Solution
Representations of Orientation

Task 1.

a) Orientation representations can be converted into each other via the orientation matrix. First, the given orientation of RPY angles is converted into an orientation matrix:

$$\underline{RPY} \longmapsto \underline{R} = \begin{bmatrix} \cos\alpha\cdot\cos\beta & -\sin\alpha\cdot\cos\gamma+\cos\alpha\cdot\sin\beta\cdot\sin\gamma & \cos\alpha\cdot\sin\beta\cdot\cos\gamma+\sin\alpha\cdot\sin\gamma \\ \sin\alpha\cdot\cos\beta & \cos\alpha\cdot\cos\gamma+\sin\alpha\cdot\sin\beta\cdot\sin\gamma & \sin\alpha\cdot\sin\beta\cdot\cos\gamma-\cos\alpha\cdot\sin\gamma \\ -\sin\beta & \cos\beta\cdot\sin\gamma & \cos\beta\cdot\cos\gamma \end{bmatrix}$$

Here, the result is: $\underline{R} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

Then the orientation matrix is converted into rotation vector and rotation angle:

$$\underline{R} \longmapsto \underline{RVA}, \underline{R} := \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Determination of the angle of rotation:

$$\Theta = atan2\left[\sqrt{(R_{32}-R_{23})^2+(R_{13}-R_{31})^2+(R_{21}-R_{12})^2}, (R_{11}+R_{22}+R_{33}-1)\right]$$

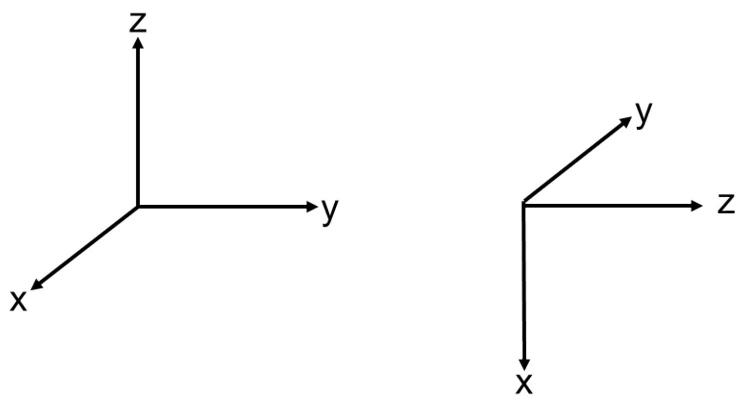
Case differentiation depending on the angle of rotation:

- $\underline{\Theta \neq n\pi, n \in N}$
  $$\begin{aligned} k_x &= \tfrac{R_{32}-R_{23}}{2\sin\Theta} \\ k_y &= \tfrac{R_{13}-R_{31}}{2\sin\Theta} \\ k_z &= \tfrac{R_{21}-R_{12}}{2\sin\Theta} \end{aligned}$$
- $\underline{\Theta = n\pi, n \in N = 0}$
  $\underline{k}$ undefined.
- $\underline{\Theta = n\pi, n \in N \neq 0}$
  $$\begin{aligned} k_x &= sign\left(R_{32}-R_{23}\right)\sqrt{\tfrac{R_{11}-\cos\Theta}{1-\cos\Theta}} \\ k_y &= sign\left(R_{13}-R_{31}\right)\sqrt{\tfrac{R_{22}-\cos\Theta}{1-\cos\Theta}} \\ k_y &= sign\left(R_{21}-R_{12}\right)\sqrt{\tfrac{R_{33}-\cos\Theta}{1-\cos\Theta}} \end{aligned}$$

Here, the result is: $\Theta = atan2\left[\sqrt{3}, -1\right] = \frac{2\pi}{3}, k_x = -\frac{1}{\sqrt{3}}, k_y = \frac{1}{\sqrt{3}}, k_z = \frac{1}{\sqrt{3}}$

Keep in mind: Vector $\underline{k}$ still has to be normalised!

Initial (left) and target (right) coordinate systems:



Task 2.

a) Orientation representations can be converted into one another via the orientation matrix.
First, the orientation given by a rotation vector and a rotation angle is converted into an orientation matrix:

$$\underline{RVA} \longmapsto \underline{R} = \begin{bmatrix} k_x k_x vers\Theta + \cos\Theta & k_x k_y vers\Theta - k_z \sin\Theta & k_x k_z vers\Theta + k_y \sin\Theta \\ k_x k_y vers\Theta + k_z \sin\Theta & k_y k_y vers\Theta + \cos\Theta & k_y k_z vers\Theta - k_x \sin\Theta \\ k_x k_z vers\Theta - k_y \sin\Theta & k_y k_z vers\Theta + k_x \sin\Theta & k_z k_z vers\Theta + \cos\Theta \end{bmatrix}, vers\Theta := (1-\cos\Theta)$$

Here, the result is: $\underline{R} = \begin{bmatrix} 0 & 0 & +1 \\ 0 & +1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Then the orientation matrix is converted into RPY angles:

$$\underline{R} \longmapsto \underline{RPY}, \underline{R} := \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Case differentiation according to the cosine of the pitch angle:

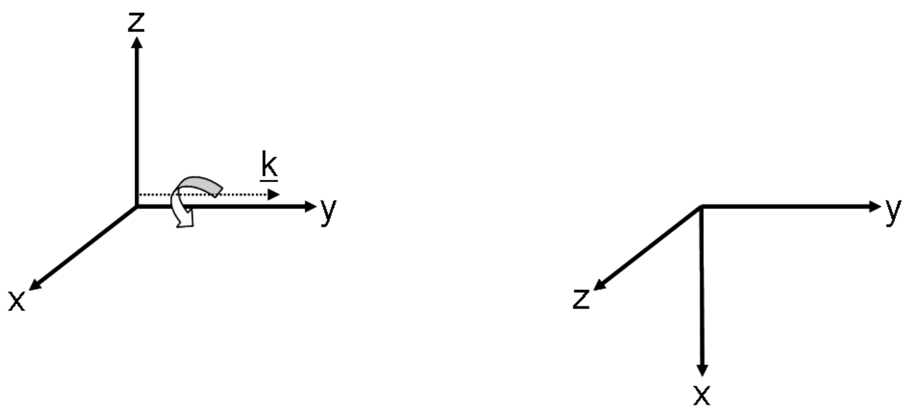
- $\underline{\cos\beta = \sqrt{1 - {R_{31}}^2} \neq 0}$

$$\begin{aligned} \alpha &= atan2\left(R_{21}, R_{11}\right) &&= atan2\left[\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \equiv T_\alpha\right] \\ \beta &= atan2\left(-R_{31}, \cos\beta\right) &&= atan2\left[\frac{\sin\beta}{\cos\beta} \equiv T_\beta\right] \\ \gamma &= atan2\left(R_{32}, R_{33}\right) &&= atan2\left[\frac{\cos\beta\sin\gamma}{\cos\beta\cos\gamma} \equiv T_\gamma\right] \end{aligned}$$

- $\underline{\cos\beta = \sqrt{1 - {R_{31}}^2} = 0}$
  $\beta = \pm n\frac{\pi}{2}, n \in N$ odd
  $\alpha - \gamma = atan2\left(-R_{12}, R_{22}\right) = atan2\left[\frac{T_\alpha - T_\gamma}{1 + T_\alpha T_\gamma}\right]$

Here, the result is: $\cos\beta = 0, \beta$ chosen as $\frac{\pi}{2} = 90°, \alpha - \gamma = atan2\left(-R_{12}, R_{22}\right) = 0$
Solution is ambiguous, since the coupling of α and γ cannot be resolved!

Initial (left) and target (right) coordinate systems:



Task 3.

a) For a rotation from a starting position $\underline{RPY}_0$ to a target position $\underline{RPY}_n$ within the mentioned n partial rotations, for the i-th partial rotation ($i \in [0 \ldots n]$) the following holds:

$$\underline{RPY}_i = \underline{RPY}\left(\alpha_0 + i\Delta\alpha, \beta_0 + i\Delta\beta, \gamma_0 + i\Delta\gamma\right)$$

If the incremental change of orientation for many partial rotations n is supposed to lead to a continuous interpolation, a "distributivity" $\left(f\left(a+b\right) \equiv f\left(a\right) + f\left(b\right)\right)$ would have to be valid for each step of the interpolation:

$\underline{RPY}\,(\alpha_i + \Delta\alpha, \beta_i + \Delta\beta, \gamma_i + \Delta\gamma) \equiv \underline{RPY}\,(\alpha_i, \beta_i, \gamma_i) \cdot \underline{RPY}\,(\Delta\alpha, \Delta\beta, \Delta\gamma)$

The element (3,1) of the two sides is compared as an example. For the left side this results:

$$[\underline{RPY}\,(\alpha_i + \Delta\alpha, \beta_i + \Delta\beta, \gamma_i + \Delta\gamma)]_{3,1} = -\sin(\beta_i + \Delta\beta) = -\sin\beta_i \cos\Delta\beta - \cos\beta_i \sin\Delta\beta$$

Considering the relevant rows and columns, the result for the right side is:

$$[\underline{RPY}\,(\alpha_i, \beta_i, \gamma_i) \cdot \underline{RPY}\,(\Delta\alpha, \Delta\beta, \Delta\gamma)]_{3,1} = \begin{bmatrix} . & . & . \\ . & . & . \\ -\sin\beta_i & \cos\beta_i \sin\gamma_i & \cos\beta_i \cos\gamma_i \end{bmatrix} \cdot \begin{bmatrix} \cos\Delta\alpha\cos\Delta\beta & . & . \\ \sin\Delta\alpha\cos\Delta\beta & . & . \\ -\sin\Delta\beta & . & . \end{bmatrix}$$

$$= -\sin\beta_i \cos\Delta\alpha\cos\Delta\beta + \cos\beta_i \sin\gamma_i \sin\Delta\alpha\cos\Delta\beta - \cos\beta_i\cos\gamma_i\sin\Delta\beta$$
$$\neq -\sin\beta_i\cos\Delta\beta - \cos\beta_i\sin\Delta\beta!$$

RPY angles hence cannot be used for interpolation, since no "distributivity" is guaranteed where an incremental change of the orientation in n partial rotations would lead to the same result as the transformation of a start orientation into a final orientation in one step.

b) With the given values for the partial rotations the following applies:

$$\Delta\alpha = \tfrac{1}{2} \cdot (0° + 90°) = +45°,\ \Delta\beta = \tfrac{1}{2} \cdot (0° - 90°) = -45°,\ \Delta\gamma = \tfrac{1}{2} \cdot (0° - 0°) = 0°$$

As shown in a), a "distributivity" must be given for an interpolatable orientation representation:
$\underline{RPY}\,(\alpha_i + \Delta\alpha, \beta_i + \Delta\beta, \gamma_i + \Delta\gamma) \equiv \underline{RPY}\,(\alpha_i, \beta_i, \gamma_i) \cdot \underline{RPY}\,(\Delta\alpha, \Delta\beta, \Delta\gamma)$

The element (3,1) of the two sides is compared as an example. For the left side the result is:

$$[\underline{RPY}\,(-90° + 45°, +90° - 45°, 0° + 0°)]_{3,1} = -\sin(+90° - 45°) = -\sin(+45°) = -\tfrac{\sqrt{2}}{2}$$

Considering the relevant rows and columns, the result for the right side is:

$$[\underline{RPY}\,(-90°, +90°, 0°) \cdot \underline{RPY}\,(+45°, -45°, 0°)]_{3,1} = \begin{bmatrix} . & . & . \\ . & . & . \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} +\frac{1}{2} & . & . \\ +\frac{1}{2} & . & . \\ +\frac{\sqrt{2}}{2} & . & . \end{bmatrix}$$

$$= -\tfrac{1}{2}$$
$$\neq -\tfrac{\sqrt{2}}{2}$$

Task 4.

a) Rotation axis: $\underline{n} := \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, angle: φ

Lecture notes p. 48 $\rightarrow Q = w + ix + jy + kz = cos\left(\frac{\varphi}{2}\right) + sin\left(\frac{\varphi}{2}\right) \cdot \underline{n} = cos\left(\frac{\varphi}{2}\right) + sin\left(\frac{\varphi}{2}\right) \cdot (in_x + jn_y + kn_z)$

Matrix representation $\rightarrow$ lecture notes p. 49 with equivalences for w, x, y, z in accordance with the equation above.

b) Rotation axis: $\underline{n} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, angle: φ

Rotation axis has to be normalised!

$\rightarrow \underline{n}_{norm} = \underline{n} \cdot \frac{1}{\|\underline{n}\|}$

Result: $Q_{rot} = cos\left(\frac{\varphi}{2}\right) + sin\left(\frac{\varphi}{2}\right) \cdot \frac{1}{\sqrt{14}}\,(i1 + j2 + k3)$

c) With $\varphi = 45° = 0.7854\ rad$

$Q_{rot} = cos\left(\frac{0.7854}{2}\right) + sin\left(\frac{0.7854}{2}\right) \cdot \frac{1}{\sqrt{14}}\left(i1 + j2 + k3\right) =: \mathbf{q}$

Given is the position vector $\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Result (cf. lecture notes p. 48, eq. (4.40) for the definition of the inverse):

$\underline{v}' = \mathbf{q}\underline{v}\mathbf{q}^{-1} = \mathbf{q}\underline{v}\frac{\mathbf{q}^*}{\mathbf{q}\mathbf{q}^*} = \begin{pmatrix} 0.6437 \\ 1.3361 \\ 0.8947 \end{pmatrix}$