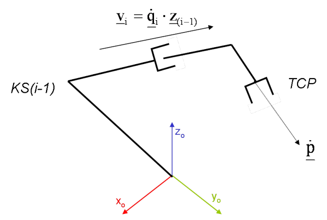
Solution
Universal transformation

The Jacobian matrix of a general robot with N axis is defined as follows:

$$\begin{aligned}\dot{\underline{l}} &= \underline{J}\dot{\underline{q}} \\ &= [\underline{J}_1 \cdots \underline{J}_N] \cdot \dot{\underline{q}}\end{aligned}$$

The elements $\underline{J}_i$ contain the influence of time changes in axis positions on time changes in Cartesian coordinates of the TCP. The elements are set up depending on the axis type.



In a **translational axis**, the TCP moves with $\dot{\underline{q}_i}$ in direction $\underline{z}_{(i-1)}$ (unit vector) in axis coordinates.

$\underline{v}_i$ is the Cartesian velocity of the TCP caused by axis i.

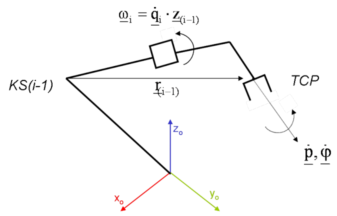
$$\dot{\underline{p}}_i = \underline{v}_i = \underline{J}_{i\underline{p}} \cdot \dot{\underline{q}}_i = \dot{\underline{q}}_i \cdot \underline{z}_{(i-1)}$$

$\underline{\omega}_i$ is the Cartesian angular velocity of the TCP caused by axis i.

$$\dot{\underline{\phi}}_i = \underline{\omega}_i = \underline{J}_{i\underline{\phi}} \cdot \dot{\underline{q}}_i = \underline{0}$$

The cloumn $\underline{J}_i$ thus results in:

$$\underline{J}_i = \begin{bmatrix} \underline{z}_{(i-1)} \\ \underline{0} \end{bmatrix}$$



In a **rotational axis**, the TCP rotates with $\dot{\underline{q}_i}$ about the direction $\underline{z}_{(i-1)}$ (unit vector) in axis coordinates.

$\underline{v}_i$ is the Cartesian velocity of the TCP caused by axis i.

$$\dot{\underline{p}}_i = \underline{v}_i = \underline{J}_{i\underline{p}} \cdot \dot{\underline{q}}_i = \underline{\omega}_i \times \underline{r}_{(i-1)}$$

$\underline{\omega}_i$ is the Cartesian angular velocity of the TCP caused by axis i.

$$\dot{\underline{\phi}}_i = \underline{\omega}_i = \underline{J}_{i\underline{\phi}} \cdot \dot{\underline{q}}_i = \dot{\underline{q}}_i \cdot \underline{z}_{(i-1)}$$

The cloumn $\underline{J}_i$ thus results in:

$$\underline{J}_i = \begin{bmatrix} \underline{z}_{(i-1)} \times \underline{r}_{(i-1)} \\ \underline{z}_{(i-1)} \end{bmatrix}$$

Task 1.

a) **Axis 1 is a rotational axis**
To establish column $\underline{J}_1$ of the Jacobian matrix, $\underline{r}_0$ and $\underline{z}_0$ are determined in world coordinates.

Let ${}^0\underline{d}_3$ be the position part of the forward transformation ${}^0\underline{T}_3$ in world coordinates:

$$\underline{r}_0 = {}^0\underline{d}_3 = \begin{bmatrix} l_1 C_{q1} + (h_2 + h_3 + q_3)\, S_{q1} S_{q2} \\ l_1 S_{q1} - (h_2 + h_3 + q_3)\, C_{q1} S_{q2} \\ h_1 + (h_2 + h_3 + q_3)\, C_{q2} \end{bmatrix}$$

Unit vector $\underline{z}_0$ coincides with the Z axis of the world coordinate system:

$$\underline{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the first column of the Jacobian matrix results in:

$$\underline{J}_1 = \begin{bmatrix} \underline{z}_0 \times \underline{r}_0 \\ \underline{z}_0 \end{bmatrix} = \begin{bmatrix} -l_1 S_{q1} + (h_2 + h_3 + q_3)\, C_{q1} S_{q2} \\ l_1 C_{q1} + (h_2 + h_3 + q_3)\, S_{q1} S_{q2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Axis 2 is a rotational axis
To establish column $\underline{J}_2$ of the Jacobian matrix, $\underline{r}_1$ and $\underline{z}_1$ are determined in world coordinates.

Let ${}^1\underline{d}_3$ be the position part of the forward transformation ${}^1\underline{T}_3$ in world coordinates:

$$\underline{r}_1 = {}^1\underline{d}_3 = {}^0\underline{d}_3 - {}^0\underline{d}_1 = {}^0\underline{d}_3 - \begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix} = \begin{bmatrix} l_1 C_{q1} + (h_2 + h_3 + q_3)\, S_{q1} S_{q2} \\ l_1 S_{q1} - (h_2 + h_3 + q_3)\, C_{q1} S_{q2} \\ (h_2 + h_3 + q_3)\, C_{q2} \end{bmatrix}$$

Unit vector $\underline{z}_1$ in world coordinates is described in the orientation part of the forward transformation ${}^0\underline{T}_1$:

$$\underline{z}_1 = {}^0\underline{R}_1^z = \begin{bmatrix} C_{q1} \\ S_{q1} \\ 0 \end{bmatrix}$$

Thus, the second column of the Jacobian matrix results in:

$$\underline{J}_2 = \begin{bmatrix} \underline{z}_1 \times \underline{r}_1 \\ \underline{z}_1 \end{bmatrix} = \begin{bmatrix} (h_2 + h_3 + q_3)\, S_{q1} C_{q2} \\ -(h_2 + h_3 + q_3)\, C_{q1} C_{q2} \\ -(h_2 + h_3 + q_3)\, S_{q2} \\ C_{q1} \\ S_{q1} \\ 0 \end{bmatrix}$$

Axis 3 is a translational axis
To establish column $\underline{J}_3$ of the Jacobian matrix, $\underline{z}_2$ is determined in world coordinates.

Unit vector $\underline{z}_2$ in world coordinates is described in the orientation part of the forward transformation ${}^0\underline{T}_2 = {}^0\underline{T}_1 \cdot {}^1\underline{T}_2$:

$$\underline{z}_2 = {}^0\underline{R}_2^z = \left[{}^0\underline{R}_1 \cdot {}^1\underline{R}_2\right]^z = \begin{bmatrix} S_{q1}S_{q2} \\ -C_{q1}S_{q2} \\ C_{q2} \end{bmatrix}$$

Thus, the third column of the Jacobian matrix results in:

$$\underline{J}_2 = \begin{bmatrix} \underline{z}_2 \\ \underline{0} \end{bmatrix} = \begin{bmatrix} S_{q1}S_{q2} \\ -C_{q1}S_{q2} \\ C_{q2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) The singularities of the robot coincide with the singularities of the Jacobian matrix. For 6-axis industrial robots, the zeros of the Jacobian determinant can be used to infer the singularities. Here we have a Jacobian matrix for a 3-axis manipulator, which is not quadratic, so that no determinant can be determined.

There are no general procedures to determine the points at which such a Jacobian matrix loses rank; however, the literature gives practical hints [M. W. Spong, M. Vidyasagar; "‘Robot Dynamics and Control"’; 1989].

At this point, a possible approach would be the following interpretation of the Jacobian matrix: by considering only the top three rows, which indicate the dependence of the TPC's Cartesian coordinates on the axis angles, the manipulator is used such that the TCP's position is chosen freely and the subsequent orientation is determined by this choice of position (This is <u>one</u> possible interpretation).

Considering only the upper three rows of the Jacobian matrix leads again to a square matrix:

$$\underline{J}_{\underline{p}} = \begin{bmatrix} \underline{z}_0 \times \underline{r}_0 & \underline{z}_1 \times \underline{r}_1 & \underline{z}_2 \end{bmatrix}$$

A determinant for $\underline{J}_{\underline{p}}$ can then be established as:

$$det\left(\underline{J}_{\underline{p}}\right) = -\left(h_2 + h_3 + q_3\right)^2 C_{q2}S_{q2}$$

The zeros of $det\left(\underline{J}_{\underline{p}}\right)$ then correspond to the singularities:

$$q_3 = -\left(h_2 + h_3\right)$$

The zero point would be taken. If the TCP was pulled back to the axis $\underline{z}_1$ by the means of the third axis, then a rotation around axis 2 would no longer have any effect on the position of the TCP and its x and y coordinates could no longer be chosen independently.