Class C:

$$P(O, Q_1|\lambda) = 1 \cdot 0.5 \cdot 0.5 \cdot 0.218 \cdot 0.127 \cdot 0.187 = 0.25 \cdot 5.2 \cdot 10^{-3} = 1.3 \cdot 10^{-3}$$
$$P(O, Q_2|\lambda) = 1 \cdot 0.2 \cdot 0.3 \cdot 0.218 \cdot 0.266 \cdot 0.187 = 0.06 \cdot 10.8 \cdot 10^{-3} = 0.65 \cdot 10^{-3}$$
$$P(O, Q_3|\lambda) = 1 \cdot 0.3 \cdot 1 \cdot 0.218 \cdot 0.127 \cdot 0.187 = 0.3 \cdot 5.2 \cdot 10^{-3} = 1.56 \cdot 10^{-3}$$

d) Class B, since path Q_2 has the highest probability.

e) Class A:

$$P(O|\lambda) = \sum_{\forall Q_i} P(O, Q_i|\lambda) = 1.41 \cdot 10^{-3}$$

Class B:

$$P(O|\lambda) = 0.73 \cdot 10^{-3} + 1.87 \cdot 10^{-3} = 2.6 \cdot 10^{-3}$$

Class C:

$$P(O|\lambda) = 1.3 \cdot 10^{-3} + 0.65 \cdot 10^{-3} + 1.56 \cdot 10^{-3} = 3.51 \cdot 10^{-3}$$

The number of possible paths depends on the class. The results are not comparable without normalization.