Solution
Inverse Kinematics

Task 1.

a) In the exercise, it is shown how to extract q_1, q_2, and q_3 from the position components of the forward transformation. However, this solution is incomplete because the orientation of the TCP is ignored. I.e. the solution noted here allows to determine a set of joint angles in order to set a desired Cartesian position for the TCP; however, any orientation is taken.

It is also possible to find solutions for q_1, q_2 and q_3 from the orientation components of the forward transformation. These solutions then have the opposite problem, that only one set of joint angles can be determined with them in order to set a desired Cartesian orientation for the TCP; however, an arbitrary position is assumed.

In practice, both types of solutions must be considered. The aim of this exercise is to convey a feeling of the complexity of such problems in robotics using the position-based backward transformation as an example.

By the forward transformation ${}^0\underline{T}_3$ the position components are given:

$$x = lC_{q_1} + (h_2 + h_3 + q_3)\,S_{q_1}S_{q_2} \tag{1}$$
$$y = lS_{q_1} - (h_2 + h_3 + q_3)\,C_{q_1}S_{q_2} \tag{2}$$
$$z = h_1 + (h_2 + h_3 + q_3)\,C_{q_2} \tag{3}$$

Solving for q_1:

$$(1)\cdot C_{q_1} \Rightarrow \quad xC_{q_1} = l{C_{q_1}}^2 + (h_2 + h_3 + q_3)\,S_{q_1}C_{q_1}S_{q_2} \tag{4}$$
$$(2)\cdot S_{q_1} \Rightarrow \quad yS_{q_1} = l{S_{q_1}}^2 - (h_2 + h_3 + q_3)\,S_{q_1}C_{q_1}S_{q_2} \tag{5}$$
$$(4)+(5) \Rightarrow \quad xC_{q_1} + yS_{q_1} = l \tag{6}$$

Two values, r and ϕ, are defined, which represent x and y in polar coordinates:

$$\left.\begin{array}{l} r = \sqrt{x^2+y^2} \\ \phi = atan2\,(y,x) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = rC_\phi \\ y = rS_\phi \end{array}\right.$$

Plugging into (6) yields:

$$\Rightarrow \quad rC_\phi C_{q_1} + rS_\phi S_{q_1} = l$$

With $C_{\phi\pm q_1} = C_\phi C_{q_1} \mp S_\phi S_{q_1}$ (addition theorem) this results to:

$$\Rightarrow \quad rC_{\phi-q_1} = l$$
$$\Rightarrow \quad C_{\phi-q_1} = \frac{l}{r}$$

With $S_{\phi-q_1} = \pm\sqrt{1-{C_{\phi-q_1}}^2}$ (Pythagoras) follows:

$$\Rightarrow \quad S_{\phi-q_1} = \pm\sqrt{1-{C_{\phi-q_1}}^2} = \pm\sqrt{1-\frac{l^2}{r^2}} = \frac{\pm\sqrt{r^2-l^2}}{r}$$

Using the tangent yields:

$$\begin{aligned}
\Rightarrow \qquad T_{\phi-q_1} &= \frac{S_{\phi-q_1}}{C_{\phi-q_1}} \\
&= \frac{\pm\sqrt{r^2-l^2}}{l} \\
\Rightarrow \qquad \phi - q_1 &= atan2\left(\pm\sqrt{r^2+l^2}, l\right)
\end{aligned}$$

With the above definition of ϕ the expression is solvable for q_1:

$$\boxed{q_1 = atan2\left(y,x\right) - atan2\left(\pm\sqrt{r^2-l^2}, l\right)} \tag{7}$$

To obtain q_2, (3) is first rearranged and then solved using (1) and (2):

$$(3) \Rightarrow \qquad (h_2+h_3+q_3)\,C_{q_2} = z - h_1 \tag{8}$$

$$(1)\cdot S_{q_1}C_{q_2} \Rightarrow \qquad \begin{aligned} xS_{q_1}C_{q_2} &= lC_{q_1}S_{q_1}C_{q_2} + (h_2+h_3+q_3)\,C_{q_2}{S_{q_1}}^2S_{q_2} \\ &= lC_{q_1}S_{q_1}C_{q_2} + (z-h_1)\,{S_{q_1}}^2S_{q_2} \end{aligned} \tag{9}$$

$$(2)\cdot C_{q_1}C_{q_2} \Rightarrow \qquad \begin{aligned} yC_{q_1}C_{q_2} &= lC_{q_1}S_{q_1}C_{q_2} - (h_2+h_3+q_3)\,C_{q_2}{C_{q_1}}^2S_{q_2} \\ &= lC_{q_1}S_{q_1}C_{q_2} - (z-h_1)\,{C_{q_1}}^2S_{q_2} \end{aligned} \tag{10}$$

$$(9)-(10) \Rightarrow \qquad \begin{aligned} xS_{q_1}C_{q_2} - yC_{q_1}C_{q_2} &= (z-h_1)\left({S_{q_1}}^2+{C_{q_1}}^2\right)S_{q_2} \\ &= (z-h_1)\,S_{q_2} \end{aligned} \tag{11}$$

Using the tangent:

$$\Rightarrow \qquad \begin{aligned} T_{q_2} &= \frac{S_{q_2}}{C_{q_2}} \\ &= \frac{xS_{q_1}-yC_{q_1}}{z-h_1} \end{aligned}$$

Then the expression is solvable for q_2:

$$\boxed{q_2 = atan2\left(xS_{q_1}-yC_{q_1}, z-h_1\right)} \tag{12}$$

A solution for q_3 can also be obtained from (1) and (2):

$$(1)\cdot S_{q_1} \Rightarrow \qquad xS_{q_1} = lC_{q_1}S_{q_1} + (h_2+h_3+q_3)\,{S_{q_1}}^2S_{q_2} \tag{13}$$

$$(2)\cdot C_{q_1} \Rightarrow \qquad yC_{q_1} = lC_{q_1}S_{q_1} - (h_2+h_3+q_3)\,{C_{q_1}}^2S_{q_2} \tag{14}$$

$$(13)-(14) \Rightarrow \qquad xS_{q_1} - yC_{q_1} = (h_2+h_3+q_3)\,S_{q_2} \tag{15}$$

$$(15)\cdot S_{q_2} \Rightarrow \qquad xS_{q_1}S_{q_2} - yC_{q_1}S_{q_2} = (h_2+h_3+q_3)\,{S_{q_2}}^2 \tag{16}$$

$$(3)\cdot C_{q_2} \Rightarrow \qquad zC_{q_2} = h_1C_{q_2} + (h_2+h_3+q_3)\,{C_{q_2}}^2 \tag{17}$$

$$(16)+(17) \Rightarrow \qquad xS_{q_1}S_{q_2} - yC_{q_1}S_{q_2} + zC_{q_2} = h_1C_{q_2} + (h_2+h_3+q_3) \tag{18}$$

This expression is solvable for q_3:

$$\boxed{q_3 = xS_{q_1}S_{q_2} - yC_{q_1}S_{q_2} + zC_{q_2} - h_1C_{q_2} - h_2 - h_3} \tag{19}$$