Solution
Velocity Profile

Velocity profile A velocity profile specifies the target velocity as a function of time and is used in robot motion control. The velocity profile is usually specified as a third-order polynomial. It is divided into three phases:

- *Acceleration phase:* Acceleration from v_i to v_{max}
- *Plateau phase:* Constant velocity of v_{max}
- *Deceleration phase:* Slowing down (negative acceleration) from v_{max} to v_f

Task 1.
The velocity is given as a third order polynomial:

$$v(t) = c_3t^3 + c_2t^2 + c_1t + c_0 \tag{1}$$

Derivation leads to the acceleration:

$$a(t) = 3c_3t^2 + 2c_2t + c_1 \tag{2}$$

a) Sketch of the velocity profile:

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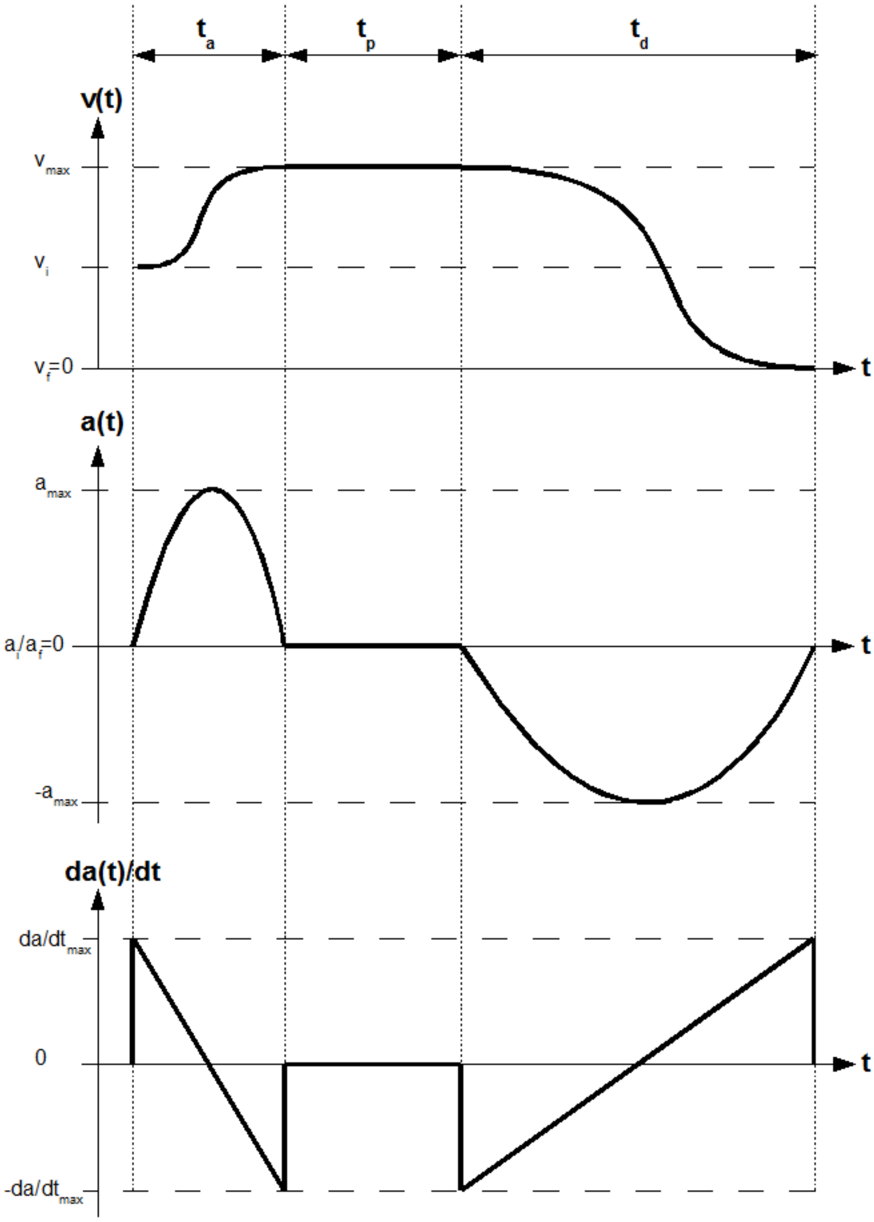

t a
t p
t d
v(t)
v max
v i
v f=0
t
a(t)
a max
a i/a f=0
t
-a max
da(t)/dt
da/dt max
0
t
-da/dt max

b) The distance s_{total} of the movement can be calculated from the given values of the movement:

$$s_{total} = |\overrightarrow{p_f} - \overrightarrow{p_i}| \tag{3}$$
$$s_{total} = \left((p_{fx} - p_{ix})^2 + (p_{fy} - p_{iy})^2 + (p_{fz} - p_{iz})^2\right)^{\frac{1}{2}} \tag{4}$$
$$s_{total} = 1215mm \tag{5}$$

Furthermore, the distance is the sum of the integrals of the velocities:

$$s_{total} = \int_0^{t_a} v_a(t)dt + \int_0^{t_p} v_p(t)dt + \int_0^{t_d} v_d(t)dt \tag{6}$$

c) Velocity functions

- Acceleration phase: As indicated in (6), the key time points of the acceleration phase are 0 and t_a. From the given data and the above formulas (1) and (2) of the polynomials, the following can be calculated:

$$v_a(0) = v_i = 1400 \tag{7}$$
$$v_a(0) = c_3 \cdot 0 + c_2 \cdot 0 + c_1 \cdot 0 + c_0 = 1400 \tag{8}$$
$$\Rightarrow c_0 = 1400 \tag{9}$$

$$a_a(0) = a_i = 0 \tag{10}$$
$$a_a(0) = 3c_3 \cdot 0 + 2c_2 \cdot 0 + c_1 = 0 \tag{11}$$
$$\Rightarrow c_1 = 0 \tag{12}$$

By definition, at the end of the acceleration phase, the acceleration is zero.

$$a_a(t_a) = 0 \tag{13}$$
$$a_a(t_a) = 3c_3t_a^2 + 2c_2t_a + 0 = 0 \tag{14}$$
$$\Rightarrow c_2 = -\frac{3}{2}c_3t_a \tag{15}$$

At the end of the acceleration phase, the maximum velocity is reached.

$$v_a(t_a) = v_{max} = 1800 \tag{16}$$
$$v_a(t_a) = c_3t_a^3 - \frac{3}{2}c_3t_a \cdot t_a^2 + 0 \cdot t_a + 1400 = 1800 \tag{17}$$
$$c_3t_a^3 - \frac{3}{2}c_3t_a^3 = 1800 - 1400 \Leftrightarrow -\frac{1}{2}c_3t_a^3 = 400 \tag{18}$$
$$\Rightarrow c_3 = -\frac{800}{t_a^3} \tag{19}$$

Thus, velocity within the acceleration phase is given as:

$$c_2 = -\frac{3}{2}c_3t_a = \frac{1200}{t_a^2} \tag{20}$$
$$v_a(t) = \left(-\frac{800}{t_a^3}\right)t^3 + \left(\frac{1200}{t_a^2}\right)t^2 + 0 + 1400 \tag{21}$$

- Plateau phase: By definition:

$$v_p(t) = v_{max} \tag{22}$$

- Deceleration phase : As given in (6), key time points of the deceleration phase are 0 and t_d. From the given data and the above formulas (1) and (2) of the polynomials, the following can be calculated:

$$v_d(0) = v_{max} = 1800 \tag{23}$$
$$v_d(0) = c_3 \cdot 0 + c_2 \cdot 0 + c_1 \cdot 0 + c_0 = 1800 \tag{24}$$
$$\Rightarrow c_0 = 1800 \tag{25}$$

By definition, at the beginning of the deceleration phase, the acceleration is zero:

$$a_d(0) = 0 \tag{26}$$
$$a_d(0) = 3c_3 \cdot 0 + 2c_2 \cdot 0 + c_1 = 0 \tag{27}$$
$$\Rightarrow c_1 = 0 \tag{28}$$

Likewise, acceleration is zero at its end:

$$a_d(t_d) = a_f = 0 \tag{29}$$
$$a_d(t_d) = 3c_3 t_d^2 + 2c_2 t_d + 0 = 0 \tag{30}$$
$$\Rightarrow c_2 = -\frac{3}{2} c_3 t_d \tag{31}$$

At the end of the deceleration phase, velocity reaches the final velocity v_f:

$$v_d(t_d) = v_f = 0 \tag{32}$$
$$v_d(t_d) = c_3 t_d^3 - \frac{3}{2} c_3 t_d \cdot t_d^2 + 0 \cdot t_d + 1800 = 0 \tag{33}$$
$$-\frac{1}{2} c_3 t_d^3 = -1800 \tag{34}$$
$$\Rightarrow c_3 = \frac{3600}{t_d^3} \tag{35}$$

Thus, velocity within the deceleration phase is given as:

$$c_2 = -\frac{3}{2} c_3 t_d = -\frac{5400}{t_d^2} \tag{36}$$
$$v_d(t) = \left(\frac{3600}{t_d^3}\right) t^3 + \left(-\frac{5400}{t_d^2}\right) t^2 + 0 + 1800 \tag{37}$$

d) Movement duration of the phases:

- Acceleration phase: Maximum acceleration is reached at the midpoint of the acceleration phase:

$$a_a\left(\frac{t_a}{2}\right) = a_{max} = 3000 \tag{38}$$
$$a_a\left(\frac{t_a}{2}\right) = 3c_3\left(\frac{t_a}{2}\right)^2 + 2c_2 \frac{t_a}{2} + c_1 = 3000 \tag{39}$$
$$\tag{40}$$

Substituting by (20), (19) and (12) yields:

$$3\left(-\frac{800}{t_a^3}\right)\left(\frac{t_a}{2}\right)^2 + 2\left(\frac{1200}{t_a^2}\right)\frac{t_a}{2} + 0 = 3000 \tag{41}$$
$$-\frac{600}{t_a} + \frac{1200}{t_a} = 3000 \tag{42}$$
$$\frac{t_a}{600} = \frac{1}{3000} \tag{43}$$
$$\Rightarrow t_a = 0.2s = 200ms \tag{44}$$

- Deceleration phase: Maximum deceleration is reached at the midpoint of the deceleration phase:

$$a_d\left(\frac{t_d}{2}\right) = -a_{max} = -3000 \tag{45}$$
$$a_d\left(\frac{t_d}{2}\right) = 3c_3\left(\frac{t_d}{2}\right)^2 + 2c_2 \frac{t_d}{2} + c_1 = -3000 \tag{46}$$
$$\tag{47}$$

Substituting by (36), (35) and (28) yields:

$$3\left(\frac{3600}{t_d^3}\right)\left(\frac{t_d}{2}\right)^2 + 2\left(-\frac{5400}{t_d^2}\right)\frac{t_d}{2} + 0 = -3000 \tag{48}$$

$$\frac{2700}{t_d} - \frac{5400}{t_d} = -3000 \tag{49}$$

$$-\frac{t_d}{2700} = -\frac{1}{3000} \tag{50}$$

$$\Rightarrow t_d = 0.9s = 900ms \tag{51}$$

- Plateau phase: By substituting (22), (6) can be simplified to:

$$s_{total} = \int_0^{t_a} v_a(t)dt + v_{max}t_p + \int_0^{t_d} v_d(t)dt \tag{52}$$

$$\Rightarrow t_p = \frac{1}{v_{max}}\left(s_{total} - \int_0^{t_a} v_a(t)dt - \int_0^{t_d} v_d(t)dt\right) \tag{53}$$

$$t_p = \frac{1}{v_{max}}\left(s_{total} - s_a - s_d\right) \tag{54}$$

By substituting (44) in (21)

$$v_a(t) = \left(-\frac{800}{0.2^3}\right)t^3 + \left(\frac{1200}{0.2^2}\right)t^2 + 0 + 1400 \tag{55}$$

$$v_a(t) = \left(-\frac{800}{0.008}\right)t^3 + \left(\frac{1200}{0.04}\right)t^2 + 1400 \tag{56}$$

$$v_a(t) = -1\cdot 10^5 t^3 + 3\cdot 10^4 t^2 + 1.4\cdot 10^3 \tag{57}$$

This yields:

$$s_a = \int_0^{t_a} v_a(t)dt \tag{58}$$

$$s_a = \int_0^{t_a} \left(-1\cdot 10^5 t^3 + 3\cdot 10^4 t^2 + 1.4\cdot 10^3\right)dt \tag{59}$$

$$s_a = \left[-2.5\cdot 10^4 t^4 + 1\cdot 10^4 t^3 + 1.4\cdot 10^3 t\right]_0^{t_a} \tag{60}$$

$$s_a = -2.5\cdot 10^4\cdot 0.2^4 + 1\cdot 10^4\cdot 0.2^3 + 1.4\cdot 10^3\cdot 0.2 - 0 \tag{61}$$

$$s_a = -40 + 80 + 280 \tag{62}$$

$$s_a = 320mm \tag{63}$$

By substituting (51) in (37):

$$v_d(t) = \left(\frac{3600}{0.9^3}\right)t^3 + \left(-\frac{5400}{0.9^2}\right)t^2 + 0 + 1800 \tag{64}$$

$$v_d(t) = \left(\frac{3600}{0.729}\right)t^3 + \left(-\frac{5400}{0.81}\right)t^2 + 1800 \tag{65}$$

$$v_d(t) = 5\cdot 10^3 t^3 + -6.7\cdot 10^3 t^2 + 1.8\cdot 10^3 \tag{66}$$

This yields:

$$s_d = \int_0^{t_d} v_d(t)dt \tag{67}$$

$$s_d = \int_0^{t_d} \left(5\cdot 10^3 t^3 + -6.7\cdot 10^3 t^2 + 1.8\cdot 10^3\right)dt \tag{68}$$

$$s_d = \left[1.25\cdot 10^3 t^4 - 2.23\cdot 10^3 t^3 + 1.8\cdot 10^3 t\right]_0^{t_d} \tag{69}$$

$$s_d = 1.25\cdot 10^3\cdot 0.9^4 - 2.23\cdot 10^3\cdot 0.9^3 + 1.8\cdot 10^3\cdot 0.9 - 0 \tag{70}$$

$$s_d = 820 - 1626 + 1620 \tag{71}$$

$$s_d = 814mm \tag{72}$$

By substituting (5), (63) and (72) in (54):

$$t_p = \frac{1}{1800}(1215 - 320 - 814) \tag{73}$$

$$t_p = \frac{81}{1800} \tag{74}$$

$$t_p = 0.045s = 45ms \tag{75}$$

e) Total velocity function (velocity profile)

- Acceleration phase

$$t_{start} = t_i = 0ms \tag{76}$$

$$t_{end} = t_a = 200ms \tag{77}$$

- Plateau phase

$$t_{start} = t_a = 200ms \tag{78}$$

$$t_{end} = t_a + t_p = 245ms \tag{79}$$

- Deceleration phase

$$t_{start} = t_a + t_p = 245ms \tag{80}$$

$$t_{end} = t_a + t_p + t_d = 1145ms \tag{81}$$

Since the velocity function of the deceleration phase was calculated in the interval between 0 and t_d, a correction must be made for the overall profile so that:

$$t' = t - 245 \tag{82}$$

- Total

$$v(t) = \begin{cases} 0 & t < 0ms \\ -1 \cdot 10^5 \frac{mm}{s^4} t^3 + 3 \cdot 10^4 \frac{mm}{s^3} t^2 + 1.4 \cdot 10^3 \frac{mm}{s} & 0ms \leq t < 200ms \\ 1800 \frac{mm}{s} & 200ms \leq t < 245ms \\ 5 \cdot 10^3 \frac{mm}{s^4} t'^3 - 6.7 \cdot 10^3 \frac{mm}{s^3} t'^2 + 1.8 \cdot 10^3 \frac{mm}{s} & 245ms \leq t < 1145ms, \text{ with } t' = t - 245ms \\ 0 & t \leq 1145ms \end{cases} \tag{83}$$

Task 2.

a) Polynom coefficients

$$\omega_a(t) = c_3t^3 + c_2t^2 + c_1t + c_0 \tag{84}$$
$$\dot{\omega}_a(t) = a_a(t) = 3c_3t^2 + 2c_2t + c_1 \tag{85}$$

At the beginning of the acceleration phase, for speed and acceleration apply:

$$\omega_0 = c_0 = 0 \tag{86}$$
$$a_a = c_1 = 0 \tag{87}$$

At the fitted time t_a, the throttled maximum speed ω_{max} is reached:

$$\omega_a(t_a) = c_3t_a^3 + c_2t_a^2 = \omega_{max} \tag{88}$$

At the fitted time t_a the acceleration phase is finished:

$$a_a(t_a) = 3c_3t_a^2 + 2c_2t_a = 0 \tag{89}$$

From conditions (88) and (89) for t_a, it follows for the coefficients:

$$c_3 = -\frac{2\omega_{max}}{t_a^3} \tag{90}$$
$$c_2 = \frac{3\omega_{max}}{t_a^2} \tag{91}$$

b) Duration of acceleration phase
At the midpoint of the fitted acceleration phase from (85) at $t = \frac{t_a}{2}$, the max acceleration $a_{max} = 100$ is reached:

$$a_a(\frac{t_a}{2}) = 3c_3(\frac{t_a}{2})^2 + 2c_2(\frac{t_a}{2}) = 100 \tag{92}$$
$$3(\frac{-2\omega_{max}}{t_a^3})(\frac{t_a^2}{4}) + 2(\frac{3\omega_{max}}{t_a^2})(\frac{t_a}{2}) = 100 \tag{93}$$
$$\frac{-6\omega_{max}}{4}\frac{1}{t_a} + \frac{6\omega_{max}}{2}\frac{1}{t_a} = 100 \tag{94}$$
$$\Rightarrow t_a = \frac{3}{200}\omega_{max} \tag{95}$$

c) Traversed angle in acceleration phase ϕ_a:

$$\phi_a = \int_0^{t_a} \omega_a(t)dt = \int_0^{t_a} c_3t_a^3 + c_2t_a^2 dt \tag{96}$$
$$\phi_a = \left[\frac{c_3}{4}t^4 + \frac{c_2}{3}t^3\right]_0^{t_a} = \frac{c_3}{4}t_a^4 + \frac{c_2}{3}t_a^3 \tag{97}$$

Substitute c_3 from (90) and c_2 from (91)

$$\phi_a = \frac{-2\omega_{max}}{4}\frac{t_a^4}{t_a^3} + \frac{3\omega_{max}}{3}\frac{t_a^3}{t_a^2} \tag{98}$$
$$\tag{99}$$

substitute t_a from (95)

$$\phi_a = \frac{1}{2}\omega_{max}t_a \tag{100}$$
$$\Rightarrow \phi_a = \frac{3}{400}\omega_{max}^2 \tag{101}$$

d) Duration of plateau phase
Exploiting the symmetry of acceleration and deceleration phase, for the fitted total duration t_{total} applies:

$$t_{total} = 2t_a + t_p = 0.5 \tag{102}$$
$$\Rightarrow t_p = 0.5 - 2t_a \tag{103}$$
$$\tag{104}$$

Analogously, for the traversed angle applies:

$$\phi_{total} = 2\phi_a + \phi_p = 3 \tag{105}$$
$$\Rightarrow \phi_p = 3 - 2\phi_a \tag{106}$$

The fitted duration t_p can then be expressed by:

$$t_p = \frac{\phi_p}{\omega_{max}} \tag{107}$$
$$\tag{108}$$

By substituting for ϕ_p by (106)

$$t_p = \frac{3 - 2\phi_a}{\omega_{max}} \tag{109}$$
$$t_p = \frac{3 - \frac{3}{200}\omega_{max}^2}{\omega_{max}} \tag{110}$$
$$t_p = \frac{3}{\omega_{max}} - \frac{3}{200}\omega_{max} \tag{111}$$

e) Maximum velocity based on the fitted total duration t_{total}:

$$t_{total} = 2t_a + t_p = 0.5 \tag{112}$$
$$\frac{3}{100}\omega_{max} + \frac{3}{\omega_{max}} - \frac{3}{200}\omega_{max} = \frac{1}{2} \tag{113}$$
$$6\omega_{max} + \frac{600}{\omega_{max}} - 3\omega_{max} = 100 \tag{114}$$
$$3\omega_{max} + \frac{600}{\omega_{max}} = 100 \tag{115}$$
$$\omega_{max}^2 - \frac{100}{3}\omega_{max} = -200 \tag{116}$$

Turn into polynomial of type $(a-b)^2 = a^2 - 2ab + b^2$

$$\omega_{max}^2 - \frac{100}{3}\omega_{max} + \left(\frac{50}{3}\right)^2 = -200 + \left(\frac{50}{3}\right)^2 \tag{117}$$
$$\left(\omega_{max} - \frac{50}{3}\right)^2 = -200 + \left(\frac{50}{3}\right)^2 \tag{118}$$
$$\omega_{max} = \pm\sqrt{77.78} + \frac{50}{3} \tag{119}$$
$$\Rightarrow \omega_{max} = 7.85\frac{rad}{s} \tag{120}$$