Solution
Introduction

Task 1. Trigonometry

a) With the sum of the squares of sine and cosine ($\sin^2 + \cos^2 = 1$) and the binomial formula ($a^2 + 2ab + b^2 = (a+b)^2$):

$$\begin{aligned}
& & 6\cos^2(x) - \sin(x) - 4 &= 0 \\
&\Rightarrow & 6(1 - \sin^2(x)) - \sin(x) - 4 &= 0 \\
&\Rightarrow & 6\sin^2(x) + \sin(x) - 2 &= 0 \\
&\Rightarrow & \sin^2(x) + (2\frac{1}{12})\sin(x) - \frac{2}{6} &= 0 \\
&\Rightarrow & \sin(x)_{1,2} &= +\frac{1}{2} \vee -\frac{2}{3} \\
&\Rightarrow & x &= \left\{\frac{\pi}{6}, \frac{5\pi}{6}, 3.871, 5.553\right\}
\end{aligned}$$

b) With addition theorem ($\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$) and the sum of the squares of sine and cosine ($\sin^2 + \cos^2 = 1$):

$$\begin{aligned}
& & \sin^2(2x) - 2\sin^2(x) + \cos(4x) - 2\cos^2(x) + 1 &= 0 \\
&\Rightarrow & \sin^2(2x) - 2 + \cos(4x) + 1 &= 0 \\
&\Rightarrow & \sin^2(2x) + \cos^2(2x) - \sin^2(2x) - 1 &= 0 \\
&\Rightarrow & \cos^2(2x) - 1 &= 0 \\
&\Rightarrow & \sin^2(2x) &= 0 \\
&\Rightarrow & x &= \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}
\end{aligned}$$

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Task 2. Vectors and Matrices

a) Vector algebra

1. Vector lengths

$$|\underline{v}_1| = \sqrt{59}$$
$$|\underline{v}_2| = \sqrt{30}$$

2. normalised vectors

$$||\underline{v}_1|| = \begin{pmatrix} \frac{1}{\sqrt{59}} \\ \\ \frac{3}{\sqrt{59}} \\ \\ \frac{7}{\sqrt{59}} \end{pmatrix}$$
$$||\underline{v}_2|| = \begin{pmatrix} \frac{2}{\sqrt{30}} \\ \\ \frac{-5}{\sqrt{30}} \\ \\ \frac{1}{\sqrt{30}} \end{pmatrix}$$

3. Dot (inner), vector and cross (outer) product

$$3\underline{v}_1 + 2\underline{v}_2 = \begin{pmatrix} 7 \\ -1 \\ 23 \end{pmatrix}$$
$$\underline{v}_1 \cdot \underline{v}_2 = -6$$
$$\underline{v}_1 \times \underline{v}_2 = \begin{pmatrix} 38 \\ 13 \\ -11 \end{pmatrix}$$

4. Orthogonality

$$(\underline{v}_1 \times \underline{v}_2) \perp \underline{v}_1 \Rightarrow (\underline{v}_1 \times \underline{v}_2) \cdot \underline{v}_1 = 0$$
$$(\underline{v}_1 \times \underline{v}_2) \perp \underline{v}_2 \Rightarrow (\underline{v}_1 \times \underline{v}_2) \cdot \underline{v}_2 = 0$$

5. Angle between vectors

$$\underline{v}_1 \cdot \underline{v}_2 = |\underline{v}_1|\,|\underline{v}_2| \cos(\Theta)$$
$$\Rightarrow \qquad \Theta = \cos^{-1}\left(\frac{\underline{v}_1 \cdot \underline{v}_2}{|\underline{v}_1|\,|\underline{v}_2|}\right)$$
$$\Rightarrow \qquad \Theta = 1.714\,rad = 98.2\,deg$$

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b) Matrix algebra

1. Determinant of a matrix

$$|\underline{A}| = -1$$

2. Transpose of a matrix

$$\underline{A}^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

3. Inverse of a matrix

$$\underline{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \underline{A}^T$$

4. Orthonormality $\underline{A}$ is an orthonormal matrix, if the equivalence of transpose and inverse holds.

Task 3. Laplace Transform

- Definition: $F(s) = \int_0^\infty f(t)e^{-st}dt$

a) Laplace Transforms

1. with integration by parts
   - $\int_0^\infty u(t)v'(t)dt = [u(t)v(t)]_0^\infty - \int_0^\infty u'(t)v(t)dt$
   - $\frac{d}{dt}\left(\frac{-1}{s}e^{-st}\right) = e^{-st}$
   - $u(t) := t$
   - $v'(t) := e^{-st}$
   - $\lim_{t\to\infty} e^{-st} = 0$
   - $\lim_{t\to 0} e^{-st} = 1$

$$\begin{aligned}
& \int_0^\infty te^{-st}dt = \left[t \cdot \frac{-1}{s}e^{-st}\right]_0^\infty - \int_0^\infty 1 \cdot \frac{-1}{s}e^{-st}dt \\
\Rightarrow \quad & = [0-0] + \frac{1}{s}\left[\frac{-1}{s}e^{-st}\right]_0^\infty \\
\Rightarrow \quad & = -\frac{1}{s^2}\left[e^{-st}\right]_0^\infty \\
\Rightarrow \quad & F(s) = \frac{1}{s^2}
\end{aligned}$$

2. by multiplication of exponential functions

$$\begin{aligned}
& \int_0^\infty e^{-\alpha t}e^{-st}dt = \int_0^\infty e^{-(s+\alpha)t}dt \\
\Rightarrow \quad & = \left[-\frac{1}{s+\alpha}e^{-(s+\alpha)t}\right]_0^\infty \\
\Rightarrow \quad & = -\frac{1}{s+\alpha}\left[e^{-(s+\alpha)t}\right]_0^\infty \\
\Rightarrow \quad & F(s) = \frac{1}{s+\alpha}
\end{aligned}$$

3. via the conversion of trigonometric functions
   - $sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$

$$\begin{aligned}
& \int_0^\infty e^{-\alpha t}\sin(\omega t)e^{-st}dt = \int_0^\infty e^{-\alpha t}\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt \\
\Rightarrow \quad & = \frac{1}{2j}\int_0^\infty e^{-(s+\alpha-j\omega)t}dt - \frac{1}{2j}\int_0^\infty e^{-(s+\alpha+j\omega)t}dt \\
\Rightarrow \quad & = \frac{1}{2j}\left[\frac{-1}{s+\alpha-j\omega}e^{-(s+\alpha-j\omega)t}\right]_0^\infty - \frac{1}{2j}\left[\frac{-1}{s+\alpha+j\omega}e^{-(s+\alpha+j\omega)t}\right]_0^\infty \\
\Rightarrow \quad & = \frac{1}{2j}\left[\frac{1}{s+\alpha-j\omega} - \frac{1}{s+\alpha+j\omega}\right] \\
\Rightarrow \quad & = \frac{1}{2j}\left[\frac{2j\omega}{s^2+2\alpha s+\alpha^2+\omega^2}\right] \\
\Rightarrow \quad & F(s) = \frac{\omega}{(s+\alpha)^2+\omega^2}
\end{aligned}$$

b) inverse Laplace transforms

1. from correspondence tables ($\frac{1}{(s-\alpha)^2} \leftrightarrow te^{\alpha t}$)

$$
\begin{aligned}
&& F(s) &= \frac{4}{s^2+6s+9} \\
\Rightarrow && &= \frac{4}{(s+3)^2} \\
\Rightarrow && f(t) &= 4te^{-3t}
\end{aligned}
$$

2. from correspondence tables ($\sin(\alpha t) \leftrightarrow \frac{\alpha}{s^2+\alpha^2}$)

$$
\begin{aligned}
&& F(s) &= \frac{4}{s^2+8} \\
\Rightarrow && &= \frac{4\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}}}{s^2+(\sqrt{8})^2} \\
\Rightarrow && f(t) &= \sqrt{2}\sin(\sqrt{8}t)
\end{aligned}
$$

3. via partial fraction decomposition

$$
\begin{aligned}
&& F(s) &= \frac{10s+8}{s(s^2+3s+2)} \\
\Rightarrow && &= \frac{10s+8}{s(s+1)(s+2)} \\
\Rightarrow && &= \frac{4}{s} + \frac{2}{s+1} - \frac{6}{s+2} \\
\Rightarrow && f(t) &= 4 + 2e^{-t} - 6e^{-2t}
\end{aligned}
$$

c) Differential equations

- $f(t) \leftrightarrow F(s)$
- $\dot{f}(t) \leftrightarrow sF(s) - f(0)$
- $\ddot{f}(t) \leftrightarrow s^2F(s) - sf(0) - \dot{f}(0)$

1. Calculation of a DE reaction to an excitation Transformation of the DE with initial value into the Laplace domain:

$$\begin{aligned}
&f(t) : \ddot{y}(t) + 2\dot{y}(t) + 2y(t) &&= 5\sin(t) \\
\Rightarrow \quad &F(s) : \left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + 2\left[sY(s) - y(0)\right] + 2\left[Y(s)\right] &&= \frac{5}{s^2+1}
\end{aligned}$$

Solve for $Y(s)$:

$$\begin{aligned}
\Rightarrow \quad s^2Y(s) + 2s + 2sY(s) + 4 + 2Y(s) &= \frac{5}{s^2+1} \\
\Rightarrow \quad \left(s^2 + 2s + 2\right) Y(s) &= \frac{5}{s^2+1} - 2s - 4 \\
\Rightarrow \quad Y(s) &= \frac{5}{(s^2+1)(s^2+2s+2)} - \frac{2s+4}{s^2+2s+2} \\
\Rightarrow \quad &= \frac{-2s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{(s+1)^2+1}
\end{aligned}$$

Inverse transform of the reaction to the excitation back into time domain:

$$y(t) = -2\cos(t) + \sin(t) - e^{-t}\sin(t)$$

2. Calculation of a DE system reaction to an excitation Transformation of the DEs with initial value into the Laplace domain:

$$\begin{aligned}
&f_1(t) : \dot{x}(t) + 2\dot{y}(t) - 3y(t) &&= 3e^t \\
\Rightarrow \quad &F_1(s) : [sX(s) - x(0)] + 2\,[sY(s) - y(0)] - 3\,[Y(s)] &&= \frac{3}{s-1} \\
&f_2(t) : \dot{x}(t) - \dot{y}(t) - 6x(t) &&= 6 \\
\Rightarrow \quad &F_2(s) : [sX(s) - x(0)] - [sY(s) - y(0)] - 6\,[X(s)] &&= \frac{6}{s}
\end{aligned}$$

Solve the equation system for $X(s)$, $Y(s)$:

$$\begin{aligned}
\Rightarrow \quad Y(s) &= \frac{-(s+4)}{(s-1)(s-2)(s-3)} \\
\Rightarrow \quad &= -\frac{\frac{5}{2}}{s-1} + \frac{6}{s-2} - \frac{\frac{7}{2}}{s-3} \\
\Rightarrow \quad X(s) &= \frac{3}{s(s-1)} - \frac{3(s+4)}{s(s-1)(s-2)(s-3)} + \frac{2(s+4)}{(s-1)(s-2)(s-3)} \\
\Rightarrow \quad &= \frac{5s^2 - 10s + 6}{s(s-1)(s-2)(s-3)} \\
\Rightarrow \quad &= -\frac{1}{s} + \frac{\frac{1}{2}}{s-1} - \frac{3}{s-2} + \frac{\frac{7}{2}}{s-3}
\end{aligned}$$

Inverse transform of the reactions to the excitation back into time domain:

$$\begin{aligned}
y(t) &= -\frac{5}{2}e^t + 6e^{2t} - \frac{7}{2}e^{3t} \\
x(t) &= -1 + \frac{1}{2}e^t - 3e^{2t} + \frac{7}{2}e^{3t}
\end{aligned}$$