Solution

Spatial Representation of Poses

Task 1.

a) A rotation of a point about the Z axis is calculated as follows:

$$P^{'} = \underline{Rot}\left(\underline{z}, 30°\right) \cdot P = \begin{bmatrix} C_{30} & -S_{30} & 0 \\ S_{30} & C_{30} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ 2 \\ 10 \end{pmatrix}$$

b) A rotation of the CS corresponds to a rotation of the point in the other direction:

$$\widetilde{P} = \underline{Rot}\left(\underline{z}, 30°\right)^{-1} \cdot P = \underline{Rot}\left(\underline{z}, -30°\right) \cdot P = \begin{bmatrix} C_{-30} & -S_{-30} & 0 \\ S_{-30} & C_{-30} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ -2 \\ 10 \end{pmatrix}$$

c) A rotation of a point about the X axis is calculated as follows:

$$P^{''} = \underline{Rot}\left(\underline{x}, 60°\right) \cdot P^{'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{60} & -S_{60} \\ 0 & S_{60} & C_{60} \end{bmatrix} \cdot P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{pmatrix} 2\sqrt{3} \\ 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ 1-5\sqrt{3} \\ \sqrt{3}+5 \end{pmatrix}$$

d) Consecutive rotations of a point about different axes are calculated as follows:

$$P^{''} = \underline{R}_{X,Z} \cdot P = \underline{Rot}\left(\underline{x}, 60°\right) \underline{Rot}\left(\underline{z}, 30°\right) \cdot P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{1}{2} \end{bmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ 1-5\sqrt{3} \\ \sqrt{3}+5 \end{pmatrix}$$

e) Be mindful with the order of consecutive rotations about different axes:

$$P^{''} = \underline{R}_{Z,X} \cdot P = \underline{Rot}\left(\underline{z}, 30°\right) \underline{Rot}\left(\underline{x}, 60°\right) \cdot P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{1}{2} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{9\sqrt{3}}{2} \\ -\frac{11}{2} \\ 5 \end{pmatrix}$$

Task 2.

a) Two consecutive rotations about the same axis are generally calculated like this:

$$\underline{R}_1 = \underline{Rot}\left(\underline{z}, \Theta_1\right) \cdot \underline{Rot}\left(\underline{z}, \Theta_2\right) \cdot = \begin{bmatrix} C_{\Theta_1} & -S_{\Theta_1} & 0 \\ S_{\Theta_1} & C_{\Theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_{\Theta_2} & -S_{\Theta_2} & 0 \\ S_{\Theta_2} & C_{\Theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\Theta_1}C_{\Theta_2} - S_{\Theta_1}S_{\Theta_2} & -C_{\Theta_1}S_{\Theta_2} - S_{\Theta_1}C_{\Theta_2} & 0 \\ S_{\Theta_1}C_{\Theta_2} + C_{\Theta_1}S_{\Theta_2} & -S_{\Theta_1}S_{\Theta_2} + C_{\Theta_1}C_{\Theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Two consecutive rotations about the same axis are calculated in reverse order as follows:

$$\underline{R}_2 = \underline{Rot}\left(\underline{z}, \Theta_2\right) \cdot \underline{Rot}\left(\underline{z}, \Theta_1\right) \cdot = \begin{bmatrix} C_{\Theta_2} & -S_{\Theta_2} & 0 \\ S_{\Theta_2} & C_{\Theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_{\Theta_1} & -S_{\Theta_1} & 0 \\ S_{\Theta_1} & C_{\Theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\Theta_2}C_{\Theta_1} - S_{\Theta_2}S_{\Theta_1} & -C_{\Theta_2}S_{\Theta_1} - S_{\Theta_2}C_{\Theta_1} & 0 \\ S_{\Theta_2}C_{\Theta_1} + C_{\Theta_2}S_{\Theta_1} & -S_{\Theta_2}S_{\Theta_1} + C_{\Theta_2}C_{\Theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) If the rotation is repeated about the same axis, the order of application is irrelevant:

$$\underline{R}_1 = \underline{R}_2$$

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d) If it is a repeated rotation about the same axis, the rotation is commutative:

$$\underline{Rot}\left(\underline{z}, \Theta_1 + \Theta_2\right) = \begin{bmatrix} C_{\Theta_1+\Theta_2} & -S_{\Theta_1+\Theta_2} & 0 \\ S_{\Theta_1+\Theta_2} & C_{\Theta_1+\Theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\Theta_1}C_{\Theta_2} - S_{\Theta_1}S_{\Theta_2} & -C_{\Theta_1}S_{\Theta_2} - S_{\Theta_1}C_{\Theta_2} & 0 \\ S_{\Theta_1}C_{\Theta_2} + C_{\Theta_1}S_{\Theta_2} & -S_{\Theta_1}S_{\Theta_2} + C_{\Theta_1}C_{\Theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{R}_1 = \underline{R}_2$$

Task 3.

a) Two consecutive rotations about different axes are generally calculated like this:

$$\underline{R}_1 = \underline{Rot}\left(\underline{z}, \Theta_1\right) \cdot \underline{Rot}\left(\underline{y}, \Theta_2\right) \cdot = \begin{bmatrix} C_{\Theta_1} & -S_{\Theta_1} & 0 \\ S_{\Theta_1} & C_{\Theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_{\Theta_2} & 0 & S_{\Theta_2} \\ 0 & 1 & 0 \\ -S_{\Theta_2} & 0 & C_{\Theta_2} \end{bmatrix}$$

$$= \begin{bmatrix} C_{\Theta_1}C_{\Theta_2} & -S_{\Theta_1} & C_{\Theta_1}S_{\Theta_2} \\ S_{\Theta_1}C_{\Theta_2} & C_{\Theta_1} & S_{\Theta_1}S_{\Theta_2} \\ -S_{\Theta_2} & 0 & C_{\Theta_2} \end{bmatrix}$$

b) Two consecutive rotations about different axes are calculated in reverse order as follows:

$$\underline{R}_2 = \underline{Rot}\left(\underline{y}, \Theta_2\right) \cdot \underline{Rot}\left(\underline{z}, \Theta_1\right) \cdot = \begin{bmatrix} C_{\Theta_2} & 0 & S_{\Theta_2} \\ 0 & 1 & 0 \\ -S_{\Theta_2} & 0 & C_{\Theta_2} \end{bmatrix} \cdot \begin{bmatrix} C_{\Theta_1} & -S_{\Theta_1} & 0 \\ S_{\Theta_1} & C_{\Theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{\Theta_1}C_{\Theta_2} & -S_{\Theta_1}C_{\Theta_2} & S_{\Theta_2} \\ S_{\Theta_1} & C_{\Theta_1} & 0 \\ -C_{\Theta_1}S_{\Theta_2} & S_{\Theta_1}S_{\Theta_2} & C_{\Theta_2} \end{bmatrix}$$

c) If consecutive rotations about different axes are involved,
the order of application must be observed; the rotation is not commutative:

$\underline{R}_1 \neq \underline{R}_2$

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